Ample Sheaves and Ample Families

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In this short note we recall the definition of an *ample sheaf* and an *ample family* of sheaves. Our notes on ample sheaves are essentially just a translation of EGA II 4.5.2, while the notes on ample families are based on Illusie's SGA 6 Exposé II 2.2.3 and parts of Thomason & Trobaugh's "Higher Algebraic K-Theory of Schemes and of Derived Categories".

Contents

1 Ample Sheaves

Let X be a scheme and $\mathscr{L}$ an invertible sheaf. Given a global section $f \in \Gamma(X, \mathscr{L})$ the set $X_f = \{x \in X \,|\, germ_x f \notin \mathfrak{m}_x \mathscr{L}_x\}$ is open (MOS,Lemma 29). The inclusion $X_f \longrightarrow X$ is affine (RAS,Lemma 6) and in particular if X is an affine scheme then X_f is itself affine. Given a sequence of global sections $f_1, \ldots, f_n$ the open sets X_{f_i} cover X if and only if the f_i generate $\mathscr{L}$ (MOS,Lemma 32).

Lemma 1. *Let $(X, \mathcal{O}_X)$ be a quasi-compact ringed space and $\mathscr{F}$ a sheaf of modules of finite type. If $\mathscr{F}$ is generated by global sections then it can be generated by a* finite *number of global sections.*

Proof. See (MOS,Definition 2) for the definition of a sheaf of modules of *finite type*. Let $\{s_i\}_{i \in I}$ be a nonempty family of global sections of $\mathscr{F}$ which generate. Let Λ be the set of all finite subsets of I and for each $\lambda \in \Lambda$ let $\mathscr{F}_\lambda$ be the submodule of $\mathscr{F}$ generated by the s_i belonging to λ. This is a direct family of submodules and clearly $\mathscr{F} = \varinjlim_\lambda \mathscr{F}_\lambda$, so it follows from (MOS,Lemma 57) that $\mathscr{F} = \mathscr{F}_\lambda$ for some λ. In other words, $\mathscr{F}$ can be generated by a finite number of global sections. $\square$

Lemma 2. *Let $(X, \mathcal{O}_X)$ be a quasi-noetherian ringed space, $\mathscr{F}$ a sheaf of modules on X and $\{\mathscr{F}_\alpha\}_{\alpha \in \Lambda}$ a direct family of submodules of $\mathscr{F}$. If $U \subseteq X$ is quasi-compact then*

$$\Gamma\left(U, \sum_\alpha \mathscr{F}_\alpha\right) = \sum_\alpha \Gamma(U, \mathscr{F}_\alpha)$$

Proof. By (COS,Proposition 23) the functor $\Gamma(U, -) : \mathfrak{Mod}(X) \longrightarrow \mathcal{O}_X(U)\mathbf{Mod}$ preserves direct limits, and since $\mathfrak{Mod}(X)$ and $\mathbf{Ab}$ are both grothendieck abelian the direct limit of a direct family of subobjects is just their categorical union (that is, their internal sum). One can also give a direct proof along the lines of (SGR,Lemma 12). $\square$

Given a scheme X and a fixed invertible sheaf $\mathscr{L}$, we define $\mathscr{F}(n) = \mathscr{F} \otimes \mathscr{L}^{\otimes n}$ for any $n \in \mathbb{Z}$. This notation does not reflect the sheaf $\mathscr{L}$, but this is unlikely to ever cause any confusion. Note that if $\mathscr{F}$ is a sheaf of modules of finite type then the same is true of $\mathscr{F}(n)$ for any $n \in \mathbb{Z}$.

Proposition 3. *Let X be a concentrated scheme and $\mathscr{L}$ an invertible sheaf on X. The following conditions are equivalent:*

(a) The open sets X_f for all $f \in \Gamma(X, \mathscr{L}^{\otimes n})$ with $n > 0$ form a basis for X.

