Superbicategories

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Abstract

We provide a concise foundation for superbicategories, lax superfunctors and lax supertransformations. A strictification result is adapted from a Yoneda-embedding based approach to this super setting, stating that any superbicategory is superbiequivalent to a super-2-category. Adjoints in (super)bicategories are treated briefly; we give a revised definition of a graded pivotal superbicategory as extra structure and data built upon a superbicategory. Functor and superfunctor bicategories are also discussed throughout; we provide simple conditions on the source and target bicategory to guarantee the existence of adjoints in these functor bicategories.

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Contents

Introduction

Super-2-categories were introduced in [KKjO14] to "supercategorify" algebraic structures. For example, in [EL16] an explicit super-2-category is constructed to categorify Kac-moody algebras using super-2-functors. Essentially, super-2-categories provide the framework for considering $\mathbb{Z}_2$-graded morphisms between 1-cells by way of *parity shift* 1-cells. An illuminating example is the bicategory of $\mathbb{Z}_2$-graded algebras with $\mathbb{Z}_2$-bimodules as 1-cells and degree zero bimodule maps as 2-cells, where a parity shift is the same bimodule but with the opposite grading. However, as noted in [EL16], this example actually lies outside the framework of super-2-categories; the tensor product as composition is not strictly associative nor unital. This is one motivation for the introduction of superbicategories, which are the main focus of this thesis.

Superbicategories allow for the weakening of the data of super-2-categories, being to them what bicategories are to 2-categories. Superbicategories were first defined in [Mur18], where it was shown that the bicategory of Landau-Ginzberg models $\mathcal{LG}_k$ [CM16] possesses a natural k-superbicategory structure, for k a commutative ring. Also formulated in *ibid.* is the notion of a superfunctor between superbicategories, but the notion of a supertransformation between those functors is missing. One of the main goals of this thesis is to lay some of the abstract foundations for superbicategories, including giving the "right" definitions of lax superfunctors and lax transformations between them. Our argument for the correctness of these definitions lies in our main theorem:

Theorem 4.6: Any k-superbicategory is superbiequivalent to a strict k-superbicategory (a super-2-category).

This strictification theorem hinges on our key result that a superfunctor between superbicategories is a superbiequivalence if and only if the underlying pseudofunctor is a biequivalence. To this end we adapt the strategies of [JY19] to produce an inverse superfunctor. As a key step towards this goal, we obtain a *super-Yoneda embedding* of any k-superbicategory $\mathcal{B}$ into the superfunctor category $[\mathcal{B}^{op}, \mathbf{sCat}_k]^{sup}$. Here $\mathbf{sCat}_k$ is the k-superbicategory of small k-supercategories, superfunctors and supertransformations.

Beyond laying the foundations for superbicategories and their morphisms, we also present some results on adjoints in (super)functor bicategories. The concept of an adjunction in a bicategory $\mathcal{B}$ generalises that of an adjoint pair of functors; the notions coincide if $\mathcal{B}$ is the bicategory **Cat** of small categories, functors and natural transformations. Having an adjoint (either left or right) is often interpreted as a *finiteness* condition on a 1-cell. For instance, in the monoidal category of vector spaces with the tensor product, viewed as a bicategory with one object, a vector space V has an adjoint if and only if it is finite dimensional. A *pivotal* bicategory has isomorphic left and right duals for each 1-cell. In [CM16] it is shown that $\mathcal{LG}_k$ is a *graded* pivotal bicategory, that is, each 1-cell has both left and right adjoints that agree up to a parity shift. We give a more concise definition in this thesis of a graded pivotal bicategory, as extra structure built upon a k-superbicategory. We utilize string diagrams heavily to place a natural graded pivotal k-superbicategory structure on the superfunctor bicategory $[\mathcal{B}, \mathcal{C}]$ given a graded pivotal structure on $\mathcal{C}$ and assuming that $\mathcal{B}$ has both left and right adjoints. To this end, we prove that any lax (resp. oplax) transformation between pseudofunctors $F, G : \mathcal{B} \to \mathcal{C}$ is actually strong, provided that $\mathcal{B}$ has left (resp. right) adjoints. This is a generalisation of a somewhat classical result about invertibility of natural transformations between monoidal functors, going back to at least [Riv72], see [DP08, Proposition 7] for a similar result.
A brief outline of each section follows.

- **Section 1:** We recall the notions of bicategories, lax functors and lax transformations before introducing the concept of an adjunction in a bicategory. We show that the pseudofunctor bicategory $[\mathcal{B}, \mathcal{C}]$ has left and right adjoints if both $\mathcal{B}$ and $\mathcal{C}$ do.

- **Section 2:** We define k-superbicategories, lax superfunctors and lax supertransformations. k-Supercategories are also discussed, and we prove that a superfunctor between k-supercategories is a superequivalence if and only if the underlying functor is an equivalence of categories. The notion of a graded pivotal bicategory is defined, and we show that $[\mathcal{B}, \mathcal{C}]^{sup}$ has a graded pivotal structure given a graded pivotal structure on $\mathcal{C}$ with $\mathcal{B}$ having left and right adjoints.

- **Section 3:** From the notion of lax superfunctors and lax transformations follows that of a superbiequivalence between superbicategories. We show that a pseudo superfunctor is a superbiequivalence if and only if the underlying pseudofunctor is a biequivalence.

- **Section 4:** We introduce the representable super-morphisms corresponding to any k-superbicategory and subsequently build a Yoneda pseudo superfunctor

$$\tilde{\mathcal{Y}} : \mathcal{B} \to [\mathcal{B}^{op}, \mathbf{sCat}_k]^{sup}$$

for any k-superbicategory $\mathcal{B}$. This leads to a strictification result for k-superbicategories by applying the results of the previous section.

Notation	Explanation	Reference
k	Always denotes a commutative ring.	
$[\mathcal{B}, \mathcal{C}]$	The bicategory with pseudofunctors $\mathcal{B} \to \mathcal{C}$ as objects, strong transformations and modifications respectively as 1- and 2-cells. Here $\mathcal{B}, \mathcal{C}$ are bicategories.	Definition 1.12
$[\mathcal{B}, \mathcal{C}]^{sup}$	The k-superbicategory with pseudo superfunctors $\mathcal{B} \to \mathcal{C}$ as objects, and strong supertransformations and modifications respectively as 1- and 2-cells. Here $\mathcal{B}, \mathcal{C}$ are k-superbicategories.	Proposition 2.34
Cat	The 2-category of small categories, functors and transformations.	Example 1.2
$\mathbf{sCat}_k$	The super-2-category of small k-supercategories, superfunctors and supertransformations.	Proposition 2.14
Ω, μ	Will always represent a parity shift transformation and parity involution modification on a k-superbicategory, respectively. There may be additional labelling, such as $\Omega^{\mathcal{B}}$ etc.	Definition 2.3
$\tilde{\Omega}, \tilde{\mu}$	Will always represent a parity shift functor and parity involution transformation on a k-supercategory, respectively. The exception being when they appear as superdata for $\mathbf{sCat}_k$, see Proposition 2.14.	Definition 2.9
f^*	Pre-composition superfunctor $\mathcal{B}(Y, A) \to \mathcal{B}(X, A)$ between local k-supercategories, for $f : X \to Y$ a 1-cell in a bicategory $\mathcal{B}$.	Definition 2.19
f_*	Post-composition superfunctor $\mathcal{B}(A, X) \to \mathcal{B}(A, Y)$ for $f : X \to Y$ a 1-cell in a bicategory $\mathcal{B}$.	Definition 2.19
$\tilde{\mathcal{Y}}$	Denotes the Yoneda pseudo superfunctor $\mathcal{B} \to [\mathcal{B}^{op}, \mathbf{sCat}_k]^{sup}$ for a k-superbicategory $\mathcal{B}$.	Subsection 4.1
$\mathrm{sStr}(F, G)$	The k-linear category of strong supertransformations and modifications between pseudo superfunctors F, G.	Just before Definition 4.3

1 Adjunctions in bicategories

1.1 Background on bicategories

The classical reference for bicategories and their morphisms is the founding paper [Bé67]. Our notation and naming schemes will follow the more modern treatment of [JY20], however.

Definition 1.1. A *bicategory* $\mathcal{B}$ consists of the follows data:

- A class of *objects* or 0-*cells*, denoted $\mathcal{B}_0$.
- For each pair of objects A, B in $\mathcal{B}$, a category $\mathcal{B}(A, B)$ called a *local category.* The objects of $\mathcal{B}(A, B)$ are referred to as 1-*cells*, the morphisms as 2-*cells.* Composition of 2-cells inside these local categories is referred to as *vertical composition* in $\mathcal{B}$.
- For each triple of objects A, B, C in $\mathcal{B}_0$, a composition functor

  $$c_{A,B,C} : \mathcal{B}(B, C) \times \mathcal{B}(A, B) \to \mathcal{B}(A, C)$$

  We write $g \circ f = c_{A,B,C}(g, f)$ or simply gf for the *composition of* 1-*cells.* We write $\gamma * \beta = c_{A,B,C}(\gamma, \beta)$ for the *horizontal composition* of 2-cells. A 1-cell will often be denoted with an arrow, such as $f : A \to B$, whereas a 2-cell may be denoted with either a double arrow $\beta : f \Rightarrow g$ or a regular arrow.
- For each object $A \in \mathcal{B}_0$, a distinguished 1-cell $1_A \in \mathcal{B}(A, A)$ called the *identity* on A.
- For 1-cells $f : A \to B, g : B \to C, h : C \to D$ in $\mathcal{B}$, a 2-cell isomorphism

  $$\alpha_{h,g,f} : (h \circ g) \circ f \to h \circ (g \circ f)$$

  called an *associator.* Often we will omit the subscripts and simply write α.
- For each 1-cell $f : A \to B$ in $\mathcal{B}$, 2-cell isomorphisms

  $$\lambda_f : 1_B \circ f \to f \qquad \rho_f : f \circ 1_A \to f$$

  called the *left unitor* and *right unitor* respectively. Again we will often omit subscripts.

The above data is required to satisfy the following conditions:

- With the following setup:

  $$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow\beta"] & B \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow\gamma"] & C \arrow[r, bend left, "h"] \arrow[r, bend right, "h'"'] \arrow[r, phantom, "\Downarrow\delta"] & D \end{tikzcd}$$

  the following diagrams commute

  $$\begin{tikzcd} (hg)f \arrow[r, "\alpha"] \arrow[d, "(\delta*\gamma)*\beta"'] & h(gf) \arrow[d, "\delta*(\gamma*\beta)"] \\ (h'g')f' \arrow[r, "\alpha"'] & h'(g'f') \end{tikzcd} \qquad \begin{tikzcd} f1_A \arrow[r, "\rho"] \arrow[d, "\beta*1"'] & f \arrow[d, "\beta"] \\ f'1_A \arrow[r, "\rho"'] & f' \end{tikzcd} \qquad \begin{tikzcd} 1_Bf \arrow[r, "\lambda"] \arrow[d, "1*\beta"'] & f \arrow[d, "\beta"] \\ 1_Bf' \arrow[r, "\lambda"'] & f' \end{tikzcd}$$

  These conditions are referred to as *naturality* of α, ρ and λ respectively.

- For each set of data $A \xrightarrow{f} B \xrightarrow{g} C$, the following diagram commutes

$$\begin{tikzcd}
(g1_B)f \arrow[rr, "\alpha"] \arrow[dr, "\rho * 1"'] & & g(1_B f) \arrow[dl, "1 * \lambda"] \\
& gf &
\end{tikzcd}$$

  This condition is referred to as the *unity axiom* for $\mathcal{B}$.

- For composable 1-cells f, g, h, k, the following diagram commutes:

$$\begin{tikzcd}
& (k(hg))f \arrow[rr, "\alpha"] & & k((hg)f) \arrow[dr, "1 * \alpha"] & \\
((kh)g)f \arrow[ur, "\alpha * 1"] \arrow[drr, "\alpha"'] & & & & k(h(gf)) \\
& & (kh)(gf) \arrow[urr, "\alpha"'] & &
\end{tikzcd}$$

  This condition is referred to as the *pentagon axiom* for $\mathcal{B}$.

A bicategory $\mathcal{B}$ such that α, ρ, λ are all identity 2-cells is called a 2-*category* or a *strict bicategory*.

Some basic properties of Bicategories:

- Functoriality of the composition functors says that $1_g * 1_f = 1_{g \circ f}$ and that

$$(\gamma' * \beta') \circ (\gamma * \beta) = (\gamma' \circ \gamma) * (\beta' \circ \beta)$$

  This is referred to as the *interchange law*, or also as the *middle four exchange law*. If one uses the interchange law twice, it is possible to show that diagrams such as the following commute in any bicategory:

$$\begin{tikzcd}
gf \arrow[r, "1 * \gamma"] \arrow[d, "\beta * 1"'] & gf' \arrow[d, "\beta * 1"] \\
g'f \arrow[r, "1 * \gamma"'] & g'f'
\end{tikzcd}$$

  We will claim the commutativity of many such diagrams throughout this thesis as being "due to the interchange law".

- The following diagrams commute in any bicategory:

$$\begin{tikzcd}
(1_C g)f \arrow[rr, "\alpha"] \arrow[dr, "\lambda * 1"'] & & 1_C(gf) \arrow[dl, "\lambda"] \\
& gf &
\end{tikzcd}
\qquad
\begin{tikzcd}
(gf)1_A \arrow[rr, "\alpha"] \arrow[dr, "\rho"'] & & g(f1_A) \arrow[dl, "1 * \rho"] \\
& gf &
\end{tikzcd}
\tag{1}$$

- We have $\rho_{1_A} = \lambda_{1_A} : 1_A \circ 1_A \to 1_A$ for any $A \in \mathcal{B}_0$.

Example 1.2. The prototypical example of a 2-category is **Cat**, with small categories as objects, functors as 1-cells and natural transformations as 2-cells. The vertical composition of natural transformations should be familiar to readers, we give a brief explanation of horizontal composition. Suppose we have the following setup

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "F'"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{D} \arrow[r, bend left, "G"] \arrow[r, bend right, "G'"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathcal{E} \end{tikzcd}$$

Here $\mathcal{C}, \mathcal{D}, \mathcal{E}$ are categories, F, F', G, G' are functors and α, β are natural transformations. The following diagram commutes by naturality of β:

$$\begin{tikzcd} GFX \arrow[r, "G(\alpha_X)"] \arrow[d, "\beta_{FX}"'] & GF'X \arrow[d, "\beta_{F'X}"] \\ G'FX \arrow[r, "G'(\alpha_X)"'] & G'F'X \end{tikzcd}$$

We define $(\beta * \alpha)_X : GFX \to G'F'X$ to be either path in the diagram above. If $\beta : G \to G$ is the identity transformation, we have

$$(1 * \alpha)_X = G(\alpha_X) \tag{2}$$

If instead $\alpha : F \to F$ is the identity transformation, we have

$$(\beta * 1)_X = \beta_{FX} \tag{3}$$

Horizontal composition with identity natural transformations is referred to as *whiskering*. The horizontal composition of 2-cells in an arbitrary bicategory with identity 2-cells is also referenced in this way. The identities (2) and (3) will be used heavily throughout this thesis without explicit mention.

Example 1.3. There is a bicategory **Bimod** whose objects are rings (unital, associative) where $\mathbf{Bimod}(R, S)$ is the category of (R, S)-bimodules and bimodule maps. The unit bimodules are the rings R considered as (R, R)-bimodules. Composition is given by the tensor product

$$\begin{aligned} \mathbf{Bimod}(S, T) \times \mathbf{Bimod}(R, S) &\to \mathbf{Bimod}(R, T) \\ ({}_S M_T, {}_R N_S) &\mapsto {}_R N \otimes_S M_T \end{aligned}$$

There are canonical associator and unitor maps comprising the rest of the bicategory structure, in particular **Bimod** is not a 2-category.

Remark 1.4. A *monoidal category* consists of the same data as a bicategory with one object. Let $\mathcal{B}$ denote such a bicategory with unique object A. Then we have a local category $\mathcal{B}(A, A)$ with a composition operation $\mathcal{B}(A, A) \times \mathcal{B}(A, A) \to \mathcal{B}(A, A)$ and a unital object $1_A \in \mathcal{B}(A, A)$. The associator and unitor comprise the rest of the data for a monoidal category. There is a monoidal category $\mathbf{Vect}_{\mathbb{F}}$ consisting of vector spaces over a field $\mathbb{F}$ along with $\mathbb{F}$-linear maps, equipped with the tensor product operation $\otimes : \mathbf{Vect}_{\mathbb{F}} \times \mathbf{Vect}_{\mathbb{F}} \to \mathbf{Vect}_{\mathbb{F}}$. The corresponding bicategory with one object shares many similarities with **Bimod**.

Definition 1.5. A 1-cell $f : X \to Y$ in a bicategory $\mathcal{B}$ is said to be an equivalence if there exists a 1-cell $g : Y \to X$ and 2-cell isomorphisms $1_X \cong g \circ f, 1_Y \cong f \circ g$. In this case we say that X and Y are equivalent.

Example 1.6. An equivalence in the bicategory **Cat** is the classical notion of an equivalence of categories.

Definition 1.7. Let $\mathcal{B}, \mathcal{C}$ be two bicategories. A *lax functor* $F : \mathcal{B} \to \mathcal{C}$ consists of the following data:

- A function $\mathcal{B}_0 \to \mathcal{C}_0$ on objects.
- For each pair of objects $A, B \in \mathcal{B}_0$ a functor

$$F_{AB} : \mathcal{B}(A, B) \to \mathcal{C}(FA, FB)$$

  called a *local functor*.
- For each object $A \in \mathcal{B}_0$, a 2-cell in $\mathcal{C}$:

$$F^0_A : 1_{FA} \to F_{AA}(1_A)$$

  These 2-cells are referred to as the *lax unity constraints* of F.
- For $f : A \to B, g : B \to C$ in $\mathcal{B}$, a 2-cell

$$F^2_{g,f} : Fg \circ Ff \to F(g \circ f)$$

  called the *lax functoriality constraints* of F. Often the subscripts will simply be omitted for the local functors and constraints.

The above data is required to satisfy the following conditions:

- The 2-cells $F^2_{g,f}$ are natural in g and f.
- The following diagram commutes, which we shall refer to as α*-compatibility* of F:

$$\begin{tikzcd}
(FhFg)Ff \arrow[r, "\alpha"] \arrow[d, "F^2 * 1"'] & Fh(FgFf) \arrow[d, "1 * F^2"] \\
F(hg)Ff \arrow[d, "F^2"'] & FhF(gf) \arrow[d, "F^2"] \\
F((hg)f) \arrow[r, "F(\alpha)"'] & F(h(gf))
\end{tikzcd}$$

- The following diagrams commute, respectively referred to as λ-compatibility and ρ-compatibility:

$$\begin{tikzcd}
1_{FB} \circ Ff \arrow[r, "\lambda"] \arrow[d, "F^0 * 1"'] & Ff \\
F1_BFf \arrow[r, "F^2"'] & F(1_Bf) \arrow[u, "F(\lambda)"']
\end{tikzcd}
\qquad
\begin{tikzcd}
Ff \circ 1_{FA} \arrow[r, "\rho"] \arrow[d, "1 * F^0"'] & Ff \\
FfF1_A \arrow[r, "F^2"'] & F(f1_A) \arrow[u, "F(\rho)"']
\end{tikzcd}$$

A *pseudofunctor* $F : \mathcal{B} \to \mathcal{C}$ is a lax functor such that all components of F^0 and F^2 are invertible 2-cells.

Definition 1.8. An *oplax functor* $F : \mathcal{B} \to \mathcal{C}$ consists of the same data as a lax functor $F : \mathcal{B} \to \mathcal{C}$, except that the directions of the 2-cells F^0, F^2 are reversed. So now we have

$$F^0_A : F1_A \to 1_{FA} \qquad F^2_{g,f} : F(g \circ f) \to Fg \circ Ff$$

They are referred to now as *oplax unity constraints* and *oplax functoriality constraints* respectively. The conditions on this data are the same conditions as in Definition 1.7 with the relevant 2-cells reversed, henceforth referred to as *oplax* coherence conditions now.

Remark 1.9. As a convention, we have asked that the constraint 2-cells of a pseudofunctor are directed the same way as those of a lax functor. It is clear however that any pseudofunctor may be considered as an oplax functor by reversing the constraint 2-cells.

Definition 1.10. Let $F, G : \mathcal{B} \to \mathcal{C}$ be lax functors between bicategories. A *lax transformation* $\beta : F \to G$ consists of the following data

- For each $A \in \mathcal{B}_0$, a 1-cell $\beta_A : FA \to GA$ called the *A-component 1-cell* of β.
- For each $f : A \to B$ in $\mathcal{B}$, a 2-cell
$$\beta_f : Gf \circ \beta_X \to \beta_Y \circ Ff$$
These 2-cells are referred to as the *lax naturality constraints* of β.

