Notes on Grassmannians (checked)

Let R be a commutative ring and

$$0 \longrightarrow U \xrightarrow{\alpha} V \xrightarrow{\beta} W \longrightarrow 0 \qquad (1.1)$$

an exact sequence of free R-modules, of ranks $k, n, n-k$ respectively. Observe that the following diagram commutes (both π's canonical)

$$\begin{tikzcd}
 & \Lambda^k V \otimes V \arrow[r, two heads, "\pi"] & \Lambda^{k+1} V \\
\Lambda^k U \otimes V \arrow[ur, "\Lambda^k \alpha \otimes 1"] & & \\
 & \Lambda^k U \otimes U \arrow[ul, "1 \otimes \alpha"] \arrow[uu, "\Lambda^k \alpha \otimes \alpha"'] \arrow[r, two heads, "\pi"'] & \Lambda^{k+1} U = 0 \arrow[uu, "\Lambda^{k+1}\alpha"']
\end{tikzcd} \qquad (1.2)$$

<u>Lemma</u> The sequence $0 \longrightarrow \Lambda^k U \otimes U \xrightarrow{1 \otimes \alpha} \Lambda^k U \otimes V \xrightarrow{\pi(\Lambda^k \alpha \otimes 1)} \Lambda^{k+1} V$ is exact.

<u>Proof</u> It is clear from (1.2) that $\pi \circ (\Lambda^k \alpha \otimes \alpha) = 0$. There is a nondegenerate pairing

$$\Lambda^{k+1} V \otimes \Lambda^{n-k-1} V \xrightarrow{\text{prod}} \Lambda^n V \cong R.$$

Suppose $t \otimes \eta \in \Lambda^k U \otimes V$ lies in $\operatorname{Ker}(\pi \circ \Lambda^k \alpha \otimes 1)$. Then

$$B := \Lambda^k U \otimes V \otimes \Lambda^{n-k-1} V \xrightarrow{\pi \circ (\Lambda^k \alpha \otimes 1) \otimes 1} \Lambda^{k+1} V \otimes \Lambda^{n-k-1} V \xrightarrow{\text{prod}} \Lambda^n V \cong R$$

vanishes on $t \otimes \eta \otimes \omega$ for all $\omega \in \Lambda^{n-k-1} V$. Choosing a splitting $V \cong U \oplus V/U$ of (1.1), with corresponding decomposition $\eta = (\eta_U, \eta_{V/U})$ we find that $B(t \otimes \eta_{V/U} \otimes \omega) = 0$ for all ω, but since $V/U \otimes \Lambda^{n-k-1} V/U \longrightarrow \Lambda^{n-k} V/U \cong R$ is also nondegenerate this implies $\eta_{V/U} = 0$ so $\eta \in U$ as claimed. $\square$

Now let $R = k$ be a field.

<u>Lemma</u> The function $P : G(k, V) \longrightarrow \mathbb{P}(\Lambda^k V)$, defined by

$$P(U \xrightarrow{\alpha} V) := [\, \mathrm{Im}(\Lambda^k \alpha)\,] \qquad (2.1)$$

is injective, and sends $\mathrm{span}(u_1, \dots, u_k)$ to $[u_1 \wedge \dots \wedge u_k]$.

<u>Proof</u> Given a line $\ell \subset \Lambda^k V$ denote by U_ℓ the kernel of $V \cong \ell \otimes V \longrightarrow \Lambda^k V \otimes V \xrightarrow{\text{prod}} \Lambda^{k+1} V$.
By the previous lemma if $\ell = \mathrm{Im}(\Lambda^k \alpha)$ then $U_\ell = U$. $\square$

Note that $x \in \Lambda^k V$ is in the image of P (or rather $k \cdot x$ is) if and only if x is <u>decomposable</u>, i.e. $x = v_1 \wedge \dots \wedge v_k$ for some $v_i \in V$, $1 \leq i \leq k$.
Let $(e_1, \dots, e_n)$ be an ordered basis of V so that a basis for $\Lambda^k V$ is given by the

$$e_I = e_{i_1} \wedge \dots \wedge e_{i_k} \qquad I = \{ i_1 < \dots < i_k \}.$$

The corresponding coordinates C_I on $\mathbb{P}(\Lambda^k V)$ are called <u>Plücker coordinates</u>.

<u>Lemma</u> The image of P is closed in $\mathbb{P}(\Lambda^k V)$.

