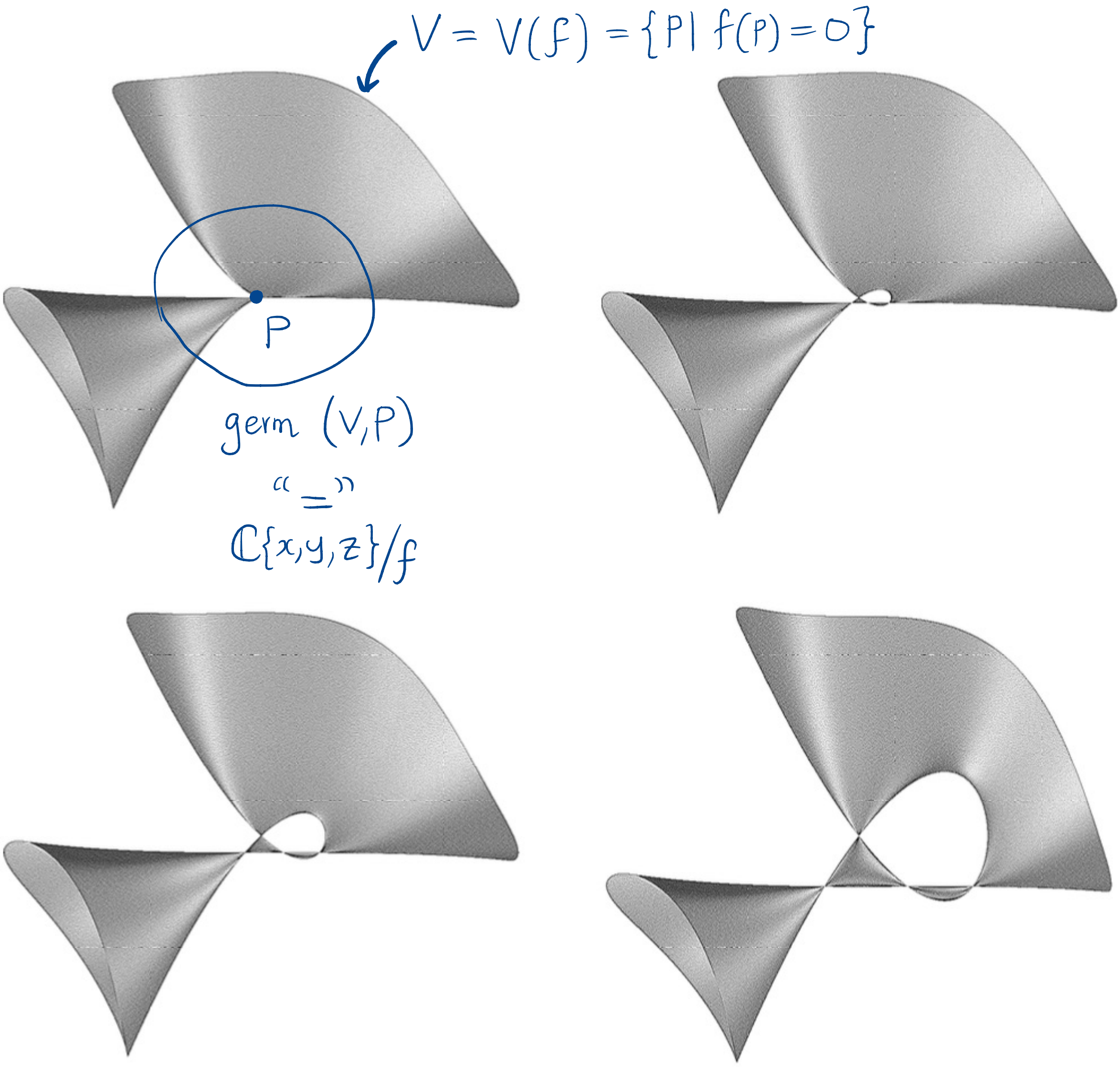
LG/CFT seminar 1 : Introduction to Singularity Theory

In this lecture I will give an introduction to singularity theory using [GLS] as the primary reference. The emphasis will be on complex analytic geometry. For a brief reminder about how critical points and singularities arise in the context of nonlinear dynamical systems see [CT1], and for a higher-categorical context [CT2, CT3]. I'm just going to completely steal the introduction of [GLS] because it is perfect.