The above data is required to satisfy the following conditions:

- The lax naturality constraints β_f are natural in f.
- The *lax naturality coherence condition*, which says that diagrams of the following form commute:
$$\begin{tikzcd}
(GgGf)\beta_A \arrow[r, "G^2*1"] \arrow[d, "\alpha"'] & G(gf)\beta_A \arrow[r, "\beta_{gf}"] & \beta_C F(gf) \\
Gg(Gf \circ \beta_A) \arrow[d, "1*\beta_f"'] & & \\
Gg(\beta_B Ff) \arrow[d, "\alpha^{-1}"'] & & \\
(Gg\beta_B)Ff \arrow[r, "\beta_g*1"'] & (\beta_C Fg)Ff \arrow[r, "\alpha"'] & \beta_C(FgFf) \arrow[uuu, "1*F^2"']
\end{tikzcd}$$
The above diagram may sometimes be referred to as the (g, f)-component of the lax naturality condition of β.
- The *lax unity coherence condition*, which says that the following diagram commutes:
$$\begin{tikzcd}
G1_A \circ \beta_A \arrow[rr, "\beta_{1_A}"] & & \beta_A \circ F1_A \\
1_{GA} \circ \beta_A \arrow[u, "G^0*1"] \arrow[r, "\lambda"'] & \beta_A \arrow[r, "\rho^{-1}"'] & \beta_A \circ 1_{FA} \arrow[u, "1*F^0"']
\end{tikzcd}$$

If each naturality constraint is an isomorphism, we say that β is a *strong transformation*. An *oplax transformation* $\beta : F \to G$ is to a lax transformation as an oplax functor is to a lax functor - reverse the directions of the constraint 2-cells. Analogous definitions of transformations exist if either F or G (or both) happen to be oplax; the only thing that changes is the structure of the coherence conditions.

Definition 1.11. A *modification* $\Gamma : \beta \to \gamma$ between lax transformations $\beta, \gamma : F \to G$ between functors consists of a family of 2-cells $\Gamma_A : \beta_A \to \gamma_A$ such that the following diagram commutes for all 1-cells $f : A \to B$ in the source bicategory:
$$\begin{tikzcd}
Gf \circ \beta_A \arrow[r, "\beta_f"] \arrow[d, "1*\Gamma_A"'] & \beta_B \circ Ff \arrow[d, "\Gamma_B*1"] \\
Gf \circ \gamma_A \arrow[r, "\gamma_f"'] & \gamma_B \circ Ff
\end{tikzcd}$$

Here F and G may be independently lax or oplax. There are analogous definitions if β or γ or both happen to be oplax transformations.

Suppose now that $F : \mathcal{B} \to \mathcal{C}$ and $G : \mathcal{C} \to \mathcal{D}$ are lax functors. The composite lax functor GF has the following data:

$$(GF)^0_A : 1_{GFA} \xrightarrow{G^0_{FA}} G1_{FA} \xrightarrow{G(F^0_A)} GF1_A$$
$$(GF)^2_{g,f} : GFg \circ GFf \xrightarrow{G^2} G(Fg \circ Ff) \xrightarrow{G(F^2)} GF(g \circ f)$$

We may also compose oplax functors in a similar fashion. Now let $F, F', F'' : \mathcal{B} \to \mathcal{C}$ be lax functors and $\beta : F \to F', \gamma : F' \to F''$ lax transformations. The composite lax transformation $\gamma \circ \beta$ has the following data:

$$(\gamma \circ \beta)_A : FA \xrightarrow{\beta_A} F'A \xrightarrow{\gamma_A} F''A$$

$$\begin{tikzcd}
F''f(\gamma_X\beta_X) \arrow[r, dashed, "(\gamma\circ\beta)_f"] \arrow[d, "\alpha^{-1}"'] & (\gamma_Y\beta_Y)Ff \\
(F''f \circ \gamma_X)\beta_X \arrow[d, "\gamma_f * 1"'] & \gamma_Y(\beta_Y \circ Ff) \arrow[u, "\alpha^{-1}"'] \\
(\gamma_Y \circ F'f)\beta_X \arrow[r, "\alpha"'] & \gamma_Y(F'f \circ \beta_X) \arrow[u, "1 * \beta_f"']
\end{tikzcd}$$

We may also compose oplax transformations in a similar fashion. We may compose modifications vertically and horizontally by applying the respective composition component-wise. We omit the proofs that these various notions of composition are well defined, see [JY20] for this.

Definition 1.12. Given two bicategories $\mathcal{B}, \mathcal{C}$ there is a bicategory $[\mathcal{B}, \mathcal{C}]$ consisting of pseudofunctors $\mathcal{B} \to \mathcal{C}$, strong transformations between them, and modifications between those transformations. The identity transformation $1_F : F \to F$ is defined by the data

- $(1_F)_X = 1_{FX}$
- $(1_F)_f : Ff \circ 1_X \xrightarrow{\rho} Ff \xrightarrow{\lambda^{-1}} 1_Y \circ Ff$

The associator and unitors on $[\mathcal{B}, \mathcal{C}]$ are now families of modifications, given component-wise by the associator and unitors in $\mathcal{C}$. See [JY20] for the details and proof of axioms, there the same bicategory is denoted $\mathrm{Bicat}^{ps}(\mathcal{B}, \mathcal{C})$.

There are also similar bicategories $[\mathcal{B}, \mathcal{C}]_{x,y}$ where x describes the types of functors, standing for either lax, oplax or pseudo and y describes the type of transformations standing for either lax, oplax or strong. The only place where such notation is used within this thesis is Theorem 1.26.

Definition 1.13. Let $\mathcal{B}$ be a bicategory. We define the *opposite bicategory*, denoted $\mathcal{B}^{op}$, to have the following data:

- The objects of $\mathcal{B}^{op}$ are the same as $\mathcal{B}$.
- $\mathcal{B}^{op}(X, Y) := \mathcal{B}(Y, X)$.
- $1^{op}_X := 1_X \in \mathcal{B}(X, X) = \mathcal{B}^{op}(X, X)$.
- Composition is defined as
$$f \circ^{op} g := g \circ f$$
$$\beta *^{op} \gamma := \gamma * \beta$$

- The components of the associator and unitors are given as

$$\alpha^{op} : (f \circ^{op} g) \circ^{op} h = h \circ (g \circ f) \xrightarrow{\alpha^{-1}} (h \circ g) \circ f = f \circ^{op} (g \circ^{op} h)$$
$$\rho^{op} : f \circ^{op} 1_Y = 1_Y \circ f \xrightarrow{\lambda} f$$
$$\lambda^{op} : 1_X \circ^{op} f = f \circ 1_X \xrightarrow{\rho} f$$

1.2 Adjunctions

We now introduce the notion of an adjunction in a bicategory. Within this section, the results of Proposition 1.18, Proposition 1.20 and Proposition 1.21 are standard and can be found in [Gra74]. The details of the various proofs will be needed for the remaining results within this section, which are less standard; in particular we show that if $\mathcal{B}, \mathcal{C}$ have both left and right adjoints, then so does $[\mathcal{B}, \mathcal{C}]$. We make heavy use of string diagrams throughout, which are introduced and justified in Appendix A.2.

Definition 1.14. An *adjunction* in a bicategory $\mathcal{B}$ is a quadruple (f, g, η, ϵ) where $f : X \to Y, g : Y \to X$ are 1-cells and

$$\eta : 1_X \to g \circ f$$
$$\epsilon : f \circ g \to 1_Y$$

are 2-cells. We require the following composites to be equal to 1_f and 1_g respectively:

$$\begin{gathered} f \xrightarrow{\rho^{-1}} f1_X \xrightarrow{1*\eta} f(gf) \xrightarrow{\alpha^{-1}} (fg)f \xrightarrow{\epsilon*1} 1_Yf \xrightarrow{\lambda} f \\ g \xrightarrow{\lambda^{-1}} 1_Xg \xrightarrow{\eta*1} (gf)g \xrightarrow{\alpha} g(fg) \xrightarrow{1*\epsilon} g1_Y \xrightarrow{\rho} g \end{gathered} \tag{4}$$

In this case f is said to be *left adjoint* to g, which is *right adjoint* to f. Sometimes we may omit the data of the 2-cells and simply write $f \dashv g$ to refer to the adjunction. Often η is referred to as the unit, or coevaluation map, and ϵ as the counit, or evaluation map of the adjunction. If both η and ϵ are isomorphisms, the data is called an *adjoint equivalence.*

Remark 1.15. Recall that we say objects X and Y are equivalent in a bicategory $\mathcal{B}$ if there exist $f : X \to Y, g : Y \to X$ and isomorphisms $\eta : 1_X \cong gf, \epsilon : fg \cong 1_Y$. An adjoint equivalence between X and Y is a slightly stronger condition, as the triangle identities must also hold. It is however a well known fact that any equivalence upgrades to an adjoint equivalence by only having to alter the 2-cells ϵ, η.

Remark 1.16. In the bicategory **Cat**, adjunctions are exactly the adjunctions of functors from regular category theory. For a 1-cell $f : X \to Y$ in an arbitrary bicategory, having an adjoint is often interpreted as a finiteness condition. For instance, in the bicategory with one object corresponding to the monoidal category $\mathbf{Vect}_{\mathbb{F}}$ of Remark 1.4, a vector space has an adjoint if and only if it is finite dimensional. On a more global level, adjoints in a bicategory $\mathcal{B}$ influence the structure of transformations between pseudofunctors $F, G : \mathcal{B} \to \mathcal{C}$, see our result Theorem 1.23.

Definition 1.17. Let $\mathcal{B}$ be a bicategory. We say that $\mathcal{B}$ *has left [resp. right] adjoints* if for every 1-cell $f : X \to Y$ there exists an adjunction (g, f, η, ϵ) [resp. (f, g, η, ϵ)]. If $\mathcal{B}$ has this property, we *fix* an adjunction

$$({}^\dagger f, f, \mathrm{coev}_f, \mathrm{ev}_f) \qquad \text{[resp. } (f, f^\dagger, \widetilde{\mathrm{coev}}_f, \widetilde{\mathrm{ev}}_f)]$$

for each 1-cell $f : X \to Y$.

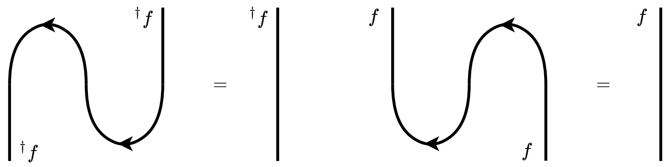


Figure 1: The Zorro moves associated to each left adjoint for f. A similar pair of moves holds for each right adjoint to f.

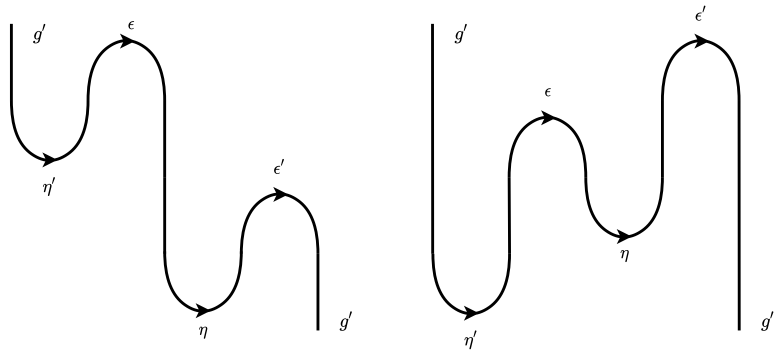
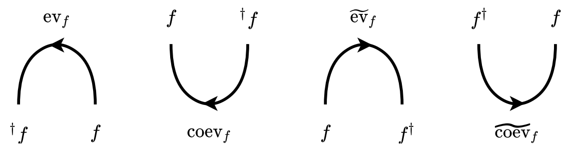


Figure 2: Starting with the left diagram, drag ϵ' up above both η' and ϵ using the interchange law, then drag η' down below η to arrive at the right diagram. We can now apply the Zorro moves twice to obtain $1_{g'} : g' \to g'$.

If $\mathcal{B}$ has left or right adjoints, we denote the (co)evaluation maps in string diagram notation as follows



The direction of each arrow is governed by the following informal rule: the arrow points away from f if f is in the domain and towards f if f is in the codomain. Before we prove some results regarding adjunctions, we note that the identities (4) have useful string diagram interpretations called the *Zorro moves*, as displayed in Figure 1.

Proposition 1.18 (Uniqueness of right adjoints)**.** *Suppose that* $(f, g, \eta, \epsilon), (f, g', \eta', \epsilon')$ *are two adjunctions in* $\mathcal{B}$*. Then there is a canonical isomorphism* $g \to g'$*.*

Proof. Define 2-cells via the compositions:

$$g \xrightarrow{\lambda^{-1}} 1_X g \xrightarrow{\eta' * 1} (g'f)g \xrightarrow{\alpha} g'(fg) \xrightarrow{1 * \epsilon} g' 1_Y \xrightarrow{\rho} g'$$

$$g' \xrightarrow{\lambda^{-1}} 1_X g' \xrightarrow{\eta * 1} (gf)g' \xrightarrow{\alpha} g(fg') \xrightarrow{1 * \epsilon'} g 1_Y \xrightarrow{\rho} g$$

If we vertically compose one with the other, we get the corresponding identity 2-cell. One such calculation is shown in Figure 2.

□

Remark 1.19. Suppose that (f, g, η, ϵ) is an adjunction in $\mathcal{B}$. It is easily checked that (g, f, η, ϵ) is an adjunction in $\mathcal{B}^{op}$, from which it follows with the above proposition that left adjoints are also unique.

Proposition 1.20 (Pseudofunctors preserve adjoints)**.** *Suppose that (f, g, η, ϵ) is an adjunction in $\mathcal{B}$, with $F : \mathcal{B} \to \mathcal{C}$ a pseudofunctor. Then $(Ff, Fg, \eta', \epsilon')$ is an adjunction in $\mathcal{C}$, where η' and ϵ' are given by the compositions:*

$$\eta' : 1_{FX} \xrightarrow{F^0} F1_X \xrightarrow{F(\eta)} F(g \circ f) \xrightarrow{(F^2)^{-1}} Fg \circ Ff$$
$$\epsilon' : Ff \circ Fg \xrightarrow{F^2} F(f \circ g) \xrightarrow{F(\epsilon)} F1_Y \xrightarrow{(F^0)^{-1}} 1_{FY}$$

Since F will map an isomorphism of 1-cells to an isomorphism of 1-cells, it is clear that $Ff \dashv Fg$ is an adjoint equivalence if $f \dashv g$ is.

Proof. We want to show that the following composition is equal to 1_{Ff}:

$$Ff \xrightarrow{\rho^{-1}} Ff \circ 1_{FX} \xrightarrow{1*\eta'} Ff(FgFf) \xrightarrow{\alpha} (FfFg)Ff \xrightarrow{\epsilon'*1} 1_{FY} \circ Ff \xrightarrow{\lambda} Ff \tag{5}$$

Observe the following diagram:

$$\begin{tikzcd}
Ff \arrow[r, "F(\rho^{-1})"] \arrow[d, "\rho^{-1}"'] & F(f1_X) \arrow[d, "(F^2)^{-1}"] \arrow[rdd, bend left, "F(1*\eta)"] & \\
(Ff)1_{FX} \arrow[r, "1*F^0"'] \arrow[d, "1*\eta'"'] & FfF1_X \arrow[d, "1*F\eta"] & \\
Ff(FgFf) \arrow[r, "1*F^2"'] \arrow[d, "\alpha^{-1}"'] & FfF(gf) \arrow[r, "F^2"'] & F(f(gf)) \arrow[d, "F(\alpha^{-1})"] \\
(FfFg)Ff \arrow[r, "F^2*1"] \arrow[d, "\epsilon'*1"'] & F(fg)Ff \arrow[r, "F^2"] \arrow[d, "F\epsilon*1"] & F((fg)f) \arrow[ldd, bend left, "F(\epsilon*1)"] \\
1_{FY}Ff \arrow[r, "F^0*1"] \arrow[d, "\lambda"'] & F1_YFf \arrow[d, "F^2"] & \\
Ff & F(1_Yf) \arrow[l, "F\lambda"] &
\end{tikzcd}$$

The leftmost vertical path from Ff to Ff is the composite (5), while the rightmost path from Ff to Ff is equal to $F1_f = 1_{Ff}$, due to the fact that the following composite is 1_f (a triangle identity):

$$f \xrightarrow{\rho^{-1}} f1_X \xrightarrow{1*\eta} f(gf) \xrightarrow{\alpha} (fg)f \xrightarrow{\epsilon*1} 1_Yf \xrightarrow{\lambda} f$$

The rectangular sub-diagram is α-compatibility of F. The uppermost and lowermost diagram are ρ-compatibility and λ-compatibility of F, respectively. The sub-diagrams with curved arrows commute due to naturality of F^2. The remaining two squares commute by definition. Similarly, the relevant composite from Fg to Fg is equal to 1_{Fg}. ☐

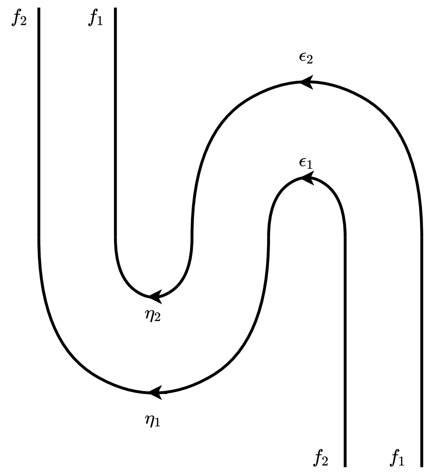


Figure 3: This string diagram evaluates to $1_{f_2 \circ f_1}$ by using the interchange law and applying two Zorro moves.

Proposition 1.21. *Suppose that* $(f_1, g_1, \eta_1, \epsilon_1), (f_2, g_2, \eta_2, \epsilon_2)$ *are two adjunctions in* $\mathcal{B}$ *as in the following diagram*

$$\begin{tikzcd} X \arrow[r, bend left, "f_1"] & Y \arrow[l, bend left, "g_1"] \arrow[r, bend left, "f_2"] & Z \arrow[l, bend left, "g_2"] \end{tikzcd}$$

Then $(f_2 \circ f_1, g_1 \circ g_2, \eta', \epsilon')$ *is an adjunction, where* η', ϵ' *are given as the following composites*

$$\eta' : 1_X \xrightarrow{\eta_1} g_1 f_1 \xrightarrow{\rho^{-1} * 1} (g_1 1_Y) f_1 \xrightarrow{1 * \eta_2 * 1} (g_1(g_2 f_2)) f_1 \xrightarrow{\alpha^{-1} * 1} ((g_1 g_2) f_2) f_1 \xrightarrow{\alpha} (g_1 g_2)(f_2 f_1)$$

$$\epsilon' : (f_2 f_1)(g_1 g_2) \xrightarrow{\alpha^{-1}} ((f_2 f_1) g_1) g_2 \xrightarrow{\alpha * 1} (f_2(f_1 g_1)) g_2 \xrightarrow{1 * \epsilon_1 * 1} (f_2 1_Y) g_2 \xrightarrow{\rho * 1} f_2 g_2 \xrightarrow{\epsilon_2} 1_Z$$

Proof. In Figure 3 we set up the relevant composition for one of the Zorro moves. From there we interchange ϵ_1 and η_2 then apply the individual Zorro moves, leaving us with $1_{f_2} * 1_{f_1} = 1_{f_2 \circ f_1}$. The other Zorro move is similar. □

Definition 1.22. Suppose that $(f, g, \eta, \epsilon), (f', g', \eta', \epsilon')$ are two adjunctions in a bicategory $\mathcal{B}$, where $f : X \to Y, f' : X' \to Y'$. Suppose we have 1-cells $a : X \to X', b : Y \to Y'$. Then given 2-cells

$$\beta : f' \circ a \to b \circ f$$
$$\gamma : a \circ g \to g' \circ b$$

we define the *mate* of β to be the 2-cell $m(\beta) : a \circ g \to g' \circ b$ given as the left diagram in Figure 4. We define the mate of γ to be the 2-cell $m(\gamma) : f' \circ a \to b \circ f$ given as the right diagram in Figure 4.

Theorem 1.23. *Let* $F, G : \mathcal{B} \to \mathcal{C}$ *be two pseudofunctors between bicategories. If either of the following two conditions hold, then any lax transformation* $\beta : F \to G$ *is strong:*

1. $\mathcal{B}$ *has left adjoints.*

2. $\mathcal{B}$ *has right adjoints and an isomorphism* $k_f : f \xrightarrow{\cong} (f^\dagger)^\dagger$ *for each* 1*-cell* $f : X \to Y$*. No naturality is required.*

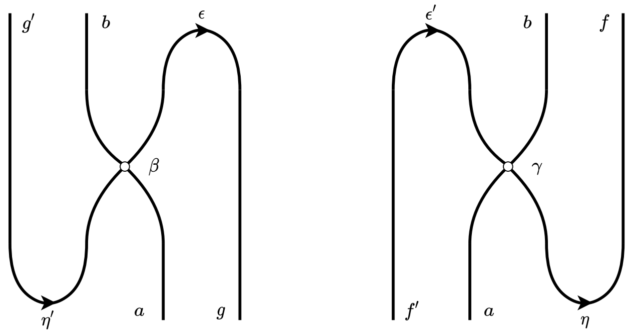


Figure 4: The left diagram represents the mate of $\beta : f' \circ a \to b \circ f$, while the right diagram represents the mate of $\gamma : a \circ g \to g' \circ b$.

