Abelian Categories

Daniel Murfet

October 5, 2006

Abelian categories are the most general category in which one can develop homological algebra. The idea and the name "abelian category" were first introduced by MacLane [Mac50], but the modern axiomitisation and first substantial applications were given by Grothendieck in his famous Tohoku paper [Gro57]. This paper was motivated by the needs of algebraic geometry, where the category of sheaves over a scheme are a central example of an abelian category. Although the purpose of this note is mainly to fix the background on abelian categories needed in our notes on algebraic geometry, we take some time to give the foundations of category theory in some detail. For a full introduction to the subject see [Bor94], [Sch72], [Mit65], [Ste75].

Contents

1 Categories

In our approach to the subject, there are two "stages" in the definition of a category. First, we define a category as an algebraic object consisting of sets and maps satisfying some properties. In this stage a category is something akin to an abelian group, and the only set-theoretic background required is standard ZFC. It is only in the second stage that we introduce grothendieck universes and prepare ourselves for constructions like the "category of all sets".

Definition 1. A *category* is an ordered tuple $\mathcal{C} = (O, M, d, c, \circ)$ consisting of a set O of *objects*, a set M of *morphisms*, and two functions $d, c : M \longrightarrow O$ called the *domain* and *codomain* functions respectively. If f is a morphism with $d(f) = A, c(f) = B$ then we write $f : A \longrightarrow B$. If we define $D = \{(f, g) \,|\, f, g \in M \text{ and } d(f) = c(g)\}$ then the final piece of data is a function $\circ : D \longrightarrow M$, called the *composition law*. We require that the following conditions be satisfied:

1. *Matching Condition*: If $f \circ g$ is defined, then $d(f \circ g) = d(g)$ and $c(f \circ g) = c(f)$.
2. *Associativity Condition*: If $f \circ g$ and $h \circ f$ are defined, then $h \circ (f \circ g) = (h \circ f) \circ g$.
3. *Identity Existence Condition*: For each object A there exists a morphism e such that $d(e) = A = c(e)$ and
   (a) $f \circ e = f$ whenever $f \circ e$ is defined, and
   (b) $e \circ g = g$ whenever $e \circ g$ is defined.

Definition 2. Let $(O, M, c, d, \circ)$ be a category. The *opposite category* of $\mathcal{C}$ is the tuple $(O, M, c, d, \circ')$ where $\circ'$ maps a pair (g, f) to $f \circ g$.

Definition 3. Let $\mathcal{C} = (O, M, d, c, \circ), \mathcal{D} = (O', M', d', c', \circ')$ be categories. A *covariant functor* $F : \mathcal{C} \longrightarrow \mathcal{D}$ is an ordered tuple $F = (\varphi, \psi)$ consisting of functions $\varphi : O \longrightarrow O'$ and $\psi : M \longrightarrow M'$ satisfying the following conditions

1. If $g : A \longrightarrow A'$ then $\psi(g) : \varphi(A) \longrightarrow \varphi(A')$.
2. If $f \circ g$ is defined in $\mathcal{C}$ then $\psi(f \circ g) = \psi(f) \circ' \psi(g)$.
3. For each $A \in O$ we have $\psi(1_A) = 1_{\varphi(A)}$.

A natural transformation $\varphi : F \longrightarrow G$ between covariant functors is a function $O \longrightarrow M'$ which assigns to every object A of O a morphism $\varphi_A : F(A) \longrightarrow G(A)$ such that for $f : A \longrightarrow B$ in M we have $G(f)\varphi_A = \varphi_B F(f)$.

Definition 4. Let $\mathcal{C} = (O, M, d, c, \circ), \mathcal{D} = (O', M', d', c', \circ')$ be categories. A *contravariant functor* $F : \mathcal{C} \longrightarrow \mathcal{D}$ is an ordered tuple $F = (\varphi, \psi)$ consisting of functions $\varphi : O \longrightarrow O'$ and $\psi : M \longrightarrow M'$ satisfying the following conditions

1. If $g : A \longrightarrow A'$ then $\psi(g) : \varphi(A') \longrightarrow \varphi(A)$.
2. If $f \circ g$ is defined in $\mathcal{C}$ then $\psi(f \circ g) = \psi(g) \circ' \psi(f)$.
3. For each $A \in O$ we have $\psi(1_A) = 1_{\varphi(A)}$.

A natural transformation $\varphi : F \longrightarrow G$ between contravariant functors is a function $O \longrightarrow M'$ which assigns to every object A of O a morphism $\varphi_A : F(A) \longrightarrow G(A)$ such that for $f : A \longrightarrow B$ in M we have $G(f)\varphi_B = \varphi_A F(f)$.

Definition 5. Let $\mathcal{C}, \mathcal{D}$ be categories. We denote by $[\mathcal{C}, \mathcal{D}]$ the category of all covariant functors $\mathcal{C} \longrightarrow \mathcal{D}$ with natural transformations as morphisms.

Note that at the level of sets, a contravariant functor $\mathcal{A} \longrightarrow \mathcal{B}$ is the *same thing* as a covariant functor $\mathcal{A}^{\mathrm{op}} \longrightarrow \mathcal{B}$. If $F, G : \mathcal{A} \longrightarrow \mathcal{B}$ are contravariant functors then a function $O \longrightarrow M'$ is a natural transformation $F \longrightarrow G$ if and only if it is a natural transformation of covariant functors $\mathcal{A}^{\mathrm{op}} \longrightarrow \mathcal{B}$. Therefore the categories of covariant functors $\mathcal{A}^{\mathrm{op}} \longrightarrow \mathcal{B}$ and contravariant functors $\mathcal{A} \longrightarrow \mathcal{B}$ are the *same categories* in a strict set-theoretic sense.

For the definition of a grothendieck universe, $\mathfrak{U}$-set and $\mathfrak{U}$-class, see (FCT,Section 4). In the next definition we introduce the concept of a $\mathfrak{U}$-category. This terminology also appears in SGA4, but means something different there. We explain in (FCT,Section 4) why we have chosen to adopt a different definition.

Definition 6. Let $\mathfrak{U}$ be a grothendieck universe and $\mathcal{C} = (O, M, d, c, \circ)$ a category. We call $\mathcal{C}$ a $\mathfrak{U}$-*category* if O, M are $\mathfrak{U}$-classes and if for every pair of objects A, B of $\mathcal{C}$ the set $Hom_{\mathcal{C}}(A, B)$ is a $\mathfrak{U}$-set.

Throughout our notes we work with the logical foundation given by the first order theory ZFCU, as described in (FCT,Section 4). We fix a universe $\mathscr{U}$ (which is always assumed to contain $\mathbb{N}$) and work with the notation of the conglomerate convention (CC) for $\mathscr{U}$ unless there is some indication to the contrary. Under this convention, the term "category" will always mean a $\mathfrak{U}$-category and the term "portly category" will always mean a category in the sense of Definition 1. The CC fixes the concepts of *set, class* and *conglomerate* which we will use freely. Just to be clear, let us restate the definition of a category (that is, a $\mathfrak{U}$-category) using the new language.

Definition 7. A *category* is an ordered conglomerate $\mathcal{C} = (O, M, d, c, \circ)$ consisting of a class O of *objects*, a class M of *morphisms*, and two functions $d, c : M \longrightarrow O$ called the *domain* and *codomain* functions respectively. If f is a morphism with $d(f) = A, c(f) = B$ then we write $f : A \longrightarrow B$. If we define the class $D = \{(f, g) \,|\, f, g \in M \text{ and } d(f) = c(g)\}$ then the final piece of data is a function $\circ : D \longrightarrow M$, called the *composition law*. We require that the following conditions be satisfied:

1. *Matching Condition*: If $f \circ g$ is defined, then $d(f \circ g) = d(g)$ and $c(f \circ g) = c(f)$.
2. *Associativity Condition*: If $f \circ g$ and $h \circ f$ are defined, then $h \circ (f \circ g) = (h \circ f) \circ g$.
3. *Identity Existence Condition*: For each object A there exists a morphism e such that $d(e) = A = c(e)$ and
   (a) $f \circ e = f$ whenever $f \circ e$ is defined, and
   (b) $e \circ g = g$ whenever $e \circ g$ is defined;
4. *Smallness of Morphism Class Condition*: For any pair (A, B) of objects, the class
   $$Hom_{\mathcal{C}}(A, B) = \{f \in M \,|\, d(f) = A \text{ and } c(f) = B\}$$
   is a set (possibly empty).

We say that a conglomerate is *small* if it is in bijection with a set. Observe that a class is small if and only if it is a set.

Definition 8. Let $\mathcal{C} = (O, M, d, c, \circ)$ be a category. We say that $\mathcal{C}$ is *small* if the class O is a set. Then since M can be written as an $O \times O$-indexed union of sets, M is a set. It follows that $d, c, \circ$ are all sets, and therefore $\mathcal{C}$ is itself a set.

Lemma 1. *Let $\mathcal{C}, \mathcal{D}$ be categories with $\mathcal{C}$ small. Then*

(i) Any functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ is a set.

(ii) The conglomerate of all functors $\mathcal{C} \longrightarrow \mathcal{D}$ is a class.

(iii) If $F, G : \mathcal{C} \longrightarrow \mathcal{D}$ are functors then the conglomerate of all natural transformations $F \longrightarrow G$ is a set.

Proof. (i) Since F is an ordered pair, it suffices by the axioms of a universe to show that if X, Y are classes with X a set, then any function $f : X \longrightarrow Y$ is a set. But we can write f as the following union
$$f = \bigcup_{x \in X} \{(x, f(x))\}$$
so it suffices to show that $(x, f(x))$ is a set for each $x \in X$. But by assumption Y is a class, so $f(x) \in Y$ is a set and therefore so is the pair $(x, f(x))$. (ii) is trivial. (iii) Suppose that F, G

are covariant (the same argument works if they are contravariant). By definition any natural transformation is a subconglomerate of the following union

$$\bigcup_{x \in O} \{\{x\} \times Hom_{\mathcal{D}}(F(x), G(x))\}$$

which is a set-indexed union of sets, and is therefore a set. Since any subconglomerate of a set is a set, it follows that the conglomerate of all natural transformations $F \longrightarrow G$ is a set. $\square$

Remark 1. Let $\mathcal{C}, \mathcal{D}$ be categories with $\mathcal{C}$ small. By Lemma 1 the portly category $[\mathcal{C}, \mathcal{D}]$ is actually a category.

Remark 2. A category may be empty, but to avoid unnecessary hypothesis we will assume all categories are nonempty unless explicitly stated otherwise.

Remark 3. We have the usual categories $\mathbf{Sets}, \mathbf{Ab}, \mathbf{Rng}, \mathbf{Top}$ and so on. One must be careful to note that **Sets** is the category of all *sets* (meaning elements of $\mathscr{U}$) *not* the category of all conglomerates. Similarly **Ab** is the category of all abelian groups built from sets, and so on. So the definition of these basic categories depends on the choice of universe $\mathscr{U}$. When we want to refer to abelian groups or rings built out of classes (or even conglomerates) we will indicate this explicitly.

The set of endomorphisms of an object A is denoted $End(A)$. If C is an object of $\mathcal{A}$, then the functor $\mathcal{A} \longrightarrow \mathbf{Sets}$ given by $D \mapsto Hom(C, D)$ we denote alternatively by H^C or $Hom_{\mathcal{A}}(C, -)$. If $\mathcal{A}$ is a category, then we denote the opposite category by $\mathcal{A}^{\text{op}}$. If $\mathcal{A}, \mathcal{B}$ are categories, then the category of covariant functors $\mathcal{A} \longrightarrow \mathcal{B}$, with natural transformations as morphisms, is denoted $\mathcal{B}^{\mathcal{A}}$. With this notation, the category of contravariant functors $\mathcal{A} \longrightarrow \mathcal{B}$ is denoted $\mathcal{B}^{\mathcal{A}^{\text{op}}}$.

Definition 9. Let $\mathcal{C}$ be a category. A *subcategory* is a functor $F : \mathcal{A} \longrightarrow \mathcal{C}$ which on objects is the inclusion of a subclass, and on morphisms is the inclusion of a subset $Hom_{\mathcal{A}}(A, B) \subseteq Hom_{\mathcal{C}}(A, B)$ for every pair of objects A, B.

Definition 10. If A is an object in a category, then a *subobject* of A is a monomorphism $A' \longrightarrow A$, and a *quotient* of A is an epimorphism $A \longrightarrow A''$. Given subobjects $\alpha : B \longrightarrow A$ and $\beta : C \longrightarrow A$, we say α *precedes* β, and write $\alpha \leq \beta$, if α factors through β.

If α, β are two subobjects of A such that $\alpha \leq \beta$ and $\beta \leq \alpha$, then α and β are isomorphic and we say that they are isomorphic as subobjects. This defines an equivalence relation on subobjects of A, and we write $\alpha = \beta$ if the two subobjects are equivalent in this way. Notice that $\alpha = -\alpha$ for any subobject α.

Definition 11. For any object A in a category $\mathcal{C}$, we let $SubA$ denote the conglomerate of equivalence classes of subobjects under this relation. This conglomerate is nonempty and is partially ordered by the relation $\leq$. If $SubA$ is in bijection with a set for each object A, we say that $\mathcal{C}$ is *locally small*. Dually, the category $\mathcal{C}$ is *colocally small* if its dual is locally small.

Definition 12. A *terminal object* in a category $\mathcal{C}$ is an object $\mathbf{1}$ such that for any $C \in \mathcal{C}$ there is precisely one morphism $C \longrightarrow \mathbf{1}$. Dually, an *initial object* is an object $\mathbf{0}$ such that for any $C \in \mathcal{C}$ there is precisely one morphism $\mathbf{0} \longrightarrow C$. A *zero object* is an object which is both a terminal and initial object. We say $\mathcal{C}$ *has a zero* if it contains a zero object.

Definition 13. Let $\mathcal{C}$ be a small category and let $F : \mathcal{C} \longrightarrow \mathbf{Sets}$ be a functor. A *subfunctor* of F is a functor $P : \mathcal{C} \longrightarrow \mathbf{Sets}$ together with a natural transformation $i : P \longrightarrow F$ such that for $C \in \mathcal{C}$ the map $i_C : P(C) \longrightarrow F(C)$ is the inclusion of a subset.

Definition 14. A *generator* for a category $\mathcal{C}$ is an object U with the following property: for any two distinct morphisms $f, g : A \longrightarrow B$ there is a morphism $x : U \longrightarrow A$ such that $fx \neq gx$. A *family of generators* for $\mathcal{C}$ is a nonempty set of objects $\{U_i\}_i$ with the following property: for any two distinct morphisms $f, g : A \longrightarrow B$ there is an object U_i in the collection and $x : U_i \longrightarrow A$ such that $fx \neq gx$.

Definition 15. A functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ which is faithful and distinct on objects is called an *embedding.*

Definition 16. A subcategory $\mathcal{A}$ of $\mathcal{C}$ is *replete* when it is closed under isomorphisms. That is, whenever $C \in \mathcal{C}$ is isomorphic to some A in $\mathcal{A}$, we must have $C \in \mathcal{A}$. The *replete closure* of a full subcategory $\mathcal{A}$ is the smallest full, replete subcategory of $\mathcal{C}$ containing $\mathcal{A}$, obtained by taking as objects all of $\mathcal{A}$ together with any object of $\mathcal{C}$ isomorphic to an object of $\mathcal{A}$.

Definition 17. Let $\mathcal{C}$ be a small category. Then there is a covariant functor called the *Yoneda embedding*

$$Hom_{\mathcal{C}}(-, C) : \mathcal{C} \longrightarrow \mathbf{Sets}^{\mathcal{C}^{\mathrm{op}}}$$

which takes an object $C \in \mathcal{C}$ to the representable functor $H_C = Hom_{\mathcal{C}}(-, C)$, and a morphism $\alpha : C \longrightarrow C'$ to the natural transformation $H_\alpha : H_C \longrightarrow H_{C'}$ defined by

$$(H_\alpha)_D(f) = \alpha f$$

It is not difficult to check that this is a full embedding.

Definition 18. Let $\mathcal{C}$ be a category and A an object of $\mathcal{C}$. We say that A is *compact* (or sometimes *small*) if whenever we have a morphism $u : A \longrightarrow \bigoplus_{i \in I} A_i$ from A into a nonempty coproduct, there is a nonempty finite subset $J \subseteq I$ and a factorisation of u of the following form

$$A \longrightarrow \bigoplus_{j \in J} A_j \longrightarrow \bigoplus_{i \in I} A_i$$

where the second morphism is canonical.

1.1 Limits and Colimits

If I is a set of indices and M is an object of a category we sometimes denote a coproduct $\coprod_{i \in I} M$ by ${}^I M$ when it exists, and a product $\prod_{i \in I} M$ by M^I when it exists.

Definition 19. A *diagram scheme* Σ is a triple (I, M, d) where I is a set whose elements are called *vertices*, M is a set whose elements are called *arrows*, and d is a function from M to $I \times I$. If $d(f) = (i, j)$ we say f begins at i and ends at j and write $f : i \longrightarrow j$. Both I, M are allowed to be empty. A *composite arrow* is a nonempty sequence $m_1, \ldots, m_n$ of arrows such that for each i, the arrow m_i ends where m_{i+1} begins. The collection of all composite arrows beginning at i and ending at j is denoted by $\kappa(i, j)$.

A *diagram* in a category $\mathcal{A}$ over a scheme Σ is a function D which assigns to each vertex $i \in I$ an object $D_i \in \mathcal{A}$ and to each arrow $m : i \longrightarrow j$ a morphism $D(m) : D_i \longrightarrow D_j$. A *morphism* of diagrams $D \longrightarrow D'$ over a scheme Σ is a morphism $D_i \longrightarrow D'_i$ for each vertex i with the property that for any arrows $m : i \longrightarrow j$ the following diagram commutes

$$\begin{array}{ccc} D_i & \longrightarrow & D'_i \\ {\scriptstyle D(m)}\downarrow & & \downarrow{\scriptstyle D'(m)} \\ D_j & \longrightarrow & D'_j \end{array}$$

A *cone* on a diagram D is an object L together with morphisms $p_i : L \longrightarrow D_i$ for each $i \in I$ such that for any arrow $m : i \longrightarrow j$ of Σ we have $D(m)p_i = p_j$. A *cocone* on D is an object C together with morphisms $u_i : D_i \longrightarrow C$ for each $i \in I$ such that for any arrow $m : i \longrightarrow j$ of Σ we have $u_j D(m) = u_i$.

A *limit* for D is a cone $p_i : L \longrightarrow D_i$ with the property that if $q_i : L' \longrightarrow D_i$ is any other cone there is a unique morphism $t : L' \longrightarrow L$ such that $p_i t = q_i$ for all i. A *colimit* for D is a cocone $u_i : D_i \longrightarrow C$ with the property that if $v_i : D_i \longrightarrow C'$ is any other cocone there is a unique morphism $t : C \longrightarrow C'$ such that $t u_i = v_i$ for all i.