A deformation of a simple surface singularity of type E_7 into four A_1-singularities. The family is defined by the equation

$$F(x,y,z;t) = z^2 - \left(x + \sqrt{\frac{4t^3}{27}}\right) \cdot \left(x^2 - y^2(y+t)\right).$$

The pictures[1] show the surface obtained for $t=0$, $t=\frac{1}{4}$, $t=\frac{1}{2}$ and $t=1$.

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Singularity Theory

> *"The theory of singularities of differentiable maps is a rapidly developing area of contemporary mathematics, being a grandiose generalization of the study of functions at maxima and minima, and having numerous applications in mathematics, the natural sciences and technology (as in the so-called theory of bifurcations and catastrophes)."*
> V.I. Arnol'd, S.M. Guzein-Zade, A.N. Varchenko [AGV].

The above citation describes in a few words the essence of what is called today often "singularity theory". A little bit more precisely, we can say that the subject of this relatively new area of mathematics is the study of systems of finitely many differentiable, or analytic, or algebraic, functions in the neighbourhood of a point where the Jacobian matrix of these functions is not of locally constant rank. The general notion of a "singularity" is, of course, much more comprehensive. Singularities appear in all parts of mathematics, for instance as zeroes of vector fields, or points at infinity, or points of indeterminacy of functions, but always refer to a situation which is not regular, that is, not the usual, or expected, one.

In the first part of this book, we are mainly studying the singularities of systems of complex analytic equations,

$$\begin{aligned} f_1(x_1,\ldots,x_n) &= 0\,, \\ \vdots \qquad\quad & \ \ \vdots \\ f_m(x_1,\ldots,x_n) &= 0\,, \end{aligned} \tag{0.0.1}$$

where the f_i are holomorphic functions in some open set of $\mathbb{C}^n$. More precisely, we investigate geometric properties of the solution set $V = V(f_1,\ldots,f_m)$ of a system (0.0.1) in a small neighbourhood of those points, where the analytic set V fails to be a complex manifold. In algebraic terms, this means to study *analytic* $\mathbb{C}$*-algebras*, that is, factor algebras of power series algebras over the field of complex numbers. Both points of view, the geometric one and the algebraic one, contribute to each other. Generally speaking, we can say that geometry provides intuition, while algebra provides rigour.

3

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Of course, the solution set of the system (0.0.1) in a small neighbourhood of some point $\boldsymbol{p} = (p_1, \dots, p_n) \in \mathbb{C}^n$ depends only on the ideal I generated by $f_1, \dots, f_m$ in $\mathbb{C}\{\boldsymbol{x} - \boldsymbol{p}\} = \mathbb{C}\{x_1 - p_1, \dots, x_n - p_n\}$. Even more, if J denotes the ideal generated by $g_1, \dots, g_\ell$ in $\mathbb{C}\{\boldsymbol{x} - \boldsymbol{p}\}$, then the Hilbert-Rückert Nullstellensatz states that $V(f_1, \dots, f_m) = V(g_1, \dots, g_\ell)$ in a small neighbourhood of $\boldsymbol{p}$ iff $\sqrt{I} = \sqrt{J}$. Here, $\sqrt{I} := \big\{ f \in \mathbb{C}\{\boldsymbol{x} - \boldsymbol{p}\} \;\big|\; f^r \in I \text{ for some } r \geq 0 \big\}$ denotes the *radical* of I.