Proof. We give the proof using condition 2) since it is slightly more complicated. The proof using 1) is very similar, we briefly comment on it towards the end. For each 1-cell $f : X \to Y$ in $\mathcal{B}$ we have fixed adjunctions

$$(f, f^\dagger, \widetilde{\mathrm{coev}}_f, \widetilde{\mathrm{ev}}_f) \qquad (f^\dagger, (f^\dagger)^\dagger, \widetilde{\mathrm{coev}}_{f^\dagger}, \widetilde{\mathrm{ev}}_{f^\dagger})$$

By applying Proposition 1.20 to this second adjunction, we have two adjunctions in $\mathcal{C}$:

$$(Ff^\dagger, Ff^{\dagger\dagger}, \eta_f^F, \epsilon_f^F) \qquad (Gf^\dagger, Gf^{\dagger\dagger}, \eta_f^G, \epsilon_f^G) \tag{6}$$

with data given by

$$\eta_f^F : \qquad 1_{FY} \xrightarrow{F^0} F1_Y \xrightarrow{F(\widetilde{\mathrm{coev}}_{f^\dagger})} F(f^{\dagger\dagger}f^\dagger) \xrightarrow{(F^2)^{-1}} Ff^{\dagger\dagger}Ff^\dagger$$

$$\epsilon_f^F : \qquad Ff^\dagger Ff^{\dagger\dagger} \xrightarrow{F^2} F(f^\dagger f^{\dagger\dagger}) \xrightarrow{F(\widetilde{\mathrm{ev}}_f)} F1_X \xrightarrow{(F^0)^{-1}} 1_{FX}$$

$$\eta_f^G : \qquad 1_{GY} \xrightarrow{G^0} G1_Y \xrightarrow{G(\widetilde{\mathrm{coev}}_{f^\dagger})} G(f^{\dagger\dagger}f^\dagger) \xrightarrow{(G^2)^{-1}} Gf^{\dagger\dagger}Gf^\dagger$$

$$\epsilon_f^G : \qquad Gf^\dagger Gf^{\dagger\dagger} \xrightarrow{G^2} G(f^\dagger f^{\dagger\dagger}) \xrightarrow{G(\widetilde{\mathrm{ev}}_f)} G1_X \xrightarrow{(G^0)^{-1}} 1_{GX}$$

Since $\beta : F \to G$ is a lax transformation, we have 1-cells $\beta_X : FX \to GX$ and 2-cells $\beta_f : Gf\beta_X \to \beta_Y Ff$. We define a 2-cell $\overline{\beta}_{f^\dagger} : \beta_Y Ff \to Gf\beta_X$ to be the composite

$$\overline{\beta}_{f^\dagger} : \quad \beta_Y Ff \xrightarrow{1 * F(k_f)} \beta_Y F(f^{\dagger\dagger}) \xrightarrow{m(\beta_{f^\dagger})} G(f^{\dagger\dagger})\beta_X \xrightarrow{G(k_f^{-1}) * 1} Gf\beta_X$$

where $m(\beta_{f^\dagger})$ is the mate of $\beta_{f^\dagger} : G(f^\dagger)\beta_X \to \beta_Y F(f^\dagger)$ with respect to the two adjunctions in (6). We claim that $\overline{\beta}_{f^\dagger} \circ \beta_f = 1_{Gf\beta_X}$. To this end, we claim that the following diagram commutes in $\mathcal{C}(FX, GX)$ (we are omitting associators for clarity):

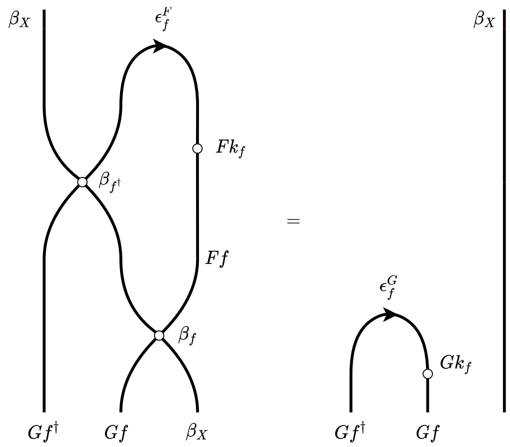


Figure 5: This equality of string diagrams represents the fact that the diagram in the proof of Theorem 1.23 commutes.

$$\begin{tikzcd}
 & Gf^\dagger Gf\beta_X \arrow[r, "1*\beta_f"] \arrow[dl, "1*G(k_f)*1"'] \arrow[d, "G^2*1"'] & Gf^\dagger\beta_Y Ff \arrow[r, "\beta_{f^\dagger}*1"] & \beta_X Ff^\dagger Ff \arrow[dl, "1*F^2"] \arrow[dr, "1*1*Fk_f"] & \\
Gf^\dagger Gf^{\dagger\dagger}\beta_X \arrow[dr, "G^2*1"'] & G(f^\dagger f)\beta_X \arrow[r, "\beta_{f^\dagger f}"'] \arrow[d, "G(1*k_f)*1"'] & \beta_X F(f^\dagger f) \arrow[dr, "1*F(1*k_f)"] & & \beta_X Ff^\dagger Ff^{\dagger\dagger} \arrow[dl, "1*F^2"] \\
 & G(f^\dagger f^{\dagger\dagger})\beta_X \arrow[rr, "\beta_{f^\dagger f^{\dagger\dagger}}"'] \arrow[d, "G(\widetilde{\mathrm{ev}}_{f^\dagger})*1"'] & & \beta_X F(f^\dagger f^{\dagger\dagger}) \arrow[d, "1*F(\widetilde{\mathrm{ev}}_{f^\dagger})"] & \\
 & G1_X\beta_X \arrow[rr, "\beta_{1_X}"'] \arrow[d, "(G^0)^{-1}*1"'] & & \beta_X F1_X \arrow[d, "1*(F^0)^{-1}"] & \\
 & 1_{GX}\beta_X \arrow[r, "\lambda"'] & \beta_X & \beta_X 1_{FX} \arrow[l, "\rho"] &
\end{tikzcd}$$

Both basic subdiagrams containing the 2-cells F^2 or G^2 commute due to naturality. The subdiagram with five vertices commutes due to the lax naturality condition on β. The bottom-most square commutes due to the lax unity condition of β. The final two squares commute due to naturality of the 2-cells of β. Since both ways around the diagram are equal, we have an equality of string diagrams as displayed in Figure 5.

Now in Figure 6, the left diagram represents the vertical composition $\overline{\beta}_{f^\dagger}\beta_f$, *after* having η_f^G pulled below Fk_f and β_f via the interchange law. We alter this diagram by substituting the equality of Figure 5. Finally we may pull down Gk_f, apply a Zorro move, then cancel the inverse 2-cells to see that $\overline{\beta}_{f^\dagger}\beta_f = 1_{Gf\beta_X}$. To show that $\beta_f\overline{\beta}_{f^\dagger} = 1_{\beta_Y Ff}$ is a similar calculation which we omit, so we are done in this case.

Now suppose that $\mathcal{B}$ has left adjoints. Similarly to before, Proposition 1.20 guarantees two adjunctions in $\mathcal{C}$

$$(F({}^\dagger f), Ff, \eta_f^F, \epsilon_f^F) \qquad (G({}^\dagger f), Gf, \eta_f^G, \epsilon_f^G)$$

corresponding to the adjunction $({}^\dagger f, f, \mathrm{coev}_f, \mathrm{ev}_f)$ in $\mathcal{B}$. With respect to these two adjunctions we let $\overline{\beta_{{}^\dagger f}}$ be the mate of $\beta_{{}^\dagger f}$. From here the claim is the same, that both $\overline{\beta_{{}^\dagger f}}\beta_f = 1_{Gf\beta_X}$ and $\beta_f\overline{\beta_{{}^\dagger f}} = 1_{\beta_Y Ff}$. Both subclaims follow using similar, but slightly simpler, string diagrams as in the above calculation. □

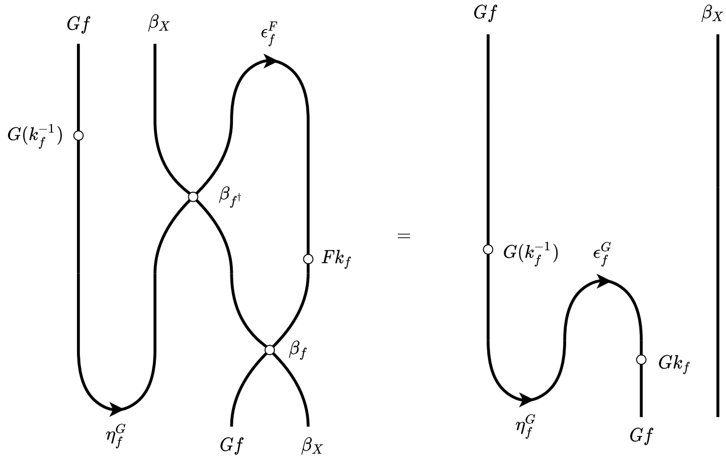


Figure 6: The main step in proving Theorem 1.23. The diagram on the right is easily seen to evaluate to $1_{Gf\circ\beta_X}$.

Corollary 1.24. *If $\mathcal{B}$ has right adjoints, then any oplax transformation $\beta : F \to G$ between pseudofunctors $F, G : \mathcal{B} \to \mathcal{C}$ is strong.*

Proof. The pseudofunctors F and G correspond to pseudofunctors $F^{op}, G^{op} : \mathcal{B}^{op} \to \mathcal{C}^{op}$, and β corresponds to a lax transformation $\beta^{op} : F^{op} \to G^{op}$. Since $\mathcal{B}$ has right adjoints, $\mathcal{B}^{op}$ has left adjoints, so the previous theorem says that β^{op} is strong. Therefore β is strong. See [JY20, Lemma 4.3.9] for details of the op-constructions. □

The results of Theorem 1.23 and Corollary 1.24 will be used to show that $[\mathcal{B}, \mathcal{C}]$ has left and right adjoints if both $\mathcal{B}$ and $\mathcal{C}$ have left and right adjoints as well, this is Corollary 1.27. Essentially, the (left or right) adjoint to a strong transformation will be given component-wise by the adjoints in $\mathcal{C}$, which will initially define an oplax transformation. The adjoints in $\mathcal{B}$ then allow us to conclude that this oplax transformation is actually strong, using the previous results.

Proposition 1.25. *Suppose $F, G : \mathcal{B} \to \mathcal{C}$ are lax functors, where $\mathcal{C}$ has left adjoints. If $\beta : F \to G$ is a lax transformation, then the following data defines an oplax transformation ${}^\dagger\beta : G \to F$:*

- *For $X \in \mathcal{B}_0$ define $({}^\dagger\beta)_X = {}^\dagger(\beta_X) : GX \to FX$.*
- *${}^\dagger\beta_f : {}^\dagger\beta_Y \circ Gf \to Ff \circ {}^\dagger\beta_X$ is the mate of $\beta_f : Gf \circ \beta_X \to \beta_Y \circ Ff$ with respect to the adjunctions*

$$({}^\dagger\beta_X, \beta_X, \mathrm{coev}_{\beta_X}, \mathrm{ev}_{\beta_X}) \qquad ({}^\dagger\beta_Y, \beta_Y, \mathrm{coev}_{\beta_Y}, \mathrm{ev}_{\beta_Y})$$

Proof. Firstly let us show that our choices of 2-cells are natural. The vertical paths in the following diagram are ${}^\dagger\beta_f, {}^\dagger\beta_{f'}$, from left to right respectively, where $\theta : f \to f'$ is a 2-cell:

$$\begin{tikzcd}
{}^\dagger\beta_Y Gf \arrow[r, "1*G\theta"] \arrow[d, "\rho^{-1}"'] & {}^\dagger\beta_Y Gf' \arrow[d, "\rho^{-1}"] \\
({}^\dagger\beta_Y Gf)1_{GX} \arrow[r, "(1*G\theta)*1"] \arrow[d, "1*\mathrm{coev}_{\beta_X}"'] & ({}^\dagger\beta_Y Gf')1_{GX} \arrow[d, "1*\mathrm{coev}_{\beta_X}"] \\
({}^\dagger\beta_Y Gf)(\beta_X {}^\dagger\beta_X) \arrow[r, "(1*G\theta)*1"'] \arrow[d] & ({}^\dagger\beta_Y Gf')(\beta_X {}^\dagger\beta_X) \arrow[d] \\
{}^\dagger\beta_Y((Gf\beta_X){}^\dagger\beta_X) \arrow[r, "1*((G\theta*1)*1)"'] \arrow[d, "1*\beta_f*1"'] & {}^\dagger\beta_Y((Gf'\beta_X){}^\dagger\beta_X) \arrow[d, "1*\beta_{f'}*1"] \\
{}^\dagger\beta_Y((\beta_Y Ff){}^\dagger\beta_X) \arrow[r, "1*((1*F\theta)*1)"'] \arrow[d] & {}^\dagger\beta_Y((\beta_Y Ff'){}^\dagger\beta_X) \arrow[d] \\
({}^\dagger\beta_Y\beta_Y)(Ff{}^\dagger\beta_X) \arrow[r, "1*(F\theta*1)"'] \arrow[d, "\mathrm{ev}_{\beta_Y}*1"'] & ({}^\dagger\beta_Y\beta_Y)(Ff'{}^\dagger\beta_X) \arrow[d, "\mathrm{ev}_{\beta_Y}*1"] \\
1_{FY}(Ff{}^\dagger\beta_X) \arrow[r, "1*(F\theta*1)"'] \arrow[d, "\lambda"'] & 1_{FY}(Ff'{}^\dagger\beta_X) \arrow[d, "\lambda"] \\
Ff\beta_X^\dagger \arrow[r, "F\theta*1"'] & Ff'{}^\dagger\beta_X
\end{tikzcd}$$

The top and bottom square commute due to naturality of the unitors. The unlabelled edges are arbitrary rebracketings using associators, whose corresponding squares commute due to naturality of the associator. The middle square commutes due to the same naturality of β that we are currently seeking for ${}^\dagger\beta$. The final two squares commute due to the interchange law. The outside diagram commuting says that our 2-cell choices are natural in f. It remains to show that the oplax unity and oplax naturality conditions hold.

Oplax naturality of ${}^\dagger\beta$ is commutativity of the following diagram:

$$\begin{tikzcd}
{}^\dagger\beta_Z(GgGf) \arrow[rr, "1*G^2"] \arrow[d, "\alpha^{-1}"'] & & {}^\dagger\beta_Z G(gf) \arrow[dddd, "{}^\dagger\beta_{gf}"] \\
({}^\dagger\beta_Z Gg)Gf \arrow[d, "{}^\dagger\beta_g*1"'] & & \\
(Fg\,{}^\dagger\beta_Y)Gf \arrow[d, "\alpha"'] & & \\
Fg({}^\dagger\beta_Y Gf) \arrow[d, "1*{}^\dagger\beta_f"'] & & \\
Fg(Ff\,{}^\dagger\beta_X) \arrow[r, "\alpha^{-1}"'] & (FgFf)\,{}^\dagger\beta_X \arrow[r, "F^2*1"'] & F(gf)\,{}^\dagger\beta_X
\end{tikzcd}$$

If we expand the left-bottom path around the above diagram and use the interchange law several times, we arrive at diagram (A). Further applications of the interchange law take us from (A) to (B), where it is clear that we may apply a Zorro move to arrive at (C). From here we replace a portion of the diagram by using lax naturality of β, as depicted in Figure 8a, bringing us to (D). This final diagram is the top-right path of the oplax condition on ${}^\dagger\beta$ above.

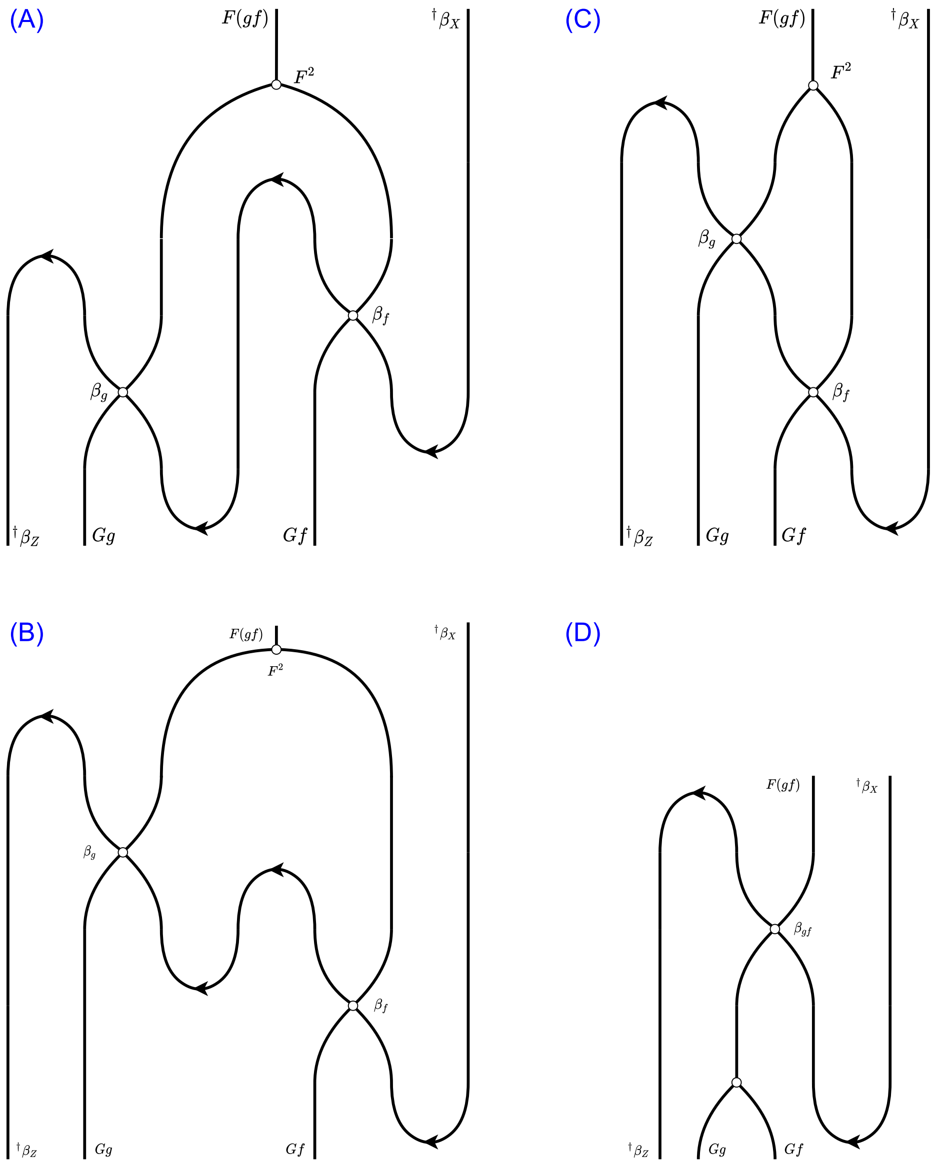


Figure 7: A sequence of equivalent string diagrams expressing the oplax naturality condition for the oplax transformation ${}^\dagger\beta$ in Proposition 1.25.

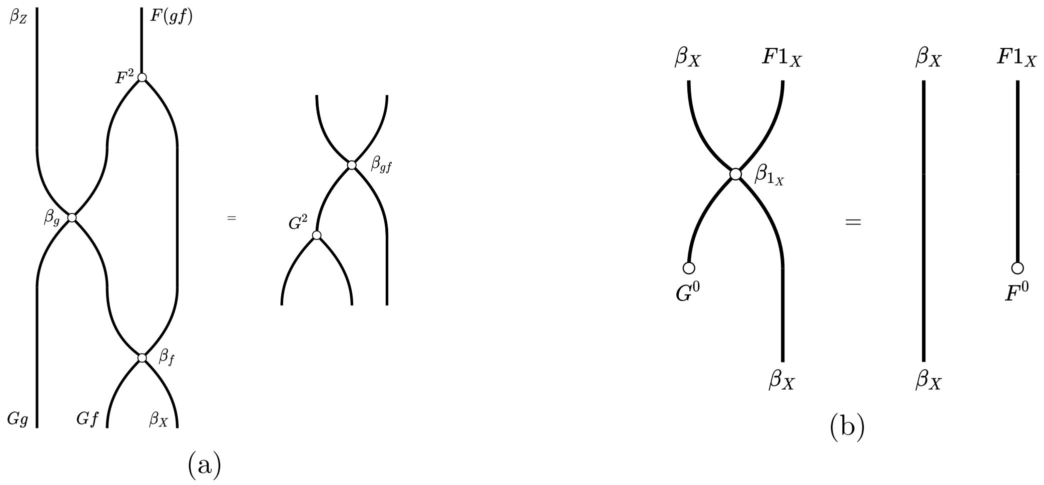


Figure 8: The naturality and unity conditions for a lax transformation $\beta : F \to G$ between lax functors.

So we have shown oplax naturality for ${}^\dagger\beta$. Figure 8b is the string diagram translation of lax unity for the transformation β, from which the oplax unity condition of $\beta^\dagger$ follows by a similar calculation.

□

Theorem 1.26. *Let $\mathcal{B}, \mathcal{C}$ be bicategories, both having left adjoints. Let $F, G : \mathcal{B} \to \mathcal{C}$ be two pseudofunctors and $\beta : F \to G$ a lax transformation. By Theorem 1.23 we may consider β as a pseudofunctor and hence an oplax transformation. Then there is an adjunction $({}^\dagger\beta, \beta, \mathrm{coev}_\beta, \mathrm{ev}_\beta)$ in $[\mathcal{B}, \mathcal{C}]_{ps,oplax}$ where we define modifications to have the following components*

$$(\mathrm{coev}_\beta)_X = \mathrm{coev}_{\beta_X} \qquad (\mathrm{ev}_\beta)_X = \mathrm{ev}_{\beta_X}$$

Proof. We need to show that the data $\{\mathrm{coev}_{\beta_X}\}_{X \in \mathcal{B}_0}$ defines a modification $\mathrm{coev}_\beta : 1_G \to \beta \circ {}^\dagger\beta$, so we need the following diagram to commute:

$$\begin{tikzcd}
1_{GY} Gf \arrow[rr, "\lambda"] \arrow[d, "\mathrm{coev}_{\beta_Y} * 1"'] & & Gf \arrow[rr, "\rho^{-1}"] & & Gf\ 1_{GX} \arrow[d, "1 * \mathrm{coev}_{\beta_X}"] \\
\beta_Y\ {}^\dagger\beta_Y\ Gf \arrow[rr, "1 * {}^\dagger\beta_f"'] & & \beta_Y Ff\ {}^\dagger\beta_X \arrow[rr, "(\beta_f)^{-1} * 1"'] & & Gf\ \beta_X\ {}^\dagger\beta_X
\end{tikzcd}$$

The left-bottom path is expanded in Figure 9. If we apply the interchange law suitably, we may apply a Zorro move and see equality with the top-right path of the diagram above after cancelling β_f with its inverse.