We associate to any diagram Σ a category $\mathcal{C}$ called the path category of Σ. The objects of $\mathcal{C}$ are the vertices of Σ (so $\mathcal{C}$ may be the empty category) and for vertices i, j the morphisms $Hom(i, j)$ are defined to be $\kappa(i, j)$ if $i \neq j$ and $\kappa(i, i) \cup \{1_i\}$ where composition is defined by concatenation with 1_i as the identities. It is easy to check that $\mathcal{C}$ is a small category.

Lemma 2. *Let Σ be a diagram scheme and $\mathcal{C}$ the path category. For a category $\mathcal{A}$ there is a bijection between diagrams in $\mathcal{A}$ over Σ and covariant functors $\mathcal{C} \longrightarrow \mathcal{A}$, in such a way that morphisms of diagrams correspond to natural transformations of functors.*

Proof. Associated to a diagram D is the functor defined on vertices by $i \mapsto D_i$, on arrows by $m \mapsto D(m)$ and on composite arrows and identities in the obvious way. Associated to a functor $F : \mathcal{C} \longrightarrow \mathcal{A}$ is the diagram $D_i = F(i), D(m) = F(m)$. This is clearly a bijection. □

Definition 20. Let $\mathcal{C}$ be a small category (possibly empty), $F : \mathcal{C} \longrightarrow \mathcal{A}$ a covariant functor. A *cone* for F is an object L together with morphisms $p_i : L \longrightarrow F(i)$ for each object $i \in \mathcal{C}$ with the property that for any morphism $m : i \longrightarrow j$ of $\mathcal{C}$ we have $F(m)p_i = p_j$. A *cocone* for F is an object C together with morphisms $u_i : F(i) \longrightarrow C$ for each object $i \in \mathcal{C}$ with the property that for any morphism $m : i \longrightarrow j$ of $\mathcal{C}$ we have $u_j F(m) = u_i$.

A *limit* for F is a cone $p_i : L \longrightarrow F(i)$ through which every other cone factors uniquely, in the above sense. A *colimit* for F is a cocone $u_i : F(i) \longrightarrow C$ through which every other cocone factors uniquely.

Lemma 3. *Let Σ be a diagram scheme and $\mathcal{C}$ the path category. If D is a diagram over Σ in $\mathcal{A}$ and $F : \mathcal{C} \longrightarrow \mathcal{A}$ the associated functor, then there is a bijection between cones, cocones, limits and colimits of D and F respectively.*

Proof. In either case a cone is a family of morphisms $p_i : L \longrightarrow F(i) = D_i$ indexed by the vertices of Σ (= objects of $\mathcal{C}$) satisfying a certain property. One checks easily that the two properties are equivalent, so a collection of morphisms is a cone for the diagram iff. it is a cone for the functor. Similarly for cocones. It is clear that a cone (cocone) is a limit (colimit) for the diagram iff. it is a limit (colimit) for the functor. □

Lemma 4. *The following statements are equivalent for a category $\mathcal{A}$*

(i) Every functor $F : \mathcal{C} \longrightarrow \mathcal{A}$ from a (possibly empty) small category $\mathcal{C}$ has a limit.

(ii) Every diagram in $\mathcal{A}$ has a limit.

In which case we say $\mathcal{A}$ is complete. *If we replace "small" by "finite" and diagrams by finite diagrams, the statements are still equivalent, and in that case we say $\mathcal{A}$ is* finitely complete. *Similarly the following two statements are equivalent*

(i) Every functor $F : \mathcal{C} \longrightarrow \mathcal{A}$ from a (possibly empty) small category $\mathcal{C}$ has a colimit.

(ii) Every diagram in $\mathcal{A}$ has a colimit.

With the statements still being equal when we replace "small" by finite and diagrams by finite diagrams.

Proof. $(i) \Rightarrow (ii)$ is trivial. For $(ii) \Rightarrow (i)$ take the diagram scheme Σ whose vertices are the objects of $\mathcal{C}$ and whose arrows are the morphisms of $\mathcal{C}$. The functor F gives a diagram in $\mathcal{A}$ over this diagram scheme, which has a limit, and this is clearly a limit for F. □

Definition 21. A *preorder* is a nonempty small category in which every morphism set has at most one element. This is equivalent to giving a set with a binary relation $\leq$ which is reflexive and transitive and we freely identify the two, writing $i \leq j$ if $Hom(i, j) \neq \emptyset$. A *directed set* is a preorder with the property that for every i, j there is k with $i \leq k$ and $j \leq k$.

If I is a directed set, a *direct system* over I in a category $\mathcal{A}$ is a functor $I \longrightarrow \mathcal{A}$. This consists of the following data: an assignment of an object of $\mathcal{A}$ to every object of I, and a morphism π_{ij} to every relation $i \leq j$ with the property that $\pi_{ii} = 1$ for all i and $\pi_{jk}\pi_{ij} = \pi_{ik}$ for all $i \leq j \leq k$. A *direct limit* of this direct system is a colimit of the functor.

Definition 22. Let $\{u_i : A_i \longrightarrow A\}_{i\in I}$ be a nonempty set of subobjects in a category $\mathcal{A}$ (i.e. monomorphisms). As usual we write $u_i \leq u_j$ if u_i factors through u_j. This defines a reflexive, transitive relation on the set I. If this makes I into a directed set (i.e. for every i, j there is k with $u_i \leq u_k, u_j \leq u_k$) we say that $\{u_i\}$ is a *direct family of subobjects.* We call a colimit of the direct system $i \mapsto A_i$ a *direct limit* of the family of subobjects.

Definition 23. Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be a functor. We say that F *preserves direct limits* if for every directed set I and functor $G : I \longrightarrow \mathcal{A}$, if the object C together with morphisms $G(i) \longrightarrow C$ is a colimit for G then the object $F(C)$ and morphisms $FG(i) \longrightarrow F(C)$ are a colimit for FG.

Remark 4. Take a direct system of groups, rings or modules $\{A_i, \pi_{ij}\}$. Let C be the set of pairs (i, a) with $a \in A_i$ subject to the equivalence relation that says $(i, a) \sim (j, b)$ iff. $\pi_{ik}(a) = \pi_{jk}(b)$ for some $i \leq k, j \leq k$. Then this can be given the structure of a group, ring or module in such a way that the canonical maps $A_i \longrightarrow C$ are all morphisms of groups, rings, or modules and are a colimit in their respective categories.

Remark 5. If M_i is a diagram of modules and $p_i : L \longrightarrow M_i$ morphisms of modules which form a limit for the diagram as a diagram of abelian groups and group morphisms, then the p_i are a limit of modules. The p_i are clearly a cone, and any other cone of module morphisms factors uniquely through L via a morphism of abelian groups. To see that this factorisation is actually a morphism of modules, use the limit morphisms.

Definition 24. A nonempty small category $\mathcal{C}$ is *filtered* if it satisfies the following conditions

(i) For any pair of objects $x, y \in \mathcal{C}$ there is an object z and morphisms $f : x \longrightarrow z, g : y \longrightarrow z$.

(ii) For any pair of parallel morphisms $f, g : x \longrightarrow y$ there is a morphism $\alpha : y \longrightarrow z$ with $\alpha f = \alpha g$.

A *filtered system* over $\mathcal{C}$ in a category $\mathcal{A}$ is a functor $\mathcal{C} \longrightarrow \mathcal{A}$, and a *filtered colimit* of this filtered system is colimit of the functor. In particular any directed set is filtered, so direct systems and direct limits are special cases of filtered systems and filtered colimits.

1.2 Functor Categories

Definition 25. Let $\mathcal{A}, \mathcal{B}$ be categories with $\mathcal{A}$ small. Then the category $[\mathcal{A}, \mathcal{B}]$ has as objects the covariant functors $\mathcal{A} \longrightarrow \mathcal{B}$ and as morphisms the natural transformations. If $\eta : S \longrightarrow T$ is a morphism in $[\mathcal{A}, \mathcal{B}]$ we say that η is a *pointwise isomorphism* (resp. *monomorphism,epimorphism*) if η_A is an isomorphism (resp. monomorphism, epimorphism) in $\mathcal{B}$ for every $A \in \mathcal{A}$. If $\mathcal{B}$ is a preadditive category and $\varphi, \psi : S \longrightarrow T$ are natural transformations, we can define a morphism $\varphi + \psi : S \longrightarrow T$ by $(\varphi + \psi)_A = \varphi_A + \psi_A$ and this makes $[\mathcal{A}, \mathcal{B}]$ a preadditive category.

Definition 26. If $\mathcal{A}, \mathcal{B}$ are preadditive categories with $\mathcal{A}$ small we denote the full subcategory of $[\mathcal{A}, \mathcal{B}]$ consisting of the additive functors by $(\mathcal{A}, \mathcal{B})$. Observe that this subcategory is replete (i.e. any functor isomorphic to an additive functor is additive). This is a preadditive category in the usual way. The category $(\mathcal{A}, \mathcal{B})$ may be empty, although it is nonempty if $\mathcal{B}$ has a zero object.

Lemma 5. *Let $\mathcal{A}$ be a small category and $\mathcal{B}$ any category. A natural transformation $\eta : T \longrightarrow S$ is an isomorphism in $[\mathcal{A}, \mathcal{B}]$ if and only if it is a pointwise isomorphism. If η is a pointwise epimorphism (resp. pointwise monomorphism) then it is an epimorphism (resp. monomorphism).*

1.2.1 Pointwise Limits and Colimits

Throughout this section $\mathcal{A}, \mathcal{B}$ are categories with $\mathcal{A}$ small.

Definition 27. For $A \in \mathcal{A}$ we define the *evaluation functor* $E_A : [\mathcal{A}, \mathcal{B}] \longrightarrow \mathcal{B}$ by $E_A(F) = F(A)$ and $E_A(\eta) = \eta_A$.

Suppose that D is a diagram in $[\mathcal{A}, \mathcal{B}]$ over a diagram scheme Σ. Then for each $A \in \mathcal{A}$ we define a diagram $D(A)$ in $\mathcal{B}$ by $D(A)_i = D_i(A)$ and $D(A)(m) = D(m)_A$. That is, we simply *evaluate* the diagram D at A. Suppose that for every $A \in \mathcal{A}$ the morphisms $\rho_{i,A} : L(A) \longrightarrow D_i(A)$ are a limit for the diagram $D(A)$. For every morphism $\alpha : A \longrightarrow A'$ of $\mathcal{A}$, the morphisms $D_i(\alpha)$ give a morphism of diagrams $D(A) \longrightarrow D(A')$, which induces a morphism of the limits $L(\alpha) : L(A) \longrightarrow L(A')$. This defines a functor $L : \mathcal{A} \longrightarrow \mathcal{B}$ and we can define natural transformations $\rho_i : L \longrightarrow D_i$ by $(\rho_i)_A = \rho_{i,A}$. One checks easily that the object L together with the morphisms ρ_i is a *limit* of the diagram D in $[\mathcal{A}, \mathcal{B}]$.

Similarly, suppose that for every $A \in \mathcal{A}$ the morphisms $\tau_{i,A} : D_i(A) \longrightarrow C(A)$ are a colimit for the diagram $D(A)$. For every $\alpha : A \longrightarrow A'$ we induce a morphism of the colimits $C(A) \longrightarrow C(A')$ as before, and this defines a functor $C : \mathcal{A} \longrightarrow \mathcal{B}$. We define natural transformations $\tau_i : D_i \longrightarrow C$ by $(\tau_i)_A = \tau_{i,A}$ and the object C together with these morphisms is a *colimit* of the diagram D in $[\mathcal{A}, \mathcal{B}]$.

Proposition 6. *Let Σ be a diagram scheme. If $\mathcal{B}$ is Σ-complete or Σ-cocomplete, so is $[\mathcal{A}, \mathcal{B}]$. In particular, if $\mathcal{B}$ is finitely complete, finitely cocomplete, complete or cocomplete then the same is true of $[\mathcal{A}, \mathcal{B}]$.*

Lemma 7. *Let D be a diagram in $[\mathcal{A}, \mathcal{B}]$ and suppose we have a functor $X : \mathcal{A} \longrightarrow \mathcal{B}$ together with morphisms $\{\alpha_i : X \longrightarrow D_i\}_{i \in D}$. If the morphisms $(\alpha_i)_A$ are a limit for the diagram $D(A)$ for every $A \in \mathcal{A}$ then the morphisms α_i are a limit in $[\mathcal{A}, \mathcal{B}]$.*

Lemma 8. *Let D be a diagram in $[\mathcal{A}, \mathcal{B}]$ and suppose we have a functor $X : \mathcal{A} \longrightarrow \mathcal{B}$ together with morphisms $\{\alpha_i : D_i \longrightarrow X\}_{i \in D}$. If the morphisms $(\alpha_i)_A$ are a colimit for the diagram $D(A)$ for every $A \in \mathcal{A}$ then the morphisms α_i are a colimit in $[\mathcal{A}, \mathcal{B}]$.*

Lemma 9. *Let D be a diagram in $[\mathcal{A}, \mathcal{B}]$ and suppose X together with morphisms $\alpha_i : X \longrightarrow D_i$ is a limit for the diagram. Suppose also that the diagram $D(A)$ has a limit for every $A \in \mathcal{A}$. Then for every $A \in \mathcal{A}$ the object $X(A)$ together with the morphisms $(\alpha_i)_A$ is a limit for $D(A)$ in $\mathcal{B}$.*

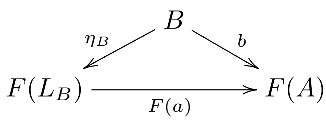
Lemma 10. *Let D be a diagram in $[\mathcal{A}, \mathcal{B}]$ and suppose X together with morphisms $\alpha_i : D_i \longrightarrow X$ is a colimit for the diagram. Suppose also that the diagram $D(A)$ has a colimit for every $A \in \mathcal{A}$. Then for every $A \in \mathcal{A}$ the object $X(A)$ together with the morphisms $(\alpha_i)_A$ is a colimit for $D(A)$ in $\mathcal{B}$.*

Remark 6. Suppose that that $\mathcal{A}, \mathcal{B}$ are preadditive and that $\mathcal{B}$ has a zero object. Let Σ be a diagram scheme and D a diagram in the category $(\mathcal{A}, \mathcal{B})$ over Σ. Suppose that $\mathcal{B}$ is Σ-complete (resp. Σ-cocomplete). Then any limit (resp. colimit) of D in $[\mathcal{A}, \mathcal{B}]$ also belongs to $(\mathcal{A}, \mathcal{B})$ and is the limit (reps. colimit) there. That is, limits (resp. colimits) of additive functors are additive, provided the codomain has the necessary limits (resp. colimits). In particular if 0 is a zero object of $\mathcal{B}$ the functor sending every object of $\mathcal{A}$ to 0 and every morphism to the identity is a zero object of $(\mathcal{A}, \mathcal{B})$, and we denote it by 0 also.

Proposition 11. *Let $\mathcal{A}, \mathcal{B}$ be preadditive categories with $\mathcal{A}$ small, and suppose that $\mathcal{B}$ has a zero object. If $\mathcal{B}$ is Σ-complete or Σ-cocomplete for some diagram scheme Σ, then so is $(\mathcal{A}, \mathcal{B})$ and the limits (resp. colimits) are computed pointwise. In particular if $\mathcal{B}$ is finitely complete, finitely cocomplete, complete or cocomplete then so is $(\mathcal{A}, \mathcal{B})$.*

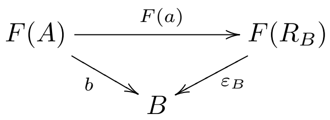
1.3 Adjoint Functors

Definition 28. Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be a functor and B an object of $\mathcal{B}$. A *reflection of B along F* is a pair (L_B, η_B) consisting of an object $L_B \in \mathcal{A}$ and a morphism $\eta_B : B \longrightarrow F(L_B)$ with the following universal property: given any $A \in \mathcal{A}$ and morphism $b : B \longrightarrow F(A)$ there is a unique morphism $a : L_B \longrightarrow A$ in $\mathcal{A}$ making the following diagram commute



Definition 29. A functor $L : \mathcal{B} \longrightarrow \mathcal{A}$ is *left adjoint* to a functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ if there exists a natural transformation $\eta : 1_{\mathcal{B}} \longrightarrow FL$ such that for every $B \in \mathcal{B}$ the pair $(L(B), \eta_B)$ is a reflection of B along F. A natural transformation η with this property is called a *left adjunction* of L to F.

Definition 30. Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be a functor and B an object of $\mathcal{B}$. A *coreflection of* B *along* F is a pair (R_B, ε_B) consisting of an object $R_B \in \mathcal{A}$ and a morphism $\varepsilon_B : F(R_B) \longrightarrow B$ with the following universal property: given any $A \in \mathcal{A}$ and morphism $b : F(A) \longrightarrow B$ there is a unique morphism $a : A \longrightarrow R_B$ in $\mathcal{A}$ making the following diagram commute



Definition 31. A functor $R : \mathcal{B} \longrightarrow \mathcal{A}$ is *right adjoint* to a functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ if there exists a natural transformation $\varepsilon : FR \longrightarrow 1_{\mathcal{B}}$ such that for every $B \in \mathcal{B}$ the pair $(R(B), \varepsilon_B)$ is a coreflection of B along F. A natural transformation ε with this property is called a *right adjunction* of F to R.

Lemma 12. *Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ and $G : \mathcal{B} \longrightarrow \mathcal{A}$ be functors. There is a bijection between left adjunctions of G to F and right adjunctions of G to F. In particular G is left adjoint to F if and only if F is right adjoint to G.*

Proof. Suppose we are given a left adjunction $\eta : 1 \longrightarrow FG$ of G to F. Given $A \in \mathcal{A}$ the morphism $1_{F(A)} : F(A) \longrightarrow F(A)$ induces a unique morphism $\varepsilon_A : GF(A) \longrightarrow A$ with $F(\varepsilon_A)\eta_{F(A)} = 1_{F(A)}$. It is easily checked that ε is a natural transformation $GF \longrightarrow 1$, and in fact it is a right adjunction of G to F.

Given a right adjunction $\varepsilon : GF \longrightarrow 1$ of G to F and $B \in \mathcal{B}$ the morphism $1_{G(B)} : G(B) \longrightarrow G(B)$ induces a unique morphism $\eta_B : B \longrightarrow FG(B)$ with $\varepsilon_{GB} G(\eta_B) = 1_{G(B)}$. Once again it is easy to check that η is a left adjunction of G to F. These two maps are inverse to one another, so we have the desired bijection. □

Definition 32. Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ and $G : \mathcal{B} \longrightarrow \mathcal{A}$ be functors. An *adjunction* $G \dashv F$ is a pair (η, ε) consisting of a left adjunction η of G to F and a right adjunction ε of G to F with η, ε corresponding under the bijection of Lemma 12.