Of course, this is analogous to Hilbert's Nullstellensatz for solution sets in $\mathbb{C}^n$ of complex polynomial equations and for ideals in the polynomial ring $\mathbb{C}[\boldsymbol{x}] = \mathbb{C}[x_1, \dots, x_n]$. The Nullstellensatz provides a bridge between algebra and geometry.

The somewhat vague formulation "a sufficiently small neighbourhood of $\boldsymbol{p}$ in V" is made precise by the concept of the *germ* $(V, \boldsymbol{p})$ *of the analytic set* V at $\boldsymbol{p}$. Then the Hilbert-Rückert Nullstellensatz can be reformulated by saying that two analytic functions, defined in some neighbourhood of $\boldsymbol{p}$ in $\mathbb{C}^n$, define the same function on the germ $(V, \boldsymbol{p})$ iff their difference belongs to $\sqrt{I}$. Thus, the algebra of complex analytic functions on the germ $(V, \boldsymbol{p})$ is identified with $\mathbb{C}\{\boldsymbol{x} - \boldsymbol{p}\}/\sqrt{I}$.

However, although I and $\sqrt{I}$ have the same solution set, we loose information when passing from I to $\sqrt{I}$. This is similar to the univariate case, where the sets $V(x)$ and $V(x^k)$ coincide, but where the zero of the polynomial x, respectively x^k, is counted with multiplicity 1, respectively with multiplicity k. The significance of the multiplicity becomes immediately clear if we slightly "deform" x, resp. x^k: while $x - t$ has only one root, $(x - t)^k$ has k different roots for small $t \neq 0$. The notion of a *complex space germ* generalizes the notion of a germ of an analytic set by taking into account these multiplicities. Formally, it is just a pair, consisting of the germ $(V, \boldsymbol{p})$ and the algebra $\mathbb{C}\{\boldsymbol{x} - \boldsymbol{p}\}/I$. As $(V, \boldsymbol{p})$ is determined by I, analytic $\mathbb{C}$-algebras and germs of complex spaces essentially carry the same information (the respective categories are equivalent). One is the algebraic, respectively the geometric, mirror of the other. In this book, the word "singularity" will be used as a synonym for "complex space germ".

Let $U \subseteq \mathbb{C}^n$ be open and $f : U \longrightarrow \mathbb{C}$ holomorphic, then for $P \in U$ we have in local coordinates $u_i = x_i - P_i$ an (absolutely) convergent series in an open neighborhood of P

$$f(u) = \sum_{\alpha \in \mathbb{N}^n} c_\alpha u^\alpha \qquad u^\alpha = u_1^{\alpha_1} \cdots u_n^{\alpha_n} \qquad (4.1)$$

A typical open neighborhood is a polydisc $W_\varepsilon = \{ z \in \mathbb{C}^n \mid |z_i - P_i| < \varepsilon \text{ for } 1 \le i \le n \}$ and so absolute convergence of (4.1) means that for all ε sufficiently small and positive $\sum_\alpha |c_\alpha| \varepsilon^\alpha < \infty$

<u>Def^n</u> Let $P \in \mathbb{C}^n$ and let $(U, f), (V, g)$ be pairs where U, V are open $P \in U \cap V$ and $f : U \to \mathbb{C}$, $g : V \to \mathbb{C}$ holomorphic. We say $(U, f) \sim (V, g)$ if there exists W open containing P with $W \subseteq U \cap V$ and $f|_W = g|_W$. Denote by $\mathcal{G}_P$ the set of equivalence classes of this relation.

<u>Exercise L1-1</u> Prove $\mathcal{G}_P$ is a $\mathbb{C}$-algebra with $[(U,f)] \cdot [(V,g)] = [(U \cap V, fg)]$ and $[(U,f)] + [(V,g)] = [(U \cap V, f+g)]$, and that $m_P = \{ [(U,f)] \in \mathcal{G}_P \mid f(P) = 0 \}$ is the unique maximal ideal. That is, $\mathcal{G}_P$ is a <u>local</u> $\mathbb{C}$-algebra.