Similarly, the data $(\mathrm{ev}_\beta)_X = \mathrm{ev}_{\beta_X}$ defines a modification $\mathrm{ev}_\beta : {}^\dagger\beta \circ \beta \to 1_F$. The Zorro moves for $({}^\dagger\beta, \beta, \mathrm{coev}_\beta, \mathrm{ev}_\beta)$ hold since they hold component-wise in $\mathcal{C}$. □

Corollary 1.27. *Suppose $\mathcal{B}$ and $\mathcal{C}$ both have left and right adjoints. Then $[\mathcal{B}, \mathcal{C}]$ has both left and right adjoints.*

Proof. Suppose $\beta : F \to G$ is a strong transformation between pseudofunctors. The ${}^\dagger\beta$ from Theorem 1.26 was only guaranteed to be oplax, but Corollary 1.24 then says that it is strong. Right adjoints in $[\mathcal{B}, \mathcal{C}]$ are constructed analogously. □

Remark 1.28. A similar result to Corollary 1.27 was obtained independently in [Ver21, Corollary 4.5], though a slightly different approach was taken.

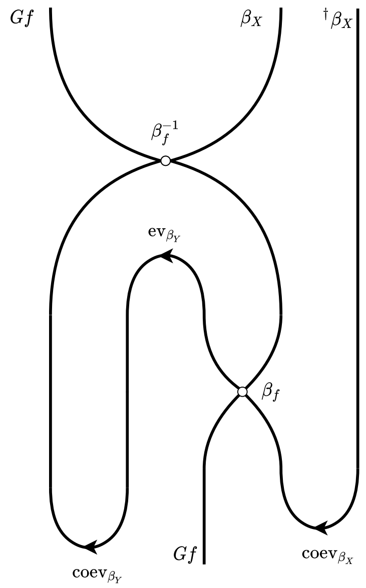


Figure 9: The left-bottom path around the modification diagram in Theorem 1.26.

2 Superbicategories

Our aim now is to finally define k-superbicategories, correcting the definition given in [Mur18], which in turn aims to generalise the definition for 2-categories as given in [EL16]. We also introduce k-supercategories and their respective morphisms, of which our main sources are [EL16, KKjO14]. We prove that a superfunctor between k-supercategories is a superequivalence if and only if the underlying functor is an equivalence, which is applied in a later section to prove Corollary 4.7. Next we refine the definition of a graded pivotal bicategory as given in [CM16]. Finally we introduce superfunctors and supertransformations between k-superbicategories, showing that the superfunctor bicategory $[\mathcal{B}, \mathcal{C}]^{sup}$ is graded pivotal if $\mathcal{C}$ is, building upon many of the results in the previous section. In this thesis, k will always denote a commutative ring.

Definition 2.1. A k-linear category is a category $\mathcal{C}$ with a k-module structure on each hom-set $\mathcal{C}(X, Y)$, such that composition is k-bilinear. A k-linear functor between k-linear categories is a functor $F : \mathcal{C} \to \mathcal{D}$ such that

$$F : \mathcal{C}(X, Y) \to \mathcal{D}(FX, FY)$$

is k-linear for $X, Y \in \mathcal{C}$. If $\mathcal{C}, \mathcal{D}, \mathcal{E}$ are k-linear, then a k-bilinear functor $F : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ is a bifunctor such that

$$F : (\mathcal{C} \times \mathcal{D})((X_1, X_2), (Y_1, Y_2)) = \mathcal{C}(X_1, Y_1) \times \mathcal{D}(X_2, Y_2) \to \mathcal{E}(F(X_1, X_2), F(Y_1, Y_2))$$

is k-bilinear.

Definition 2.2. A k-linear bicategory is a bicategory $\mathcal{B}$ such that each hom-category has the structure of a k-linear category, where composition $c_{ABC} : \mathcal{B}(A, B) \times \mathcal{B}(B, C) \to \mathcal{B}(A, C)$ is now a k-bilinear functor. A k-linear lax functor between k-linear bicategories is a lax functor $F : \mathcal{B} \to \mathcal{C}$ such that each local functor

$$F_{XY} : \mathcal{B}(X, Y) \to \mathcal{C}(FX, FY)$$

is k-linear.

Explanation 2.1. In a k-linear bicategory we may treat 2-cells as elements of a k-module, provided their (co)domains match. Bilinearity of vertical composition says for instance that $(\beta + \beta') \circ \gamma = \beta \circ \gamma + \beta' \circ \gamma$. Since composition c_{ABC} is a k-bilinear functor we also have bilinearity of horizontal composition, for instance $(\beta + \beta') * \delta = \beta * \delta + \beta' * \delta$.

A natural transformation between k-linear functors is simply a natural transformation between the underlying functors. Similarly for lax transformations between k-linear lax functors, and modifications too.

Definition 2.3. A *k-superbicategory* is a k-linear bicategory $\mathcal{B}$ with two pieces of data:

- A strong transformation $\Omega : 1_{\mathcal{B}} \to 1_{\mathcal{B}}$ of the identity functor on $\mathcal{B}$, called the *parity shift* transformation.
- An invertible modification $\mu : \Omega \circ \Omega \to 1_{1_{\mathcal{B}}}$, called the *parity involution* modification. Here $1_{1_{\mathcal{B}}}$ is the identity transformation on the identity functor $1_{\mathcal{B}}$.

There are two requirements on this data:

- $(\Omega, \Omega, \mu^{-1}, \mu)$ is an adjoint equivalence in $[\mathcal{B}, \mathcal{B}]$.
- The 2-cell $\Omega_{\Omega_X} : \Omega_X \Omega_X \to \Omega_X \Omega_X$ is equal to $-1_{\Omega_X^2}$. [1]

Often we may simply write $\mathcal{B}$ to represent a k-superbicategory $(\mathcal{B}, \Omega^{\mathcal{B}}, \mu^{\mathcal{B}})$. If the underlying bicategory is strict, then the above data defines a *strict k-superbicategory* or a *k-super-2-category*.

Explanation 2.2. The transformation Ω has component 1-cells $\Omega_X : X \to X$ with natural 2-isomorphisms

$$\Omega_f : f \circ \Omega_X \to \Omega_Y \circ f$$

The lax unity coherence condition says that $\Omega_{1_X} = \rho^{-1} \circ \lambda$. Lax naturality says that diagrams of the following form commute:

$$\begin{tikzcd}
(gf) \circ \Omega_X \arrow[rrr, "\Omega_{gf}"] \arrow[d, "\alpha"'] & & & \Omega_Z \circ (gf) \\
g \circ (f \circ \Omega_X) \arrow[r, "1 * \Omega_f"'] & g \circ (\Omega_Y \circ f) \arrow[r, "\alpha^{-1}"'] & (g \circ \Omega_Y) \circ f \arrow[r, "\Omega_g * 1"'] & (\Omega_Z \circ g) \circ f \arrow[u, "\alpha"']
\end{tikzcd}$$

Since μ is an invertible modification, we have 2-isomorphisms $\mu_X : \Omega_X \circ \Omega_X \to 1_X$. Since $(\Omega, \Omega, \mu^{-1}, \mu)$ is an adjoint equivalence, we have adjoint equivalences $(\Omega_X, \Omega_X, \mu_X^{-1}, \mu_X)$ in $\mathcal{B}$. We borrow a stylistic choice from [EL16] and [CM16] expressing the 1-cells Ω_X as blue dotted lines in string diagrams. Furthermore, we keep the 2-cells μ_X, μ_X^{-1} directionless when displayed as evaluation or coevaluation maps. These choices are showcased in Figure 10, which expresses the modification condition for μ.

Remark 2.4. The point of a k-superbicategory is that we now have a $\mathbb{Z}_2$-grading on the 2-cells between 1-cells, where we consider a 2-cell $\beta : f \to g$ to be of even parity, and a 2-cell $f \to \Omega \circ g \cong g \circ \Omega$ (equivalently $\Omega \circ f \cong f \circ \Omega \to g$) to be of odd parity. The k-linearity was only needed in the definition of a k-superbicategory to allow taking the negative 2-cell $\Omega_{\Omega_A} = -1_{\Omega_A \circ \Omega_A}$. A negative sign will appear in at least two other definitions in this thesis, signifying to some degree that two parity shift 1-cells have "switched places". See the definition of the opposite k-superbicategory corresponding to any k-superbicategory in Definition 2.8, and also the definition of local k-supercategories in Proposition 2.17.

[1] This last requirement was omitted in [Mur18].

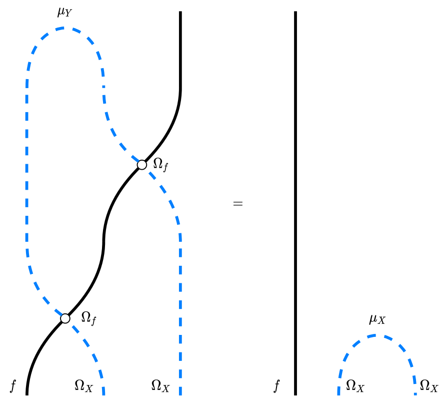


Figure 10: The modification condition for $\mu : \Omega \circ \Omega \to 1_{1_{\mathcal{B}}}$.

Example 2.5 (Super bimodules)**.** There is a bicategory consisting of $\mathbb{Z}_2$-graded k-algebras (also called superalgebras) as objects, $\mathbb{Z}_2$-graded bimodules as 1-cells and degree zero bimodule homomorphisms as 2-cells. The parity shift bimodule $\Omega_R : R \to R$ is the (R, R)-bimodule R with the opposite grading, with left and right action given on homogeneous elements by

$$r \cdot m = (-1)^{|r|} rm \qquad m \cdot r = (-1)^{|r|} mr$$

The isomorphism $\Omega_M : M \otimes \Omega_R \to \Omega_S \otimes M$ is extended linearly from the following mapping, given on homogeneous elements

$$m \otimes 1_R \mapsto (-1)^{|m|} 1_S \otimes m$$

Here $1_R \in \Omega_R$ refers to the unit of the ring R. Considering R as a bimodule it must be true that 1_R is an even element, so that $1_R \in \Omega_R$ is odd. Thus the isomorphism Ω_{Ω_R} is equal to $-1_{\Omega_R \otimes \Omega_R}$ as needed.

Remark 2.6. In many applications k will be commutative ring such that 2 is invertible, for instance this is required in the definition of a supercategory in [KKjO14]. Assuming that $2 \neq 0$ in k, then it is not possible to put a super structure on a bicategory $\mathcal{B}$ by defining $\Omega_X = 1_X$ for any object X. This will lead to the equality $-1_{1_X \circ 1_X} = 1_{1_X \circ 1_X}$, implying that $2 = 0$. Of course instead it could be true that $1_{1_X \circ 1_X} = 0$, but this is unlikely in any interesting k-linear bicategory.

We now put a superstructure on $[\mathcal{B}, \mathcal{C}]$ provided one on $\mathcal{C}$. Once we have defined superfunctors of k-superbicategories and supertransformations between them, we will put a similar super structure on the bicategory of superfunctors, see Proposition 2.34.

Proposition 2.7. *If $(\mathcal{C}, \Omega, \mu)$ is a k-superbicategory, then so is $([\mathcal{B}, \mathcal{C}], \Omega', \mu')$ for $\mathcal{B}$ a bicategory. Here Ω', μ' are defined as follows:*

- *$(\Omega')_F : F \to F$ is a strong transformation for $F : \mathcal{B} \to \mathcal{C}$ a pseudofunctor, with component 1-cells $((\Omega')_F)_X = \Omega_{FX}$ and 2-cells $((\Omega')_F)_f = \Omega_{Ff}$.*
- *$(\Omega')_\beta : \beta \circ (\Omega')_F \to (\Omega')_G \circ \beta$ is a modification for $\beta : F \to G$ a strong transformation, with component 2-cells $((\Omega')_\beta)_X = \Omega_{\beta_X}$.*
- *$(\mu')_F : (\Omega')_F \circ (\Omega')_F \to 1_F$ is a modification for $F : \mathcal{B} \to \mathcal{C}$ a pseudofunctor, with component 2-cells $((\mu')_F)_X = \mu_{FX}$.*

Proof. The data of a parity shift transformation Ω' on $[\mathcal{B}, \mathcal{C}]$ will have 1-cells $(\Omega')_F$ and 2-cells $(\Omega')_\beta$ as above - strong transformations and modifications in $[\mathcal{B}, \mathcal{C}]$ respectively, while being itself a strong transformation $\Omega' : 1_{[\mathcal{B},\mathcal{C}]} \to 1_{[\mathcal{B},\mathcal{C}]}$. Let Ω' be defined as in the statement of the proposition. Naturality in f of the 2-cells $((\Omega')_F)_f$ follows from naturality of Ω. Each 2-cell is an isomorphism, so we at least have the *data* for a strong transformation. Lax unity for $(\Omega')_F$ is commutativity of the outside of the following diagram

$$\begin{tikzcd}
F1_X \circ \Omega_{FX} \arrow[rr, "\Omega_{F1_X}"] & & \Omega_{FX} \circ F1_X \\
1_{FX} \circ \Omega_{FX} \arrow[u, "F^0 * 1"] \arrow[rr, "\Omega_{1_{FX}}"] & & \Omega_{FX} \circ 1_{FX} \arrow[u, "1 * F^0"'] \\
& \Omega_{FX} \arrow[ul, "\lambda^{-1}"] \arrow[ur, "\rho^{-1}"'] &
\end{tikzcd}$$

The square commutes by naturality of the 2-cells of Ω, the triangle due to lax unity of Ω. The lax naturality coherence condition follows similarly, so $(\Omega')_F$ is a strong transformation.
We move on to describing the modifications $(\Omega')_\beta$. Define a family of 2-cells as in the statement of the proposition:

$$((\Omega')_\beta)_X = \Omega_{\beta_X} : \beta_X \circ \Omega_{FX} \to \Omega_{GX} \circ \beta_X$$

After expanding the modication axiom for this data and omitting associators, we get the outside diagram below:

$$\begin{tikzcd}
Gf \circ \beta_X \circ \Omega_{FX} \arrow[r, "\beta_f * 1"] \arrow[d, "1 * \Omega_{\beta_X}"'] \arrow[dr, "\Omega_{Gf \circ \beta_X}"] & \beta_Y \circ Ff \circ \Omega_{FX} \arrow[r, "1 * \Omega_{Ff}"] \arrow[dr, "\Omega_{\beta_Y \circ Ff}"] & \beta_Y \circ \Omega_{FY} \circ Ff \arrow[d, "\Omega_{\beta_Y} * 1"] \\
Gf \circ \Omega_{GX} \circ \beta_X \arrow[r, "\Omega_{Gf} * 1"'] & \Omega_{GY} \circ Gf \circ \beta_X \arrow[r, "1 * \beta_f"'] & \Omega_{GY} \circ \beta_Y \circ Ff
\end{tikzcd}$$

Each triangle commutes due to lax naturality of Ω, while the square commutes due to naturality of the 2-cells of Ω. So the data $((\Omega')_\beta)_X$ makes $(\Omega')_\beta$ a modification which is invertible since each component is invertible. Furthermore, naturality in β follows from component-wise naturality of Ω in $\mathcal{C}$, as can be checked.

So we have 1-cells $(\Omega')_F$ and natural 2-cells $(\Omega')_\beta$ in $[\mathcal{B}, \mathcal{C}]$, which we want to coalesce into a strong transformation Ω' of the identity pseudofunctor. What remains is to show the lax unity and lax naturality conditions hold, but this follows easily component-wise by the associated conditions on Ω.

We move on to the definition of μ'. Define a family of 2-cells (modifications) $(\mu')_F : (\Omega')_F \circ (\Omega')_F \to 1_F$ to have components

$$((\mu')_F)_X = \mu_{FX} : \Omega_{FX} \circ \Omega_{FX} \to 1_{FX}$$

The modification condition for $(\mu')_F$ holds due to the modification condition for μ. The modification condition for μ' asks that the following diagram commutes, for $\beta : F \to G$ a strong transformation:

$$\begin{tikzcd}
\beta \circ (\Omega')_F \circ (\Omega')_F \arrow[r, "(\Omega')_\beta * 1"] \arrow[d, "1 * (\mu')_F"'] & (\Omega')_G \circ \beta \circ (\Omega')_F \arrow[r, "1 * (\Omega')_\beta"] & (\Omega')_G \circ (\Omega')_G \circ \beta \arrow[d, "(\mu')_G * 1"] \\
\beta \circ 1_F \arrow[rr, "\lambda^{-1} \circ \rho"'] & & 1_G \circ \beta
\end{tikzcd}$$

Component-wise this commutes, again due to the modifiction axiom for μ. We have finally defined the data Ω', μ' of a k-superbicategory structure on $[\mathcal{B}, \mathcal{C}]$. All that is missing is to show that $(\Omega', \Omega', (\mu')^{-1}, \mu')$ is an adjunction in $[\mathcal{D}, \mathcal{D}]$ where $\mathcal{D} = [\mathcal{B}, \mathcal{C}]$. But this follows easily component-wise from the associated adjunction $(\Omega, \Omega, \mu^{-1}, \mu)$ in $[\mathcal{C}, \mathcal{C}]$. $\square$

Definition 2.8. Let $(\mathcal{B}, \Omega, \mu)$ be a k-superbicategory. The *opposite k-superbicategory* is $(\mathcal{B}^{op}, \Omega^{op}, \mu^{op})$ where $\Omega_X^{op} = \Omega_X$ and Ω_f^{op} is given by

$$f \circ^{op} \Omega_Y = \Omega_Y \circ f \xrightarrow{\Omega_f^{-1}} f \circ \Omega_X = \Omega_X \circ^{op} f$$

and μ_X^{op} is given by

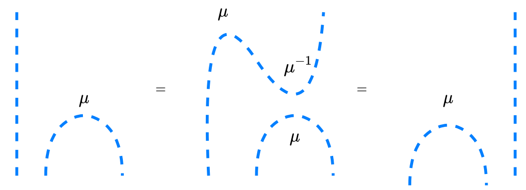
$$\Omega_X \circ^{op} \Omega_X = \Omega_X \circ \Omega_X \xrightarrow{-\mu_X} 1_X = 1_X^{op}$$

noticing the sign. We denote the opposite k-superbicategory by $(\mathcal{B}, \Omega, \mu)^{op}$ or even $\mathcal{B}^{op}$ if no confusion arises.

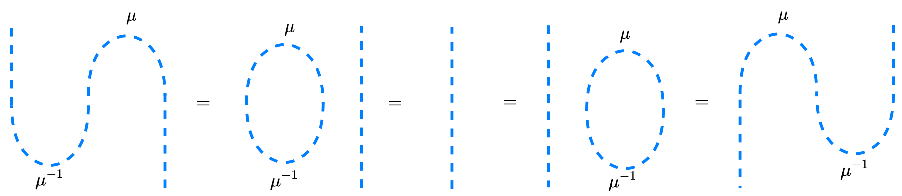
2.1 Supercategories

Definition 2.9. A *k-supercategory* is a k-linear category $\mathcal{C}$ together with a k-linear functor $\tilde{\Omega} : \mathcal{C} \to \mathcal{C}$ and natural isomorphism $\tilde{\mu} : \tilde{\Omega} \circ \tilde{\Omega} \to 1_{\mathcal{C}}$ such that the data $(\tilde{\Omega}, \tilde{\Omega}, \tilde{\mu}^{-1}, \tilde{\mu})$ defines an adjoint equivalence. Here $\tilde{\Omega}$ is referred to as the *shift* or *parity shift* functor of the k-supercategory.

Remark 2.10. The Zorro moves for the adjoint equivalence in the definition of a k-supercategory imply that $\tilde{\mu}_{\mathcal{C}} * 1 = 1 * \tilde{\mu}_{\mathcal{C}}$:



Conversely if $\tilde{\mu}_{\mathcal{C}} * 1 = 1 * \tilde{\mu}_{\mathcal{C}}$, then the Zorro moves hold:



Therefore, our definition of a k-supercategory is equivalent to that given in [EL16].