Lemma 13. *Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ and $G : \mathcal{B} \longrightarrow \mathcal{A}$ be functors and let (η, ε) be an adjunction $G \dashv F$. Then the pair $(\varepsilon^{op}, \eta^{op})$ is an adjunction $F^{op} \dashv G^{op}$.*

Proof. That is, we have functors $F^{\text{op}} : \mathcal{A}^{\text{op}} \longrightarrow \mathcal{B}^{\text{op}}$ and $G^{\text{op}} : \mathcal{B}^{\text{op}} \longrightarrow \mathcal{A}^{\text{op}}$ and natural transformations $\varepsilon^{\text{op}} : 1 \longrightarrow G^{\text{op}} F^{\text{op}}$ and $\eta^{\text{op}} : F^{\text{op}} G^{\text{op}} \longrightarrow 1$, and one checks easily that this data defines an adjunction. □

Theorem 14. *Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ and $G : \mathcal{B} \longrightarrow \mathcal{A}$ be functors. Then the following are equivalent*

1. *G is left adjoint to F.*

2. *F is right adjoint to G.*

3. *There exist natural transformations $\eta : 1_{\mathcal{B}} \longrightarrow FG$ and $\varepsilon : GF \longrightarrow 1_{\mathcal{A}}$ such that*

$$F\varepsilon \circ \eta F = 1_F, \qquad \varepsilon G \circ G\eta = 1_G$$

4. *There exists a family of bijections $\{\theta_{A,B}\}_{A \in \mathcal{A}, B \in \mathcal{B}}$*

$$\theta_{A,B} : Hom_{\mathcal{A}}(GB, A) \longrightarrow Hom_{\mathcal{B}}(B, FA)$$

which is natural in both variables.

Proof. In fact we will show that there is a bijection between (a) adjunctions $G \dashv F$, (b) pairs of natural transformations η, ε with the property of (3) and (c) families of bijections θ with the property of (4).

Given an adjunction $(\eta, \varepsilon) : G \dashv F$ it is clear that η, ε satisfy the condition of (3). Conversely, suppose that a pair of natural transformations η, ε is given satisfying this condition. It is not difficult to check that η is a left adjunction of G to F and ε is a right adjunction of G to F, with η corresponding to ε under the bijection of Lemma 12. In other words, the pair (η, ε) is an adjunction. This proves the bijection $(a) \Leftrightarrow (b)$.

Let $(\eta, \varepsilon) : G \dashv F$ be an adjunction. For $A \in \mathcal{A}, B \in \mathcal{B}$ define define maps

$$\begin{aligned} \theta_{A,B} &: Hom_{\mathcal{A}}(GB, A) \longrightarrow Hom_{\mathcal{B}}(B, FA), & \theta_{A,B}(a) &= F(a)\eta_B \\ \tau_{A,B} &: Hom_{\mathcal{B}}(B, FA) \longrightarrow Hom_{\mathcal{A}}(GB, A), & \tau_{A,B}(b) &= \varepsilon_A G(b) \end{aligned}$$

One checks that $\tau_{A,B} = \theta_{A,B}^{-1}$ and that θ is natural in both variables. Conversely, given the natural family of bijections θ, define $\eta_B = \theta_{GB,B}(1_{GB})$ and $\varepsilon_A = \theta^{-1}_{A,FA}(1_{F(A)})$. One checks that (η, ε) is an adjunction, and that these assignments are inverse to one another, establishing the bijection $(a) \Leftrightarrow (c)$ and completing the proof. □

Lemma 15. *Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be a functor, and suppose $G_1, G_2 : \mathcal{B} \longrightarrow \mathcal{A}$ are both left adjoint to F, with adjunctions η_1, η_2. Then there is a canonical natural equivalence $\rho : G_1 \longrightarrow G_2$.*

Proof. For every $B \in \mathcal{B}$ the following diagram

$$\begin{array}{ccc} & B & \\ {\scriptstyle \eta_{2,B}}\swarrow & & \searrow{\scriptstyle \eta_{1,B}} \\ FG_2(B) & & FG_1(B) \end{array}$$

induces morphisms $\rho_B : G_1(B) \longrightarrow G_2(B)$ and $\tau_B : G_2(B) \longrightarrow G_1(B)$ which are respectively unique such that $F(\rho_B)\eta_{1,B} = \eta_{2,B}$ and $F(\tau_B)\eta_{2,B} = \eta_{1,B}$. One checks that $\rho : G_1 \longrightarrow G_2$ and $\tau : G_2 \longrightarrow G_1$ are natural transformations with $\rho\tau = 1$ and $\tau\rho = 1$, so the proof is complete. □

Remark 7. In particular if $F : \mathcal{A} \longrightarrow \mathcal{B}$ and $G : \mathcal{B} \longrightarrow \mathcal{A}$ are functors and η, η' are the unit transformations of two adjunctions $G \dashv F$ then there is a natural equivalence $\rho : G \longrightarrow G$ such that $F(\rho_B)\eta_B = \eta'_B$. So up to canonical isomorphism, there is only one adjunction between a given pair of functors.

Lemma 16. *Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be a functor, and suppose $G_1, G_2 : \mathcal{B} \longrightarrow \mathcal{A}$ are both right adjoint to F, with adjunctions $\varepsilon_1, \varepsilon_2$. Then there is a canonical natural equivalence $\rho : G_1 \longrightarrow G_2$.*

Proof. For every $B \in \mathcal{B}$ the following diagram

$$\begin{array}{ccc} FG_1(B) & & FG_2(B) \\ {\scriptstyle \varepsilon_{1,B}}\searrow & & \swarrow{\scriptstyle \varepsilon_{2,B}} \\ & B & \end{array}$$

induces morphisms $\rho_B : G_1(B) \longrightarrow G_2(B)$ and $\tau_B : G_2(B) \longrightarrow G_1(B)$ which are respectively unique such that $\varepsilon_{2,B}F(\rho_B) = \varepsilon_{1,B}$ and $\varepsilon_{1,B}F(\tau_B) = \varepsilon_{2,B}$. One checks that $\rho : G_1 \longrightarrow G_2$ and $\tau : G_2 \longrightarrow G_1$ are natural transformations with $\rho\tau = 1$ and $\tau\rho = 1$, so the proof is complete. □

Lemma 17. *Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ and $G : \mathcal{B} \longrightarrow \mathcal{A}$ be functors with F left adjoint to G. If $F \cong F'$ then F' is left adjoint to G, and if $G \cong G'$ then G' is right adjoint to F.*

Proof. Let (η, ε) be an adjunction $F \dashv G$ and $\rho : F \longrightarrow F'$ a natural equivalence. Then $\rho G \circ \eta : 1 \longrightarrow F'G$ is a natural transformation and it is not difficult to check that it is a left adjunction of F' to G. On the other hand if $\rho : G' \longrightarrow G$ is a natural equivalence then $\varepsilon \circ F\rho : FG' \longrightarrow 1$ is a natural transformation and it is not difficult to check that it is a right adjunction of F to G'. □

Proposition 18. *Consider the following diagram of functors*

$$\mathcal{A} \underset{F}{\overset{G}{\rightleftarrows}} \mathcal{B} \underset{H}{\overset{K}{\rightleftarrows}} \mathcal{C}$$

where G is left adjoint to F and K is left adjoint to H. Then GK is left adjoint to HF.

Proof. Choose adjunctions $G \dashv F$ and $K \dashv H$ represented by natural families of bijections μ and θ respectively. For $A \in \mathcal{A}, C \in \mathcal{C}$ we have the following bijection

$$Hom_{\mathcal{A}}(GKC, A) \xrightarrow{\mu_{A,KC}} Hom_{\mathcal{B}}(KC, FA) \xrightarrow{\theta_{FA,C}} Hom_{\mathcal{C}}(C, HFA)$$

which is easily checked to be natural in both variables. This defines the required adjunction between GK and HF. □

Lemma 19. *Let $\mathcal{C}, \mathcal{A}$ be nonempty categories with $\mathcal{C}$ small. The functor $\Delta : \mathcal{A} \longrightarrow [\mathcal{C}, \mathcal{A}]$ defined by $\Delta(A)(C) = A$ has a right adjoint if and only if every functor $F : \mathcal{C} \longrightarrow \mathcal{A}$ has a limit.*

Proof. Suppose every functor $F : \mathcal{C} \longrightarrow \mathcal{A}$ has a limit. Choose one and denote it by $L(F)$. A natural transformation of functors $F \longrightarrow F'$ induces a morphism of the limits $L(F) \longrightarrow L(F')$ and this defines the functor $L : [\mathcal{C}, \mathcal{A}] \longrightarrow \mathcal{A}$ (which is unique up to natural equivalence). The natural transformation $\varepsilon_F : \Delta L(F) \longrightarrow F$ defined to be pointwise the projection morphisms $L(F) \longrightarrow F(C)$ of the limit, shows that L is right adjoint to Δ. On the other hand if Δ has a right adjoint L the object $L(F)$ together with the natural transformation $\varepsilon_F : \Delta L(F) \longrightarrow F$ is a limit for F. □

Lemma 20. *Let $\mathcal{C}, \mathcal{A}$ be nonempty categories with $\mathcal{C}$ small. The functor $\Delta : \mathcal{A} \longrightarrow [\mathcal{C}, \mathcal{A}]$ of the previous Lemma has a left adjoint if and only if every functor $F : \mathcal{C} \longrightarrow \mathcal{A}$ has a colimit.*

Proof. As before. If $\mathcal{A}$ admits colimits for all these functors, the functor $[\mathcal{C}, \mathcal{A}] \longrightarrow \mathcal{A}$ which takes colimits is the left adjoint to Δ (this functor is not unique, but any other choice is naturally equivalent). □

Proposition 21. *Let $\mathcal{A}, \mathcal{B}$ be categories and suppose we have functors $F : \mathcal{A} \longrightarrow \mathcal{B}$ and $G : \mathcal{B} \longrightarrow \mathcal{A}$ and an adjunction $G \dashv F$ with unit $\eta : 1 \longrightarrow FG$ and counit $\varepsilon : GF \longrightarrow 1$. Then*

(i) F is full iff. ε is a pointwise coretraction

(ii) F is faithful iff. ε is a pointwise epimorphism

and dually

(iii) G is full iff. η is a pointwise retraction

(iv) G is faithful iff. η is a pointwise monomorphism

Proof. By duality we need only prove (i) and (ii). For two objects C, D of $\mathcal{A}$, consider the diagram

$$\begin{array}{ccc} GF(C) & \dashrightarrow & GF(D) \\ {\scriptstyle \varepsilon_C}\downarrow & & \downarrow{\scriptstyle \varepsilon_D} \\ C & \dashrightarrow & D \end{array}$$

and denote the natural isomorphism $Hom(G(-), -) \cong Hom(-, F(-))$ by θ. Recall that for $\alpha : C \longrightarrow D$, we have $F(\alpha) = \theta(\alpha\varepsilon_C)$. If we put $D = GF(C)$, then $F(z) = \eta_{F(C)}$ for some z, and it follows that $z\varepsilon_C = 1_C$. Conversely, if $q : F(C) \longrightarrow F(D)$ is given, let $m = \theta^{-1}(q)z$ and observe that $F(m) = q$. This proves (i).

For (ii), suppose $\alpha\varepsilon_C = \beta\varepsilon_C$. Then $F(\alpha) = \theta(\alpha\varepsilon_C) = \theta(\beta\varepsilon_C) = F(\beta)$, so $\alpha = \beta$. Conversely, if $F(\alpha) = F(\beta)$ then since θ is bijective, $\alpha\varepsilon_C = \beta\varepsilon_C$, and if ε is pointwise epi, it follows that $\alpha = \beta$ and F is faithful. □

Proofs of the following results can be found in any decent reference on category theory.

Theorem 22. *If $\mathcal{A}$ is a complete and locally small category with a cogenerator, then a functor $T : \mathcal{A} \longrightarrow \mathcal{B}$ has a left adjoint if and only if it is limit preserving.*

Theorem 23. *If $\mathcal{A}$ is a cocomplete and colocally small category with a generator, then a functor $T : \mathcal{A} \longrightarrow \mathcal{B}$ has a right adjoint if and only if it is colimit preserving.*

Corollary 24. *Let $\mathcal{A}$ be a complete locally small category with a cogenerator. Then $\mathcal{A}$ is cocomplete. Dually, any cocomplete colocally small category $\mathcal{A}$ with a generator is complete.*

Proposition 25. *Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be a functor between abelian categories. Then*

(i) If F has an exact left adjoint then F preserves injectives.

(ii) If F has an exact right adjoint then F preserves projectives.

Theorem 26. *Let $\mathcal{C}, \mathcal{D}$ be abelian categories, and assume that $\mathcal{D}$ has enough injectives. Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ have right adjoint G. Then the following are equivalent:*

(i) F is an exact functor.

(ii) G preserves injectives.

2 Abelian Categories

Definition 33. A *preadditive category* is a category $\mathcal{C}$ which has an abelian group structure on each of its morphism sets such that composition is bilinear:

$$\gamma(\alpha + \beta) = \gamma\alpha + \gamma\beta$$
$$(\alpha + \beta)\gamma' = \alpha\gamma' + \beta\gamma'$$

These equations must be satisfied whenever the compositions make sense. The zero element of a morphism set $Hom_{\mathcal{C}}(B, C)$ is denoted by 0_{BC} or more often just by 0. If A is an object of a preadditive category $\mathcal{C}$, then H_C and H^C denote the group valued functors defined respectively by $D \mapsto Hom(C, D)$ and $D \mapsto Hom(D, C)$.

Definition 34. An *additive category* is a preadditive category with finite products and coproducts.

We have the following simple consequences of the definition:

Lemma 27. *The following hold in any preadditive category:*

(i) For any objects A, B and C, $0_{BC}0_{AB} = 0_{AC}$;

(ii) For any $\alpha : A \longrightarrow B$, $0\alpha = 0$ and $\alpha 0 = 0$;

(iii) If the category has a zero object, the zero elements 0_{AB} are the zero morphisms;

(iv) A morphism is monic iff. the zero morphism is its kernel, and epi iff. the zero morphism is its cokernel;

(v) For any morphisms α, β,

$$\alpha(-\beta) = -\alpha\beta$$
$$(-\alpha)\beta = -\alpha\beta$$
$$(-\alpha)(-\beta) = \alpha\beta$$
$$\alpha(-1) = (-1)\alpha = -\alpha$$

(vi) The kernel of $\alpha - \beta$ is the equaliser of α and β, and similarly the cokernel is the coequaliser.

Hence any preadditive category with zero and kernels (cokernels) has equalisers (coequalisers). Notice that a *ring* is precisely a preadditive category with one object. In particular, the ring of endomorphisms of any object in a preadditive category is a ring.

Definition 35. Given a commutative ring k a *k-linear category* is a preadditive category $\mathcal{C}$ together with a left k-module structure on each abelian group $Hom_{\mathcal{C}}(X,Y)$, such that the composition is bilinear. That is, so that we have

$$\gamma \circ (r \cdot \alpha) = r \cdot (\gamma \circ \alpha) = (r \cdot \gamma) \circ \alpha$$

for any $r \in k$ and composable morphisms γ, α.

Let $\{A_i\}_{i \in I}$ be a nonempty family of objects in a preadditive category. If $\bigoplus A_i$ is a coproduct with injections u_i, the morphisms $1_{A_i} : A_i \longrightarrow A_i$ and $0 : A_j \longrightarrow A_i$, $j \neq i$, induce a morphism $p_i : \bigoplus A_i \longrightarrow A_i$ unique with the property that $p_i u_j = \delta_{ij}$ for any i, j. The morphism δ_{ij} is defined to be 1 if $i = j$ and the zero morphism otherwise.

Definition 36. An object A and two nonempty families of morphisms $u_i : A_i \longrightarrow A$, $p_i : A \longrightarrow A_i$ in a category is called a *biproduct* of the A_i if the u_i are a coproduct, the p_i a product, and $p_i u_j = \delta_{ij}$ for all i, j.

Proposition 28. *Let $A_1, \ldots, A_n$ be a finite collection of objects in a preadditive category. A family of morphisms $u_i : A_i \longrightarrow A$ is a coproduct for the family iff. there is a family of morphisms $p_i : A \longrightarrow A_i$ such that $p_i u_j = \delta_{ij}$ and $\sum_{k=1}^n u_k p_k = 1_A$.*

Proof. If $A = \bigoplus A_i$ with injections u_i, the p_j clearly exist. Also $(\sum u_k p_k) u_i = u_i = 1_A u_i$ so the above equation holds. Conversely, if such p_j exist, given morphisms $f_i : A_i \longrightarrow R$, define $f : A \longrightarrow R$ by $f = \sum f_k p_k$. Then $f u_k = f_k$ and if $f' u_k = f_k$,

$$f' = f' 1_A = f' \sum u_k p_k = \sum f_k p_k = f$$

as required. □

The dual result is

Corollary 29. *Let $A_1, \ldots, A_n$ be a finite collection of objects in a preadditive category. A family of morphisms $p_i : A \longrightarrow A_i$ is product for the family iff. there is a family of morphisms $u_i : A_i \longrightarrow A$ such that $p_i u_j = \delta_{ij}$ and $\sum_{k=1}^n u_k p_k = 1_A$.*

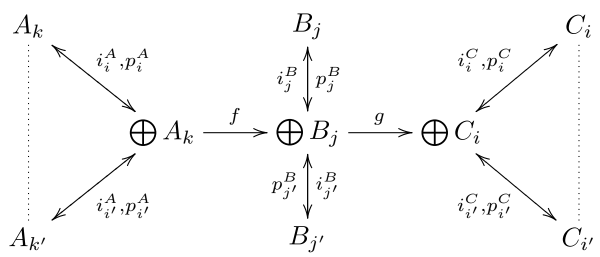
Corollary 30. *In a preadditive category every finite nonempty product and coproduct is a biproduct.*

Let $\mathcal{C}$ be a preadditive category. Given a positive integer n and an object A, we will use A^n to denote a biproduct of n copies of A. Let I, J, K be finite nonempty sets. Consider a collection of morphisms $g_{ij} : B_j \longrightarrow C_i$ for $i \in I, j \in J$ and suppose that biproducts $\bigoplus B_j, \bigoplus C_i$ exist. We can fix j and then let the morphisms g_{ij} induce a morphism $g_j : B_j \longrightarrow \bigoplus C_i$. The collection g_j then induce a morphism $g : \bigoplus B_j \longrightarrow \bigoplus C_i$. This morphism is uniquely defined by the property:

$$p_i^C g i_j^B = g_{ij} \qquad \forall i, j$$

Where p_i^C, i_i^C and p_j^B, i_j^B denote the projections and injections for $\bigoplus C_i$ and $\bigoplus B_j$ respectively.

Let $f_{jk} : A_k \longrightarrow B_j$ be another collection of morphisms, suppose that the biproduct $\bigoplus A_k$ exists and let $f : \bigoplus A_k \longrightarrow \bigoplus B_j$ the induced morphism. Let h be the composite $h = gf$. These morphisms fit into the following diagram:



As a morphism between a coproduct and a product, h is uniquely determined by its components:

$$\begin{aligned} h_{ik} &= p_i^C h u_k^A = p_i^C g f u_k^A \\ &= p_i^C g(\sum_j u_j^B p_j^B) f u_k^A = \sum_j g_{ij} f_{jk} \end{aligned} \tag{1}$$

If we take the components g_{ij} of the morphism g and put them in an $I \times J$ matrix, where the morphism in row i and column j is the component $p_i^C g i_j^B$, and do the same with the components of f to produce a $J \times K$ matrix, then (1) shows that the components of the composite gf are just the entries in the matrix product.