(b) There are sections $f \in \Gamma(X, \mathscr{L}^{\otimes n})$ with $n > 0$ such that the X_f form an affine basis for X.

(c) There are sections $f \in \Gamma(X, \mathscr{L}^{\otimes n})$ with $n > 0$ such that the X_f form an affine cover for X.

(d) For any quasi-coherent sheaf $\mathscr{F}$ and $n > 0$ let $\mathscr{F}_n$ denote the submodule of $\mathscr{F}(n)$ generated by elements of $\Gamma(X, \mathscr{F}(n))$. Then $\mathscr{F}$ is the sum of the submodules $\mathscr{F}_n(-n)$ for $n > 0$.

(d') The property (d) for every quasi-coherent sheaf of ideals *on X.*

(e) For any quasi-coherent sheaf $\mathscr{F}$ of finite type, there exists $N > 0$ such that for all $n \geq N$ the sheaf $\mathscr{F}(n)$ is generated by global sections.

(f) For any quasi-coherent sheaf $\mathscr{F}$ of finite type, there exist integers $n > 0$ and $k > 0$ such that $\mathscr{F}$ is a quotient of $\mathscr{L}^{\otimes(-n)} \otimes \mathcal{O}_X^k$.

(f') The property (f) for every quasi-coherent sheaf of ideals *of finite type.*

Proof. $(a) \Rightarrow (b)$ Given a point x find an affine open neighborhood $x \in U$ and $f \in \Gamma(X, \mathscr{L}^{\otimes n})$ with $x \in X_f \subseteq U$. The inclusion $X_f \longrightarrow X$ is affine so X_f must be affine. $(b) \Rightarrow (c)$ is trivial. $(c) \Rightarrow (d)$ The invertible sheaf $\mathscr{L}^{\otimes(-n)}$ is flat, so $\mathscr{F}_n(-n)$ is a submodule of $\mathscr{F}(n)(-n) \cong \mathscr{F}$. Let global sections $f_i \in \Gamma(X, \mathscr{L}^{\otimes k_i})$ for various $k_i \geq 1$ be given, such that the X_{f_i} are an affine open cover of X. Fix one of these global sections $f \in \Gamma(X, \mathscr{L}^{\otimes k})$. Any quasi-coherent sheaf on an affine scheme is generated by its global sections, so $\mathscr{F}|_{X_f}$ can be generated by global sections. Since X is concentrated we can apply (H, II.5.14) (see also EGAI 9.3.1) to see that every element of $\Gamma(X_f, \mathscr{F})$ corresponds under the canonical isomorphism $\mathscr{F} \cong \mathscr{F}(km)(-km)$ to a section of the form

$$t|_{X_f} \dot{\otimes} (f|_{X_f}^{-1})^{\otimes m}$$

for some $m > 0$ and $t \in \Gamma(X, \mathscr{F}(km))$. In other words, every element of $\Gamma(X_f, \mathscr{F})$ belongs to $\mathscr{F}_{km}(-km)$ for some $m > 0$. It follows that $\mathscr{F}$ is the sum of the submodules $\mathscr{F}_n(-n), n > 0$, as required. $(d) \Rightarrow (d')$ is trivial. $(d') \Rightarrow (a)$ Given an open set $U \subseteq X$ and $x \in U$ let $\mathscr{I}$ be the ideal sheaf of the closed set $X \setminus U$ (SI,Definition 1). This is quasi-coherent and moreover

$$X \setminus U = Supp(\mathcal{O}_X / \mathscr{I})$$

so $\mathscr{I}_x = \mathcal{O}_{X,x}$. We can therefore find $n > 0$ and $f \in \Gamma(X, \mathscr{I}(n))$ such that $f(x) \neq 0$ (meaning $germ_x f \notin \mathfrak{m}_x \mathscr{I}(n)_x$). Since $\mathscr{I}(n) \subseteq \mathscr{L}^{\otimes n}$ this f can be considered as a global section in $\Gamma(X, \mathscr{L}^{\otimes n})$. By construction $x \in X_f \subseteq U$ so the proof is complete (we have $X_f \subseteq U$ since outside U, $\mathscr{I}_y \subseteq \mathfrak{m}_y$).