Definition 2.11. A *superfunctor* $(\mathcal{C}, \tilde{\Omega}_{\mathcal{C}}, \tilde{\mu}_{\mathcal{C}}) \to (\mathcal{D}, \tilde{\Omega}_{\mathcal{D}}, \tilde{\mu}_{\mathcal{D}})$ between k-supercategories is a k-linear functor $F : \mathcal{C} \to \mathcal{D}$ together with a natural isomorphism $\tilde{\Omega}_F : F \circ \tilde{\Omega}_{\mathcal{C}} \cong \tilde{\Omega}_{\mathcal{D}} \circ F$ such the following diagram commutes

$$\begin{tikzcd}
F \circ \tilde{\Omega}_{\mathcal{C}} \circ \tilde{\Omega}_{\mathcal{C}} \arrow[r, "\tilde{\Omega}_F * 1"] \arrow[d, "1*\tilde{\mu}_{\mathcal{C}}"'] & \tilde{\Omega}_{\mathcal{D}} \circ F \circ \tilde{\Omega}_{\mathcal{C}} \arrow[d, "1*\tilde{\Omega}_F"] \\
F & \tilde{\Omega}_{\mathcal{D}} \circ \tilde{\Omega}_{\mathcal{D}} \circ F \arrow[l, "\tilde{\mu}_{\mathcal{D}} * 1"]
\end{tikzcd} \tag{7}$$

We define the composite of superfunctors $\mathcal{C} \xrightarrow{(F, \tilde{\Omega}_F)} \mathcal{D} \xrightarrow{(G, \tilde{\Omega}_G)} \mathcal{E}$ to be $(GF, \tilde{\Omega}_{GF})$ where $\tilde{\Omega}_{GF}$ is defined as

$$
\begin{tikzcd}
 & G\tilde{\Omega}_{\mathcal{D}}F \arrow[dr, "\tilde{\Omega}_G * 1"] & \\
GF\tilde{\Omega}_{\mathcal{C}} \arrow[ur, "1*\tilde{\Omega}_F"] \arrow[rr, dashed, "\tilde{\Omega}_{GF}"'] & & \tilde{\Omega}_{\mathcal{E}}GF
\end{tikzcd}
\tag{8}
$$

The *identity superfunctor* on $(\mathcal{C}, \tilde{\Omega}_{\mathcal{C}}, \tilde{\mu}_{\mathcal{C}})$ is defined to be $(1_{\mathcal{C}}, 1_{\tilde{\Omega}_{\mathcal{C}}})$, where $1_{\tilde{\Omega}_{\mathcal{C}}}$ is the identity natural transformation on $\tilde{\Omega}_{\mathcal{C}}$.

Definition 2.12. A *supertransformation* $(F, \tilde{\Omega}_F) \to (G, \tilde{\Omega}_G)$ is a natural transformation $\alpha : F \to G$ such that the following commutes

$$
\begin{tikzcd}
F \circ \tilde{\Omega}_{\mathcal{C}} \arrow[r, "\tilde{\Omega}_F"] \arrow[d, "\alpha * 1"'] & \tilde{\Omega}_{\mathcal{D}} \circ F \arrow[d, "1*\alpha"] \\
G \circ \tilde{\Omega}_{\mathcal{C}} \arrow[r, "\tilde{\Omega}_G"'] & \tilde{\Omega}_{\mathcal{D}} \circ G
\end{tikzcd}
\tag{9}
$$

Horizontal and vertical composition of supertransformations is performed on the underlying natural transformations.

Remark 2.13. When we consider $\tilde{\Omega}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ as a superfunctor, rather than a functor, we equip it with the isomorphism $-1_{\tilde{\Omega}_{\mathcal{C}} \circ \tilde{\Omega}_{\mathcal{C}}}$, noticing the sign. The superfunctor axiom holds because $\tilde{\mu}_{\mathcal{C}} * 1 = 1 * \tilde{\mu}_{\mathcal{C}} : \tilde{\Omega}_{\mathcal{C}}^3 \to \tilde{\Omega}_{\mathcal{C}}$ as shown in Remark 2.10. With this superfunctor definition, $\tilde{\mu}_{\mathcal{C}}$ then becomes a supertransformation as one may check.

Proposition 2.14. *Small k-supercategories $(\mathcal{C}, \tilde{\Omega}_{\mathcal{C}}, \tilde{\mu}_{\mathcal{C}})$, superfunctors and supernatural transformations assemble into a k-superbicategory $(\boldsymbol{sCat}_k, \tilde{\Omega}, \tilde{\mu})$ with the obvious structure given above.*

Proof. It is straightforward to see that k-supercategories, superfunctors and supernatural transformations form a k-linear 2-category, where vertical and horizontal composition of transformations, composition of superfunctors, and identity superfunctors are all as discussed above. Define a strong transformation $\tilde{\Omega} : 1_{\mathbf{sCat}_k} \to 1_{\mathbf{sCat}_k}$ to have component 1-cells $\tilde{\Omega}_{\mathcal{C}}$ and 2-cells $\tilde{\Omega}_F$, as the structure given on any k-supercategory and superfunctor. Naturality of the 2-cells is captured in the diagram (9). Lax unity and lax naturality follow directly from our definitions of identity superfunctor and composition of superfunctors, respectively. Next we define an invertible modification $\tilde{\mu} : \tilde{\Omega} \circ \tilde{\Omega} \to 1_{1_{\mathbf{sCat}_k}}$ to have $\mathcal{C}$-component $\tilde{\mu}_{\mathcal{C}}$, the parity involution transformation on $\mathcal{C}$. The modification condition is captured exactly in (7). Lastly, $(\tilde{\Omega}, \tilde{\Omega}, \tilde{\mu}^{-1}, \tilde{\mu})$ is an adjoint equivalence in $[\mathbf{sCat}_k, \mathbf{sCat}_k]$ since each $(\tilde{\Omega}_{\mathcal{C}}, \tilde{\Omega}_{\mathcal{C}}, \tilde{\mu}_{\mathcal{C}}^{-1}, \tilde{\mu}_{\mathcal{C}})$ is an adjoint equivalence. □

Definition 2.15. An equivalence in $\mathbf{sCat}_k$ is referred to as a superequivalence.

Recall that a functor $F : \mathcal{C} \to \mathcal{D}$ between categories is an equivalence if and only if F is fully faithful and essentially surjective, see for instance [Rie16, Theorem 1.5.9].

Proposition 2.16. *A superfunctor $(F, \tilde{\Omega}_F) : (\mathcal{C}, \tilde{\Omega}_{\mathcal{C}}, \tilde{\mu}_{\mathcal{C}}) \to (\mathcal{D}, \tilde{\Omega}_{\mathcal{D}}, \tilde{\mu}_{\mathcal{D}})$ is a superequivalence of k-supercategories if and only if $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories.*

Proof. A superequivalence must be an equivalence on the underlying categorical structure, so we move on to the other direction, assuming now that F is an equivalence and hence essentially surjective and fully faithful. We proceed with the construction of a functor $G : \mathcal{D} \to \mathcal{C}$ and natural transformations η, ϵ as in the proof of [Rie16, Theorem 1.5.9]. By essential surjectivity we choose an object $GY \in \mathcal{C}$ for all $Y \in \mathcal{D}$ and an isomorphism $\epsilon_Y : F(GY) \cong Y$. This is possible since F is essentially surjective. Since F is fully faithful, there exists a unique map $GY \to GY'$, denoted by Gf, such that the following diagram commutes:

$$\begin{tikzcd}
F(GY) \arrow[r, "\epsilon_Y"] \arrow[d, dashed, "F(Gf)"'] & Y \arrow[d, "f"] \\
F(GY') \arrow[r, "\epsilon_{Y'}"'] & Y'
\end{tikzcd}$$

Functoriality of G follows from uniqueness. Notice then that the maps ϵ_Y assemble to form a natural isomorphism $F \circ G \to 1_{\mathcal{D}}$. We can also see that G is k-linear, since $G : \mathcal{D}(Y, Y') \to \mathcal{C}(GY, GY')$ is given by the following composition of k-module isomorphisms:

$$\mathcal{D}(Y, Y') \xrightarrow{(\epsilon_Y)^*} \mathcal{D}(F(GY), Y') \xrightarrow{(\epsilon_{Y'})_*} \mathcal{D}(F(GY), F(GY')) \xrightarrow{F^{-1}} \mathcal{C}(GY, GY')$$

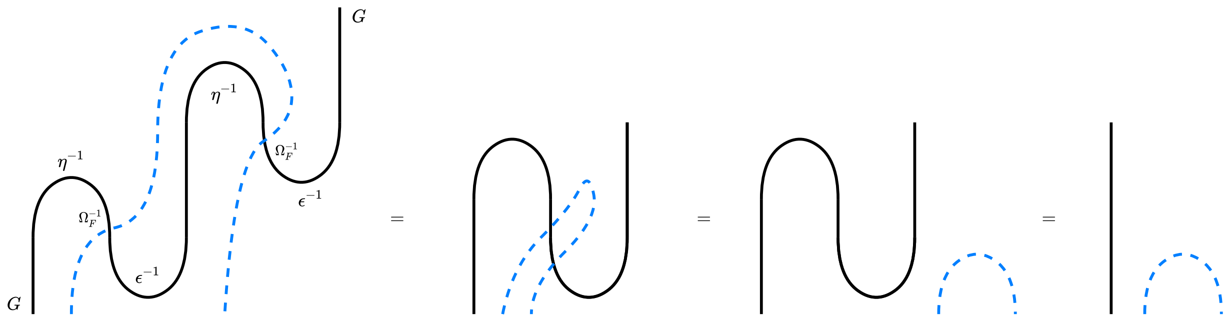
Notice that we have $\epsilon_{FX}^{-1} : FX \to FGFX$, so there exists a unique map $\eta_X : X \to GFX$ such that $F\eta_X = \epsilon_{FX}^{-1}$. It follows that η is a natural isomorphism and that $1_F * \eta = \epsilon^{-1} * 1_F$, so (F, G, η, ϵ) is actually an *adjoint* equivalence. To form a superequivalence we need to give G the structure of a superfunctor and then show that η and ϵ are supertransformations. For this first goal define $\tilde{\Omega}_G$ to be the composite:

$$G\tilde{\Omega}_{\mathcal{D}} \xrightarrow{1*\epsilon^{-1}} G\tilde{\Omega}_{\mathcal{D}}FG \xrightarrow{1*\tilde{\Omega}_F*1} GF\tilde{\Omega}_{\mathcal{C}}G \xrightarrow{\eta^{-1}*1} \tilde{\Omega}_{\mathcal{C}}G$$

To show that $(G, \tilde{\Omega}_G)$ is a supertransformation requires commutativity of

$$\begin{tikzcd}
G\tilde{\Omega}_{\mathcal{D}}\tilde{\Omega}_{\mathcal{D}} \arrow[r, "\tilde{\Omega}_G*1"] \arrow[d, "1*\tilde{\mu}_{\mathcal{D}}"'] & \tilde{\Omega}_{\mathcal{C}}G\tilde{\Omega}_{\mathcal{C}} \arrow[d, "1*\tilde{\Omega}_G"] \\
G & \tilde{\Omega}_{\mathcal{C}}\tilde{\Omega}_{\mathcal{C}}G \arrow[l, "\tilde{\mu}_{\mathcal{C}}*1"]
\end{tikzcd}$$

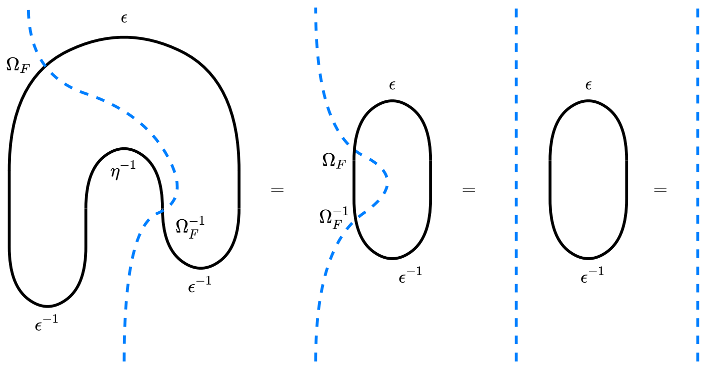
If we expand the longest path we get the first diagram below



The first and last equalities arise from Zorro moves while the middle equality is superfunctoriality of F. The right-most diagram is the shorter path expanded and so G is a superfunctor. Supernaturality of ϵ requires the following diagram to commute:

$$\begin{tikzcd}
FG\tilde{\Omega}_{\mathcal{D}} \arrow[r, "1*\tilde{\Omega}_G"] \arrow[d, "\epsilon*1"'] & F\tilde{\Omega}_{\mathcal{C}}G \arrow[r, "\tilde{\Omega}_F*1"] & \tilde{\Omega}_{\mathcal{D}}FG \arrow[d, "1*\epsilon"] \\
\tilde{\Omega}_{\mathcal{D}} \arrow[rr, "="'] & & \tilde{\Omega}_{\mathcal{D}}
\end{tikzcd}$$

If we reverse the left vertical isomorphism, then the composite is represented as the first diagram below



Here we applied a Zorro move and cancelled isomorphisms. A very similar argument shows that η is a supertransformation, so we are done. □

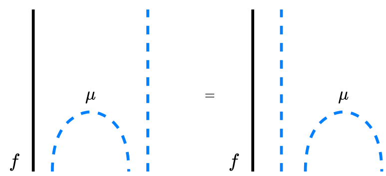
The earliest reference we found for the definition of a super-2-category is [KKjO14]. The definition there takes the approach of defining a supercategory $\mathcal{B}(X,Y)$ for objects X, Y and forcing the composition functors to respect this structure in a particular way. Subsequent papers such as [EL16] take an approach more similar to our definition of superbicategories, with local categories $\mathcal{B}(X,Y)$ together other additional data on a global level. We shall not make an effort to imitate the definition of [KKjO14], followed by a discussion about why the two definitions are equivalent. Instead we are content with the following, which doubles as a definition and proposition:

Proposition 2.17 (Local k-supercategories)**.** *Let $(\mathcal{B}, \Omega, \mu)$ be a k-superbicategory. Then each local category $\mathcal{B}(X,Y)$ has the structure of a k-supercategory $\tilde{\Omega}_{X,Y}, \tilde{\mu}_{X,Y}$ where $\tilde{\Omega}_{X,Y}(f) = f \circ \Omega_X$ and $\tilde{\Omega}_{X,Y}(\theta) = \theta * 1_{\Omega_X}$. The f-component of $\tilde{\mu}_{X,Y}$ is the composite*

$$(f\Omega_X)\Omega_X \xrightarrow{\alpha} f(\Omega_X\Omega_X) \xrightarrow{-1*\mu_X} f1_X \xrightarrow{\rho} f$$

noticing the sign.

Proof. Now $\tilde{\Omega}_{X,Y}$ is functorial due to the interchange law and the identity $1_{\gamma*\beta} = 1_\gamma * 1_\beta$ for 2-cells γ, β. It is clear that $\tilde{\mu}_{X,Y}$ is natural. We lastly need to show that we have an adjoint equivalence $(\tilde{\Omega}_{X,Y}, \tilde{\Omega}_{X,Y}, \tilde{\mu}_{X,Y}^{-1}, \tilde{\mu}_{X,Y})$. For this it suffices to show that $\tilde{\mu}_{X,Y} * 1 = 1 * \tilde{\mu}_{X,Y}$ by Remark 2.10. The f-component of this equality is shown, with minus signs cancelled:



Equality holds since $(\Omega_X, \Omega_X, \mu_X^{-1}, \mu_X)$ is an adjoint equivalence in $\mathcal{B}$, together with the logic in Remark 2.10. □

Remark 2.18. In the above proposition we could have chosen $\tilde{\Omega}_{X,Y}(f) = \Omega_Y \circ f$ instead. This would be naturally isomorphic to our original definition due to the natural 2-isomorphisms $\Omega_f : f \circ \Omega_X \to \Omega_Y \circ f$.

Recall that in any bicategory $\mathcal{B}$ we have post- and pre-composition functors

$$f_* : \mathcal{B}(A,X) \to \mathcal{B}(A,Y) \qquad f^* : \mathcal{B}(Y,A) \to \mathcal{B}(X,A)$$

where $f_*(g) = f \circ g$ and $f_*(\beta) = 1_f * \beta$. Here $f : X \to Y$ is a 1-cell in $\mathcal{B}$. The pre-composition functor f^* is defined similarly. Now Proposition 2.17 puts a k-supercategory structure on each local category in a k-superbicategory. We can upgrade the pre- and post-composition functors to superfunctors, which will be used when discussing the (super)representable morphisms in Section 4:

Definition 2.19. Let $(\mathcal{B}, \Omega, \mu)$ be a k-superbicategory, $A \in \mathcal{B}_0$ and $f : X \to Y$. We define *pre-composition* and *post-composition superfunctors* respectively to have underlying functor

$$f^* : \mathcal{B}(Y, A) \to \mathcal{B}(X, A) \qquad f_* : \mathcal{B}(A, X) \to \mathcal{B}(A, Y)$$

each paired with the following natural isomorphisms, respectively:

$$\tilde{\Omega}_{f^*} : f^* \circ \tilde{\Omega}_{Y,A} \to \tilde{\Omega}_{X,A} \circ f^* \qquad \tilde{\Omega}_{f_*} : f_* \circ \tilde{\Omega}_{A,X} \to \tilde{\Omega}_{A,Y} \circ f_*$$

with given components

$$\begin{aligned} (\tilde{\Omega}_{f^*})_g : &\quad (g\ \Omega_Y)f \xrightarrow{\alpha} g(\Omega_Y f) \xrightarrow{1*\Omega_f^{-1}} g(f\ \Omega_X) \xrightarrow{\alpha^{-1}} (gf)\Omega_X \\ (\tilde{\Omega}_{f_*})_h : &\quad f(h\ \Omega_A) \xrightarrow{\alpha^{-1}} (fh)\Omega_A \end{aligned}$$

Remark 2.20. Recall that for a k-supercategory $(\mathcal{C}, \tilde{\Omega}_\mathcal{C}, \tilde{\mu}_\mathcal{C})$ the functor $\tilde{\Omega}_\mathcal{C} : \mathcal{C} \to \mathcal{C}$ is considered as a superfunctor with isomorphism $-1_{\tilde{\Omega}^2_\mathcal{C}} : \tilde{\Omega}^2_\mathcal{C} \to \tilde{\Omega}^2_\mathcal{C}$. If $(\mathcal{B}, \Omega, \mu)$ is a k-superbicategory, then the parity shift functors $\tilde{\Omega}_{X,Y} : \mathcal{B}(X, Y) \to \mathcal{B}(X, Y)$ on the local categories of $\mathcal{B}$ as given in Proposition 2.17 have the same underlying functor as $\Omega^*_X : \mathcal{B}(X, Y) \to \mathcal{B}(X, Y)$. The induced natural isomorphism on $\Omega^*_X \circ \Omega^*_X$ according to the above definition has g-component

$$(g\ \Omega_X)\Omega_X \xrightarrow{\alpha} g(\Omega_X \Omega_X) \xrightarrow{1*(\Omega_{\Omega_X})^{-1} = -1_{g(\Omega^2_X)}} g(\Omega_X \Omega_X) \xrightarrow{\alpha^{-1}} (g\ \Omega_X)\Omega_X$$

This composition is equal to $-1_{(g\ \Omega_X)\Omega_X}$, so the natural isomorphism on $\Omega^*_X \circ \Omega^*_X$ is $-1_{\Omega^*_X \circ \Omega^*_X}$. Therefore we may identify $\tilde{\Omega}_{X,Y}$ with Ω^*_X as *super*functors (see Remark 2.13).

Definition 2.21. Let $f, g : X \to Y$ be 1-cells in a k-superbicategory $(\mathcal{B}, \Omega, \mu)$. For $\beta : f \to g$ a 2-cell, define a natural transformation $\beta^* : f^* \to g^* : \mathcal{B}(Y, A) \to \mathcal{B}(X, A)$ to have h-component $\beta^*_h = 1_h * \beta$. This is easily checked to be natural. Considering f^*, g^* as superfunctors now as in the previous definition, the natural transformation β^* is easily seen to be supernatural.

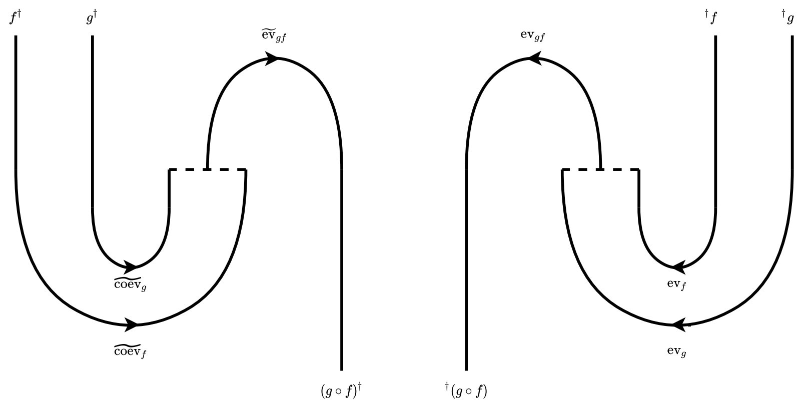
Similarly we define a super natural transformation $\beta_* : f_* \to g_* : \mathcal{B}(A, X) \to \mathcal{B}(A, Y)$.

2.2 Graded pivotal bicategories

We now aim to define a *graded pivotal* structure on a k-superbicategory $(\mathcal{B}, \Omega, \mu)$, this is a refinement of the definition given in [CM16]. The word "pivotal" used to describe a bicategory with left and right adjoints generally refers to a natural isomorphism between each left and right adjoint[2], whereas "graded pivotal" will mean left and right adjoints that agree only up to a parity shift. Pivotal bicategories often arise in the context of topological quantum field theory, see for instance [CR16, CR12, CMS20].