Proposition 31. *Let $\mathcal{C}$ be a preadditive category, A an object of $\mathcal{C}$ and $n \geq 1$. Let A^n be a biproduct of n copies of A. Then there is a canonical isomorphism of rings*

$$\Phi : End_{\mathcal{C}}(A^n) \longrightarrow M_n(R)$$
$$\Phi(h)_{ij} = p_i h u_j$$

where R is the endomorphism ring of A and p_i, u_i are the projections and injections respectively.

Let $\{A_i\}_{i \in I}$ be a nonempty family of objects in a cocomplete preadditive category, and let the coproduct $\bigoplus_i A_i$ have injections u_i and projections p_i. For any nonempty finite subset $J \subseteq I$ with coproduct $\bigoplus_{j \in J} A_j$, injections $\widehat{u}_j$ and projections $\widehat{p}_j$, it is not hard to check that the induced monomorphism $\bigoplus_j A_j \longrightarrow \bigoplus_i A_i$ is given by the sum

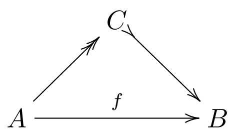
$$\sum_{j \in J} u_j \widehat{p}_j$$

Definition 37. Let $\mathcal{C}$ be a category. An endomorphism $\theta : A \longrightarrow A$ is called *idempotent* if $\theta\theta = \theta$. Clearly if θ is an idempotent in a preadditive category, then $1_A - \theta$ is also idempotent.

Proposition 32. *Let $u_1 : A_1 \longrightarrow A_1 \bigoplus A_2$ and $u_2 : A_2 \longrightarrow A_1 \bigoplus A_2$ be the injections into the coproduct in a preadditive category. Then $u_1 p_1 = \theta$ is an idempotent, as is $u_2 p_2 = 1_A - \theta$. Also, u_1 is the kernel of $1_A - \theta$ and u_2 is the kernel of θ.*

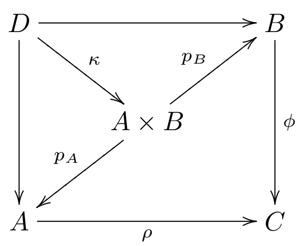
Definition 38. A category $\mathcal{C}$ is *normal* if every monomorphism is the kernel of some morphism, and is *conormal* if every epimorphism is the cokernel of some morphism.

Definition 39. A category $\mathcal{C}$ *has epi-mono factorisations* if we can write any morphism $f : A \longrightarrow B$ as an epimorphism followed by a monomorphism, as in a commutative triangle



Definition 40. An *abelian* category is a preadditive category with zero, finite products, kernels and cokernels, which is normal and conormal and has epi-mono factorisations.

Lemma 33. *If $\rho : A \longrightarrow C$ and $\phi : B \longrightarrow C$ are two morphisms in an arbitrary category, and if the product $A \times B$ with projections p_A, p_B exists, then a morphism $\kappa : D \longrightarrow A \times B$ as in the diagram*



is the equaliser of ρp_A and ϕp_B if and only if the outside square is a pullback.

Hence any abelian category has pullbacks, and consequently all finite limits. Given a morphism $f : A \longrightarrow B$ and an epi-mono factorisation $A \longrightarrow I \longrightarrow B$ of f, we call I an *image* of f, and notice that it is the smallest subobject of B through which f factors, since

Lemma 34. *Let $\mathcal{A}$ be an abelian category. Let $f : A \longrightarrow B$ be a morphism with epi-mono factorisation $\theta\psi$. If f factors through any subobject B' of B, $I \leq B'$.*

Proof. Let $K \longrightarrow A$ be the kernel of f. Hence ψ is the cokernel of $K \longrightarrow A$. If f factors through a subobject $B' \longrightarrow B$, we must have $K \longrightarrow A \longrightarrow B' = 0$, and hence $I \longrightarrow B'$ such that $A \longrightarrow B' = A \longrightarrow I \longrightarrow B'$. Since ψ is an epimorphism, it follows that $I \leq B'$. $\square$

Definition 41. If $f : A \longrightarrow B$ is a morphism and $\alpha : A' \longrightarrow A$ is a monomorphism, we denote by $f(\alpha)$ or $f(A')$ the image of the composite $f\alpha$.

Lemma 35. *Let $\mathcal{A}$ be an abelian category and suppose we have a diagram*

$$\begin{array}{ccc} A' & \longrightarrow & A \\ & & \downarrow \\ B' & \longrightarrow & B \end{array}$$

where $B' \longrightarrow B$ is a monomorphism. The diagram can be completed to a pullback if and only if $A' \longrightarrow A$ is the kernel of $A \longrightarrow B \longrightarrow B/B'$.

Definition 42. Let A be an object of an abelian category. If $\{\alpha_i : A_i \longrightarrow A\}_{i \in I}$ is a family of subobjects of A (possibly empty), then a subobject $\gamma : C \longrightarrow A$ is an *intersection* of the collection if

(i) For all $i \in I$ we have $\gamma \leq \alpha_i$;

(ii) If δ is another subobject with $\delta \leq \alpha_i$ for all i, then $\delta \leq \gamma$.

The intersection is uniquely determined up to equivalence of subobjects, and is denoted $\cap \alpha_i$ or by abuse of notation $\cap A_i$.

An abelian category has finite intersections since it has finite limits (take the limit of the diagram consisting of all the morphisms $\alpha_i : A_i \longrightarrow A$. The morphism $L \longrightarrow A$ out of the limit is the intersection). If $\mathcal{A}$ is complete then it has intersections over any family of subobjects. A subobject $\gamma : C \longrightarrow A$ is an intersection in this sense iff. it is an intersection in the sense of [Mit65] Chapter 1. If we replace the α_i by equivalent subobjects then γ is still an intersection, so "taking intersections" associates to any finite subset of $SubA$ a well-defined element of $SubA$ (if $\mathcal{A}$ has all limits, then it associates an intersection to any subset). Notice that the intersection of the empty family is the improper subobject 1_A.

Definition 43. Let A be an object of an abelian category. If $\{\alpha_i : A_i \longrightarrow A\}_{i \in I}$ is a family of subobjects of A (possibly empty), then a subobject $\gamma : C \longrightarrow A$ is a *union* of the collection if

(i) For all i we have $\alpha_i \leq \gamma$;

(ii) If δ is another subobject with $\alpha_i \leq \delta$ for all i, then $\gamma \leq \delta$.

The union is uniquely determined up to equivalence of subobjects, and is denoted $\cup \alpha_i$ or by abuse of notation $\cup A_i$.

Any morphism $0 \longrightarrow A$ out of a zero object gives a union of the empty family. For a nonempty family $\{\alpha_i : A_i \longrightarrow A\}_{i \in I}$ take a coproduct $\bigoplus_i A_i$ and induce $\bigoplus_i A_i \longrightarrow A$. The image $\gamma : I \longrightarrow A$ of this morphism gives a union for the α_i. Hence an abelian category has finite unions, and if $\mathcal{A}$ is cocomplete it has arbitrary unions. It follows from [Mit65] II,2.8 that γ is also a union in the more general sense of [Mit65] Chapter 1. Hence a subobject $C \longrightarrow A$ is a union of a family $\{\alpha_i\}$ in our sense iff. it is a union in the sense of [Mit65]. This only works in an abelian category. If we replace the α_i by equivalent subobjects then γ is still a union, so "taking unions" associates to any subset of $SubA$ a well-defined element of $SubA$.

Lemma 36. *Suppose an object A of an abelian category has a nonempty family of subobjects $u_i : A_i \longrightarrow A$ whose union is all of A. If $\lambda : A \longrightarrow B$ is a morphism such that $\lambda u_i = 0$ for all i, then $\lambda = 0$.*

Proof. The conditions imply that $A_i \leq Ker\lambda$ for each i. Hence $A = \bigcup A_i \leq Ker\lambda$, implying that $\lambda = 0$. $\square$

Proposition 37. *Consider a pullback diagram*

$$\begin{array}{ccc} P & \xrightarrow{\pi_2} & C_2 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \alpha_2} \\ C_1 & \xrightarrow[\alpha_1]{} & C \end{array}$$

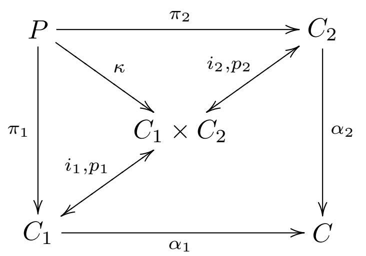
in an abelian category. Then

(i) If α_1 is a monomorphism, so is π_2;

(ii) If α_1 is an epimorphism, so is π_2;

(iii) If α_1 is the kernel of a morphism $\beta : C \longrightarrow D$, then π_2 is the kernel of $\beta\alpha_2$.

Proof. We prove (ii) and leave (i) and (iii) as exercises. Form the product $C_1 \times C_2$ and let $\kappa : P \longrightarrow C_1 \times C_2$ be $Ker(\alpha_1 p_1 - \alpha_2 p_2)$:



notice that $\mu = \alpha_1 p_1 - \alpha_2 p_2$ is the morphism induced out of the coproduct by α_1 and $-\alpha_2$. Hence there is an exact sequence

$$0 \longrightarrow P \xrightarrow{\kappa} C_1 \times C_2 \xrightarrow{\mu} C \longrightarrow 0$$

where μ is an epimorphism because $\mu i_1 = \alpha_1$ is an epimorphism. Suppose $\xi : C_2 \longrightarrow X$ is a morphism such that $\xi\pi_2 = 0$. Then $0 = \xi\pi_2 = \xi p_2 \kappa$ implies that there is $\eta : C \longrightarrow X$ with $\eta\mu = \xi p_2$. This gives $\eta\alpha_1 = \eta\mu i_1 = 0$, and α_1 an epimorphism implies $\eta = 0$. But then $\xi p_2 = 0$, and so $\xi = 0$. $\square$

Consider the particular case of the above where α_1, α_2 are monomorphisms. Then P is what we called the intersection, and the image of the induced morphism $C_1 \times C_2 \longrightarrow C$ is the union of α_1 and α_2. Since $\alpha_1 = -\alpha_1$ as subobjects, and since unions of equivalent subobjects are equivalent, we may as well take the union to be the image of the morphism $\alpha_1 p_1 - \alpha_2 p_2$. This means we have an exact sequence

$$0 \longrightarrow C_1 \cap C_2 \longrightarrow C_1 \bigoplus C_2 \xrightarrow{\mu} C_1 + C_2 \longrightarrow 0$$

We say that the sum $C_1 + C_2$ is *direct* if the morphism $C_1 \oplus C_2 \longrightarrow C$ induced by the inclusions is monic, and it follows that a union is direct iff. $C_1 \cap C_2 = 0$.

Lemma 38. *Let $\mathcal{A}, \mathcal{B}$ be abelian categories. The following conditions on a functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ are equivalent:*

(i) F is additive;

(ii) F preserves finite products;

(iii) F preserves finite coproducts.

Proof. Suppose F is additive. Then F preserves zero objects (since a zero object is characterised by having its identity equal to its zero endomorphism), so it preserves the empty product and coproduct. If $\{p_i : A \longrightarrow A_i\}_{i=1}^n$ is a product with associated coproduct $\{u_i : A_i \longrightarrow A\}_{i=1}^n$, then $p_j u_i = \delta_{ij}$ and $\sum u_k p_k = 1_A$. Since F is additive we have $F(p_j)F(u_i) = \delta_{ij}$ and $\sum F(u_k)F(p_k) = 1_A$, which means that the $F(u_i)$ are a coproduct and the $F(p_j)$ a product. It follows that F preserves both finite products and coproducts.

Next we show that $(ii) \Leftrightarrow (iii)$. Suppose F preserves finite products. Then F preserves zero objects, and it preserves nonempty finite coproducts by the same argument used above. Similarly, if F preserves finite coproducts then it preserves finite products.

It only remains to show $(ii) \Rightarrow (i)$. Given $\alpha, \beta : A \longrightarrow B$ let $\Delta : A \longrightarrow A \oplus A$ be the diagonal (so $p_i \Delta = 1$). Then $\alpha + \beta$ is the composite of Δ with $(\alpha, \beta) : A \oplus A \longrightarrow B$. Since F preserves finite products and coproducts, $F(\Delta) : F(A) \longrightarrow F(A \oplus A)$ is the diagonal and $F((\alpha,\beta)) = (F(\alpha), F(\beta))$. So in fact $F(\alpha+\beta) = F(\alpha) + F(\beta)$. □

Let $\mathcal{A}$ be an abelian category with objects $C_1, \ldots, C_n, D_1, \ldots, D_n$ and suppose we have morphisms $\varphi_i : C_i \longrightarrow D_i$. Take a biproduct $\{u_i, p_i\}$ for the C_i and $\{v_i, q_i\}$ for the D_i. The *coproduct* of the morphisms is the morphism induced $C_1 \oplus \cdots \oplus C_n \longrightarrow D_1 \oplus \cdots D_n$ out of the coproduct by the composites $v_i \varphi_i$ and the *product* of the morphisms is the morphism into the product induced by the composites $\varphi_i p_i$.

It is not difficult to check that $\sum_i v_i \varphi_i p_i$ satisfies both the properties which uniquely identify the product and coproduct of the φ_i, so $\coprod_i \varphi_i = \prod_i \varphi_i$ and we denote both simply by $\varphi_1 \oplus \cdots \oplus \varphi_n$.

Remark 8. Let $\mathcal{A}$ be an abelian category. The product of monomorphisms is a monomorphism, and the coproduct of epimorphisms is an epimorphism, so since finite products and coproducts agree in $\mathcal{A}$ it follows that finite products and coproducts preserve both monomorphisms and epimorphisms. Suppose we are given morphisms $\varphi : A \longrightarrow B$ and $\psi : A' \longrightarrow B'$. Then one checks easily that $Im(\varphi \oplus \psi) = Im(\varphi) \oplus Im(\psi)$ and $Ker(\varphi \oplus \psi) = Ker(\varphi) \oplus Ker(\psi)$. Moreover $\varphi \oplus \psi$ is an epimorphism (resp.monomorphism, isomorphism) if and only if both φ, ψ are epimorphisms (resp. monomorphisms, isomorphisms). It follows that if we have two sequences

$$A \xrightarrow{\varphi} B \xrightarrow{\psi} C$$

$$A' \xrightarrow{\varphi'} B' \xrightarrow{\psi'} C'$$

such that the following sequence is exact

$$A \oplus A' \xrightarrow{\varphi \oplus \varphi'} B \oplus B' \xrightarrow{\psi \oplus \psi'} C \oplus C'$$

Then each of the original sequences is exact.

Definition 44. Let $\mathcal{A}$ be an abelian category. A subcategory $\mathcal{B}$ of $\mathcal{A}$ is an *abelian subcategory* if it is abelian and the inclusion $\mathcal{B} \longrightarrow \mathcal{A}$ is exact. In particular the inclusion preserves all finite limits and colimits, monomorphisms and epimorphisms. It follows that a morphism in $\mathcal{B}$ is a monomorphism (epimorphism) if and only if it is a monomorphism (epimorphism) in $\mathcal{A}$, and a sequence in $\mathcal{B}$ is exact in $\mathcal{B}$ if and only if it is exact in $\mathcal{A}$.

Lemma 39. *Let $\mathcal{A}$ be an abelian category, and let $\mathcal{C}$ be a full subcategory. Then $\mathcal{C}$ is an abelian subcategory if and only if*

(i) $\mathcal{C}$ contains a zero object of $\mathcal{A}$;

(ii) For every morphism $\varphi : A \longrightarrow B$ of objects of $\mathcal{C}$, $\mathcal{C}$ contains some kernel and cokernel of φ considered as a morphism of $\mathcal{A}$;

(iii) For every pair A, B of objects of $\mathcal{C}$, some coproduct $A \oplus B$ in $\mathcal{A}$ belongs to $\mathcal{C}$.

Proof. If $\mathcal{C}$ is an abelian subcategory, it is clear that these conditions must be satisfied. Conversely, if $\mathcal{C}$ is a full subcategory with these properties, then $\mathcal{C}$ clearly has zero, finite products, kernels and cokernels. If $\varphi : A \longrightarrow B$ is a monomorphism in $\mathcal{C}$, then its kernel in $\mathcal{A}$ is zero, so it must also be a monomorphism in $\mathcal{A}$. Similarly for epimorphisms. So $\mathcal{C}$ is trivially normal, conormal, and has epi-mono factorisations. □

Proposition 40. *Let $\mathcal{C}$ be an abelian category. If C_1 and C_2 are subobjects of C, then there is a canonical isomorphism*

$$\frac{C_1 + C_2}{C_1} \cong \frac{C_2}{C_1 \cap C_2}$$

Proof. With the use of Proposition 37 *(iii)* we get a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_1 \cap C_2 & \longrightarrow & C_2 & \longrightarrow & (C_1 + C_2)/C_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & C_1 & \longrightarrow & C_1 + C_2 & \longrightarrow & (C_1 + C_2)/C_1 & \longrightarrow & 0
\end{array}$$

since the left square is a pullback. The exactness of the upper row gives the desired result. □

Proposition 41. *Let $\mathcal{C}$ be an abelian category, $\alpha_2 : C_2 \longrightarrow C$ a morphism and*

$$0 \longrightarrow C_0 \xrightarrow{\alpha_0} C_1 \xrightarrow{\alpha_1} C \longrightarrow 0$$

an exact sequence. Then we can complete this to a commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_0 & \xrightarrow{\beta_0} & P & \xrightarrow{\beta} & C_2 & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \gamma} & & \downarrow & & \\
0 & \longrightarrow & C_0 & \xrightarrow[\alpha_0]{} & C_1 & \xrightarrow[\alpha_1]{} & C & \longrightarrow & 0
\end{array}$$

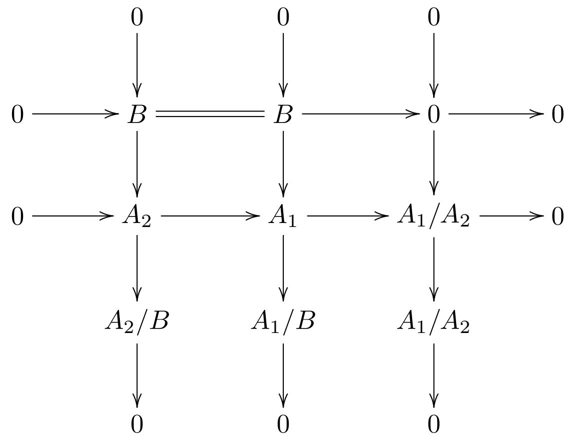
Proof. The right hand square is the pullback of α_2 and α_1, and β_0 is induced into the pullback by α_0 and the zero morphism $C_0 \longrightarrow C_2$. To see that β_0 is the kernel of β, let $\xi : X \longrightarrow P$ be such that $\beta\xi = 0$. Then $\alpha_1\gamma\xi = 0$, so $\gamma\xi = \alpha_0\lambda$ for some $\lambda : X \longrightarrow C_0$. Now $\beta_0\lambda$ and ξ give the same results when composed with γ and β, and hence $\xi = \beta_0\lambda$. This shows that the upper row is exact. □

Theorem 42. *Let $B \leq A_2 \leq A_1$ be subobjects of A_1 in an abelian category. Then we have a commutative diagram with exact rows:*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_2 & \longrightarrow & A_1 & \longrightarrow & A_1/A_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & A_2/B & \longrightarrow & A_1/B & \longrightarrow & A_1/A_2 & \longrightarrow & 0
\end{array}$$

In other words, A_2/B is a subobject of A_1/B and $(A_1/B)/(A_2/B) = A_1/A_2$.