<u>Def^n</u> A formal power series $p(x) = \sum_{\alpha \in \mathbb{N}^n} c_\alpha x^\alpha \in \mathbb{C}[[x]] = \mathbb{C}[[x_1, \ldots, x_n]]$ is called <u>convergent</u> if there exists $\varepsilon \in (\mathbb{R}_{>0})^n$ such that $\sum_\alpha |c_\alpha| \varepsilon^\alpha < \infty$. The $\mathbb{C}$-subalgebra of convergent power series is denoted $\mathbb{C}\{x\} \subseteq \mathbb{C}[[x]]$.

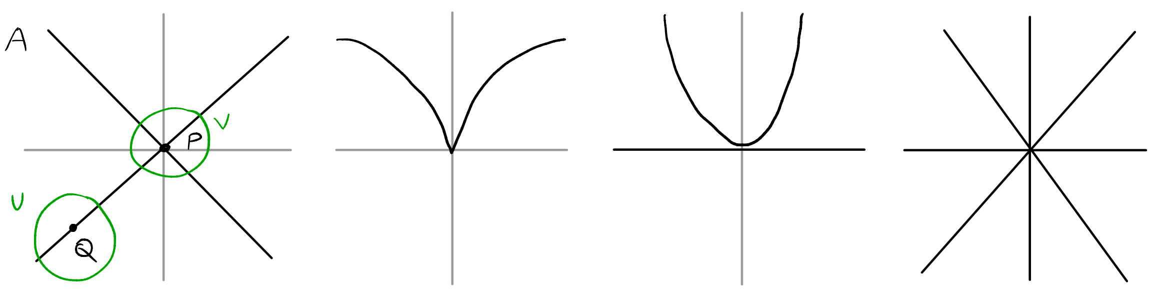
<u>Lemma L1-1</u> The function $\mathcal{G}_P \longrightarrow \mathbb{C}[[x]]$ sending (U, f) to the formal power series $\sum_\alpha c_\alpha x^\alpha$ with c_α as in (4.1) is well-defined and injective and the image is $\mathbb{C}\{x\}$. Hence as local $\mathbb{C}$-algebras $\mathcal{G}_P \cong \mathbb{C}\{x\}$.

Identical statements apply for real analytic functions, and $\mathcal{G}_P^{\mathbb{R}} \cong \mathbb{R}\{x\}$.

With $K=\mathbb{R}$ or $\mathbb{C}$ let $K\langle x\rangle$ denote $\mathbb{R}\{x\}$ or $\mathbb{C}\{x\}$. This is a local Noetherian K-algebra [GLS, Theorem 1.15] and an integral domain (in fact a factorial ring).
A K-algebra A is called <u>analytic</u> if it is of the form $K\langle x\rangle/I$ for $n\geq 0$ and an ideal I.
Hence analytic K-algebras are local and Noetherian.

Analytic spaces vs analytic algebras

Given $D\subseteq\mathbb{C}^n$ open an <u>analytic set</u> $A\subseteq D$ is a subset locally given by the vanishing of finitely many holomorphic functions. The simplest examples are <u>hypersurfaces</u> $V(f)=\{P\in D \mid f(P)=0\}$ defined by a single function



$x^2-y^2=0$ ① $\qquad x^2-y^3=0$ ② $\qquad x^2y-y^2=0$ ③ $\qquad x^3-xy^2=0$ ④

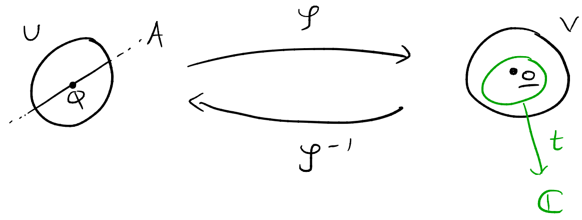
Notice that in the first example near Q, the set A is a complex submanifold (the picture is of course of the real points, but the claim is still true), but near P it is not.
What does this actually mean, concretely? Suppose $Q=(a,a)\in\mathbb{C}^2$ and $f=x^2-y^2$, with $a\neq 0$. Then with $u=x-a$, $v=y-a$

$$\begin{aligned} f &= (x-y)(x+y) = ([x-a]-[y-a])([x-a]+[y-a]+2a) \\ &= (u-v)(u+v+2a) \\ &= u^2-v^2+2a(u-v) \end{aligned}$$

$$\frac{\partial f}{\partial u} = 2u+2a \qquad \frac{\partial f}{\partial v} = -2v-2a \qquad \therefore\ \nabla f(Q)\neq 0$$