Remark 2.22. Recall that for a bicategory $\mathcal{B}$ with right adjoints we *fix* the data of an adjunction $(f, f^\dagger, \widetilde{\mathrm{coev}}_f, \widetilde{\mathrm{ev}}_f)$ for each 1-cell $f : X \to Y$. Suppose we have $X \xrightarrow{f} Y \xrightarrow{g} Z$, then Proposition 1.21 tells us that there exists an adjunction $(gf, f^\dagger g^\dagger, \eta', \epsilon')$. This adjunction may not match the data $(gf, (gf)^\dagger, \widetilde{\mathrm{coev}}_{gf}, \widetilde{\mathrm{ev}}_{gf})$ that we have fixed. Proposition 1.18 comes to the rescue with an isomorphism $\mathcal{R}_{gf} : (gf)^\dagger \cong f^\dagger g^\dagger$. If $\mathcal{B}$ has left adjoints we are also guaranteed isomorphisms $\mathcal{L}_{gf} : {}^\dagger(gf) \cong {}^\dagger f^\dagger g$. Both types of isomorphisms are depicted below

[2] In the context of monoidal categories, one might see "autonomous" or "sovereign" used instead of "pivotal".



The horizontal dotted lines simply represent the identity 2-cell $1_{g\circ f}$, yet they convey a shift in perspective from considering $g \circ f$ as a singular 1-cell to considering it as a composition of two 1-cells.

Definition 2.23. A *graded pivotal* k-superbicategory is a k-superbicategory $(\mathcal{B}, \Omega, \mu)$ with, for every 1-cell $f : X \to Y$, a 1-cell $f^\vee : Y \to X$ and adjunctions

$$(f^\vee \circ \Omega_Y, f, \mathrm{coev}_f, \mathrm{ev}_f) \qquad (f, \Omega_X \circ f^\vee, \widetilde{\mathrm{coev}}_f, \widetilde{\mathrm{ev}}_f)$$

This yields isomorphisms

$$q_f : \Omega_X \circ f^\dagger \circ \Omega_Y = \Omega_X \circ \Omega_X \circ f^\vee \circ \Omega_Y \xrightarrow{\mu_X * 1} f^\vee \circ \Omega_Y = {}^\dagger f$$

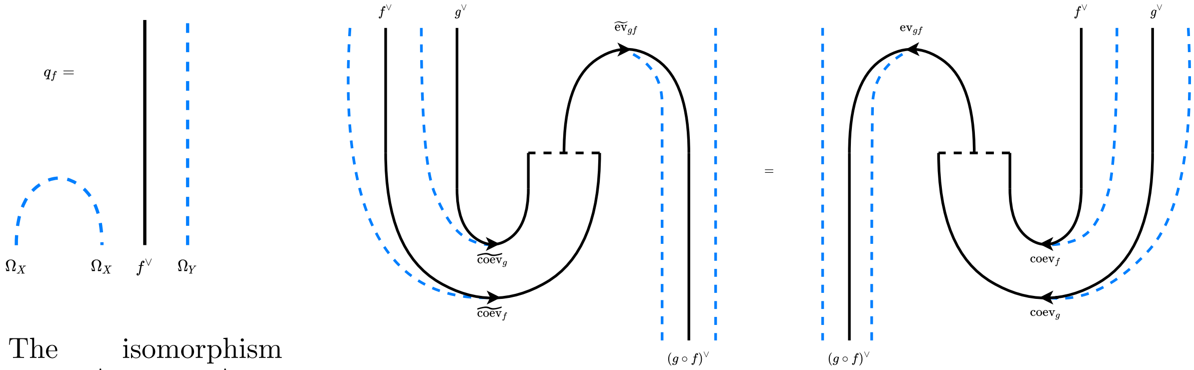
as depicted in Figure 11a. This data is required to satisfy the *graded pivotality condition* - the following diagram commutes:

$$\begin{tikzcd}
\Omega_X \circ (g \circ f)^\dagger \circ \Omega_Z \arrow[r, "q_{gf}"] \arrow[d, "1*\mathcal{R}_{gf}*1"'] & {}^\dagger(g \circ f) \arrow[ddd, "\mathcal{L}_{gf}"] \\
\Omega_X \circ f^\dagger \circ g^\dagger \circ \Omega_Z \arrow[d] & \\
\Omega_X \circ f^\dagger \circ 1_Y \circ g^\dagger \circ \Omega_Z \arrow[d, "1*\mu_Y^{-1}*1"'] & \\
\Omega_X \circ f^\dagger \circ \Omega_Y \circ \Omega_Y \circ g^\dagger \circ \Omega_Z \arrow[r, "q_f * q_g"'] & {}^\dagger f \circ {}^\dagger g
\end{tikzcd}$$

Remark 2.24. Here we have omitted bracketings and associators for clarity, the unlabelled vertical map is some choice of a whiskered unitor depending on whatever bracketing is chosen. It can be shown with minimal effort that this condition is equivalent to the string diagram equality in Figure 11b. Notice that if $\mathcal{B}$ is a graded pivotal k-superbicategory, then in particular it has both left and right adjoints for any 1-cell, where $f^\dagger = \Omega_X \circ f^\vee$ and ${}^\dagger f = f^\vee \circ \Omega_Y$.

Example 2.25. In [CM16] it is shown that the bicategory $\mathcal{LG}_k$ of Landau-Ginzburg models has both left and right adjoints. Further, although the language of superbicategories is not used there, it is shown in fact that $\mathcal{LG}_k$ is graded pivotal in the sense of Definition 2.23. It was later shown in [Mur18] that $\mathcal{LG}_k$ is a k-superbicategory.

We move on to inducing a graded pivotal structure on $[\mathcal{B}, \mathcal{C}]$ supposing such a structure on $\mathcal{C}$, building on Proposition 2.7 with the further hypothesis that $\mathcal{B}$ has both left and right adjoints. For this we need a strong transformation $\beta^\vee : G \to F$ for each strong transformation



(a) The isomorphism $\Omega_X \circ f^\dagger \circ \Omega_Y \to {}^\dagger f$.

(b) The graded pivotality condition.

Figure 11

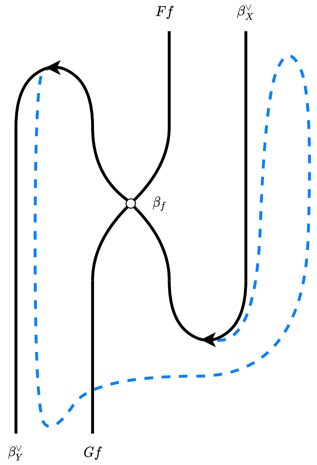


Figure 12: A string diagram representing the composition in (10).

$\beta : F \to G$. We define the 1-cell components to be given as $(\beta^\vee)_X = \beta_X^\vee$. Though $\beta^\vee$ will end up being a strong transformation, initially it is oplax with $(\beta^\vee)_f$ defined as the composite:

$$\begin{tikzcd}
\beta_Y^\vee \circ Gf \arrow[r, "1*\mu_{GY}^{-1}*1"] \arrow[dd, dashed] & {}^\dagger\beta_Y \circ \Omega_{GY} \circ Gf \arrow[d, "1*\Omega_{Gf}^{-1}"] \\
 & {}^\dagger\beta_Y \circ Gf \circ \Omega_{GX} \arrow[d, "m(\beta_f)"] \\
Ff \circ \beta_X^\vee & Ff \circ {}^\dagger\beta_X \circ \Omega_{GX} \arrow[l, "1*1*\mu_{GX}"]
\end{tikzcd} \tag{10}$$

Here $m(\beta_f)$ denotes the mate of $\beta_f : Gf \circ \beta_X \to \beta_Y \circ Ff$ under the pair of adjunctions

$$(\beta_X^\vee \circ \Omega_{GX}, \beta_X, \mathrm{coev}_{\beta_X}, \mathrm{ev}_{\beta_X}) \qquad (\beta_Y^\vee \circ \Omega_{GY}, \beta_Y, \mathrm{coev}_{\beta_Y}, \mathrm{ev}_{\beta_Y}) \tag{11}$$

See Figure 12 for a string diagram illustration. The next result is very similar to Proposition 1.25, except that now we have the added complexity of blue dotted lines:

Proposition 2.26. *$\beta^\vee : F \to G$ as defined above is an oplax transformation.*

Proof. Naturality of the 2-cells $\beta_f^\vee$ is clear. The oplax naturality coherence condition asks that the following diagram commutes:

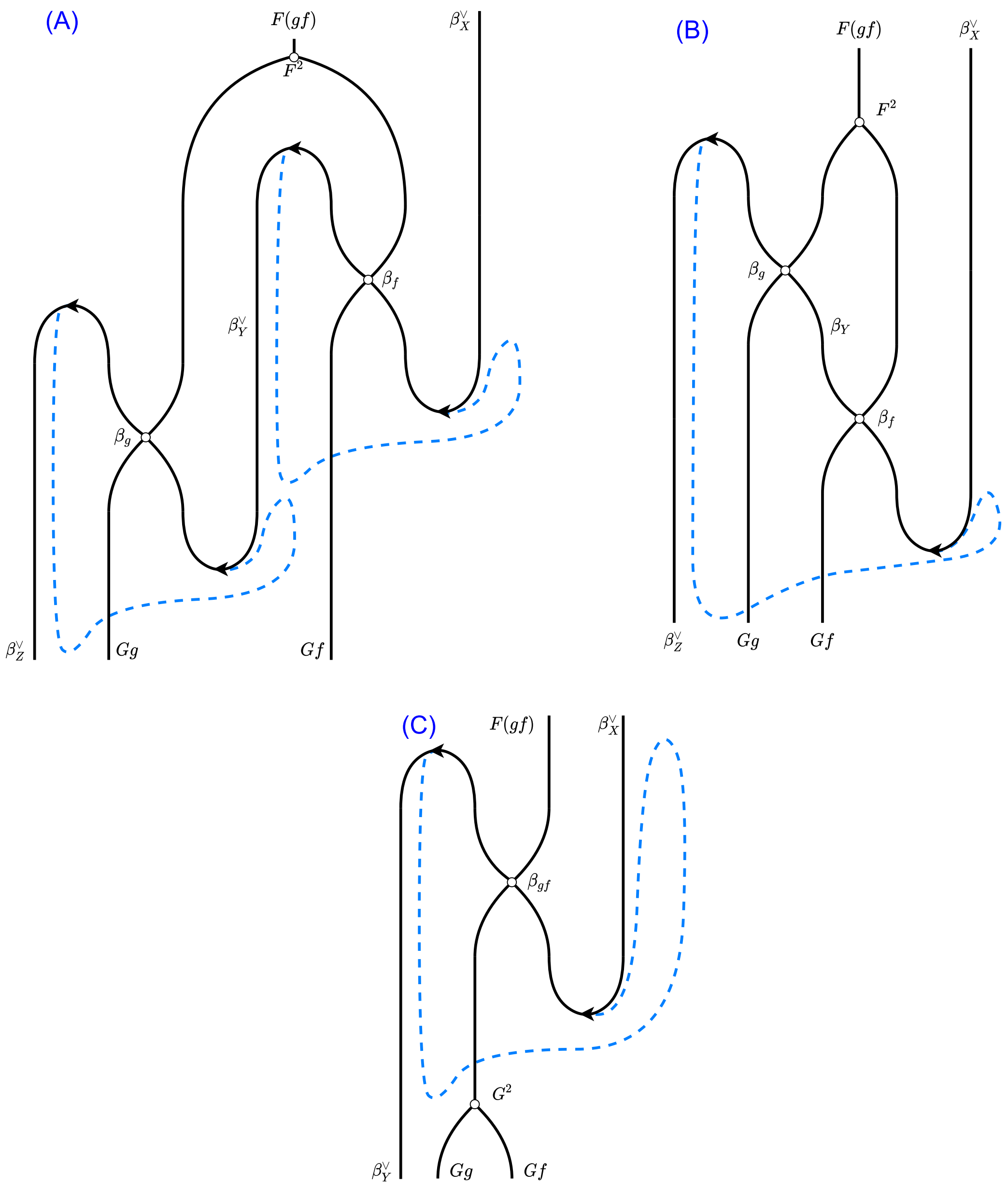
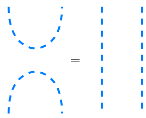


Figure 13: Diagrams used in the proof of Proposition 2.26 to show that $\beta^\vee$ satisfies oplax naturality.

$$
\begin{tikzcd}
\beta_Z^\vee GgGf \arrow[r, "1*G^2"] \arrow[d, "\beta_g^\vee*1"'] & \beta_Z^\vee G(gf) \arrow[dd, "\beta_{gf}^\vee"] \\
Fg\beta_Y^\vee Gf \arrow[d, "1*\beta_f^\vee"'] & \\
FgFf\beta_X^\vee \arrow[r, "F^2*1"'] & F(gf)\beta_X^\vee
\end{tikzcd}
$$

The left-bottom path is expressed in diagram (A) of Figure 13, up to applications of the interchange law. That $\mu_X^{-1}\mu_X = 1_{\Omega_X \circ \Omega_X}$ is expressed as



(12)

Applying this to (A) and executing a Zorro move will take us to (B). From here we apply lax naturality of β, which is depicted in Figure 8a. To finally get to (C) we need to pull the blue dotted string "over" G^2, justified by the following commutative diagram:

$$\begin{tikzcd}
 & GgGf\Omega_{GX} \arrow[r, "G^2*1"] \arrow[dl, "1*\Omega_{Gf}"'] \arrow[dd, "\Omega_{GgGf}"] & G(gf)\Omega_{GX} \arrow[dd, "\Omega_{G(gf)}"] \\
Gg\Omega_{GY}Gf \arrow[dr, "\Omega_{Gg}*1"'] & & \\
 & \Omega_{GZ}GgGf \arrow[r, "1*G^2"'] & \Omega_{GZ}G(gf)
\end{tikzcd}$$

The proof of the oplax unity condition is similar. □

If we further assume that our source bicategory $\mathcal{B}$ has left and right adjoints then we can conclude that $\beta^\vee$ is a strong transformation by Corollary 1.24. Having constructed such a dual for each β, we wish to form adjunctions

$$(\beta^\vee \circ \Omega_G, \beta, \mathrm{coev}_\beta, \mathrm{ev}_\beta) \qquad (\beta, \Omega_F \circ \beta^\vee, \widetilde{\mathrm{coev}}_\beta, \widetilde{\mathrm{ev}}_\beta) \tag{13}$$

and ultimately show that this data satisfies the *graded pivotality condition*.

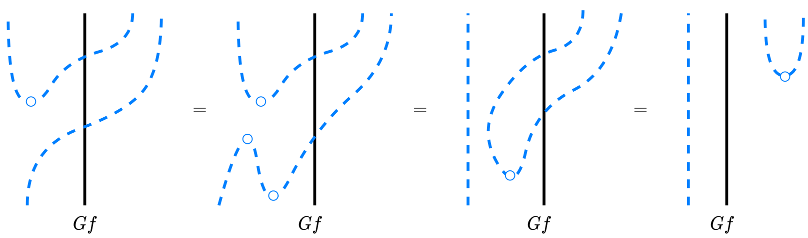
Proposition 2.27. *The following components define a modification* $\mathrm{coev}_\beta : 1_G \to \beta \circ \beta^\vee \circ \Omega_G$*:*

$$(\mathrm{coev}_\beta)_X = \mathrm{coev}_{\beta_X}$$

Proof. We need to show the following diagram commutes:

$$\begin{tikzcd}
 & Gf \arrow[dl, "\lambda^{-1}"'] \arrow[dr, "\rho^{-1}"] & \\
1_{GY} \circ Gf \arrow[d, "\mathrm{coev}_{\beta_Y}*1"'] & & Gf \circ 1_{GX} \arrow[d, "1*\mathrm{coev}_{\beta_X}"] \\
\beta_Y \beta_Y^\vee \Omega_{GY} Gf \arrow[d, "1*\Omega_{Gf}^{-1}"'] & & Gf \beta_X \beta_X^\vee \Omega_{GX} \\
\beta_Y \beta_Y^\vee Gf \Omega_{GX} \arrow[rr, "1*\beta_f^\vee*1"'] & & \beta_Y Ff \beta_X^\vee \Omega_{GX} \arrow[u, "(\beta_f)^{-1}*1"']
\end{tikzcd} \tag{14}$$

The left-bottom path is depicted in the first diagram of Figure 14. Focusing on the string diagram enclosed within the red dotted region, we see the following:



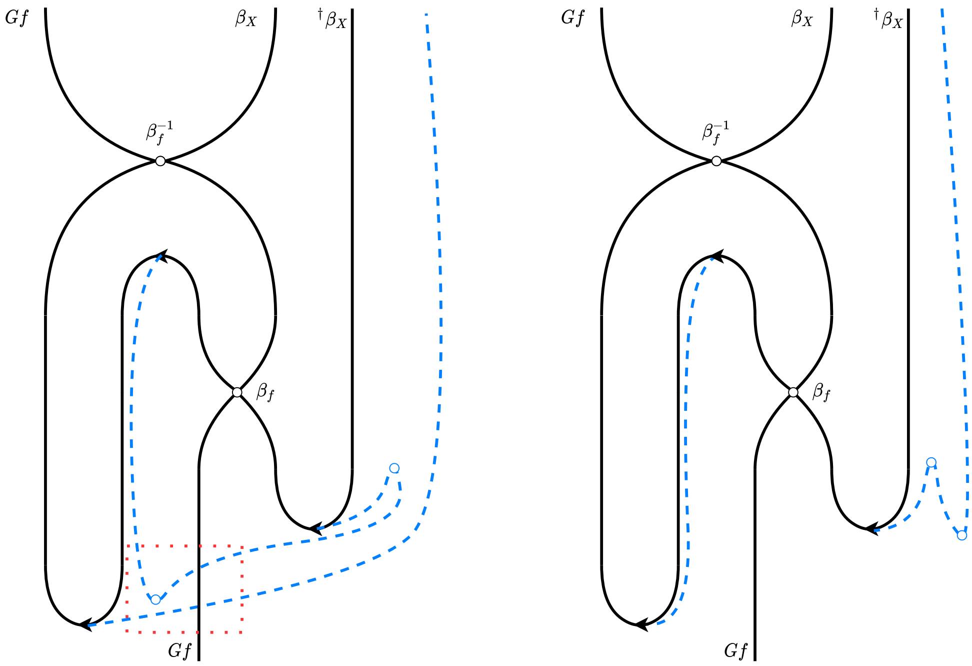


Figure 14: The left represents the left-bottom path around the diagram (14).

Here we used (12), a Zorro move, and the modification condition for μ in that order. This brings us to the second diagram of Figure 14, where we may apply two Zorro moves, proceeding on to cancel inverse 2-cells and finally arriving at the right side path of (14). □

Proposition 2.28. *If $\mathcal{C}$ is a graded pivotal k-superbicategory and $\mathcal{B}$ has left and right adjoints, then $[\mathcal{B}, \mathcal{C}]$ is a graded pivotal k-superbicategory also.*

Proof. So far we have specified a k-superbicategory structure Ω', μ' on $[\mathcal{B}, \mathcal{C}]$, along with dual transformations $\beta^\vee$ for each strong transformation $\beta : F \to G$. Similarly to Proposition 2.27 we also get modifications $\mathrm{ev}_\beta, \widetilde{\mathrm{coev}}_\beta, \widetilde{\mathrm{ev}}_\beta$. These form adjunctions as in (13) due to the relevant conditions holding component-wise. So we get isomorphisms $q_\beta : \Omega_G \circ \beta^\dagger \circ \Omega_F \to {}^\dagger\beta$ as in Figure 11a. All that remains is to show that this data satisfies the graded pivotality condition. Since $(\beta^\vee)_X = \beta_X^\vee$ and $(\mathrm{coev}_\beta)_X = \mathrm{coev}_{\beta_X}$ etc., the graded pivotality condition holds precisely because it holds component-wise in $\mathcal{C}$. □

2.3 Super morphisms

Following the definition of a k-superbicategory, it is natural to consider the notion of morphisms between them. Superfunctors between k-superbicategories are defined in [Mur18] and we follow this definition. We go beyond by defining supertransformations between these superfunctors and establish the more or less obvious composition on both structures. A "supermodification" is nothing more than a modification between the underlying transformations.