<u>Proof</u> Given $x \in \Lambda^k V$ we have $x \wedge (-) : V \longrightarrow \Lambda^{k+1} V$ defined in coordinates by $x = \sum_I C_I(x) e_I$ and

$$x \wedge e_j = \sum_I C_I(x)\, e_I \wedge e_j = \sum_{j \notin I} \pm C_I(x)\, e_{I \cup \{j\}}. \qquad (2.2)$$

We may assume $e_1, \dots, e_s$ span $\mathrm{Ker}(x \wedge -) \subseteq V$. Then $C_I(x) = 0$ whenever $j \notin I$ and $1 \leq j \leq s$. Since $|I| = k$ we must have $s = \dim \mathrm{Ker}(x \wedge -) \leq k$.

Thus every I with $C_I(x) \neq 0$ contains $\{1, \dots, s\}$.

This shows $x = e_1 \wedge \cdots \wedge e_s \wedge y$ for some $y \in \Lambda^{k-s} V$ and so $\dim \mathrm{Ker}(x \wedge -) = k$ iff. x is decomposable. But then (writing x also for the line $k \cdot x$)

$$\begin{aligned} x \in \mathrm{Im}(P) &\iff x \text{ decomposable} \\ &\iff \dim \mathrm{Ker}(x\wedge -) > k-1 \\ &\iff \dim \mathrm{Im}(x \wedge -) < n-k+1 \\ &\iff \text{every } (n-k+1)\times(n-k+1) \text{ minor of the} \\ &\qquad \text{matrix of } x\wedge - \text{ in } M_{\binom{n}{k+1}, n}(k) \text{ vanishes.} \end{aligned}$$

naturally this may be checked in any coordinates

Now the $c_I \in (\Lambda^k V)^*$ are Plücker coordinates (for an arbitrary basis $e_1, \dots, e_n$ now) and the matrix of $x \wedge -$ is given by (2.2) as a matrix of linear forms in the c_I, hence the above minors are homogeneous polynomials of degree $n-k+1$ in the c_I. By construction $\mathrm{Im}(P) \subseteq \mathbb{P}(\Lambda^k V)$ is the vanishing locus of these polynomials, so we are done. $\square$

<u>Def^n</u> $G(k,V)$ is made into a projective variety so that $G(k,V) \cong \mathrm{Im} P \subseteq \mathbb{P}(\Lambda^k V)$ as varieties.

<u>Remarks</u> (1) We may write a point of $\mathbb{P}(\Lambda^k V)$ in Plücker coordinates as

$$[\cdots : x_I : \cdots] \quad I \text{ ranging over } \{ I \subseteq \{1, \dots, n\} \mid |I| = k \}.$$

Then the standard open affines are $\mathcal{U}_J := \{ [\{x_I\}_I] \mid x_J \neq 0 \}$, i.e. where $c_I \neq 0$. Given a subspace $U \subseteq V$ of dimension k, spanned by $\beta = (u_1, \dots, u_k)$ we write U as a matrix $[U]_\beta$ with rows u_i,

$$[U]_\beta = \begin{pmatrix} u_1^T \\ \vdots \\ u_k^T \end{pmatrix} = \begin{pmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ \vdots & & & \\ u_{k1} & u_{k2} & \cdots & u_{kn} \end{pmatrix}.$$

Then under P, we have

$$\begin{aligned} P(U) &= [u_1 \wedge \cdots \wedge u_k] \quad \text{means orbit of } \mathbb{k}^* \\ &= \left[\left(\textstyle\sum_{i_1} u_{1i_1} e_{i_1}\right) \wedge \cdots \wedge \left(\textstyle\sum_{i_k} u_{ki_k} e_{i_k}\right)\right] \\ &= \left[\sum_{i_1,\dots,i_k} u_{1i_1} \cdots u_{ki_k}\, e_{i_1} \wedge \cdots \wedge e_{i_k}\right] \\ &= \left[\sum_{i_1<\cdots<i_k} \sum_{\sigma \in S_k} (-1)^{|\sigma|} u_{1i_{\sigma 1}} \cdots u_{ki_{\sigma k}}\, e_{i_1} \wedge \cdots \wedge e_{i_k}\right] \\ &= \left[\sum_{\substack{I \\ |I|=k}} \det\left([U]_\beta \text{ with only cols from } I\right) e_I\right] \end{aligned}$$

the $k\times k$ minor of $[U]_\beta$ corresponding to I. write as $\det([U]_\beta^I)$

The $k\times k$ minors <u>do</u> depend on the basis, but if β' is a different basis there is $\gamma_{\beta',\beta} \in \mathbb{k}^*$ such that $\det([U]_{\beta'}^I) = \gamma_{\beta',\beta} \det([U]_\beta^I)$ for all I.