Proof. Apply the Nine Lemma [Mit65] I.16 to the diagram



∎

Lemma 43. *Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be an additive functor between abelian categories. If F preserves direct limits then it preserves coproducts.*

Proof. Suppose we are given a coproduct $\{u_i : X_i \longrightarrow X\}_{i \in I}$ in $\mathcal{A}$. Let Λ be the set of all finite subsets of I ordered by inclusion. Then Λ is a directed set and the finite coproducts $\bigoplus_{j \in J} X_j$ for every $J \in \Lambda$ form a direct system over Λ, with the induced morphism $\bigoplus_{j \in J} X_j \longrightarrow \bigoplus_{k \in K} X_k$ for $J \subseteq K$. The canonical morphisms $\bigoplus_{j \in J} X_j \longrightarrow X$ are a direct limit. Therefore we have

$$F(X) = F(\varinjlim_{J \in \Lambda} \bigoplus_{j \in J} X_j) \cong \varinjlim_{J \in \Lambda} \bigoplus_{j \in J} F(X_j) \cong \bigoplus_{i \in I} F(X_i)$$

as required. ∎

2.1 Functor Categories

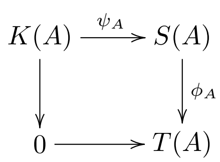
Proposition 44. *Let $\mathcal{A}, \mathcal{B}$ be categories with $\mathcal{A}$ small and $\mathcal{B}$ abelian. Then the preadditive category $[\mathcal{A}, \mathcal{B}]$ is abelian and a sequence of functors*

$$F' \longrightarrow F \longrightarrow F''$$

is exact in $[\mathcal{A}, \mathcal{B}]$ if and only if for every $A \in \mathcal{A}$ the following sequence is exact in $\mathcal{B}$

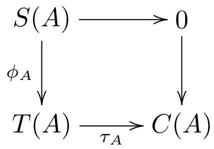
$$F'(A) \longrightarrow F(A) \longrightarrow F''(A)$$

Proof. We already know that $[\mathcal{A}, \mathcal{B}]$ is preadditive with zero. It is finitely complete and cocomplete since $\mathcal{B}$ is. Let $\phi : S \longrightarrow T$ be a morphism in $[\mathcal{A}, \mathcal{B}]$ and for each $A \in \mathcal{A}$ let $\psi_A : K(A) \longrightarrow S(A)$ be a kernel of ϕ_A. Then for each A we have a pullback diagram



For a morphism $\alpha : A \longrightarrow A'$ of $\mathcal{A}$ there is an induced morphism $K(\alpha) : K(A) \longrightarrow K(A')$ between the kernels, and defines a functor $K : \mathcal{A} \longrightarrow \mathcal{B}$ together with a natural transformation

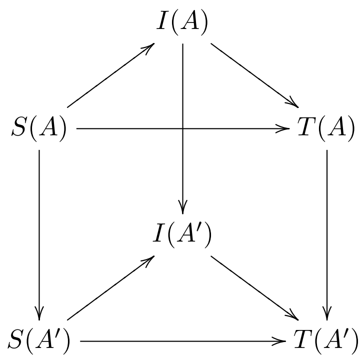
$\psi : K \longrightarrow S$. By our notes on pointwise limits, ψ is a kernel of ϕ. Similarly, for each A let $\tau_A : T(A) \longrightarrow C(A)$ be a cokernel of ϕ_A. Then we have a pushout diagram



We induce C on morphisms as above, and the functor C together with the morphism τ is a cokernel for ϕ. This also proves the following facts

- If $\phi : S \to T$ and $\psi : K \longrightarrow S$ are morphisms in $[\mathcal{A}, \mathcal{B}]$ then ψ is a kernel of ϕ if and only if ψ_A is a kernel of ϕ_A for every $A \in \mathcal{A}$.
- If $\phi : S \to T$ and $\tau : T \longrightarrow C$ are morphisms in $[\mathcal{A}, \mathcal{B}]$ then τ is a cokernel of ϕ if and only if τ_A is a cokernel of ϕ_A for every $A \in \mathcal{A}$.

Since kernels and cokernels are computed pointwise, it is clear that a morphism ϕ in $[\mathcal{A}, \mathcal{B}]$ is a monomorphism (resp. epimorphism) iff. it is a pointwise monomorphism (resp. pointwise epimorphism). Since $\mathcal{B}$ is normal and conormal, it follows easily that the same is true of $[\mathcal{A}, \mathcal{B}]$. To show that $[\mathcal{A}, \mathcal{B}]$ is abelian, it only remains to show that every morphism $\phi : S \longrightarrow T$ has an epi-mono factorisation. For every $A \in \mathcal{A}$ let $i_A : I(A) \longrightarrow T(A)$ be the image of ϕ_A and let $j_A : S(A) \longrightarrow I(A)$ be the factorisation, so that $\phi_A = i_A j_A$. For a morphism $\alpha : A \longrightarrow A'$ there is a unique morphism $I(A) \longrightarrow I(A')$ making the following diagram commute



This defines a functor I and natural transformations $j : S \longrightarrow I, i : I \longrightarrow T$. It is clear that this is an epi-mono factorisation of ϕ, as required. The statement about exact sequences is now easily checked. □

Corollary 45. *Let $\mathcal{A}, \mathcal{B}$ be preadditive categories with $\mathcal{A}$ small and $\mathcal{B}$ abelian. Then the preadditive category $(\mathcal{A}, \mathcal{B})$ is an abelian subcategory of $[\mathcal{A}, \mathcal{B}]$. In particular a sequence in $(\mathcal{A}, \mathcal{B})$ is exact if and only if it is pointwise exact.*

Proof. We already know that $(\mathcal{A}, \mathcal{B})$ is preadditive with zero. It is finitely complete and cocomplete since $\mathcal{B}$ is. If $\phi : S \longrightarrow T$ is a morphism in $(\mathcal{A}, \mathcal{B})$ then any kernel $K \longrightarrow S$ in $[\mathcal{A}, \mathcal{B}]$ is additive and is a kernel in $(\mathcal{A}, \mathcal{B})$. Similarly for cokernels. It follows that a morphism $K \longrightarrow S$ in $(\mathcal{A}, \mathcal{B})$ is a kernel of ϕ in $(\mathcal{A}, \mathcal{B})$ if and only if it is the kernel in $[\mathcal{A}, \mathcal{B}]$ (same for cokernels). In particular ϕ is a monomorphism (resp. epimorphism) in $(\mathcal{A}, \mathcal{B})$ if and only if it is a pointwise monomorphism (resp. pointwise epimorphism). Normality and conormality are easily checked, and any image of ϕ in $[\mathcal{A}, \mathcal{B}]$ is also additive, so we get epi-mono factorisations. Therefore $(\mathcal{A}, \mathcal{B})$ is abelian. We have also showed that the inclusion $(\mathcal{A}, \mathcal{B}) \longrightarrow [\mathcal{A}, \mathcal{B}]$ is exact, so $(\mathcal{A}, \mathcal{B})$ is an abelian subcategory of $[\mathcal{A}, \mathcal{B}]$. □

Lemma 46. *Let $\mathcal{A}$ be an abelian category and $\{\varphi_i : A_i \longrightarrow B_i\}_{1 \le i \le n}$ a finite set of morphisms. Then*

(a) $Ker(\prod_{i=1}^n \varphi_i) = \prod_{i=1}^n Ker(\varphi_i)$. *If $\mathcal{A}$ is complete this is true for any product.*

(b) $Coker(\bigoplus_{i=1}^n \varphi_i) = \bigoplus_{i=1}^n Coker(\varphi_i)$. *If $\mathcal{A}$ is cocomplete this is true for any coproduct.*

(c) $Im(\bigoplus_{i=1}^n \varphi_i) = \bigoplus_{i=1}^n Im(\varphi_i)$.

Proof. (a) is a special case of Lemma 19, while (b) is a special case of Lemma 20. (c) follows from the fact that taking finite coproducts (which are also finite products) preserves monomorphism and epimorphisms. □

Let $\mathcal{A}$ be a cocomplete abelian category and I a directed set. The functor category $[I, \mathcal{A}]$ is abelian and the colimit functor $C : [I, \mathcal{A}] \longrightarrow \mathcal{A}$ preserves all colimits. A morphism of $[I, \mathcal{A}]$ is a monic (resp. epi) iff. it is pointwise monic (resp. epi).

2.2 Grothendieck's Conditions

In [Gro57] Grothendieck introduced several conditions on abelian categories, which have since found a central place in homological algebra. The only conditions we will use are Ab4 and Ab5 (since Ab3 simply says that an abelian category $\mathcal{A}$ is cocomplete, and we prefer this terminology).

Definition 45. We say an abelian category $\mathcal{A}$ satisfies Ab4, or *has exact coproducts*, if it is cocomplete and if for every nonempty set I and family of monomorphisms $\{u_i : A_i \longrightarrow B_i\}_{i \in I}$ the induced morphism between the coproducts $\bigoplus_i A_i \longrightarrow \bigoplus_i B_i$ is a monomorphism. Equivalently, the "take coproducts" functor $[I, \mathcal{A}] \longrightarrow \mathcal{A}$, which is always right exact, is exact. This condition is equivalent to the condition C_1 given in [Mit65] Chapter 3.

Dually we say that $\mathcal{A}$ satisfies Ab4* if the dual abelian category $\mathcal{A}^{\mathrm{op}}$ satisfies Ab4. To be clear, we write this definition out in full.

Definition 46. We say an abelian category $\mathcal{A}$ satisfies Ab4*, or *has exact products*, if it is complete and if for every nonempty set I and family of epimorphisms $\{u_i : A_i \longrightarrow B_i\}_{i \in I}$ the induced morphism $\prod_i A_i \longrightarrow \prod_i B_i$ is an epimorphism. Equivalently, the "take products" functor $[I, \mathcal{A}] \longrightarrow \mathcal{A}$, which is always left exact, is exact.

Example 1. The categories **Ab** and R**Mod**, **Mod**R for a ring R clearly have exact coproducts and products. That is, they satisfy Ab4 and Ab4*.

Definition 47. We say an abelian category $\mathcal{A}$ satisfies Ab5, or *has exact direct limits*, if it is cocomplete and if for every directed set I the colimit functor $[I, \mathcal{A}] \longrightarrow \mathcal{A}$ is exact. Equivalently, this functor preserves monomorphisms for all directed sets I. This condition is equivalent to the condition C_3 given in [Mit65] Chapter 3.

Lemma 47. *Let $\mathcal{A}$ be a small category and $\mathcal{B}$ an abelian category. If $\mathcal{B}$ has exact products, exact coproducts or exact direct limits then so does the abelian category $[\mathcal{A}, \mathcal{B}]$. If in addition $\mathcal{A}$ is preadditive then the same statements apply to $(\mathcal{A}, \mathcal{B})$.*

Proof. Using the fact that limits and colimits in these categories are computed pointwise, these statements are easily checked. □

2.3 Grothendieck Categories

Definition 48. An abelian category $\mathcal{C}$ is *grothendieck* if it satisfies Ab5 and has a generator. This property is stable under equivalence of categories.

Remark 9. In a cocomplete abelian category $\mathcal{C}$ the existence of a generator U is equivalent to the existence of a generating set $\{U_i\}_{i \in I}$, since the coproduct of a set of generators is a generator.

Theorem 48. *Any grothendieck abelian category $\mathcal{A}$ has the following properties*

(i) *$\mathcal{A}$ is locally small and colocally small.*

(ii) $\mathcal{A}$ has enough injectives.

(iii) $\mathcal{A}$ has an injective cogenerator.

(iv) $\mathcal{A}$ is complete.

Proof. (i) [Mit65] Theorem II 15.1 and Proposition I 14.2. (ii) [Mit65] Theorem III 3.2. (iii) [Mit65] Corollary III, 3.4. (iv) follows from (LOR,Corollary 27) or alternatively Corollary 24. □

Theorem 49. *Let $T : \mathcal{A} \longrightarrow \mathcal{B}$ be a functor between grothendieck abelian categories. Then*

(i) T has a right adjoint if and only if it is colimit preserving.

(ii) T has a left adjoint if and only if it is limit preserving.

Proof. Theorem 22 and its dual Theorem 23 with Theorem 48. □

Proposition 50. *Let $\mathcal{A}$ be a grothendieck abelian category with a set of generators $\{U_i\}_i$. An object E is injective if and only if for every monomorphism $\alpha : C \longrightarrow U_i$ and morphism $\varphi : C \longrightarrow E$, there exists $\varphi' : U_i \longrightarrow E$ such that $\varphi'\alpha = \varphi$.*

Proof. For a proof see [Ste75] V. 2.9. □

Proposition 51. *Let $\mathcal{A}$ be a grothendieck abelian category. Given any family of objects $\{A_i\}_{i\in I}$ the induced morphism $\bigoplus_{i\in I} A_i \longrightarrow \prod_{i\in I} A_i$ is a monomorphism.*

Proof. For a proof see [Mit65] III. 1.3. □

Proposition 52. *Let $\mathcal{A}$ be a grothendieck abelian category. If $\{A_i\}_i$ is a direct family of subobjects of A and $f : B \longrightarrow A$ is a morphism, then*

$$f^{-1}(\bigcup A_i) = \bigcup f^{-1}A_i$$

Proof. For a proof see [Mit65] III. 1.6. □

Proposition 53. *Let $\mathcal{A}$ be a grothendieck abelian category, $\{A_i, \mu_{ij}\}_{i\in I}$ a direct system in $\mathcal{A}$, and $\{\pi_i : A_i \longrightarrow A\}_{i\in I}$ a direct limit of the system. Then for $i \in I$ we have*

$$Ker(\pi_i) = \bigcup_{i\leq j} Ker(\mu_{ij})$$

Proof. For a proof see [Mit65] III. 1.7. □

Corollary 54. *Let $\mathcal{A}$ be a grothendieck abelian category, $\{A_i, \mu_{ij}\}_{i\in I}$ a direct system in $\mathcal{A}$, and $\{\pi_i : A_i \longrightarrow A\}_{i\in I}$ a direct limit of the system. If each μ_{ij} is a monomorphisms, so is each π_i.*

Proposition 55. *Let $\mathcal{A}, \mathcal{B}$ be categories with $\mathcal{A}$ small and $\mathcal{B}$ grothendieck abelian. Then $[\mathcal{A}, \mathcal{B}]$ is grothendieck abelian.*

Proof. We already know that $[\mathcal{A}, \mathcal{B}]$ is cocomplete abelian by Proposition 44 and Proposition 6. Using the fact that direct limits are computed pointwise, it is not hard to check that $[\mathcal{A}, \mathcal{B}]$ satisfies Ab5, so it only remains to show that this category has a set of generators. Choose a generator U of $\mathcal{B}$ and define for every $A \in \mathcal{A}$ a functor $S_A : \mathcal{A} \longrightarrow \mathcal{B}$ by

$$S_A(Q) = {}^{Hom(A,Q)}U$$

That is, take a coproduct of copies of U indexed by the elements of the set $Hom(A, Q)$ (if this set is empty, then $S_A(Q)$ is the empty coproduct 0). We denote the injection of the copy of U corresponding to a morphism $f : A \longrightarrow Q$ by $u_f : U \longrightarrow S_A(Q)$. Given a morphism $\varphi : Q \longrightarrow Q'$ let $S_A(\varphi) : S_A(Q) \longrightarrow S_A(Q')$ be the unique morphism with $S_A(\varphi) \circ u_f = u_{\varphi f}$ for every morphism

$f : A \longrightarrow Q$. It is not difficult to check that this defines a functor $S_A : \mathcal{A} \longrightarrow \mathcal{B}$, and we claim that $\{S_A\}_{A \in \mathcal{A}}$ is a set of generators for $[\mathcal{A}, \mathcal{B}]$.

Let a nonzero morphism $f : S \longrightarrow T$ of $[\mathcal{A}, \mathcal{B}]$ be given, and let $A \in \mathcal{A}$ be such that $f_A : S(A) \longrightarrow T(A)$ is nonzero. Then by definition there is a morphism $g : U \longrightarrow S(A)$ in $\mathcal{B}$ with $f_A g \neq 0$. For each $Q \in \mathcal{A}$ define a morphism $t_Q : S_A(Q) \longrightarrow S(Q)$ to be the unique morphism with $t_Q \circ u_f = S(f)g$ for every morphism $f : A \longrightarrow Q$. This defines a natural transformation $t : S_A \longrightarrow S$ and it is easy to see that $ft \neq 0$, as required. □

Proposition 56. *Let $\mathcal{A}, \mathcal{B}$ be preadditive categories with $\mathcal{A}$ small and $\mathcal{B}$ grothendieck abelian. Then $(\mathcal{A}, \mathcal{B})$ is an abelian category satisfying Ab5.*

Proof. We already know that $(\mathcal{A}, \mathcal{B})$ is a cocomplete abelian category by Corollary 45 and Proposition 11. One checks the condition Ab5 as in Proposition 55. □

Lemma 57. *Any grothendieck abelian category has exact coproducts.*

Proof. A grothendieck abelian category is complete by Theorem 48, so the result is [Mit65] Corollary III, 1.3. □

Example 2. It follows from Lemma 57 that for any topological space X the category $\mathfrak{Ab}(X)$ of sheaves of abelian groups on X has exact coproducts. Similarly if $(X, \mathcal{O}_X)$ is a ringed space then the category $\mathfrak{Mod}(X)$ of sheaves of $\mathcal{O}_X$-modules has exact coproducts (one can show this directly by looking at stalks). In general these categories do not have exact products.

2.4 Portly Abelian Categories

Let $\mathcal{C}$ be a portly category. It is clear what we mean if we say that $\mathcal{C}$ is *preadditive*, *additive*, or even *abelian*. If $\mathcal{C}, \mathcal{D}$ are portly categories then from Definition 5 we have the portly category $[\mathcal{C}, \mathcal{D}]$, which has a canonical preadditive structure if $\mathcal{D}$ is preadditive. The analogues of Proposition 44 and Corollary 45 are also true:

Proposition 58. *Let $\mathcal{A}, \mathcal{B}$ be portly categories with $\mathcal{B}$ abelian. Then the preadditive portly category $[\mathcal{A}, \mathcal{B}]$ is abelian and a sequence is exact if and only if it is pointwise exact.*

Corollary 59. *Let $\mathcal{A}, \mathcal{B}$ be preadditive portly categories with $\mathcal{B}$ abelian. Then the preadditive portly category $(\mathcal{A}, \mathcal{B})$ is an abelian subcategory of $[\mathcal{A}, \mathcal{B}]$. In particular a sequence in $(\mathcal{A}, \mathcal{B})$ is exact if and only if it is pointwise exact.*

3 Reflective Subcategories

Definition 49. Let $\mathcal{C}$ be a category. A *reflective* subcategory of $\mathcal{C}$ is a full, replete subcategory $\mathcal{A}$ such that the inclusion $\mathbf{i} : \mathcal{A} \longrightarrow \mathcal{C}$ has a left adjoint. The left adjoint, generally denoted by $\mathbf{a} : \mathcal{C} \longrightarrow \mathcal{A}$, is called the *reflection*.