Definition 2.29. A *lax superfunctor* $(F, \{k_X\}_{X\in\mathcal{B}_0}) : (\mathcal{B}, \Omega^{\mathcal{B}}, \mu^{\mathcal{B}}) \to (\mathcal{C}, \Omega^{\mathcal{C}}, \mu^{\mathcal{C}})$ between k-superbicategories consists of a k-linear lax functor $F : \mathcal{B} \to \mathcal{C}$ together with an isomorphism $k_X : \Omega^{\mathcal{D}}_{FX} \to F\Omega^{\mathcal{C}}_X$ for each object X in $\mathcal{C}$, such that the following diagrams commute:

$$\begin{tikzcd}
\Omega_{FX}\Omega_{FX} \arrow[r, "k_X * k_X"] \arrow[dd, "\mu_{FX}"'] & F(\Omega_X)F(\Omega_X) \arrow[d, "F^2"] \\
 & F(\Omega_X\Omega_X) \arrow[d, "F\mu_X"] \\
1_{FX} \arrow[r, "F^0"'] & F1_X
\end{tikzcd}
\qquad
\begin{tikzcd}
Ff \circ \Omega_{FX} \arrow[r, "\Omega_{Ff}"] \arrow[d, "1*k_X"'] & \Omega_{FY} \circ Ff \arrow[d, "k_Y*1"] \\
Ff \circ F\Omega_X \arrow[d, "F^2"'] & F\Omega_Y \circ Ff \arrow[d, "F^2"] \\
F(f \circ \Omega_X) \arrow[r, "F(\Omega_f)"'] & F(\Omega_Y \circ f)
\end{tikzcd}$$

The condition that these diagrams commute will be referred to as *lax super functoriality conditions*. The isomorphisms k_X are referred to as *parity constraints*. We may often refer to a lax pseudo superfunctor $(F, \{k_X\}_{X\in\mathcal{B}_0})$ by simply writing F. To compose lax superfunctors

$$\mathcal{B} \xrightarrow{(F, \{k^F_X\}_{X\in\mathcal{B}_0})} \mathcal{C} \xrightarrow{(G, \{k^G_Y\}_{Y\in\mathcal{C}_0})} \mathcal{D}$$

compose F and G in the usual way and set k^{GF}_X to be the composite

$$\Omega_{GFX} \xrightarrow{k^G_{FX}} G(\Omega_{FX}) \xrightarrow{G(k^F_X)} GF(\Omega_X)$$

Proposition 2.30. *The data $(GF, \{k^{GF}_X\}_{X\in\mathcal{B}_0})$ defines a lax superfunctor.*

Proof. The first super lax naturality condition asks that the outside diagram below commutes:

$$\begin{tikzcd}
\Omega_{GFX}\Omega_{GFX} \arrow[r, "k_{FX} * k_{FX}"] \arrow[ddd, "\mu_{GFX}"'] & G(\Omega_{FX})G(\Omega_{FX}) \arrow[r, "G(k_X) * G(k_X)"] \arrow[d, "G^2"] & GF(\Omega_X)GF(\Omega_X) \arrow[d, "G^2"] \\
 & G(\Omega_{FX}\Omega_{FX}) \arrow[r, "G(k_X * k_X)"] \arrow[dd, "G(\mu_{FX})"] & G(F\Omega_X F\Omega_X) \arrow[d, "G(F^2)"] \\
 & & GF(\Omega_X\Omega_X) \arrow[d, "G(F\mu_X)"] \\
1_{GFX} \arrow[r, "G^0"'] & G1_{FX} \arrow[r, "G(F^0)"'] & GF(1_X)
\end{tikzcd}$$

The square commutes due to naturality of G^2. The left sub-diagram commutes since $(G, \{k^G_Y\}_{Y\in\mathcal{C}_0})$ is a lax superfunctor. The right-bottom sub-diagram also commutes, as it is the image under G of the commuting diagram corresponding to the first super lax condition on F. [3] The second

[3]Functors preserve commuting diagrams. Here the functor in question is the local functor $G : \mathcal{B}(X, Y) \to \mathcal{D}(GFX, GFY)$.

lax naturality condition asks that the outside diagram below commutes:

$$
\begin{tikzcd}
 & GFf \circ \Omega_{GFX} \arrow[r, "\Omega_{GFf}"] \arrow[d, "1 * k_{FX}"'] & \Omega_{GFX} \circ GFf \arrow[d, "k_{FX} * 1"] & \\
 & GFf \circ G(\Omega_{FX}) \arrow[dl, "1 * G(k_X)"'] \arrow[d, "G^2"'] & G(\Omega_{FY}) \circ GFf \arrow[d, "G^2"'] \arrow[dr, "G(k_Y) * 1"] & \\
GFf \circ GF(\Omega_X) \arrow[dr, "G^2"'] & G(Ff \circ \Omega_{FX}) \arrow[r, "G(\Omega_{Ff})"] \arrow[d, "G(1 * k_X)"] & G(\Omega_{FY} \circ Ff) \arrow[d, "G(k_Y * 1)"'] & GF(\Omega_Y) \circ GFf \arrow[dl, "G^2"] \\
 & G(Ff \circ F(\Omega_X)) \arrow[d, "G(F^2)"'] & G(F(\Omega_Y) \circ Ff) \arrow[d, "G(F^2)"] & \\
 & GF(f \circ \Omega_X) \arrow[r, "GF(\Omega_f)"'] & GF(\Omega_Y \circ f) &
\end{tikzcd}
$$

The sub-diagrams commute by similar logic to the previous condition. □

Definition 2.31. A lax supertransformation $\beta : (F, \{k_X^F\}_{X \in \mathcal{B}_0}) \to (G, \{k_X^G\}_{X \in \mathcal{B}_0})$ between lax superfunctors is a lax transformation $\beta : F \to G$ such that the following diagram commutes:

$$
\begin{tikzcd}
G\Omega_X \circ \beta_X \arrow[r, "\beta_{\Omega_X}"] & \beta_X \circ F\Omega_X \\
\Omega_{GX} \circ \beta_X \arrow[u, "k_X^G * 1"] & \beta_X \circ \Omega_{FX} \arrow[l, "\Omega_{\beta_X}"] \arrow[u, "1 * k_X^F"']
\end{tikzcd}
$$

Commutativity of this diagram will be referred to as the *lax super naturality condition.* The composition of lax supertransformations is just the composition of the underlying lax transformations.

Proposition 2.32. *Given lax supertransformations as in the data below*

$$
\begin{tikzcd}
(\mathcal{B}, \Omega^{\mathcal{B}}, \mu^{\mathcal{B}}) \arrow[rr, bend left, "(F, \{k_X^F\}_{X \in \mathcal{B}_0})"{name=U}] \arrow[rr, "(G, \{k_X^G\}_{X \in \mathcal{B}_0})"'{name=M}] \arrow[rr, bend right, "(H, \{k_X^H\}_{X \in \mathcal{B}_0})"'{name=D}] & & (\mathcal{C}, \Omega^{\mathcal{C}}, \mu^{\mathcal{C}})
\arrow[from=U, to=M, phantom, "\Downarrow \beta"]
\arrow[from=M, to=D, phantom, "\Downarrow \gamma"]
\end{tikzcd}
$$

then $\gamma \circ \beta : (F, \{k_X^F\}_{X \in \mathcal{B}_0}) \to (H, \{k_X^H\}_{X \in \mathcal{B}_0})$ *defines a lax supertransformation.*

Proof. The lax super naturality condition asks that the outside diagram below commutes, where we have omitted bracketings and associators:

$$
\begin{tikzcd}
H(\Omega_X)\gamma_X\beta_X \arrow[r, "\gamma_{\Omega_X} * 1"] & \gamma_X G(\Omega_X)\beta_X \arrow[r, "1 * \beta_{\Omega_X}"] & \gamma_X \beta_X F(\Omega_X) \\
\Omega_{HX}\gamma_X\beta_X \arrow[u, "k_X^H * 1"'] & \gamma_X \Omega_{GX} \beta_X \arrow[l, "\Omega_{\gamma_X} * 1"] \arrow[u, "1 * k_X^G * 1"'] & \gamma_X \beta_X \Omega_{FX} \arrow[l, "1 * \Omega_{\beta_X}"] \arrow[u, "1 * k_X^F"'] \arrow[ll, bend left, "\Omega_{\gamma_X \beta_X}"]
\end{tikzcd}
$$

The squares commute since β and γ are super lax transformations. The bottom sub-diagram commutes due to lax naturality of the strong transformation $\Omega^{\mathcal{C}} : 1_{\mathcal{C}} \to 1_{\mathcal{C}}$. Bracketings and associators may be added in the obvious ways to conclude the result more meticulously. □

Remark 2.33. A pseudo/oplax superfunctor and a strong/oplax supertransformation have the obvious definitions. Recall that a natural k-supercategory structure is induced on the local categories of a k-superbicategory as in Proposition 2.17. If $(F, \{k_X^F\}_{X \in \mathcal{B}_0}) : (\mathcal{B}, \Omega^{\mathcal{B}}, \mu^{\mathcal{B}}) \to (\mathcal{C}, \Omega^{\mathcal{C}}, \mu^{\mathcal{C}})$ is a pseudo superfunctor, then the local functors $F : \mathcal{B}(X, Y) \to \mathcal{C}(FX, FY)$ each have the structure of a superfunctor in the form of a natural transformation

$$F \circ \tilde{\Omega}^{\mathcal{B}}_{X,Y} \to \tilde{\Omega}^{\mathcal{C}}_{FX,FY} \circ F$$

with f-component

$$F(f \circ \Omega_X) \xrightarrow{(F^2)^{-1}} FfF\Omega_X \xrightarrow{1*(k_X^F)^{-1}} Ff \circ \Omega_{FX}$$

If $\mathcal{B}$ is a bicategory, Proposition 2.7 describes a k-superbicategory structure on the bicategory $[\mathcal{B}, \mathcal{C}]$ where $\mathcal{C}$ is a k-superbicategory. If $\mathcal{B}$ is a k-superbicategory too, we now describe a k-superbicategory $[\mathcal{B}, \mathcal{C}]^{sup}$ consisting of pseudo superfunctors, strong supertransformations and modifications:

Proposition 2.34. *Let $(\mathcal{B}, \Omega^{\mathcal{B}}, \mu^{\mathcal{B}}), (\mathcal{C}, \Omega^{\mathcal{C}}, \mu^{\mathcal{C}})$ be two k-superbicategories. The following data defines a k-superbicategory $([\mathcal{B}, \mathcal{C}]^{sup}, \Omega', \mu')$:*

- *Objects are pseudo superfunctors $(\mathcal{B}, \Omega^{\mathcal{B}}, \mu^{\mathcal{B}}) \to (\mathcal{C}, \Omega^{\mathcal{C}}, \mu^{\mathcal{C}})$.*
- *Let $\tilde{F}, \tilde{G}$ denote pseudo superfunctors $\mathcal{B} \to \mathcal{C}$. Then $[\mathcal{B}, \mathcal{C}]^{sup}(\tilde{F}, \tilde{G})$ is the category consisting of strong supertransformations $\tilde{F} \to \tilde{G}$ and modifications between them. Composition of modifications and unit modifications are as usual. This category is k-linear by defining the k-module structure pointwise on modifications.*
- *The strong identity supertransformation $1_{\tilde{F}} : \tilde{F} \to \tilde{F}$ is the identity transformation $F \to F$.*
- *The unitor and associator modifications are as usual, given componentwise by the unitors and associator in $\mathcal{C}$.*

We describe now the parity shift transformation and parity involution modification:

- *$\Omega'_{\tilde{F}} : \tilde{F} \to \tilde{F}$ is a strong supertransformation with $(\Omega'_{\tilde{F}})_X = \Omega_{FX} : FX \to FX$ and $(\Omega'_{\tilde{F}})_f = \Omega_{Ff} : Ff\Omega_{FX} \to \Omega_{FY}Ff$.*
- *$\Omega'_\alpha : \alpha \circ \Omega'_{\tilde{F}} \to \Omega'_{\tilde{G}} \circ \alpha$ is a modification with 2-cells $(\Omega'_\alpha)_X = \Omega_{\alpha_X}$. Here $\alpha : \tilde{F} \to \tilde{G}$ is a strong supertransformation.*
- *$\mu'_{\tilde{F}} : \Omega'_{\tilde{F}} \circ \Omega'_{\tilde{F}} \to 1_{\tilde{F}}$ is a modification with $(\mu'_{\tilde{F}})_X = \mu_{FX}$.*

Proof. For $1_{\tilde{F}} : \tilde{F} \to \tilde{F}$ to be a strong transformation requires the outside diagram below to commute:

$$\begin{tikzcd}
F\Omega_X \circ 1_{FX} \arrow[r, "\rho"] \arrow[rr, bend left, "(1_F)_{\Omega_X}"] & F\Omega_X \arrow[r, "\lambda^{-1}"] & 1_{FX} \circ F\Omega_X \\
\Omega_{FX} \circ 1_{FX} \arrow[u, "k_X * 1"] & \Omega_{FX} \arrow[l, "\rho^{-1}"] \arrow[u, "k_X"] & 1_{FX} \circ \Omega_{FX} \arrow[l, "\lambda"] \arrow[u, "1 * k_X"'] \arrow[ll, bend left, "\Omega_{1_X}"]
\end{tikzcd}$$

Naturality of the unitors makes the squares commute. The bottom-most sub-diagram commutes by lax unity of $\Omega^{\mathcal{C}} : 1_{\mathcal{C}} \to 1_{\mathcal{C}}$ while the top sub-diagram commutes by definition of the unit transformation. That this intermediate data defines a bicategory structure is clear.

That $\Omega'_{\tilde{F}} : \tilde{F} \to \tilde{F}$ defines a strong transformation is proven in Proposition 2.7. The super naturality condition asks that the following diagram commutes:

$$\begin{tikzcd}
F\Omega_X \circ \Omega_{FX} \arrow[r, "\Omega_{F\Omega_X}"] & \Omega_{FX} \circ F\Omega_X \\
\Omega_{FX}\Omega_{FX} \arrow[u, "k_X * 1"] & \Omega_{FX}\Omega_{FX} \arrow[l, "\Omega_{\Omega_{FX}}"] \arrow[u, "1 * k_X"']
\end{tikzcd}$$

But $\Omega_{\Omega_{FX}} = -1_{\Omega_{FX}\Omega_{FX}}$ in $\mathcal{C}$, in particular we may reverse the direction of the bottom arrow while keeping the same label. The resulting diagram commutes by naturality of the 2-cells of $\Omega^{\mathcal{C}}$. The remaining two points in the statement of the proposition are essentially proven in Proposition 2.7. $\square$

The following proposition is highly analogous to Proposition 2.28, where now everything in sight is "super":

Proposition 2.35. *Let $(\mathcal{B}, \Omega^{\mathcal{B}}, \mu^{\mathcal{B}}), (\mathcal{C}, \Omega^{\mathcal{C}}, \mu^{\mathcal{C}})$ be two k-superbicategories, where $\mathcal{C}$ is graded pivotal and $\mathcal{B}$ has both left and right adjoints. Then the k-superbicategory $[\mathcal{B}, \mathcal{C}]^{sup}$ as defined in Proposition 2.34 is graded pivotal as well.*

Proof. Let $\beta : \tilde{F} \to \tilde{G}$ be a strong supertransformation. We construct a strong supertransformation $\beta^\vee : \tilde{G} \to \tilde{F}$ to have underlying strong transformation as defined in the discussion preceding Proposition 2.26. To recall, we have $(\beta^\vee)_X = (\beta_X)^\vee$ and $(\beta^\vee)_f$ is defined as the following composite:

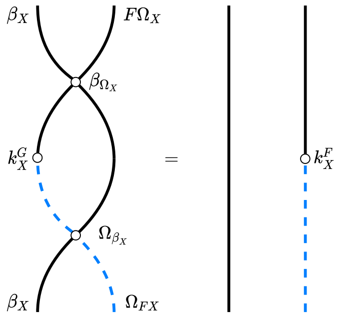
$$\begin{tikzcd}
\beta_Y^\vee \circ Gf \arrow[r, "1*\mu_{GY}^{-1}*1"] \arrow[dd, dashed] & {}^\dagger\beta_Y \circ \Omega_{GY} \circ Gf \arrow[d, "1*\Omega_{Gf}^{-1}"] \\
 & {}^\dagger\beta_Y \circ Gf \circ \Omega_{GX} \arrow[d, "m(\beta_f)"] \\
Ff \circ \beta_X^\vee & Ff \circ {}^\dagger\beta_X \circ \Omega_{GX} \arrow[l, "1*1*\mu_{GX}"]
\end{tikzcd}$$

This is represented string-diagrammatically in Figure 12.

In Proposition 2.28 it is essentially proven that $\beta^\vee$ is a strong transformation that is dual to the underlying transformation of β and that graded pivotality is induced on the functor category using these duals. What remains to be shown is that $\beta^\vee$ is actually a *super*transformation. For this we employ string diagrams and use the fact that β is a supertransformation, which says the following diagram commutes:

$$\begin{tikzcd}
G\Omega_X\beta_X \arrow[r, "\beta_{\Omega_X}"] & \beta_X F\Omega_X \\
\Omega_{GX}\beta_X \arrow[u, "k_X^G * 1"] & \beta_X\Omega_{FX} \arrow[l, "\Omega_{\beta_X}"] \arrow[u, "1 * k_X^F"']
\end{tikzcd}$$

In terms of string diagrams this says

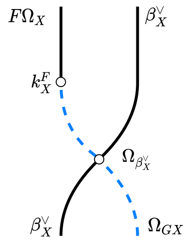


(15)

Since $\beta^\vee$ is constructed in an oplax way, we want the following diagram to commute:

$$\begin{tikzcd}
G\Omega_X\beta^\vee_X & \beta^\vee_X F\Omega_X \arrow[l, "(\beta^\vee)_{\Omega_X}"'] \\
\Omega_{GX}\beta^\vee_X \arrow[u, "k^G_X*1"] & \beta^\vee_X\Omega_{FX} \arrow[l, "\Omega_{\beta^\vee_X}"] \arrow[u, "1*k^F_X"']
\end{tikzcd}$$

The left-bottom path is represented as



The proof is finished when we show equality with the top-right path, which is represented initially in (A). Below the 2-cell β_{Ω_X}, we insert the following composition, which is equal to $1_{G\Omega_X\circ\beta_X}$:

$$G\Omega_X\ \beta_X \xrightarrow{(k^G_X)^{-1}*1} \Omega_{GX}\beta_X \xrightarrow{\Omega^{-1}_{\beta_X}} \beta_X\Omega_{GX} \xrightarrow{\Omega_{\beta_X}} \Omega_{GX}\beta_X \xrightarrow{k^G_X*1} G\Omega_X\ \beta_X$$

With this insertion, we arrive at (B) (ignore the red dotted boxes initially). The upper red box may be altered using the coherence condition for β shown in (15). The lower red box may be replaced by $\Omega_{\Omega_{GX}\Omega_{GX}}$, since the following diagram commutes by naturality of $\Omega^{\mathcal{C}}$:

$$\begin{tikzcd}
\Omega_{GX}\Omega_{GX} \arrow[r, "1*k^G_X"] \arrow[d, "\Omega^{-1}_{\Omega_{GX}}=\Omega_{\Omega_{GX}}"'] & \Omega_{GX}G\Omega_X \arrow[d, "\Omega^{-1}_{G\Omega_X}"] \\
\Omega_{GX}\Omega_{GX} \arrow[r, "k^G_X*1"'] & G\Omega_X\Omega_{GX}
\end{tikzcd}$$

These two alterations take us to (C). Observe the following

$$\begin{tikzcd}
\Omega_{GX} \arrow[r, "\rho^{-1}"] \arrow[d, "\lambda^{-1}"'] & \Omega_{GX}1_{GX} \arrow[r, "1*\mathrm{coev}"] & \Omega_{GX}\beta_X\beta^\vee_X\Omega_{GX} \arrow[r, "\Omega^{-1}_{\beta_X}*1*1"] & \beta_X\Omega_{FX}\beta^\vee_X\Omega_{GX} \\
1_{GX}\Omega_{GX} \arrow[r, "\mathrm{coev}*1"'] \arrow[ur, "\Omega_{1_{GX}}"] & \beta_X\beta^\vee_X\Omega_{GX}\Omega_{GX} \arrow[ur, "\Omega_{\beta_X\beta^\vee_X\Omega_{GX}}"] \arrow[urr, "1*\Omega_{\beta^\vee_X\Omega_{GX}}"'] \arrow[rr, "1*1*\Omega_{\Omega_{GX}}"'] & & \beta_X\beta^\vee_X\Omega_{GX}\Omega_{GX} \arrow[u, "1*\Omega_{\beta^\vee_X}*1"']
\end{tikzcd}$$

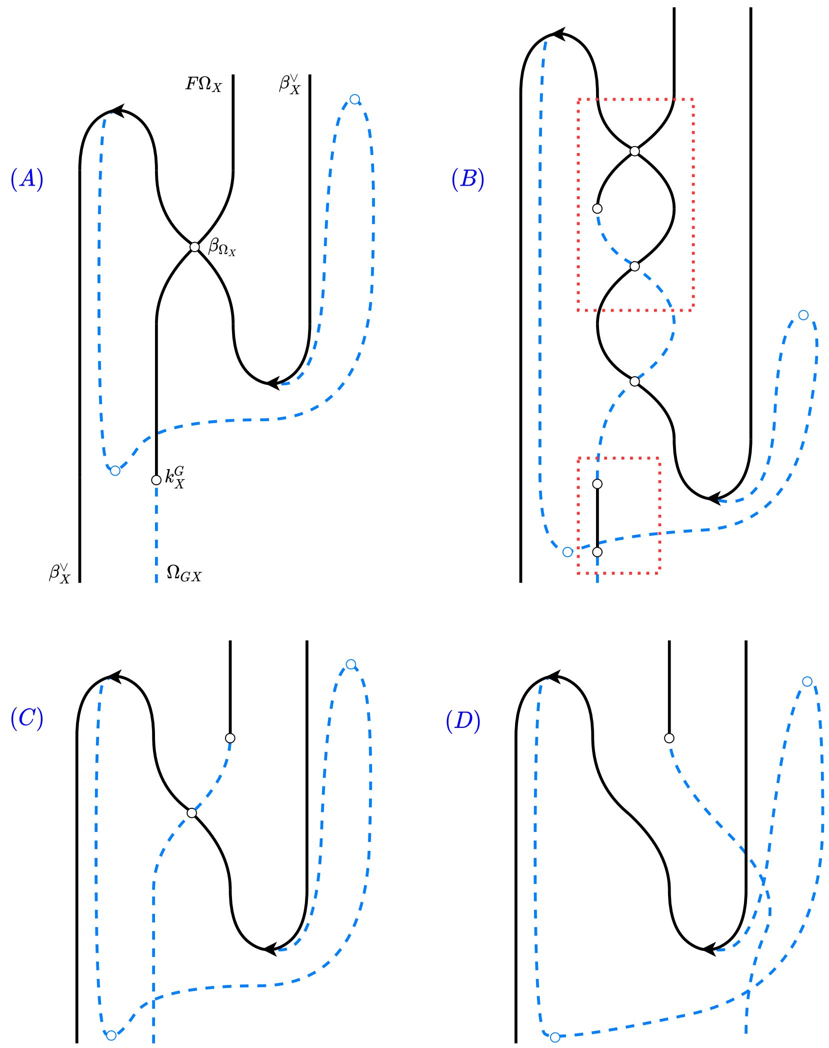
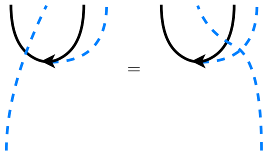
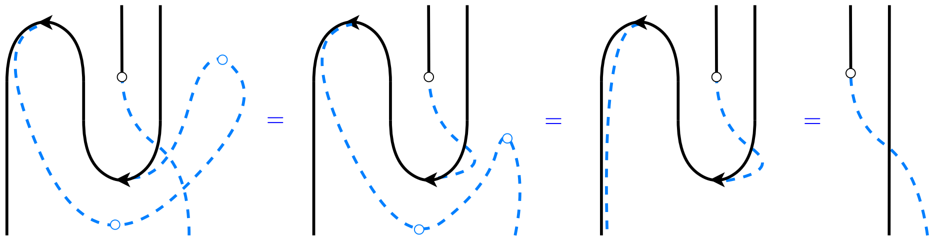


Figure 15: Various stages of a string diagram movie used in the proof of Proposition 2.35

From left to right, the sub-diagrams commute due to lax unity of Ω, naturality of the 2-cells of Ω, lax naturality and lax naturality again of Ω, respectively. Hence we have an equality of string diagrams:



Applying this to (C) we finally arrive at (D). From here a few simple moves give us the desired result:



In the first equality we replaced $\Omega_{\Omega_X} : \Omega_X^2 \to \Omega_X^2$ by $-1_{\Omega_X^2}$ in two places, thus cancelling the minus sign. The final two equalities are two different applications of Zorro moves. $\square$

3 Superbiequivalences

The Whitehead theorem for bicategories provides necessary and sufficient conditions for a pseudofunctor between bicategories to be a biequivalence, see [JY19, JY20]. This is a generalisation of the classical result that a functor between categories is an equivalence if and only if it is essentially surjective and fully faithful. The goal of this section is to formulate the notion of a superbiequivalence between superbicategories and to classify them in terms of whether or not the underlying pseudofunctor is a biequivalence. To prove this result we adapt strategies from [JY19].