$(c) \Rightarrow (e)$ Since X is quasi-compact we can find a finite number of $f_i \in \Gamma(X, \mathscr{L}^{\otimes n_i})$ such that the X_{f_i} are affine and cover X. By (MOS,Lemma 38) we may as well assume all these powers n_i are equal to a single integer $k > 0$. For each i the restriction $\mathscr{F}|_{X_{f_i}}$ can by Lemma 1 be generated by a finite number of global sections h_{ij}, and by EGAI 9.3.1 there is $m_{ij} > 0$ such that $h_{ij} \dot{\otimes} (f_i|_{X_{f_i}})^{\otimes m_{ij}}$ can be lifted to a global section t_{ij} of $\mathscr{F}(km_{ij})$. Once again we can assume the m_{ij} do not vary with i or j, so they are all equal to some fixed $m_0 > 0$. The germ of f_i at any point of X_{f_i} is a unit, so it follows that the global sections $t_{ij} \in \Gamma(X, \mathscr{F}(km_0))$ generate $\mathscr{F}(km_0)$. In fact the argument shows that $\mathscr{F}(km)$ is generated by global sections for any $m \geq m_0$.

The sheaves $\mathscr{F}(1), \ldots, \mathscr{F}(k-1)$ are also quasi-coherent of finite type, so we can apply the same process to these sheaves and increase m_0 if necessary to work for all of them simultaneously. That is, for $m \geq m_0$ the sheaves

$$\mathscr{F}(km), \mathscr{F}(km+1), \ldots, \mathscr{F}(km+k-1)$$

are generated by global sections. Clearly then $\mathscr{F}(t)$ is generated by global sections for all $t \geq km_0$, as required. $(e) \Rightarrow (f)$ and $(f) \Rightarrow (f')$ are trivial.

$(f') \Rightarrow (d')$ Suppose that (d') holds for all quasi-coherent sheaves of ideals of finite type, and let $\mathscr{I}$ be a quasi-coherent sheaf of ideals. By (MOS,Corollary 64) we can write $\mathscr{I}$ as the sum $\sum_\beta \mathscr{I}_\beta$ of its quasi-coherent submodules of finite type. It follows that for any $n > 0$

$$\mathscr{I}(n) = \sum_\beta \mathscr{I}_\beta(n)$$

and hence by Lemma 2 we have $\mathscr{I}_n = \sum_\beta \mathscr{I}_{\beta,n}$. Twisting back we deduce $\mathscr{I}_n(-n) = \sum_\beta \mathscr{I}_{\beta,n}(-n)$ and summing over n shows that $\mathscr{I}$ is the sum of its submodules $\mathscr{I}_n(-n)$. So it suffices to prove (d') for ideals of finite type. But in this case by (f') there is $n > 0$ such that $\mathscr{I}(n)$ is generated by a finite number of global sections, so $\mathscr{I}_n(-n) = \mathscr{I}(n)(-n) = \mathscr{I}$ and $\mathscr{I}$ is trivially the sum of all its submodules $\mathscr{I}_n(-n)$ for $n > 0$. $\square$

Definition 1. Let X be a concentrated scheme and $\mathscr{L}$ an invertible sheaf on X. We say that $\mathscr{L}$ is *ample* if it satisfies the equivalent conditions of Proposition 3. This property is stable under isomorphism.

It is worth checking that the present definition of an ample sheaf agrees with the one given in Hartshorne, which occurs in our notes as (PM,Definition 6).