That is, $\mathcal{A}$ is full, closed under isomorphisms, and for each $C \in \mathcal{C}$ there is a canonical choice of $\mathbf{a}C \in \mathcal{A}$ and a morphism $\eta_C : C \longrightarrow \mathbf{a}C$ such that any other morphism from C to an element of $\mathcal{A}$ factors uniquely through $C \longrightarrow \mathbf{a}C$. Since $\mathcal{A}$ is full, we can assume that the reflection $\mathbf{a}A$ of any $A \in \mathcal{A}$ is itself, and hence that $\eta_A = 1_A$ for all $A \in \mathcal{A}$. It then follows that $\mathbf{a}\alpha = \alpha$ for any morphism α between objects of $\mathcal{A}$.

Some authors do not require repleteness in Definition 49. To see that this is unnecessary, suppose that $\mathcal{A}$ is a full subcategory of $\mathcal{C}$ whose inclusion has a left adjoint. Let $\mathcal{A}'$ be the replete closure of $\mathcal{A}$. To see that this is a reflective subcategory, we assign to $C \in \mathcal{C}$ the same object $\mathbf{a}C \in \mathcal{A} \subseteq \mathcal{A}'$ and the same morphism $C \longrightarrow \mathbf{a}C$. It is easy to see that the necessary uniqueness property still holds, and hence that $\mathcal{A}'$ is reflective.

Lemma 60. *Let $\mathcal{A}$ be a subcategory of $\mathcal{C}$ whose inclusion has a left adjoint. Let $\mathcal{A}'$ be the full, replete subcategory formed from $\mathcal{A}$ by adding any object isomorphic to an object of $\mathcal{A}$. Then $\mathcal{A}'$ is a reflective subcategory of $\mathcal{C}$.*

One reason reflective subcategories are nice because of the relationship between their limits, and limits in the ambient category:

Lemma 61. *A morphism in a reflective subcategory $\mathcal{A} \subseteq \mathcal{C}$ is a monomorphism if and only if it is a monomorphism in $\mathcal{C}$.*

Proposition 62. *Let $\mathcal{A}$ be a reflective subcategory of $\mathcal{C}$. If a diagram D in $\mathcal{A}$ has a limit in $\mathcal{C}$, then this limit is in $\mathcal{A}$ and is the limit for D in $\mathcal{A}$.*

Proof. Let $\alpha_i : L \longrightarrow D_i$ be a limit for D in $\mathcal{C}$ and let $L' = \mathbf{a}(L)$. Then by definition of a reflective subcategory, there are morphisms $\alpha_i' : L' \longrightarrow D_i$ such that $\alpha_i'\eta_L = \alpha_i$, where $\eta : 1 \longrightarrow \mathbf{ia}$ is canonical. It is easy to see that α_i' defines a compatible family for the diagram in $\mathcal{C}$, and thus induces a morphism $\alpha : L' \longrightarrow L$. For each i we have $\alpha_i\alpha\eta_L = \alpha_i'\eta_L = \alpha_i$, and consequently $\alpha\eta_L = 1_L$. We wish to show that $\eta_L\alpha = 1_L$. Since $\eta_{L'} = 1_{L'}$, $\mathbf{a}(\eta_L) = 1_{L'}$. Then we have

$$\eta_L\alpha = \mathbf{a}(\alpha)\eta_{L'} = \mathbf{a}(\alpha\eta_L) = \mathbf{a}(1_L) = 1_{L'}$$

this proves that η_L is an isomorphism, and consequently $\alpha_i' : L' \longrightarrow D_i$ is a limit for D in $\mathcal{A}$. Since $\mathcal{A}$ is replete, it also follows that $L \in \mathcal{A}$ and is the limit for D there. □

Proposition 63. *Let $\mathcal{A}$ be a reflective subcategory of $\mathcal{C}$. If a diagram D in $\mathcal{A}$ has a colimit $D_i \longrightarrow L$ in $\mathcal{C}$, then a colimit for D in $\mathcal{A}$ is given by the family $D_i \longrightarrow L \longrightarrow \mathbf{a}L$.*

Proof. Since $\mathbf{a}$ has a right adjoint, it preserves colimits. Since by assumption $\mathbf{a}$ is the identity on $\mathcal{A}$, and $\eta_A = 1_A$ for $A \in \mathcal{A}$, the family $\mathbf{a}D_i \longrightarrow \mathbf{a}L$ is a colimit for the diagram $\mathbf{a}(D) = D$ in $\mathcal{A}$. Since

$$\mathbf{a}D_i \longrightarrow \mathbf{a}L = D_i \longrightarrow L \longrightarrow \mathbf{a}L$$

we are done. □

In particular, any terminal object of the ambient category ends up in $\mathcal{A}$, and the reflection of an initial object is an initial object. Reflective subcategories are intimately related to localisation, both in abelian categories and topoi. In these cases, it is a special type of reflective subcategory that is of interest:

Definition 50. Let $\mathcal{C}$ be a category. A *giraud* subcategory of $\mathcal{C}$ is a reflective subcategory for which the left adjoint $\mathbf{a}$ to the inclusion functor preserves finite limits.

Recall ([Mit65] II, 6.7) that a functor between abelian categories preserves finite limits if and only if it preserves kernels. Such functors are called *left exact*. Hence in this case a giraud subcategory is a full replete subcategory for which the inclusion has an exact left adjoint.

Theorem 64. *Let $\mathcal{A}$ be a giraud subcategory of an abelian category $\mathcal{C}$. Then $\mathcal{A}$ is an abelian category. If $\mathcal{C}$ is Ab5 then so is $\mathcal{A}$.*

Proof. The zero object of $\mathcal{C}$ is a terminal object, hence belongs to $\mathcal{A}$, and is clearly a zero object there. By Propositions 62 and 63 $\mathcal{A}$ has kernels, cokernels and finite biproducts. To prove $\mathcal{A}$ abelian it suffices to show that $\mathcal{A}$ is normal and conormal (see [Mit65] I, 20.1). Let $A \longrightarrow B$ be a monomorphism in $\mathcal{A}$, hence in $\mathcal{C}$. Then $A \longrightarrow B$ is the kernel of some morphism $B \longrightarrow B'$ in $\mathcal{C}$. By assumption on $\mathbf{a}$, this means that $\mathbf{a}A \longrightarrow \mathbf{a}B$ is the kernel in $\mathcal{A}$ of $\mathbf{a}B \longrightarrow \mathbf{a}B'$. But $\mathbf{a}A \longrightarrow \mathbf{a}B$ is just $A \longrightarrow B$. This shows that $\mathcal{A}$ is normal.

For conormality, let $A \longrightarrow B$ be an epimorphism in $\mathcal{A}$. Then its cokernel in $\mathcal{A}$ is zero. By Proposition 63 this cokernel is the composition $B \longrightarrow B' \longrightarrow \mathbf{a}B'$ where $B \longrightarrow B'$ is the cokernel in $\mathcal{C}$. Hence $\mathbf{a}B' = 0$. Consider the sequence

$$A' \longrightarrow A \longrightarrow I \longrightarrow B \longrightarrow B'$$

where I is the image of $A \longrightarrow B$ in $\mathcal{A}$ and $A' \longrightarrow A$ is the kernel in either $\mathcal{A}$ or $\mathcal{C}$. Then $\mathbf{a}I \longrightarrow \mathbf{a}B$ is the kernel of $\mathbf{a}B \longrightarrow \mathbf{a}B'$ by assumption on $\mathbf{a}$. But since $\mathbf{a}B' = 0$, this shows that $\mathbf{a}I \longrightarrow \mathbf{a}B$

is an isomorphism. Since $\mathbf{a}$ is cokernel preserving, $\mathbf{a}A \longrightarrow \mathbf{a}I$ is the cokernel in $\mathcal{A}$ of $\mathbf{a}A' \longrightarrow \mathbf{a}A$. Since $\mathbf{a}A \longrightarrow \mathbf{a}I \longrightarrow \mathbf{a}B = \mathbf{a}A \longrightarrow \mathbf{a}B = A \longrightarrow B$ and

$$\mathbf{a}A' \longrightarrow \mathbf{a}A = A' \longrightarrow A$$

this establishes that $\mathcal{A}$ is conormal.

Now suppose that $\mathcal{C}$ is $Ab5$. To prove that $\mathcal{A}$ is $Ab5$, it suffices to show that if $D \longrightarrow D'$ is a monomorphism of direct systems in $\mathcal{A}$, then the induced morphism of the colimits in $\mathcal{A}$ is a monomorphism. We know that the induced morphism $L \longrightarrow L'$ of the colimits in $\mathcal{C}$ is a monomorphism, since $\mathcal{C}$ is $Ab5$. But the induced morphism in $\mathcal{A}$ is just $\mathbf{a}L \longrightarrow \mathbf{a}L'$, which is a monomorphism since $\mathbf{a}$ preserves kernels. Hence $\mathcal{A}$ is $Ab5$. □

Corollary 65. *A giraud subcategory of a Grothendieck category is itself a Grothendieck category.*

Proof. One need only show that the reflection of a generator is a generator, which is trivial. □

Again, we gain no extra generality by considering subcategories which are not replete:

Lemma 66. *Let $\mathcal{A}$ be a subcategory of $\mathcal{C}$ whose inclusion has an exact left adjoint. Let $\mathcal{A}'$ be the full, replete subcategory formed from $\mathcal{A}$ by adding any object isomorphic to an object of $\mathcal{A}$. Then $\mathcal{A}'$ is a giraud subcategory of $\mathcal{C}$.*

Proof. We already know that $\mathcal{A}'$ is reflective, where the reflection of $\mathcal{C}$ into $\mathcal{A}'$ is simply the composite of $\mathcal{C} \longrightarrow \mathcal{A}$ followed by the inclusion $\mathcal{A} \longrightarrow \mathcal{A}'$. This composite is exact, since it is clear that if a morphism is monic in $\mathcal{A}$ it is also monic in $\mathcal{A}'$. □

4 Finiteness Conditions

In the theory of modules, objects satisfying certain finiteness conditions play a central role. For example: noetherian modules, artinian modules, modules of finite length and finitely generated modules. To generalise these conditions to an arbitrary grothendieck abelian category, we must first understand the subobject lattices of objects. As we observed in Definition 11 the conglomerate $SubA$ of subobjects up to equivalence is not necessarily a set. So in this section we temporarily *drop the conglomerate convention*. That is, there will no reference to "classes" or "conglomerates" and the term "set" has its usual meaning in ZFC. In particular, a "category" is just a special kind of tuple of sets, as defined in Definition 1.

Definition 51. Let P be a partially ordered set and $S \subseteq P$ a subset (possibly empty). An *intersection* for the set S is an element $z \in P$ with $z \leq s$ for every $s \in S$, with the property that if $t \leq s$ for every $s \in S$ then $t \leq z$. If an intersection exists it is unique, and we denote it by $\bigwedge S$. A *union* for the set S is an element $z \in P$ with $s \leq z$ for every $s \in S$, with the property that if $s \leq t$ for every $s \in S$ then $z \leq t$. If a union exists it is unique, and we denote it by $\bigvee S$.

Definition 52. A *lattice* P is a nonempty partially ordered set with binary unions and intersections. Equivalently, P has all nonempty finite unions and intersections. A *morphism* of lattices $f : P \longrightarrow Q$ is a function with $f(x \vee y) = f(x) \vee f(y)$ and $f(x \wedge y) = f(x) \wedge f(y)$ for every pair $x, y \in L$. It is clear that the operations $\wedge$ and $\vee$ are commutative and associative.

Remark 10. Let P be a lattice, and observe that given $x, y \in P$ we have $x \leq y$ if and only if $x \wedge y = x$ if and only if $x \vee y = y$. In particular any lattice morphism $f : P \longrightarrow Q$ has the property that if $x \leq y$ then $f(x) \leq f(y)$. It is clear that a morphism of lattices is an isomorphism if and only if it is a bijection with the property that $f(x) \leq f(y)$ implies $x \leq y$.

Definition 53. A *sublattice* of P is a nonempty subset Q of P with the induced partial order, which is closed under binary unions and intersections. Then Q is clearly a lattice, and the inclusion $Q \longrightarrow P$ is a morphism of lattices.

Any nonempty partially ordered set P becomes a category in the usual way, and we will refer to it as a category without further mention. In this context, an intersection for a (possibly empty) subset $S \subseteq P$ is the same thing as a product, and a union is a coproduct. So a lattice is a nonempty partially ordered set with binary products and coproducts, and a morphism of lattices is as a functor preserving binary products and coproducts. If a lattice has an initial (resp. terminal) object we denote it by 0 (resp. 1).

Definition 54. Let L be a lattice with initial and terminal objects. If $a \in L$, then a *complement* of a in L is an element $c \in L$ such that $a \wedge c = 0$ and $a \vee c = 1$. If every element of L has a complement we say that L *has complements.*

Example 3. Let L be a lattice and a, b elements of L with $a \leq b$. Then

$$[a, b] = \{x \in L \,|\, a \leq x \leq b\}$$

is a sublattice of L, called the *interval* between a and b.

4.1 Modular Lattices

Definition 55. Let L be a lattice. Given elements $x, a, b \in L$ it is clear that

$$(x \wedge b) \vee a \leq (x \vee a) \wedge b \qquad \text{for all } x \in L \text{ and } a \leq b$$

We say that L is a *modular lattice* if this is always an equality. It is clear that any interval in a modular lattice is a modular lattice. The property of being modular is stable under lattice isomorphism.

Proposition 67. *Let a and b be elements of a modular lattice. Then there is a lattice isomorphism $[a \wedge b, a] \longrightarrow [b, a \vee b]$.*

Proof. Define $\alpha : [a \wedge b, a] \longrightarrow [b, a \vee b]$ as $\alpha(x) = x \vee b$ and define $\beta : [b, a \vee b] \longrightarrow [a \wedge b, a]$ as $\beta(y) = y \wedge a$. Then $\beta\alpha(x) = (x \vee b) \wedge a = (a \wedge b) \vee x = x$ by modularity, since $x \leq a$. Dually, $\alpha\beta(y) = (y \wedge a) \vee b = (a \vee b) \wedge y = y$. The map β is thus the inverse of α. As order isomorphisms, α, β are also lattice isomorphisms. □

Proposition 68. *If L is a modular lattice with complements, then every interval of L also has complements.*

Proof. Let $a \leq b$ in L and $d \in [a, b]$ and suppose d has a complement c in L. Then one verifies that $a \vee (c \wedge b) = b \wedge (a \vee c)$ is a complement of d in $[a, b]$. □

We now give a very useful characterisation of modular lattices:

Proposition 69. *A lattice L is modular if and only if every interval I of L has the following property: if $c \in I$ has two complements a, b in I with $a \leq b$, then $a = b$.*

Proof. If L is modular then so is every interval, so for the necessity part of the proof we may assume $I = L$. Then

$$b = b \wedge 1 = b \wedge (a \vee c) = a \vee (b \wedge c) = a \vee 0 = a$$

Conversely, if a, b, c are elements of L with $a \leq b$, then we have the modular inequality

$$a_1 = (c \wedge b) \vee a \leq (c \vee a) \wedge b = a_2$$

Then $a_1 \wedge c = ((c \wedge b) \vee a) \wedge c \geq (c \wedge a) \vee (c \wedge b) = c \wedge b$, and $a_1 \leq b$ implies $a_1 \wedge c = c \wedge b$. Also, $a_2 \wedge c = (c \vee a) \wedge b \wedge c = b \wedge c$. Further we have $a_1 \vee c = (c \wedge b) \vee a \vee c = a \vee c$ and finally $a_2 \vee c = ((c \vee a) \wedge b) \vee c \leq (b \vee c) \wedge (c \vee a) = a \vee c$, and $a \leq a_2$ implies $a_2 \vee c = a \vee c$. Thus a_1 and a_2 are complements of c in $[b \wedge c, a \vee c]$, and by hypothesis they must be equal. This proves that L is modular. □

4.2 Subobject Lattices

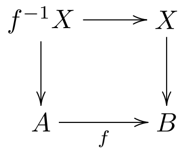
In this section we return to the conglomerate convention. In this notation, a lattice is a special kind of partially ordered conglomerate.

Example 4. Let $\mathcal{C}$ be an abelian category. We have seen how to define the union and intersection of two subobjects of an object A. The formation of these intersections and unions depends only on the equivalence class of the subobjects under the "equivalent subobject" relation. Hence the conglomerate $SubA$ of these equivalence classes forms a lattice. The 0 subobject of A is an initial object for the lattice, and the subobject $1_A : A \longrightarrow A$ is a terminal object.

Let $\mathcal{A}$ be an abelian category. For any morphism $f : A \longrightarrow B$ we can define two functors

$$f^{-1}(-) : SubB \longrightarrow SubA$$
$$f(-) : SubA \longrightarrow SubB$$

The first is defined by pullback along f: given a subobject $X \longrightarrow B$, $f^{-1}B'$ is the left hand side of the pullback



If $X \leq Y$ as subobjects of B then $f^{-1}X \leq f^{-1}Y$ as subobjects of A, so f^{-1} is a well-defined functor $SubB \longrightarrow SubA$. The second functor $f(-)$ is defined on a subobject $X \longrightarrow A$ to be the image of the composite $X \longrightarrow A \longrightarrow B$, as in the diagram. Once again, if $X \leq Y$ then $f(X) \leq f(Y)$, so this is a well-defined functor. Given subobjects $X \longrightarrow A$ and $Y \longrightarrow B$ it is easy to see that

$$X \leq f^{-1}Y \Longleftrightarrow f(X) \leq Y$$

In other words, the functor $f(-)$ is left adjoint to $f^{-1}(-)$. It is trivial that both of these functors are faithful.

Corollary 70. *Let $\mathcal{A}$ be an abelian category and $A_i \longrightarrow A$ a nonempty family of subobjects of an object A. If $f : A \longrightarrow B$ is a morphism, then*

$$f\left(\bigcup A_i\right) = \bigcup f(A_i)$$
$$f^{-1}\left(\bigcap A_i\right) = \bigcap f^{-1}(A_i)$$

whenever these unions and intersections exist.

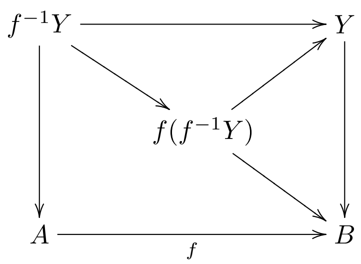
Proof. Since $f(-)$ has a right adjoint, it preserves all coproducts as a functor $SubA \longrightarrow SubB$, and therefore must preserve all unions that exist. A similar argument applies to $f^{-1}(-)$. □

Proposition 71. *Let $\mathcal{A}$ be an abelian category and $f : A \longrightarrow B$ a morphism. Then*

(i) f is an epimorphism if and only if $f^{-1}(-)$ is a full embedding.