By standard arguments up to a biholomorphism (i.e. change of coordinates) A is cut out near Q by the vanishing of a single coordinate function – it is a complex submanifold.

Let's dig into this a bit more: such a biholomorphism is a holomorphic function $\varphi: U \longrightarrow V$ from some neighborhood U of a in $\mathbb{C}^2$ to $V \ni \underline{0}$ with $\varphi(Q) = \underline{0}$



(6.1)

Given a germ of holomorphic functions (W, t) with $\underline{0} \in W$, we may as well assume $W \subseteq V$ and so $(\varphi^{-1}W, t \circ \varphi)$ is a germ of holomorphic functions at Q. This operation defines an isomorphism of local $\mathbb{C}$-algebras $\mathcal{G}_{\underline{0}} \longrightarrow \mathcal{G}_Q$ and hence

$$\mathbb{C}\{x, y\} \xrightarrow{\varphi_Q} \mathbb{C}\{u, v\}. \qquad (6.2)$$

The statement that "up to biholomorphic change of coordinate f is near Q a coordinate function" is precisely the statement that φ may be chosen so that $\varphi_Q(x) = f$. This means in particular that (6.2) induces an isomorphism of $\mathbb{C}$-algebras

$$\mathbb{C}\{y\} \cong \frac{\mathbb{C}\{x, y\}}{(x)} \cong \mathbb{C}\{u, v\}/(f). \qquad (6.3)$$

Def^n A point $P \in A = V(f)$ is regular if $\nabla f(P) \neq 0$. (in any dimension)

We have just seen that if P is regular then $\mathcal{G}_P/(f) \cong \mathbb{C}\{x-a, x-b\}/(f)$ is isomorphic to $\mathbb{C}\{t\}$ where $P = (a, b) \in \mathbb{C}^2$. Is the converse true? That is, can we detect regularity abstractly at the level of analytic algebras?

Lemma L1-2 If $\mathbb{C}\{x-a, y-b\}/(f) \cong \mathbb{C}\{t\}$ as $\mathbb{C}$-algebras then $\nabla f(P) \neq 0$ where $P = (a, b)$.

This follows immediately from

<u>Lemma L1-3</u> (Lifting lemma, [GLS, Lemma 1.23]) Let $\varphi : R \longrightarrow S$ be a morphism of analytic K-algebras, where $R = K\langle x_1, \dots, x_n\rangle / I$, $S = K\langle y_1, \dots, y_m\rangle / J$. Then φ has a lifting $\tilde{\varphi} : K\langle x\rangle \longrightarrow K\langle y\rangle$ which can be chosen as an isomorphism if φ is and $m = n$, respectively as an epimorphism if φ is and $n \geqslant m$.

<u>Proof of Lemma L1-2</u> Let $\varphi : \mathbb{C}\{t\} \longrightarrow \mathbb{C}\{x-a, y-b\}/(f)$ be an isomorphism and lift it to an isomorphism $\tilde{\varphi}$ as in the following commutative diagram

$$\begin{tikzcd}
\mathbb{C}\{s,t\} \arrow[r, "\tilde{\varphi}", "\cong"'] \arrow[d] & \mathbb{C}\{x-a, y-b\} \arrow[d] \\
\mathbb{C}\{t\} = \mathbb{C}\{s,t\}/(s) \arrow[r, "\cong", "\varphi"'] & \mathbb{C}\{x-a, y-b\}/(f)
\end{tikzcd}$$