Hence $P^{-1}(\mathcal{U}_J)$ is the set of subspaces U s.t. the J-minor of $[U]_\beta$ is nonzero. If $U \in P^{-1}(\mathcal{U}_J)$ there is (by Gaussian elim) a unique basis $\beta_{U,J}$ of U such that $[U]_{\beta_{U,J}}$ has the identity matrix in the positions of J, i.e. s.t. if $J = \{j_1 < \cdots < j_k\}$ then column j_t of $[U]_{\beta_{U,J}}$ is $(0 \cdots \overset{t}{1} \cdots 0)^T$. The remaining columns we denote $\lambda_{U,J} \in M_{k,n-k}(\mathbb{k})$.

<u>Example</u> If $J = \{1,\dots,k\} \subseteq \{1,\dots,n\}$ and $U \in P^{-1}(\mathcal{U}_J)$ then

$$[U]_{\beta_{U,J}} = \begin{pmatrix} \overbrace{\begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix}}^{k} & \overbrace{\boxed{\begin{matrix} * & \cdots & * \\ \vdots & & \vdots \\ * & \cdots & * \end{matrix}}}^{n-k} \end{pmatrix} \quad \text{call this } \lambda_{U,J}$$

This establishes a bijection $P^{-1}(\mathcal{U}_J) \cong M_{k,n-k}(\mathbb{k}) \cong \mathbb{A}^{k(n-k)}$ sending $[U]$ to $\lambda_{U,J}$. We want to check this is actually an isomorphism of <u>varieties</u>.

Observe that for $1 \le i \le k$, $1 \le j \le n-k$ (still with $J = \{1, \dots, k\}$)

$$
\begin{aligned}
(\lambda_{U,J})_{ij} &= -\det\left([U]_{\beta_{U,J}}^{\{1,\dots,k\}\setminus\{i\} \cup \{k+j\}}\right) \\
&= -\gamma_{\beta_{U,J},\beta} \det\left([U]_{\beta}^{\{1,\dots,k\}\setminus\{i\}\cup\{k+j\}}\right)
\end{aligned}
$$

But we can fix $\gamma_{\beta_{U,J},\beta}$ from

$$
1 = \det\left([U]_{\beta_{U,J}}^{\{1,\dots,k\}}\right) = \gamma_{\beta_{U,J},\beta} \det\left([U]_{\beta}^{\{1,\dots,k\}}\right)
$$

$$
\Rightarrow \quad \gamma_{\beta_{U,J},\beta} = \det\left([U]_{\beta}^{\{1,\dots,k\}}\right)^{-1}
$$

Hence in terms of our original basis

$$
\begin{aligned}
(\lambda_{U,J})_{ij} &= -\gamma_{\beta_{U,J},\beta} \det\left([U]_{\beta}^{\{1,\dots,k\}\setminus\{i\}\cup\{k+j\}}\right) \\
&= -\frac{\det\left([U]_{\beta}^{\{1,\dots,k\}\setminus\{i\}\cup\{k+j\}}\right)}{\det\left([U]_{\beta}^{\{1,\dots,k\}}\right)}
\end{aligned}
$$

This shows that in the commutative diagram

$$
\begin{tikzcd}
{[U]_\beta \in M_{k,n}(k)/GL_k} \arrow[r, phantom, "\cong"] & {U \in G(k,V)} \arrow[rr, hook, "p"] & & {[\Lambda^k \alpha] \in \mathbb{P}(\Lambda^k V)} \\
{\lambda_{U,J} \in M_{k,n-k}(k)} \arrow[u, hook] \arrow[r, phantom, "\overset{\circledast}{\cong}"] \arrow[d, phantom, "\cong" rotate=90] & {p^{-1}(\mathcal{U}_J) \ni U} \arrow[u, hook] \arrow[rr, hook] & & \mathcal{U}_J \arrow[u, hook] \arrow[d, phantom, "\cong" rotate=90] \\
\mathbb{A}^{k(n-k)} \arrow[rrr, dashed, "J"] & & & \mathbb{A}^{\binom{n}{k}-1} \ni \left(\underbrace{\det\left([U]_\beta^I\right) / \det\left([U]_\beta^J\right)}_{\text{denote this coordinate } Y^I} \right)_{I \neq J}
\end{tikzcd}
$$