Lemma 4. *Let X be a noetherian scheme and $\mathscr{L}$ an invertible sheaf on X. Then $\mathscr{L}$ is ample if and only if for every coherent sheaf $\mathscr{F}$ there is $N > 0$ such that $\mathscr{F} \otimes \mathscr{L}^{\otimes n}$ is generated by global sections for $n \geq N$.*

Proof. Suppose that $\mathscr{L}$ is ample in the sense of Definition 1. Any coherent sheaf $\mathscr{F}$ is quasi-coherent of finite type because it is locally finitely presented (MOS,Lemma 34), so Proposition 3(e) gives the desired property. Conversely, suppose that $\mathscr{L}$ is ample in the sense of Hartshorne and let $\mathscr{I}$ be a quasi-coherent sheaf of ideals. This is trivially coherent, so it is easy to check that condition (f) of Proposition 3 is satisfied for the sheaf $\mathscr{I}$. Hence $\mathscr{L}$ is ample in the new sense, and the proof is complete. $\square$

Example 1. Here are some examples of ample invertible sheaves:

- On an affine scheme any invertible sheaf is ample, because any quasi-coherent sheaf is generated by its global sections.
- Let $X = \mathbb{P}^n_k$ where k is a field. Up to isomorphism the only invertible sheaves on X are the twisting sheaves $\mathcal{O}(\ell)$ for $\ell \in \mathbb{Z}$ (DIV,Corollary 47), and of these it is precisely the ones with $\ell > 0$ that are ample (PM,Example 2).
- Any quasi-projective scheme over a noetherian ring has an ample invertible sheaf (BU,Lemma 17).

2 Ample Families

Those schemes which admit ample invertible sheaves have many good properties. By generalising the notion of an ample sheaf to an *ample family* of sheaves, we can extend many of these good properties to a wider class of schemes. For the duration of the next proof, if we are given a family of invertible sheaves $\{\mathscr{L}_\alpha\}_{\alpha\in\Lambda}$ then we write $\mathscr{F}(\alpha, n)$ for the sheaf $\mathscr{F} \otimes \mathscr{L}_\alpha^{\otimes n}$.

Proposition 5. *Let X be a concentrated scheme and $\{\mathscr{L}_\alpha\}_{\alpha\in\Lambda}$ a nonempty family of invertible sheaves on X. The following conditions are equivalent:*

(a) The open sets X_f for all $f \in \Gamma(X, \mathscr{L}_\alpha^{\otimes n})$ with $\alpha \in \Lambda, n > 0$ form a basis for X.

(b) There is a family of sections $f \in \Gamma(X, \mathscr{L}_\alpha^{\otimes n})$ with $\alpha \in \Lambda, n > 0$ such that the X_f form an affine basis for X.

(c) There is a family of sections $f \in \Gamma(X, \mathscr{L}_\alpha^{\otimes n})$ with $\alpha \in \Lambda, n > 0$ such that the X_f form an affine cover for X.

(d) For any quasi-coherent sheaf $\mathscr{F}$ and $\alpha \in \Lambda, n > 0$ let $\mathscr{F}_{\alpha,n}$ denote the submodule of $\mathscr{F}(\alpha, n)$ generated by the elements of $\Gamma(X, \mathscr{F}(\alpha, n))$. Then $\mathscr{F}$ is the sum of the submodules $\mathscr{F}_{\alpha,n}(\alpha, -n)$ for $\alpha \in \Lambda, n > 0$.

(d') The property (d) for every quasi-coherent sheaf of ideals *on X.*

(e) For any quasi-coherent sheaf $\mathscr{F}$ of finite type there exist integers $n_\alpha, k_\alpha > 0$ and morphisms $\varphi_\alpha : \mathcal{O}_X^{k_\alpha} \longrightarrow \mathscr{F}(\alpha, n_\alpha)$ such that for every $x \in X$ some $\varphi_{\alpha,x}$ is surjective.