(ii) f is a monomorphism if and only if $f(-)$ is a full embedding.

Proof. Observe that to show either functor is a full embedding it suffices to show that it is full, since then distinctness on objects is immediate. (i) If f is an epimorphism, then for any subobject $Y \longrightarrow B$ consider the diagram



Since the pullback of an epimorphism is an epimorphism the induced morphism $f(f^{-1}Y) \longrightarrow Y$ is an isomorphism. In other words, we have an equality of subobjects $Y = f(f^{-1}Y)$ from which it follows that the functor $f^{-1}(-)$ is a full embedding.

Conversely, if $f^{-1}(-)$ is a full embedding then $f^{-1}(Im(f)) = A = f^{-1}B$ so we have $B = Im(f)$, which shows that f is an epimorphism.

(ii) If f is a monomorphism then $f(-)$ acts by composition with f. If $X \longrightarrow A$ and $X' \longrightarrow A$ are subobjects and $f(X) \leq f(X')$ then it is easy to see that $X \leq X'$. Therefore $f(-)$ is a full embedding. Conversely, suppose that $f(-)$ is a full embedding and let $K \longrightarrow A$ be the kernel of f. Then $f(K) = 0 = f(0)$, and hence $K = 0$ which implies that f is a monomorphism. □

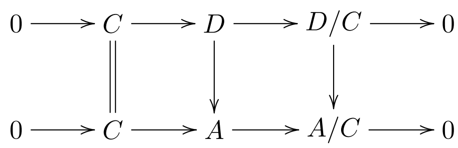
Remark 11. Let $\mathcal{A}$ be an abelian category. If $f : A \longrightarrow B$ is an isomorphism then $f(-)$ and $f^{-1}(-)$ are mutually inverse, and define a lattice isomorphism $SubA \longrightarrow SubB$.

Corollary 72. *Let $\mathcal{A}$ be an abelian category and $C \longrightarrow A$ a subobject. There is a canonical lattice isomorphism of $SubC$ with the interval $[0, C]$ of $SubA$, and of $Sub(A/C)$ with the interval $[C, A]$.*

Proof. Let $f : C \longrightarrow A$ be a monomorphism. By Proposition 71 the functor $f(-) : SubC \longrightarrow SubA$ is a full embedding whose image is clearly the interval $[0, C]$. It therefore induces a lattice isomorphism $SubC \longrightarrow [0, C]$ as required.

Let $g : A \longrightarrow A/C$ be a cokernel of f. Then $g^{-1}(-) : Sub(A/C) \longrightarrow SubA$ is a full embedding, so it only remains to show that a subobject of A is the pullback of a subobject of A/C precisely when it contains C. Firstly, it follows from Proposition 41 that the pullback of a subobject of A/C contains C.

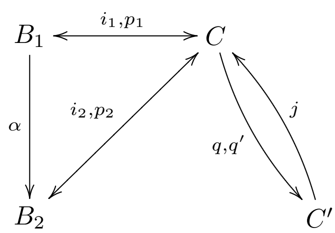
For the converse, suppose $C \leq D$. Then the intersection $C \cap D$ is just C, and so from Proposition 37 (iii) we conclude that $C \longrightarrow D$ is the kernel of $D \longrightarrow A \longrightarrow A/C$. Hence it is the kernel of $Im(D \longrightarrow A/C)$. This is summarised by the diagram



where D/C unambiguously denotes both the cokernel of the monomorphism $C \longrightarrow D$ and the image of the morphism $D \longrightarrow A/C$. From Theorem 42 we deduce that the cokernel of $D/C \longrightarrow A/C$ is $A/C \longrightarrow A/D$, and $D \longrightarrow A$ is the kernel of $A \longrightarrow A/C \longrightarrow A/D$. By Proposition 37 the right hand square is a pullback, which shows that $g^{-1}(-)$ induces a lattice isomorphism of $Sub(A/C)$ with $[C, A]$ as required. □

Proposition 73. *Let $\mathcal{A}$ be an abelian category. For any object $A \in \mathcal{A}$ the subobject lattice $SubA$ is modular.*

Proof. We use Proposition 69. Since by Proposition 72 an interval $[B_1, B_2]$ of $Sub(C)$ is isomorphic to the lattice $Sub(B_1/B_2)$, it suffices to consider subobjects $B_1 \leq B_2$ of C with common complement C'. Let $i_1 : B_1 \longrightarrow C$, $i_2 : B_2 \longrightarrow C$ and $j : C' \longrightarrow C$ be the inclusions, and let i_1 factor through i_2 as $i_1 = i_2\alpha$. Then there are two morphisms $q, q' : C \longrightarrow C'$ and morphisms $p_1 : C \longrightarrow B_1, p_2 : C \longrightarrow B_2$ such that i_1, p_1, j, q and i_2, p_2, j, q' form biproducts, as in the diagram



Then $i_1p_1 + jq = 1_C = i_2p_2 + jq'$. On composing both sides with p_2 and replacing i_1 with $i_2\alpha$ we have

$$p_2i_2\alpha p_1 + p_2jq = p_2i_2p_2 + p_2jq'$$
$$\alpha p_1 = p_2$$

Since p_1, p_2 are both cokernels of j it follows that α is an isomorphism, and hence $B_1 = B_2$ as required. □

4.3 Finiteness Conditions

Definition 56. Let L be a modular lattice. We say two intervals of L are *similar* if there exist elements $a, b \in L$ such that one of the intervals is $[a \wedge b, b]$ and the other is $[b, a \vee b]$. Proposition 67 shows that similar intervals are isomorphic lattices.

We also say two intervals I and J are *projective* if there exists a chain $I = I_0, I_1, \ldots, I_n = J$ of intervals with $n \geq 1$ such that I_{i-1} and I_i are similar for all i. Projective intervals are clearly isomorphic lattices.

Example 5. Let $\mathcal{A}$ be an abelian category and $A \in \mathcal{A}$. In the modular lattice $L = SubA$ if two intervals $[C_1, C_2]$ and $[D_1, D_2]$ are similar in $SubA$, it follows from Proposition 40 that

$$C_2/C_1 \cong D_2/D_2$$

Definition 57. Let L be a modular lattice. We call two chains $(m, n \geq 1)$

$$a = a_0 \leq a_1 \leq \ldots \leq a_m = b \quad (2)$$
$$a = b_0 \leq b_1 \leq \ldots \leq b_n = b \quad (3)$$

between the same pair of elements of L *equivalent* if $m = n$ and there is a permutation π of $\{1, \ldots, n\}$ such that the intervals $[a_{i-1}, a_i]$ and $[b_{\pi(i)-1}, b_{\pi(i)}]$ are projective. A *refinement* of a chain is obtained by inserting further elements in the chain.

The following result is known as the "Schreier refinement theorem".

Proposition 74. *Any two finite chains between the same pair of elements in a modular lattice have equivalent refinements.*

Proof. See [Ste75] III 3.1. □

Definition 58. Let L be a modular lattice. A *composition chain* between elements $a, b \in L$ is a chain $a = a_0 < a_1 < \ldots < a_m = b$ with $m \geq 1$ which has no nontrivial refinement. The integer m is the *length* of the chain.

Corollary 75. *Any two composition chains between the same pair of elements in a modular lattice are equivalent.*

Definition 59. A modular lattice L with 0 and 1 has *finite length* if there is a composition chain between 0 and 1. We define the *length* of L to be the uniquely determined length of such a composition chain. This property is stable under lattice isomorphism.

Proposition 76. *In a modular lattice of finite length, every chain can be extended to a composition chain between* 0 *and* 1.

Proof. Immediate from Proposition 74. □

Definition 60. A lattice L is *noetherian* (or *satisfies the ascending chain condition*) if there are no infinite stricly ascending chains $a_0 < a_1 < \cdots$ in L, and is *artinian* (or *satisfies the descending chain condition*) if there are no infinite strictly decreasing chains $a_0 > a_1 > \cdots$ in L. These properties are stable under lattice isomorphism.

These chain conditions have equivalent formulations as maximum (minimum) conditions. Recall a *maximal* element of a nonempty subset $S \subseteq L$ of a partially ordered conglomerate L is an element $a \in S$ such that if $a \leq x$ for any $x \in S$, then $x = a$. Similarly one defines a *minimal* element.

Proposition 77. *A lattice L is noetherian (artinian) if and only if every nonempty subset of L has a maximal (minimal) element.*

Proposition 78. *A modular lattice with 0 and 1 is of finite length if and only if it is both noetherian and artinian.*

Proof. If L has finite length m, then every strictly ascending (descending) chain in L consists of at most $m+1$ elements, so L is noetherian and artinian. Suppose conversely that L is noetherian. For every $a \neq 0$ in L there exists by Proposition 77 a maximal element b such that $b < a$. By repeated use of this observation we get a descending chain $1 > a_1 > a_2 > \dots$. If L is also artinian, this chain stops after a finite number of steps, and we obtain a composition chain between 1 and 0, showing that L has finite length. □

Proposition 79. *Let a be an element of a modular lattice L with 0 and 1. Then L is noetherian (artinian) if and only if both intervals $[0, a]$ and $[a, 1]$ are noetherian (artinian).*

Proof. If L is noetherian or artinian, then clearly every interval of L is likewise. Suppose conversely that the intervals $[0, a]$ and $[a, 1]$ are noetherian, and let $b_1 < b_2 < \dots$ be a strictly ascending chain in L. Then there exists an integer n such that

$$b_n \wedge a = b_{n+1} \wedge a = c$$
$$b_n \vee a = b_{n+1} \vee a = d$$

Applying Proposition 69 to the element a in $[c, d]$, we obtain $b_n = b_{n+1}$. Hence L must be notherian. Similarly for the artinian case. □

There are many different finiteness conditions that we can place on objects of an abelian category. Many of these conditions have an internal version (which is some condition on the subobject lattice) and an external or "functorial" version, which is a condition on the Hom functors associated to the object.

Definition 61. Let $\mathcal{A}$ be an abelian category and $C \in \mathcal{A}$ an object. We say an object C is

- *Noetherian* if the lattice $SubC$ is noetherian. That is, there are no strictly ascending infinite chains of subobjects $C_0 < C_1 < C_2 < \cdots$.
- *Artinian* if the lattice $SubC$ is artinian. That is, there are no strictly descending infinite chains of subobjects $C_0 > C_1 > C_2 > \cdots$.
- *Finite length* if the lattice $SubC$ is of finite length, or equivalently if C is both noetherian and artinian.
- *Finitely generated* if whenever $C = \sum_{i \in I} C_i$ for a direct family of subobjects C_i of C, there is an index $i_0 \in I$ such that $C = C_{i_0}$.
- *Finitely presented* if the additive functor $Hom(C, -) : \mathcal{A} \longrightarrow \mathbf{Ab}$ preserves direct limits.
- *Compact* if any morphism from C to a nonempty coproduct $\bigoplus_{i \in I} A_i$ factors through some finite subcoproduct $\bigoplus_{i=1}^n A_i$.

These properties are all stable under isomorphism. Any zero object has all of these properties.

Lemma 80. *Let $\mathcal{A}$ be a grothendieck abelian category and suppose we have an exact sequence $0 \longrightarrow C' \longrightarrow C \longrightarrow C'' \longrightarrow 0$. Then*

(i) If C is finitely generated, so is C''.

(ii) If C' and C'' are finitely generated, so is C.

Proof. (i) Suppose C'' is equal to the direct union $\sum C''_i$. Each C''_i is C_i/C' for some subobject C_i of C containing C'. Since $\mathcal{C}$ is Ab5 we can pullback direct unions to see that $C = \sum C_i$, and since C is finitely generated, $C = C_{i_0}$ for some index i_0. Hence $C'' = C/C' = C_{i_0}/C' = C''_{i_0}$.

(ii) Let C_i be a directed family of subobjects of C with $C = \sum C_i$. Then $C_i \cap C'$ is a directed family of subobjects of C' with $C' = C' \cap C = C' \cap \sum C_i = \sum C' \cap C_i$. If $\mu : C \longrightarrow C''$ is the cokernel, then $(C_i + C')/C' = \mu(C_i)$ form a direct family of subobjects of C'' with $C'' = \mu(C) = \mu(\sum C_i) = \sum \mu(C_i) = \sum (C_i + C')/C'$. Since C' and C'' are finitely generated and the sums are directed, we may find a single index k with $C' = C' \cap C_k$ and $C'' = \mu(C_k)$. Hence by Corollary 72 we have $C = C_k$. □

Corollary 81. *Let $\mathcal{A}$ be a grothendieck abelian category. Then*

(i) Any finite direct sum of finitely generated objects is finitely generated.

(ii) Any finite sum of finitely generated subobjects is finitely generated.

Proof. Follows immediately from Lemma 80. □

Lemma 82. *Let $\mathcal{A}$ be a grothendieck abelian category. An object C is finitely generated if and only if for any direct family of subobjects D_i of an object D, any morphism $C \longrightarrow \sum D_i$ factors through some D_k.*

Proof. Suppose C is finitely generated. Then if $\alpha : C \longrightarrow \sum D_i$, $Im\alpha$ is a finitely generated subobject of $\sum D_i$ by Lemma 80. But then

$$Im\alpha = Im\alpha \cap \sum D_i = \sum Im\alpha \cap D_i$$

and it follows that $Im\alpha$ is contained in some D_k. The converse is trivial. □

Proposition 83. *Let $\mathcal{A}$ be a grothendieck abelian category. An object C is finitely generated if and only if the functor $Hom_{\mathcal{A}}(C, -) : \mathcal{A} \longrightarrow$ **Ab** preserves direct unions.*

Proof. More precisely, C is finitely generated if and only if the canonical homomorphism

$$\Phi : \varinjlim Hom(C, D_i) \longrightarrow Hom(C, \sum D_i)$$

is an isomorphism for every direct family D_i of subobjects of any object D. Since $Hom(C, -)$ preserves monomorphisms and direct limits are exact, Φ is a monomorphism. Considering the definition of direct limits in **Ab**, we see that Φ is an epimorphism iff. every morphism $C \longrightarrow \sum D_i$ factors through some D_k, which is iff. C is finitely generated by Lemma 82. □

Corollary 84. *Let $\mathcal{A}$ be a grothendieck abelian category. If an object C is finitely presented then it is finitely generated.*

Lemma 85. *Let $\mathcal{A}$ be an abelian category. Given finitely presented objects X, Y and an exact sequence $X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ it follows that Z is finitely presented.*

Proof. Let a direct system $\{A_i, \varphi_{ij}\}_{i \in I}$ in $\mathcal{A}$ be given together with a direct limit $A = \varinjlim_i A_i$. Using exactness of direct limits we have a commutative diagram of abelian groups with exact rows

$$\begin{array}{ccccccc}
0 \longrightarrow & \varinjlim_i Hom(Z, A_i) & \longrightarrow & \varinjlim_i Hom(Y, A_i) & \longrightarrow & \varinjlim_i Hom(X, A_i) \\
& \downarrow \gamma & & \downarrow \alpha & & \downarrow \beta \\
0 \longrightarrow & Hom(Z, A) & \longrightarrow & Hom(Y, A) & \longrightarrow & Hom(X, A)
\end{array}$$

By hypothesis α, β are isomorphisms and therefore so is γ, which completes the proof. □

Lemma 86. *Let $\mathcal{C}$ be a preadditive category and suppose we have a morphism $\alpha : A \longrightarrow \bigoplus_{i\in I} A_i$. Given a nonempty finite subset $J \subseteq I$ and a coproduct $\bigoplus_{j\in J} A_j$ the morphism α factors through $\bigoplus_{j\in J} A_j \longrightarrow \bigoplus_{i\in I} A_i$ if and only if $\alpha = (\sum_{j\in J} u_j p_j)\alpha$ where $\{u_i, p_i\}_{i\in I}$ are the canonical injections and projections respectively.*

Proof. Denote by u'_j, p'_j the injections and projections into $\bigoplus_{j\in J} A_j$. The induced morphism $u_J : \bigoplus_{j\in J} A_j \longrightarrow \bigoplus_{i\in I} A_i$ is easily checked to be $\sum_{j\in J} u_j p'_j$. So if α factors through u_J we have $\alpha = (\sum_{j\in J} u_j p'_j)\beta$ for some morphism β. Composing both sides with p_j for $j \in J$ we deduce that $p_j\alpha = p'_j\beta$. Therefore $\alpha = (\sum_{j\in J} u_j p_j)\alpha$. If conversely $\alpha = (\sum_{j\in J} u_j p_j)\alpha$ then $\beta = \sum_{j\in J} u'_j p_j \alpha$ is a factorisation of α through the finite coproduct. □

Proposition 87. *Let $\mathcal{C}$ be an additive category. An object A is compact if and only if the functor $Hom(A, -) : \mathcal{C} \longrightarrow$ **Ab** preserves coproducts.*

Proof. Let $\{C_i\}_{i\in I}$ be a nonempty family of objects in $\mathcal{C}$ for which a coproduct $\bigoplus_i C_i$ exists. The induced morphism $\tau : \bigoplus_{i\in I} Hom(A, C_i) \longrightarrow Hom(A, \bigoplus_{i\in I} C_i)$ given by $(\alpha_i)_{i\in I} \mapsto \sum_i u_i\alpha_i$ is always injective. Suppose that A is compact, then by Lemma 86 any morphism $\beta : A \longrightarrow \bigoplus_i C_i$ is $\sum_{j\in J} u_j p_j \beta$ for some finite nonempty subset $J \subseteq I$, so τ is surjective and therefore an isomorphism. This shows that $Hom(A, -)$ preserves coproducts.

Conversely if $Hom(A, -)$ preserves coproducts, then τ must be an isomorphism. It is then easy to see that any morphism $\beta : A \longrightarrow \bigoplus_i C_i$ must factor through a finite subcoproduct. □

Lemma 88. *In any abelian category a finite coproduct of compact objects is compact.*

Proof. It suffices to prove that if C, D are compact objects, then so is their coproduct $C \oplus D$. If $\alpha : C \oplus D \longrightarrow \bigoplus_{i\in I} A_i$ is a morphism of their coproduct into another coproduct, notice that by Corollary 70 we have $Im\alpha = \alpha(C \cup D) = \alpha C \cup \alpha D$. But both αC and αD are subobjects of some finite subcoproduct of the A_i. Hence so is $Im\alpha$, as required. □

Definition 62. A grothendieck abelian category $\mathcal{A}$ is *locally finitely generated* if it has a set of finitely generated generators.

Lemma 89. *A grothendieck abelian category $\mathcal{A}$ is locally finitely generated if and only if every object is the union of finitely generated subobjects.*

Proof. Suppose that $\{G_i\}_{i\in I}$ is a generating set of finitely generated objects. Then for $C \in \mathcal{A}$ we have $C = \sum Im(\alpha)$ where α ranges over all morphisms $\alpha : G_i \longrightarrow C$ for $i \in I$. If not, the quotient $\mu : C \longrightarrow C/\sum Im(\alpha)$ would be a nonzero morphism for which there exists no $\alpha : G_i \longrightarrow C$ with $\mu\alpha \neq 0$. As quotients of finitely generated objects the $Im(\alpha)$ are all finitely generated, so the condition is necessary.