From $\tilde{\varphi}$ we easily deduce a biholomorphism as (6.1). $\square$

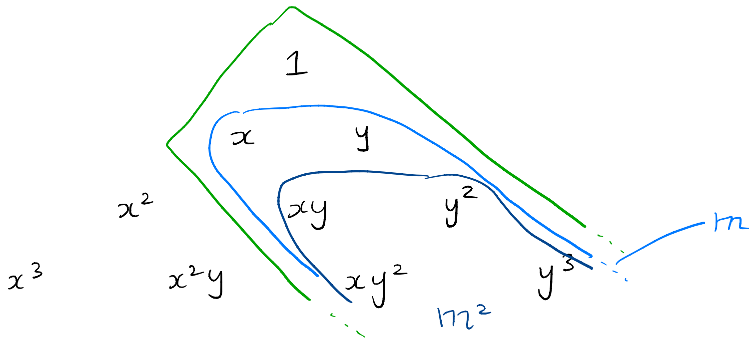
<u>Lemma L1-4</u> If $P = (a,b)$ is a regular point of $V(f)$ then with $\mathfrak{m} \subseteq \mathbb{C}\{x-a, y-b\}/(f)$ the maximal ideal we have $\mathfrak{m}/\mathfrak{m}^2 \cong \mathbb{C}$.

<u>Proof</u> If $A = \mathbb{C}\{x-a, y-b\}/(f)$ then from $A \cong \mathbb{C}\{t\}$ we deduce $\mathfrak{m}/\mathfrak{m}^2 \cong (t)/(t^2) \cong \mathbb{C}$. $\square$

Fine, so what do these local algebras $\mathbb{C}\{x-a, y-b\}/(f)$ look like at <u>singular</u> points $P = (a,b) \in V(f)$, meaning points where $\nabla f(P) = 0$, i.e. points which are not regular? Going through the examples

1. $\mathbb{C}\{x,y\}/(x^2-y^2)$ has zero divisors: $(x-y)(x+y) = 0$ so cannot be isomorphic to a ring $\mathbb{C}\{t\}$ which is a domain. Hence $V(f)$ must be singular at $\underline{0}$.

② In $\mathbb{C}\{x,y\}/(x^2-y^3)$ we have a $\mathbb{C}$-basis $\{x^i y^j\}_{0\le i\le 1, j\geq 0}$



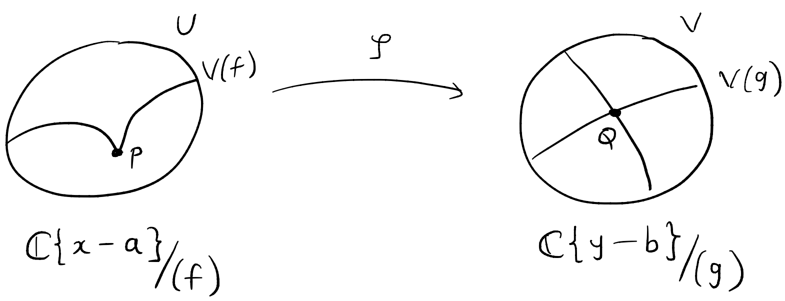
(8.1)

and hence $\mathfrak{m}/\mathfrak{m}^2 \cong \mathbb{C}x \oplus \mathbb{C}y$, so $V(f)$ is singular at $\underline{0}$.

③ In $\mathbb{C}\{x,y\}/(x^2y-y^2)$ we have a $\mathbb{C}$-basis $\{x^i y^j\}_{i\geq 0, 0\le j\le 1}$, and again $\mathfrak{m}/\mathfrak{m}^2 \cong \mathbb{C}x \oplus \mathbb{C}y$ so $V(f)$ is singular at $\underline{0}$.