(e') For any quasi-coherent sheaf $\mathscr{F}$ of finite type there exist integers $n_\alpha, k_\alpha > 0$ such that $\mathscr{F}$ is a quotient of $\oplus_\alpha(\mathscr{L}_\alpha^{\otimes -n_\alpha} \otimes \mathcal{O}_X^{k_\alpha})$.

(e") The property (e') for every quasi-coherent sheaf of ideals *of finite type.*

Proof. The implications $(a) \Leftrightarrow (b) \Leftrightarrow (c) \Leftrightarrow (d) \Leftrightarrow (d')$ and $(e') \Rightarrow (e'') \Rightarrow (d')$ are either trivial or follow in exactly the same way as in the proof of Proposition 3.

$(c) \Rightarrow (e)$ Since X is quasi-compact we can find a finite number of $f_i \in \Gamma(X, \mathscr{L}_{\alpha_i}^{\otimes n_i})$ such that the X_{f_i} are affine and cover X. We can assume the n_i are all equal to a fixed integer $t > 0$. Fix an index $\alpha \in \Lambda$ occurring among the α_i and argue as in Proposition 3 part $(c) \Rightarrow (e)$ to see that we can find an integer $m_\alpha > 0$ together with a finite number of global sections of $\mathscr{F}(\alpha, tm_\alpha)$ which generate this sheaf over every open set X_{f_i} for which $\alpha_i = \alpha$. In other words, we have an integer $k_\alpha > 0$ and a morphism

$$\varphi_\alpha : \mathcal{O}_X^{k_\alpha} \longrightarrow \mathscr{F}(\alpha, tm_\alpha)$$

which is an epimorphism on stalks for every $x \in X_{f_i}$ with $\alpha_i = \alpha$. Setting $n_\alpha = tm_\alpha$ we have (e). Twisting back we have a morphism $\mathscr{L}_\alpha^{\otimes -n_\alpha} \otimes \mathcal{O}_X^{k_\alpha} \longrightarrow \mathscr{F}$ with the same property, and so the induced morphism out of the coproduct over all $\alpha \in \Lambda$ is an epimorphism, which shows that $(e) \Rightarrow (e')$ and completes the proof. ☐

Definition 2. Let X be a concentrated scheme and $\{\mathscr{L}_\alpha\}_{\alpha \in \Lambda}$ a nonempty family of invertible sheaves on X. We say that this is an *ample family* if it satisfies the equivalent conditions of Proposition 5. Clearly a single invertible sheaf $\mathscr{L}$ is ample if and only if it is an ample family.

Definition 3. We say that a concentrated scheme X is *divisorial* if there exists an ample family of invertible sheaves on X. In particular a scheme admitting an ample invertible sheaf is divisorial.

Remark 1. Recall from (DIV,Definition 12) the definition of a *locally factorial scheme*. We refer the reader to SGA for the proof of the following result: a separated noetherian scheme which is locally factorial is divisorial. In particular a regular separated noetherian scheme is divisorial, and therefore so is any nonsingular variety over a field.

Remark 2. Let X be a concentrated scheme and $\{\mathscr{L}_\alpha\}_{\alpha \in \Lambda}$ an ample family of invertible sheaves. It follows from Proposition 5(c) that we can find a finite subset $\mathscr{L}_{\alpha_1}, \ldots, \mathscr{L}_{\alpha_n}$ which is also an ample family. The advantage of having a finite family is that we can cover X with open sets on which the $\mathscr{L}_{\alpha_i}$ are simultaneously free. Hence an *arbitrary* coproduct of tensor powers of the $\mathscr{L}_{\alpha_i}$ will be a locally free sheaf (locally finitely free if the coproduct is finite).

A famous result of Serre says that on a projective scheme over a noetherian ring, every coherent sheaf is a quotient of a finite direct sum of twisting sheaves $\mathcal{O}(n)$. This result is crucial in the calculation of cohomology on a projective scheme, because it allows us to reduce to these twisting sheaves which are very well-behaved. On an arbitrary scheme with an ample family of invertible sheaves we have a similar result.