The induced map $\mathcal{J}$ is precisely sending the (i,j) coordinate of $\mathbb{A}^{k(n-k)}$ to the $Y^{\{1,\dots,k\}\setminus\{i\}\cup\{k+j\}}$ coordinate of $\mathbb{A}^{\binom{n}{k}-1}$, so that $\mathcal{J}$ is the closed embedding of the vanishing of all the other Y^I. The same works for $J \neq \{1,\dots,k\}$, so

<u>Lemma</u> The projective variety $G(k,V)$ is covered by open affines $P^{-1}(\mathcal{U}_J) \cong \mathbb{A}^{k(n-k)}$. i.e. as varieties

<u>Remark</u> $P: G(k,V) \longrightarrow \mathbb{P}(\Lambda^k V)$ does not depend on a choice of basis for V, but the open cover $\{\mathcal{U}_J\}_{|J|=k}$ certainly does, hence so does the cover $P^{-1}(\mathcal{U}_J)$ of $G(k,V)$.

<u>Lemma</u> $GL(V)$ acts transitively on $G(k,V)$, as an algebraic group.

If we choose a basis $(e_1,\dots,e_n)$ of V and set $V_0 = \mathrm{span}(e_1,\dots,e_k)$ then the stabiliser is

$$G_{V_0} = \{ g \in GL(V) \mid gV_0 \subseteq V_0 \} = \left\{ \left(\begin{array}{c|c} A & B \\ \hline 0 & C \end{array} \right) \in GL(V) \;\middle|\; \begin{array}{l} A \in M_k(\mathbb{k}) \\ B \in M_{k,n-k}(\mathbb{k}) \\ C \in M_{n-k,n-k}(\mathbb{k}) \end{array} \right\}$$

This is Zariski closed in $G = GL(V)$ and $G/G_{V_0} \xrightarrow{\cong} G(k,V)$ is a projective algebraic variety. Such subgroups are called <u>parabolic</u>.

<u>Def^n</u> The <u>complete flag variety</u> $F(V)$ of V is, as a set,

$$F(V) = \{ (V_1,\dots,V_{n-1}) \mid 0 \subseteq V_1 \subseteq \cdots \subseteq V_{n-1} \subseteq V \text{ and } \dim V_i = i \text{ all } i \}$$

<u>Ex</u> The obvious injective map $F(V) \longrightarrow G(1,V) \times G(2,V) \times \cdots \times G(n-1,V)$ has closed image, using which we make $F(V)$ a projective variety. There is an action of $GL(V)$ on $F(V)$ via $(V_1,\dots,V_{n-1}) \mapsto (gV_1,\dots,gV_{n-1})$.

If we set $V_i^\circ = \mathrm{span}(e_1, \dots, e_i)$ and $F_0 = (V_1^\circ, \dots, V_{n-1}^\circ)$ then the action of $GL(V)$ on $F(V)$ is transitive and hence

$$G/G_{F_0} \xrightarrow{\cong} F(V)$$

where $G_{F_0} = \left\{ \begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ 0 & & * \end{pmatrix} \in GL(V) \right\}$. This is called a <u>Borel subgroup</u>.

<u>Remark</u> For a fixed basis as above, and corresponding open cover $P^{-1}(\mathcal{U}_J)$ of $G(k,V)$, observe that for $V_k^\circ = \mathrm{span}(e_1, \dots, e_k)$ we have the subgroup

$$H = \begin{pmatrix} I_k & 0 \\ * & I_{n-k} \end{pmatrix} \subseteq GL(V)$$

acting on $V_k^\circ \in G(k,V)$ by

$$\begin{pmatrix} I_k & 0 \\ * & I_{n-k} \end{pmatrix} \cdot V_k^\circ = \begin{pmatrix} I_k & 0 \\ * & I_{n-k} \end{pmatrix} \begin{pmatrix} I_k \\ 0 \end{pmatrix} = \begin{pmatrix} I_k \\ * \end{pmatrix}$$

In our earlier notation these were all the points of $P^{-1}(\mathcal{U}_J) \cong M_{k,n-k}(\mathbb{k})$ (now transposed to $M_{n-k,k}(\mathbb{k})$), that is, the orbit $H \cdot V_k^\circ$ is $P^{-1}(\mathcal{U}_J)$, for $J = \{1, \dots, k\}$. The induced bijection $H \xrightarrow{\cong} P^{-1}(\mathcal{U}_J)$, $h \mapsto h \cdot V_k^\circ$ is actually an isomorphism of varieties but identifies the group structure on H with the (abelian, additive) structure on $M_{n-k,k}(\mathbb{k})$.