<u>A category of singularities</u>

Consider two holomorphic functions $f: U \longrightarrow \mathbb{C}$, $g: V \longrightarrow \mathbb{C}$ where $U \subseteq \mathbb{C}^n$, $V \subseteq \mathbb{C}^m$ are open and take the sets of solutions $V(f) \subseteq U$, $V(g) \subseteq V$ and choose points $P \in V(f)$, $Q \in V(g)$. We write $\mathcal{O}_{U,P}$ for the ring $\mathcal{G}_P$ of germs of holomorphic functions at P, and similarly $\mathcal{O}_{V,Q}$, so that $\mathcal{O}_{U,P} \cong \mathbb{C}\{x_1-a_1, \dots, x_n-a_n\}$ and $\mathcal{O}_{V,Q} \cong \mathbb{C}\{y_1-b_1,\dots,y_m-b_m\}$ where $P=(a_1,\dots,a_n)$, $Q=(b_1,\dots,b_m)$. The local analytic algebras associated to $(V(f),P)$, $(V(g),Q)$ are therefore as below:



(8.2)

<u>Question</u>: what is a reasonable notion of <u>morphism</u> between $V(f)$ and $V(g)$ <u>locally</u> near P and Q?

A natural choice would be to let $\varphi : U \longrightarrow V$ be holomorphic with $\varphi(P) = Q$ and $\varphi(V(f)) \subseteq V(g)$, and indeed let us use this definition:

<u>Def^n</u> The category <u>Hyp</u> has as objects tuples (U, f, P) where $U \subseteq \mathbb{C}^n$ is open, $f : U \longrightarrow \mathbb{C}$ is holomorphic and $P \in V(f)$ with morphisms as defined above.

As explained above this naturally induces a morphism of $\mathbb{C}$-algebras (by precomposition of germs with φ)

$$\mathbb{C}\{x-a\} \xleftarrow{\ \Phi\ } \mathbb{C}\{y-b\}$$

<u>Lemma L1-5</u> For some $l > 0$ we have $\Phi(g)^l \in (f)$ in $\mathbb{C}\{x-a\}$.

<u>Proof</u> Given any point $X \in V(f) \subseteq U$ we have $(g\varphi)(x) = g(\varphi(x)) = 0$ since by hypothesis $\varphi(V(f)) \subseteq V(g)$. Hence $g\varphi$ is a holomorphic function vanishing on $V(f)$ and hence by the Hilbert-Rückert Nullstellensatz [GLS, Theorem 1.72] we have that a power of g lies in the ideal generated by f in analytic functions on U. Passing to germs at P gives the claim. $\square$

Recall that given an ideal I the <u>radical</u> is $\sqrt{I} = \{ t \mid t^k \in I \text{ for some } k > 0\}$. From the lemma we deduce a morphism of <u>reduced</u> $\mathbb{C}$-algebras

$\sqrt{g}$ meaning $\sqrt{(g)}$

$$\mathbb{C}\{x-a\}/\sqrt{f} \xleftarrow{\ \Phi\ } \mathbb{C}\{y-b\}/\sqrt{g}$$

<u>Def^n</u> The category <u>HypRing</u> as as objects triples $(\mathbb{C}\{x_1, \dots, x_n\}, f)$ consisting of a convergent power series ring and element $f \in \mathbb{C}\{x_1, \dots, x_n\}$. A morphism $(\mathbb{C}\{x\}, f) \longrightarrow (\mathbb{C}\{y\}, g)$ is a local morphism of $\mathbb{C}$-algebras $\mathbb{C}\{y\}/\sqrt{(g)} \longrightarrow \mathbb{C}\{x\}/\sqrt{(f)}$.