To that it is also sufficient, let U be an arbitrary generator for $\mathcal{A}$. Then U can be written as the union $U = \sum_i V_i$ of finitely generated subobjects, and it is clear that the V_i form a generating family for $\mathcal{A}$. □

Proposition 90. *Let $\mathcal{A}$ be an abelian category and $0 \longrightarrow C' \longrightarrow C \longrightarrow C'' \longrightarrow 0$ an exact sequence. Then C is noetherian if and only if both C', C'' are noetherian.*

Proof. The subobject lattices $SubC', SubC$ and $SubC''$ are modular by Proposition 73, and we have lattice isomorphisms $SubC' \cong [0, C']$ and $SubC'' \cong [C', C]$ by Corollary 72. Therefore the desired result follows from Proposition 79. □

In particular any finite direct sum of noetherian objects is noetherian, and therefore any finite sum of noetherian subobjects is noetherian.

Lemma 91. *If $\mathcal{A}$ is an abelian category then the full replete subcategory* noeth$(\mathcal{A})$ *of noetherian objects is an abelian subcategory of $\mathcal{A}$.*

Proof. Follows from Lemma 39 and Proposition 90. □

Lemma 92. *Let $\mathcal{A}$ be an abelian category. An object C is noetherian if and only if every subobject of C is finitely generated.*

Proof. First we observe that any noetherian object is finitely generated, since if we can write $C = \sum_{i \in I} C_i$ for a direct family of subobjects C_i, then this direct family has a maximal element C_{i_0}. Therefore $C = C_{i_0}$ and C is finitely generated. It now follows from Proposition 90 that if C is noetherian then every subobject is noetherian, therefore finitely generated.

Conversely, suppose every subobject of C is finitely generated and let $C_0 < C_1 < \cdots$ be a strictly ascending infinite chain of subobjects. This is a direct family of subobjects of the union $\sum_i C_i$, which is therefore equal to some C_k since it is finitely generated. This contradiction shows that C is noetherian. □

Proposition 93. *Let $\mathcal{A}$ be a grothendieck abelian category. Then*

(i) Every finitely generated object is compact.

(ii) Any quotient of a compact object is compact.

Definition 63. A grothendieck abelian category $\mathcal{A}$ is *locally noetherian* if it has a set of noetherian generators. By Lemma 92 a locally noetherian category is locally finitely generated. If $\mathcal{A}$ is locally noetherian then every object is the direct union of noetherian subobjects, and every finitely generated object is noetherian.

4.4 Finiteness Conditions for Modules

Throughout this section let R be an arbitrary ring (not necessarily commutative) and let $\mathcal{A}$ be either $R\mathbf{Mod}$ or $\mathbf{Mod}R$. In this case we already have definitions of finiteness conditions in $\mathcal{A}$ and we want to check they agree with the ones given in the previous section. To avoid confusion we refer to the finiteness conditions of Definition 61 by saying that an object X is "categorically finitely generated" or "categorically finitely presented".

Lemma 94. *An object in $\mathcal{A}$ is categorically finitely generated if and only if it is a finitely generated R-module in the usual sense.*

Proof. Let M be a categorically finitely generated R-module. The finitely generated submodules form a direct system with union M, so by definition M is equal to one of them and thus finitely generated. Conversely if M is a finitely generated R-module with $M = \sum_{i \in I} C_i$ for some direct system $\{C_i\}_{i \in I}$ of submodules then a finite generating set of M clearly belongs to some C_k, from which we deduce that $C_k = M$ as required. □

Lemma 95. *Suppose we have an exact sequence in $\mathcal{A}$ of the form*

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

where Y is finitely presented and X finitely generated. Then Z is finitely presented.

Proof. We can find a short exact sequence $0 \longrightarrow K \longrightarrow F \longrightarrow Y \longrightarrow 0$ with F a finite coproduct of copies of R and K finitely generated. If $f : F \longrightarrow Y$ is the second morphism in this sequence then it is not difficult to check that we have two exact sequences

$$0 \longrightarrow K \longrightarrow f^{-1}X \longrightarrow X \longrightarrow 0$$
$$0 \longrightarrow f^{-1}X \longrightarrow F \longrightarrow Z \longrightarrow 0$$

From the first we deduce that $f^{-1}X$ is finitely generated and therefore the second shows that Z is finitely presented. □

Lemma 96. *Any direct summand of a finitely presented R-module is finitely presented.*

Proof. Let X be a finitely presented R-module and suppose we have a split exact sequence

$$0 \longrightarrow K \longrightarrow X \longrightarrow Y \longrightarrow 0$$

Since this is split we deduce an epimorphism $X \longrightarrow K$ which implies that K is finitely generated. Therefore Lemma 95 applies to show Y finitely presented. ☐

Proposition 97. *An object in $\mathcal{A}$ is categorically finitely presented if and only if it is a finitely presented R-module in the usual sense.*

Proof. If M is a finitely presented R-module then there is an exact sequence in $\mathcal{A}$ of the form

$$R^m \longrightarrow R^n \longrightarrow M \longrightarrow 0$$

Clearly any free module is categorically finitely presented, so we deduce from Lemma 85 that M is categorically finitely presented. Now let M be a categorically finitely presented object of $\mathcal{A}$. From Lemma 94 and Corollary 84 we deduce that M is finitely generated, so there is an exact sequence

$$0 \longrightarrow K \longrightarrow R^m \longrightarrow M \longrightarrow 0$$

for some finite $m \geq 1$. Writing K as the direct limit of its finitely generated submodules K_λ, taking cokernels of the inclusions $K_\lambda \longrightarrow R^m$ and using exactness of direct limits we can write M as the direct limit of a family of finitely presented R-modules M_λ. From

$$Hom(M, M) = Hom(M, \varinjlim_\lambda M_\lambda) \cong \varinjlim_\lambda Hom(M, M_\lambda)$$

we deduce that one of the colimit morphisms $M_\mu \longrightarrow M$ is a retraction. It now follows from Lemma 96 that M is finitely presented. ☐

5 Simple objects

Definition 64. A nonzero object A in an abelian category is *simple* if its only subobjects are 0 and A. Equivalently, A is simple if and only if the modular lattice $SubA$ contains precisely two elements. An object is *semisimple* if it is the coproduct of a nonempty collection of simple objects.

Let S, S' be simple objects in an abelian category $\mathcal{A}$. If $\alpha : S \longrightarrow S'$ is a morphism then $Ker\alpha$ is a subobject of S and $Im\alpha$ is a subobject of S'. It follows that either $\alpha = 0$ or α is an isomorphism. From this we deduce the following useful result

Lemma 98 (Schur). *The endomorphism ring of a simple object in an abelian category is a division ring.*

Lemma 99. *Let $\{S_i\}_{i \in I}$ be a nonempty family of simple objects in a grothendieck abelian category. If S is simple subobject of $\bigoplus_i S_i$, then S is isomorphic to some S_i.*

Proof. Let $\gamma : S \longrightarrow \bigoplus_i S_i$ be a monomorphism. Since S is nonzero and we are working in a grothendieck abelian category, $p_i\gamma$ is nonzero for some projection $p_i : \bigoplus_i S_i \longrightarrow S_i$. Hence S is isomorphic to S_i. ☐

Lemma 100. *Let S, T, L be subobjects of an object X in an abelian category $\mathcal{A}$, such that*

$$S \cap (L + T) = 0 \quad \text{and} \quad L \cap T = 0$$

then $S \cap T = 0$ and $L \cap (S + T) = 0$.

Proof. Since $T \leq L + T$ certainly $S \cap T = 0$. Since $L \cap T = S \cap T = 0$, $L + T$ and $S + T$ are biproducts, and we get a monic $L \cap (S + T) \longrightarrow S \cap (L + T) = 0$. Hence $L \cap (S + T) = 0$. ☐

Recall that the sum of a collection of subobjects $C_i \longrightarrow C$ is said to be *direct* if the induced morphism out of the coproduct $\bigoplus_{i\in I} C_i \longrightarrow C$ is a monomorphism. In that case we have a canonical isomorphism $\bigoplus_{i\in I} C_i \longrightarrow \sum_{i\in I} C_i$ and we also say that the sum $\sum_{i\in I} C_i$ is direct. Equivalently, the morphisms $C_i \longrightarrow \sum_{i\in I} C_i$ are a coproduct.

Proposition 101. *Let $C_1, \ldots, C_n$ be subobjects of C in an abelian category $\mathcal{A}$. The sum is direct if and only if for $i = 1, \ldots, n$*

$$C_i \cap \left(\sum_{i\neq j} C_j \right) = 0$$

Corollary 102. *Let $\mathcal{A}$ be a grothendieck abelian category and $\{C_i\}_{i\in I}$ a family of subobjects of C (not necessarily finite). Then the sum $\sum C_i$ is direct if and only if for each finite subset $J \subseteq I$ and $k \notin J$,*

$$C_k \cap \left(\sum_{j\in J} C_j \right) = 0$$

Proposition 103. *Let $\mathcal{A}$ be a grothendieck abelian category and suppose that $M = \sum_{i\in I} S_i$ is a nonempty sum of simple subobjects. If L is a subobject of M then there is a nonempty subset $J \subseteq I$ such that*

$$M = L \oplus \bigoplus_{j\in J} S_j$$

Proof. Let $\mathcal{S}$ be the collection of all subsets $J \subseteq I$ with the following list of properties:

(i) The sum $\sum_{j\in J} S_j$ is direct.

(ii) The sum of L and $\sum_{j\in J} S_j$ is direct.

Partially order the set $\mathcal{S}$ by inclusion. The empty set clearly belongs to $\mathcal{S}$, so $\mathcal{S}$ is nonempty. If the collection $\{J_k\}_{k\in K}$ form a chain in $\mathcal{S}$, put $J' = \bigcup_k J_k$. By the previous Corollary, the sum $\sum_{j\in J'} S_j$ is direct, so J' has property (i). To see that it satisfies (ii), notice that the sums $\sum_{j\in J_k} S_j$ form a directed family of subobjects of M whose union is $\sum_{j\in J'} S_j$.

$$L \cap \sum_{j\in J'} S_j = L \cap \sum_k \sum_{j\in J_k} S_j = \sum_k \left(L \cap \sum_{j\in J_k} S_j \right) = 0$$

Hence $J' \in \mathcal{S}$. By Zorn's lemma there is a maximal element J of $\mathcal{S}$. If $L + \sum_{j\in J} S_j$ were not equal to M, then there would be some S_q not contained in it. Hence

$$S_q \cap \left(L + \sum_{j\in J} S_j \right) = 0$$

since S_q is simple. It follows from Lemma 100 that

$$L \cap \left(\sum_{j\in J'} S_j + S_q \right) = 0$$

which contradicts the maximality of J. Hence M is the sum of L and $\sum_{j\in J} S_j$. Since all the involved sums are direct, this is same as $M = L \oplus \bigoplus_{j\in J'} S_j$. □

Corollary 104. *Let $\mathcal{A}$ be a grothendieck abelian category and suppose that $M = \sum_{i\in I} S_i$ is a nonempty sum of simple subobjects. Then there is a nonempty subset $J \subseteq I$ such that*

$$M = \bigoplus_{j\in J} S_j$$

Proof. That is, we can find a subcollection of the S_i for which the sum is direct, and is still all of M. For the proof, just put $L = 0$ in Proposition 103. □

6 Injectives

Throughout this section $\mathcal{A}$ is a fixed abelian category. For omitted proofs see [Mit65].

Definition 65. A monomorphism $X \longrightarrow A$ is an *essential extension* if for any nonzero subobject $Y \longrightarrow A$ we have $X \cap Y \neq 0$. We say $X \longrightarrow A$ is a *proper extension* if it is not an isomorphism.

Lemma 105. *If $\mathcal{A}$ is grothendieck abelian then an object Q is injective if and only if it admits no proper essential extensions.*

Definition 66. An *injective envelope* for an object A is an essential extension $A \longrightarrow Q$ with Q injective. In light of the next result, this object is unique up to (noncanonical) isomorphism, and we will sometimes denote it by $E(A)$.

Lemma 106. *Let $u : A \longrightarrow Q$ and $u' : A \longrightarrow Q'$ be injective envelopes of A. Then there is an isomorphism (not necessarily unique) $\theta : Q \longrightarrow Q'$ such that $\theta u = u'$.*

Proposition 107. *If $\mathcal{A}$ is grothendieck abelian then every object has an injective envelope.*

Remark 12. Let $A \longrightarrow Q$ be an injective envelope, and $A \longrightarrow I$ a monomorphism with I injective. Then there is a morphism $Q \longrightarrow I$ such that the following diagram commutes

$$\begin{array}{ccc} A & \longrightarrow & Q \\ & \searrow & \downarrow \\ & & I \end{array}$$

and $Q \longrightarrow I$ is a monomorphism since $A \longrightarrow Q$ is essential. So an injective envelope embeds in any injective object containing A.

Lemma 108. *Let $\mathcal{A}$ be a grothendieck abelian category, $M \longrightarrow M'$ and $N \longrightarrow N'$ subobjects, and let $p_1 : M' \oplus N' \longrightarrow M'$ and $p_2 : N' \oplus M' \longrightarrow N'$ be the projections. Then*

$$p_1^{-1}(M) \cap p_2^{-1}(N) = M \oplus N$$

Lemma 109. *Suppose we are given a morphism $f : A \longrightarrow B$ and subobjects $A' \longrightarrow A, B' \longrightarrow B$. Then*

$$f(A') \cap B' = 0 \Longleftrightarrow f^{-1}(B') \cap A' = 0$$

Proposition 110. *Let $\mathcal{A}$ be grothendieck abelian and suppose we are given objects $C_1, \ldots, C_n$. The monomorphism $C_1 \oplus \cdots \oplus C_n \longrightarrow E(C_1) \oplus \cdots \oplus E(C_n)$ induces an isomorphism*

$$E(C_1 \oplus \cdots \oplus C_n) \longrightarrow E(C_1) \oplus \cdots \oplus E(C_n)$$

Definition 67. An nonzero object $C \in \mathcal{A}$ is called *indecomposable* if it cannot be written as the direct sum of two nonzero subobjects. A subobject $B \longrightarrow C$ is *irreducible in* C if it cannot be written as the intersection of two strictly bigger subobjects of C. We say that C is *coirreducible* if any two nonzero subobjects of C have a nonzero intersection. It is clear that B is irreducible in C if and only if C/B is a coirreducible object.

Proposition 111. *Let $\mathcal{A}$ be grothendieck abelian and E injective. Then the following are equivalent:*

(a) E is indecomposable.

(b) Each subobject of E is coirreducible.

(c) E is an injective envelope of a coirreducible object.

(d) E is an injective envelope of each one of its nonzero subobjects.

For locally noetherian categories there is a nice decomposition theory for injective objects.

Proposition 112. *Let $\mathcal{A}$ be a locally finitely generated category. Then $\mathcal{A}$ is locally noetherian if and only if every coproduct of injective objects is injective.*

Proof. Suppose that $\mathcal{A}$ is locally noetherian and let $\{E_i\}_{i \in I}$ be a family of injective objects. To show that $\oplus_i E_i$ is injective it suffices by (AC,Proposition 50) to consider a monomorphism $\alpha : B \longrightarrow C$ of noetherian objects and extend every morphism $\varphi : B \longrightarrow \oplus_i E_i$. But B is finitely generated so φ factors through a finite subcoproduct $B \longrightarrow \oplus_{i=1}^n E_{i_n}$. This can clearly be extended to a morphism on C, and the composite $C \longrightarrow \oplus_{i=1}^n E_{i_n} \longrightarrow \oplus_i E_i$ extends φ as required.

Assume conversely that every coproduct of injective objects is injective. We will show that every finitely generated object C is noetherian. Suppose there exists a strictly ascending chain $C_1 < C_2 < \cdots$ of subobjects of C and let $E_i = E(C/C_i)$ be the injective envelopes. By hypothesis the object $E = \oplus_{j=1}^\infty E_i$ is injective. For each C_n and $j \leq n$ we let $\varphi_{nj} : C_n \longrightarrow E$ denote the composite

$$C_n \longrightarrow C \longrightarrow C/C_j \longrightarrow E_j \longrightarrow E$$

and set $\varphi_n = \sum_{j=1}^n \varphi_{nj}$. These morphisms are compatible with the inclusions in the ascending sequence, so we have an induced morphism $\varphi : \sum_n C_n \longrightarrow E$ which lifts by injectivity to a morphism $\Phi : C \longrightarrow E$. By hypothesis C is finitely generated, therefore compact, so Φ factors through a finite subcoproduct $E_1 \oplus \cdots \oplus E_k$. It follows that $C_{k+1} = C_{k+2}$ as subobjects of C, which is the desired contradiction. □

Definition 68. Let $\mathcal{C}$ be a category and X an object of $\mathcal{C}$. We say that X is *compact for injectives* if every morphism $X \longrightarrow \oplus_\lambda I_\lambda$ from X to an arbitrary nonempty coproduct of injective objects in $\mathcal{C}$ factors through a finite subcoproduct.

Corollary 113. *Let $\mathcal{A}$ be a locally noetherian category. For $C \in \mathcal{A}$ the following conditions are equivalent:*

(i) C is finitely generated.

(ii) C is noetherian.

(iii) C is compact.

(iv) C is compact for injectives.

Proof. The only nontrivial implication is $(iv) \Rightarrow (ii)$. If C is compact for injectives then the second part of the proof of Proposition 112 shows that C is noetherian. □

Proposition 114 (Matlis). *If $\mathcal{A}$ is a locally noetherian category then every nonzero injective object is a coproduct of indecomposable injective objects.*

References

[Bor94] Francis Borceux, *Handbook of categorical algebra. 1*, Encyclopedia of Mathematics and its Applications, vol. 50, Cambridge University Press, Cambridge, 1994, Basic category theory. MR MR1291599 (96g:18001a)

[Gro57] Alexander Grothendieck, *Sur quelques points d'algèbre homologique*, Tôhoku Math. J. (2) **9** (1957), 119–221. MR MR0102537 (21 #1328)

[Mac50] Saunders MacLane, *Duality for groups*, Bull. Amer. Math. Soc. **56** (1950), 485–516. MR MR0049192 (14,133d)

[Mit65] Barry Mitchell, *Theory of categories*, Pure and Applied Mathematics, Vol. XVII, Academic Press, New York, 1965. MR MR0202787 (34 #2647)

[Sch72] Horst Schubert, *Categories*, Springer-Verlag, New York, 1972, Translated from the German by Eva Gray. MR MR0349793 (50 #2286)

[Ste75] Bo Stenström, *Rings of quotients*, Springer-Verlag, New York, 1975, Die Grundlehren der Mathematischen Wissenschaften, Band 217, An introduction to methods of ring theory. MR MR0389953 (52 #10782)