Proposition 6. *Let X be a divisorial scheme. Then*

(a) For any quasi-coherent sheaf $\mathscr{F}$ there is an epimorphism $\mathscr{E} \longrightarrow \mathscr{F}$ with $\mathscr{E}$ locally free.

(b) *For any quasi-coherent sheaf $\mathscr{F}$ of finite type there is an epimorphism $\mathscr{E} \longrightarrow \mathscr{F}$ with $\mathscr{E}$ locally finitely free.*

If $\{\mathscr{L}_\alpha\}_{\alpha\in\Lambda}$ is a finite ample family of invertible sheaves then in both cases $\mathscr{E}$ may be taken to be a coproduct of tensor powers of sheaves in the ample family.

Proof. We can by Remark 2 find a finite ample family $\{\mathscr{L}_\alpha\}_{\alpha\in\Lambda}$. If $\mathscr{F}$ is a quasi-coherent sheaf of finite type then it is immediate from Proposition 5(e') that $\mathscr{F}$ is a quotient of a finite coproduct of (negative) tensor powers of sheaves from the ample family. This is by Remark 2 a locally finitely free sheaf, which proves (b).

We can by (MOS,Corollary 64) write any quasi-coherent sheaf $\mathscr{F}$ as the sum of all its quasi-coherent submodules $\mathscr{F}_\beta$ of finite type. If we write each $\mathscr{F}_\beta$ as a quotient of a locally free sheaf $\mathscr{E}_\beta$ using (b), then it is clear that the canonical morphism $\mathscr{E} = \oplus_\beta \mathscr{E}_\beta \longrightarrow \mathscr{F}$ is an epimorphism. We can assume that $\mathscr{E}$ is a coproduct of tensor powers of the sheaves in the ample family, which ensures that $\mathscr{E}$ is locally free and proves (a). □

Corollary 7. *If X is a divisorial scheme and $\mathscr{F}$ a quasi-coherent sheaf then there is an epimorphism $\mathscr{P} \longrightarrow \mathscr{F}$ with $\mathscr{P}$ quasi-coherent and flat.*

If X is a scheme then the abelian category $\mathfrak{Mod}(X)$ is generated by the sheaves $\mathcal{O}_U$ corresponding to open subsets $U \subseteq X$ (MRS,Corollary 31). If X is concentrated then $\mathfrak{Qco}(X)$ is grothendieck abelian, and it is generated by a representative set of quasi-coherent sheaves of finite type (MOS,Proposition 66). This is a very large and impersonal set of generators, which is improved on on the next result.

Lemma 8. *Let X be a concentrated scheme and $\{\mathscr{L}_\alpha\}_{\alpha\in\Lambda}$ an ample family of invertible sheaves. The following set of quasi-coherent sheaves*

$$\mathcal{C} = \{\mathscr{L}_\alpha^{\otimes n} \,|\, \alpha \in \Lambda, n \in \mathbb{Z}\}$$

generates $\mathfrak{Qco}(X)$.

Proof. Let $\varphi : \mathscr{F} \longrightarrow \mathscr{G}$ be a nonzero morphism of quasi-coherent sheaves. By Proposition 6 there is an epimorphism $\mathscr{E} \longrightarrow \mathscr{F}$ where $\mathscr{E}$ is a coproduct of objects from $\mathcal{C}$. We deduce a morphism $\mu : \mathscr{L}_\alpha^{\otimes n} \longrightarrow \mathscr{F}$ for some α, n with $\varphi\mu \neq 0$, as required. Actually when you study the proof of Proposition 6 it is clear that the set $\{\mathscr{L}_\alpha^{\otimes n}\}_{\alpha\in\Lambda, n<0}$ actually generates $\mathfrak{Qco}(X)$, as the non-negative tensor powers are not necessary. □