<u>Theorem L1-6</u> There is an equivalence of categories

$$\begin{array}{rcl} \underline{\mathrm{Hyp}}^{op} & \longrightarrow & \underline{\mathrm{HypRng}} \\ (U, f, P) & \longmapsto & \mathcal{O}_{U,P}/\sqrt{(f)} \end{array} \qquad (10.1)$$

If we denote by $\underline{\mathrm{Sng}} \subseteq \underline{\mathrm{Hyp}}$ the fullsubcategory whose objects are triples (U, f, P) where $\nabla f(P) = 0$, or equivalently $V(f)$ is not a complex submanifold at P, that is, P is a <u>singularity</u> of f, then we get an equivalence

$$\underline{\mathrm{Sng}}^{op} \xrightarrow{\cong} \underline{\mathrm{SngRng}} \qquad (10.2)$$

where object on the RHS are pairs $(\mathbb{C}\{x\}, f)$ where $\mathbb{C}\{x\}/\sqrt{(f)}$ is a <u>singular</u> ring, or what is the same, in $\mathbb{C}\{x\}/(f)$ we have $\dim_{\mathbb{C}} \mathfrak{m}/\mathfrak{m}^2 \neq n-1$.

<u>Upshot</u> To a first approximation the study of hypersurface singularities is the study of the category $\underline{\mathrm{Sng}}$ or $\underline{\mathrm{SngRng}}$.

<u>Remark</u> There is a pseudofunctor $\underline{\mathrm{SngRng}} \longrightarrow \mathrm{DGcat}$, $(\mathbb{C}\{x\}, f) \longmapsto \mathrm{hmf}(f)$.

<u>Question</u> (Functoriality in Singular Learning Theory) Let (W, p, q, φ), (W', p', q', φ') be two learning machines in the sense of [W] so $W \subseteq \mathbb{R}^n$, $W' \subseteq \mathbb{R}^m$ are semi-analytic. Let $w_0 \in W$, $w_0' \in W'$ be true parameters, with open neighborhoods U, U'. Then (U, K, w_0), (U', K', w_0') are objects of the real analytic analogue $\underline{\mathrm{Hyp}}^{\mathbb{R}}$ of the above. What are natural examples of <u>morphisms</u> $(U, K, w_0) \longrightarrow (U', K', w_0')$?

How singular is it?

Given a singular local K-algebra $R = \mathbb{C}\{x\}/(f)$ we can ask for invariants that measure "how far" R is from being regular. The part of singularity theory that lies within commutative algebra could be viewed as almost entirely devoted to finding answers to this question using homological algebra in $\mathrm{Mod}R$ (including matrix factorisations).

Here are the basic examples of such invariants for <u>isolated singularities</u>:

- <u>Milnor number</u> $\mu_p(f) = \dim_{\mathbb{C}} \mathbb{C}\{x_1,\dots,x_n\}/\left(\frac{\partial f}{\partial x_1},\dots,\frac{\partial f}{\partial x_n}\right)$ $\quad$ (x_i local coords)
  - e.g. $\mu(x^2-y^2) = 1$, $\mu(x^2-y^k) = \dim_{\mathbb{C}} \mathbb{C}\{y\}/(y^{k-1}) = k-1$
  - Larger numbers mean more complex singularities
  - f is singular at P iff. $\mu_P(f) > 0$.

- <u>Order</u> $\operatorname{ord}_p(f) = \inf\left\{ |\alpha| \;\middle|\; \frac{\partial^\alpha f}{\partial x^\alpha}(P) \neq 0 \right\}$
  - f is singular at P iff. $\operatorname{ord}_P(f) \geq 2$
  - Larger numbers mean more complex singularities

- <u>Log canonical threshold</u> $\operatorname{lct}_p(f) = \sup\left\{ s \in \mathbb{R}_{>0} \;\middle|\; \frac{1}{|f|^{2s}} \text{ is integrable around } P \right\}$.
  - A refinement of the reciprocal of the order, since if $f = \sum_{i=1}^n x_i^{a_i}$ then $\operatorname{lct}_0(f) = \min\left\{1, \sum_{i=1}^n \frac{1}{a_i}\right\}$.
  - Smaller numbers mean more complex singularities.

<u>Question</u> *What is the meaning of the LCT in physics, i.e. Landau-Ginzburg models? It should be of deep importance.*

References

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