3.1 The Whitehead theorem for bicategories

Definition 3.1. A strong transformation $\beta : F \to G$ between pseudofunctors is *invertible* if there exists a strong transformation $\beta^\bullet : G \to F$ and invertible modifications

$$1_F \cong \beta^\bullet \beta \qquad 1_G \cong \beta\beta^\bullet$$

Definition 3.2. Let $\mathcal{B}, \mathcal{C}$ be two bicategories. A pseudofunctor $F : \mathcal{B} \to \mathcal{C}$ is said to be a *biequivalence* if there exists a pseudofunctor $G : \mathcal{C} \to \mathcal{B}$ and strong invertible transformations $\epsilon : FG \to 1_\mathcal{C}, \eta : 1_\mathcal{B} \to GF$.

In [JY19] it is proven that if $F : \mathcal{B} \to \mathcal{C}$ is a pseudofunctor such that

- F is essentially surjective on adjoint equivalence classes of objects.
- F is essentially full on 1-cells.
- F is fully faithful on 2-cells.

then F is a biequivalence. The converse is also true with little effort. Both directions together form the so-called *Whitehead theorem for bicategories.* We outline the construction of $G : \mathcal{C} \to \mathcal{B}$ as given in [JY19], although our exposition will be of a more explicit nature as we do not care about the extra machinery described there (lax slice bicategories). Since F is essentially surjective on adjoint equivalence classes of objects then to each $X \in \mathcal{C}_0$ there exists $GX \in \mathcal{B}_0$ and adjoint equivalence data

$$\epsilon_X : F(GX) \to X$$
$$\epsilon_X^\bullet : X \to F(GX)$$

We will have no need to specify symbolic naming for the unit or counit of these adjoint equivalences, where necessary we will simply write $\cong$: $\epsilon_X^\bullet \epsilon_X \to 1_{FGX}$ for example. For a 1-cell $f : X \to Y$ in $\mathcal{C}$ we have

$$FGX \xrightarrow{\epsilon_X} X \xrightarrow{f} Y \xrightarrow{\epsilon_Y^\bullet} FGY$$

Since F is essentially full on 1-cells, there exists a 1-cell $GX \to GY$, which we denote Gf, and 2-isomorphism

$$\epsilon_f^\dagger : \epsilon_Y^\bullet(f\epsilon_X) \to F(Gf)$$

By taking mates, this corresponds to a 2-isomorphism

$$\epsilon_f : f\epsilon_X \to \epsilon_Y FGf$$

For a 2-cell $\beta : f \to g$ in $\mathcal{C}$, define $G\beta$ to be the unique 2-cell such that the following diagram commutes:

$$\begin{tikzcd}
f\epsilon_X \arrow[r, "\epsilon_f"] \arrow[d, "\beta * 1"'] & \epsilon_Y FGf \arrow[d, "1 * F(G\beta)"] \\
g\epsilon_X \arrow[r, "\epsilon_g"'] & \epsilon_Y FGg
\end{tikzcd}$$

Many arbitrary choices have been made here, in the next section we will specify some of these arbitrary choices for the purposes of giving G the structure of a pseudo *super*functor.

Explanation 3.1. Why can we claim that a 2-cell making the above diagram commute exists and is unique? Since ϵ_f is an isomorphism, we have a composite of 2-cells:

$$\epsilon_Y FGf \xrightarrow{\epsilon_f^{-1}} f\epsilon_Y \xrightarrow{\beta * 1} g\epsilon_Y \xrightarrow{\epsilon_g} \epsilon_Y FGg$$

One may check easily that the post-composition functor $(\epsilon_Y)_* : \mathcal{C}(FGX, FGY) \to \mathcal{C}(FGX, Y)$ is fully faithful, due to the fact that ϵ_Y is part of an adjoint equivalence. Therefore there corresponds a unique 2-cell $\beta' : FGf \to FGg$ such that $1 * \beta' : \epsilon_Y FGf \to \epsilon_Y FGg$ is the composite above. This β' in turn corresponds to a unique 2-cell $G\beta : Gf \to Gg$ such that $F(G\beta) = \beta'$, since F is fully faithful on 2-cells. The same logic can be applied to show that if we want the left diagram below in $\mathcal{B}(GX, GY)$ to commute, it is equivalent to check that the right diagram in $\mathcal{C}(FGX, Y)$ commutes:

$$\begin{tikzcd}
f \arrow[r, "\beta"] \arrow[d, "\delta"'] & g \arrow[d, "\gamma"] \\
h \arrow[r, "\omega"'] & i
\end{tikzcd}
\qquad
\begin{tikzcd}
\epsilon_Y \circ Ff \arrow[r, "1 * F\beta"] \arrow[d, "1 * F\delta"'] & \epsilon_Y \circ Fg \arrow[d, "1 * F\gamma"] \\
\epsilon_Y \circ Fh \arrow[r, "1 * F\omega"'] & \epsilon_Y \circ Fi
\end{tikzcd}$$

Back to the construction of G now. The uniqueness condition satisfied by G on 2-cells implies in particular that G is functorial on 2-cells. If $\mathcal{B}, \mathcal{C}$ happen to be k-linear with F a k-linear pseudofunctor, then uniqueness also determines k-linearity of G. For example $\beta = G(\gamma + \gamma')$ makes the following diagram commute:

$$\begin{tikzcd} f\epsilon_X \arrow[r, "\epsilon_f"] \arrow[d, "(\gamma+\gamma')*1=\gamma*1+\gamma'*1"'] & \epsilon_Y FGf \arrow[d, "1*F(\beta)"] \\ g\epsilon_X \arrow[r, "\epsilon_g"'] & \epsilon_Y FGf \end{tikzcd}$$

But using k-linearity of F it is easy to see that $\beta = G(\gamma) + G(\gamma')$ also yields commutativity. By uniqueness $G(\gamma + \gamma') = G(\gamma) + G(\gamma')$.

For $X \in \mathcal{C}_0$ we define $G^0_X : 1_{GX} \to G1_X$ to be the unique 2-cell making the following diagram commute:

$$\begin{tikzcd} \epsilon_X \arrow[r, "\rho^{-1}"] \arrow[d, "\lambda^{-1}"'] & \epsilon_X 1_{FGX} \arrow[r, "1*F^0_{GX}"] & \epsilon_X F1_{GX} \arrow[d, "1*F(G^0_X)"] \\ 1_X\epsilon_X \arrow[rr, "\epsilon_{1_X}"'] & & \epsilon_X FG1_X \end{tikzcd}$$

Notice that this looks exactly like the lax unity condition for the soon-to-be strong transformation ϵ. Naturality of the 2-cells of ϵ is captured above in the definition of G on 2-cells, and the lax naturality condition of ϵ will be captured in the definition of the functorial constraints of G next:

Given $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{C}$, define $G^2_{g,f} : Gg \circ Gf \to G(g \circ f)$ to be the unique 2-cell making the following diagram commute:

$$\begin{tikzcd} g(\epsilon_Y FGf) \arrow[r, "\alpha^{-1}"] & (g\ \epsilon_Y)FGf \arrow[r, "\epsilon_g*1"] & (\epsilon_Z FGg)FGf \arrow[r, "\alpha"] & \epsilon_Z(FGgFGf) \arrow[r, "1*F^2"] & \epsilon_Z F(GgGf) \arrow[dd, "1*F(G^2_{g,f})"] \\ g(f\epsilon_X) \arrow[u, "1*\epsilon_f"] & & & & \\ (gf)\epsilon_X \arrow[u, "\alpha"] \arrow[rrrr, "\epsilon_{gf}"'] & & & & \epsilon_Z FG(gf) \end{tikzcd}$$

We have defined the data of a pseudofunctor G for which the coherence conditions remain unproven. The λ-compatibility condition of G asks that the following diagram commute:

$$\begin{tikzcd} 1_{GY}Gf \arrow[r, "\lambda"] \arrow[d, "G^0*1"'] & Gf \\ G1_XGf \arrow[r, "G^2"'] & G(1_Y f) \arrow[u, "G(\lambda)"'] \end{tikzcd}$$

By Explanation 3.1 it suffices to show that the outside diagram below commutes:

$$
\begin{tikzcd}
\epsilon_Y F(1_{GY}Gf) \arrow[rrrrrr, "1*F(\lambda)"] \arrow[rrrrrr, phantom, "(1)"', shift right=4ex] \arrow[dddd, "1*F(G^0*1)"'] & & & & & & \epsilon_Y FGf \\
 & & \epsilon_Y F1_{GY}FGf \arrow[ull, "1*F^2"'] \arrow[dd, "1*FG^0*1"] & \epsilon_Y 1_{FGY}FGf \arrow[l, "1*F^0*1"'] \arrow[urrr, "1*\lambda"] & (5) & & \\
 & (2) & \arrow[r, phantom, "(7)"] & \epsilon_Y FGf \arrow[u, "\rho^{-1}*1"] \arrow[uurrr, "="] & (6) & f\epsilon_X \arrow[uur, "\epsilon_f"'] \arrow[r, phantom, "(3)"] & \\
 & & \epsilon_Y FG1_Y FGf \arrow[dll, "1*F^2"'] & 1_Y\epsilon_Y FGf \arrow[l, "\epsilon_{1_Y}*1"] \arrow[u, "\lambda*1"] & & 1_Y f\epsilon_X \arrow[ll, "1*\epsilon_f"] \arrow[u, "\lambda*1"'] \arrow[dr, "\epsilon_{1_Y f}"'] & \\
\epsilon_Y F(G1_Y Gf) \arrow[rrrrrr, "1*F(G^2)"'] \arrow[rrrrrr, phantom, "(4)", shift left=4ex] & & & & & & \epsilon_Y FG(1_Y f) \arrow[uuuu, "1*FG(\lambda)"']
\end{tikzcd}
$$

Here we have omitted bracketings and associators to reduce the size and complexity of the diagrams, alternatively one could use pasting or string diagrams. Each corresponding sub-diagram commutes due to the following reasons

1. This follows from the corresponding λ-compatibility on F.
2. Naturality of F^2.
3. Naturality of the 2-cells of ϵ.
4. Lax naturality of ϵ.
5. Without associators this looks odd, but it is simply the unity axiom in $\mathcal{C}$.
6. Naturality of λ in $\mathcal{C}$.
7. Lax unity of ϵ.

There are three remaining conditions on G, those being lax functoriality, naturality of G^2 and ρ-compatibility. All are proven in [JY19]. We must now construct a strong transformation $\eta : GF \to 1_{\mathcal{B}}$. For each $X \in \mathcal{B}_0$ we have a 2-cell

$$\epsilon^{\bullet}_{FX} : FX \to FGFX$$

Since F is essentially full on 1-cells we may choose $\eta_X : X \to GFX$ together with a 2-cell isomorphism $\epsilon^{\bullet}_{FX} \cong F\eta_X$. We then have isomorphisms

$$1_{FX} \xrightarrow{\cong} \epsilon_X \epsilon^{\bullet}_X \xrightarrow{1*\cong} \epsilon_X F\eta_X$$

We define $\eta_f : GF(f) \circ \eta_X \to \eta_Y \circ f$ to be the unique 2-cell making the following diagram commute:

$$
\begin{tikzcd}
Ff \circ 1_{FX} \arrow[r, "1*\cong"] \arrow[d, "\rho"'] & Ff(\epsilon_{FX} F\eta_X) \arrow[r, "\alpha^{-1}"] & (Ff\ \epsilon_{FX})F\eta_X \arrow[r, "\epsilon_{Ff}*1"] & (\epsilon_{FY} FGFf)F\eta_X \arrow[r, "\alpha"] & \epsilon_{FY}(FGF(f)F\eta_X) \arrow[d, "1*F^2"] \\
Ff \arrow[d, "\lambda^{-1}"'] & & & & \epsilon_{FY} F(GF(f)\ \eta_X) \arrow[d, "1*F(\eta_f)"] \\
1_{FY} Ff \arrow[r, "\cong*1"'] & (\epsilon_{FY} F\eta_Y)Ff \arrow[r, "\alpha"'] & \epsilon_{FY}(F\eta_Y Ff) \arrow[rr, "1*F^2"'] & & \epsilon_{FY} F(\eta_Y f)
\end{tikzcd}
$$

Let us show that with this definition, the 2-cells of η are natural. For this it suffices to show that the outside diagram below commutes for $\beta : f \to g$ a 2-cell in $\mathcal{B}$:

$$\begin{tikzcd}
\epsilon_{FY}F(GF(f)\eta_X) \arrow[rrrr, "1*F(\eta_f)"] \arrow[ddd, "1*F(GF(\beta)*1)"'] & & {} \arrow[d, phantom, "(1)"] & & \epsilon_{FY}F(\eta_Y\ f) \arrow[ddd, "1*F(1*\beta)"] \\
{} \arrow[d, phantom, "(2)"] & \epsilon_{FY}FGF(f)F\eta_X \arrow[ul, "1*F^2"'] \arrow[d, "1*FGF(\beta)*1"'] \arrow[dr, phantom, "(5)"] & F(f)\epsilon_{FX}F\eta_X \arrow[l, "\epsilon_{Ff}*1"'] \arrow[r] \arrow[d, "F(\beta)*1*1"] \arrow[dr, phantom, "(6)"] & \epsilon_{FY}F\eta_YFf \arrow[ur, "1*F^2"] \arrow[d, "1*1*F(\beta)"] \arrow[dr, phantom, "(4)"] & \\
{} & \epsilon_{FY}FGF(g)F\eta_X \arrow[dl, "1*F^2"] & F(g)\epsilon_{FX}F\eta_X \arrow[l, "\epsilon_{Fg}*1"] \arrow[r] \arrow[d, phantom, "(3)"] & \epsilon_{FY}F\eta_YFg \arrow[dr, "1*F^2"'] & {} \\
\epsilon_{FY}F(GF(g)\eta_X) \arrow[rrrr, "1*F(\eta_g)"'] & & {} & & \epsilon_{FY}F(\eta_Y g)
\end{tikzcd}$$

Sub-diagram (6) is expanded as:

$$\begin{tikzcd}
F(f)\epsilon_{FX}F\eta_X \arrow[r, "1*\cong"] \arrow[d, "F(\beta)*1*1"'] & F(f)1_{FX} \arrow[r, "\rho"] \arrow[d, "F(\beta)*1"'] & Ff \arrow[r, "\lambda^{-1}"] \arrow[d, "F(\beta)"] & 1_{FY}Ff \arrow[r, "\cong*1"] \arrow[d, "1*F(\beta)"] & \epsilon_{FY}F\eta_YFf \arrow[d, "1*1*F(\beta)"] \\
F(g)\epsilon_{FX}F\eta_X \arrow[r, "1*\cong"'] & F(g)1_{FX} \arrow[r, "\rho"'] & Fg \arrow[r, "\lambda^{-1}"'] & 1_{FY}Fg \arrow[r, "\cong*1"'] & \epsilon_{FY}F\eta_YFg
\end{tikzcd}$$

In this expanded diagram, naturality of the unitors and the interchange law show commutativity of each sub-diagram. For the remaining sub-diagrams:

1. η_f is the unique 2-cell making this commute.
2. Naturality of F^2.
3. η_g is the unique 2-cell making this commute.
4. Naturality of F^2.
5. Naturality of the 2-cells of ϵ.

The Lax naturality and lax unity condition of η also hold and are proven in [JY19]. From here we only need to check that ϵ, η are invertible, which follows from the fact that they have invertible components, see [JY19, Proposition 2.25] for the proof of this fact. We have outlined a proof of the Whitehead theorem for bicategories, which we build upon quite explicitly to classify superbiequivalences:

3.2 Classifying superbiequivalences

Definition 3.3. A pseudo superfunctor $(F, \{k_X^F\}_{X\in\mathcal{B}_0}) : (\mathcal{B}, \Omega^{\mathcal{B}}, \mu^{\mathcal{B}}) \to (\mathcal{C}, \Omega^{\mathcal{C}}, \mu^{\mathcal{C}})$ between two k-superbicategories is said to be a *superbiequivalence* if there exists a pseudo superfunctor $(G, \{k_X^G\}_{X\in\mathcal{C}_0}) : \mathcal{C} \to \mathcal{B}$ and invertible strong supertransformations $\epsilon : FG \to 1_{\mathcal{C}}, \eta : 1_{\mathcal{B}} \to GF$.

Theorem 3.4 (Classifying superbiequivalences)**.** *A pseudo superfunctor*

$$(F, \{k_X^F\}_{X\in\mathcal{B}_0}) : (\mathcal{B}, \Omega^{\mathcal{B}}, \mu^{\mathcal{B}}) \to (\mathcal{C}, \Omega^{\mathcal{C}}, \mu^{\mathcal{C}})$$

between two k-superbicategories is a superbiequivalence if and only if F is a biequivalence on the underlying bicategories.

If $(F, \{k^F_X\}_{X\in\mathcal{B}_0})$ is a superbiequivalence, it is clear that F is a biequivalence. The rest of this section is devoted to the proof of the other direction, so suppose F is a biequivalence $\mathcal{B} \to \mathcal{C}$. We firstly construct a pseudo superfunctor $(G, \{k^G_X\}_{X\in\mathcal{C}_0})$ in the opposite direction. Let $G : \mathcal{C} \to \mathcal{B}$, $\epsilon : 1_\mathcal{C} \to FG$ and $\eta : GF \to 1_\mathcal{B}$ be as constructed in the previous section, save for a slight alteration:

- We have 1-cells $\Omega^\mathcal{C}_X : X \to X$ in $\mathcal{C}$. Assuming we have already chosen where G sends objects of $\mathcal{C}$, define G to send $\Omega^\mathcal{C}_X$ to $\Omega^\mathcal{B}_{GX}$, rather than letting $G(\Omega^\mathcal{C}_X)$ be chosen arbitrarily. That is, $G\Omega_X = \Omega_{GX}$. This choice must be accompanied by an isomorphism

  $$\epsilon_{\Omega_X} : \Omega_X\epsilon_X \to \epsilon_X FG\Omega_X = \epsilon_X F\Omega_{GX}$$

  Which we define to be the following composite:

  $$\Omega_X\epsilon_X \xrightarrow{\Omega^{-1}_{\epsilon_X}} \epsilon_X\Omega_{FGX} \xrightarrow{1*k^F_{GX}} \epsilon_X F\Omega_{GX}$$

  This choice actually guarantees that ϵ is a strong superfunctor, once of course it has been shown that G is a pseudo superfunctor. In the previous section the above choices were made arbitrarily, yet here we specify them so as to easily give G the structure of a pseudo superfunctor.

We then define parity constraints $k^G_X := 1_{G\Omega_X} = 1_{\Omega_{GX}}$. To show that $(G, \{k^G_X\}_{X\in\mathcal{C}_0})$ defines a pseudo superfunctor we need to check the two super coherence conditions. As before this can be done by showing that the images of those diagrams commute after applying F and composing with ϵ_X. To wit, the first condition asks that the outside diagram below commutes:

$$\begin{tikzcd}
\epsilon_X F(\Omega_{GX}\Omega_{GX}) \arrow[r, "="] \arrow[dddd, "1*F(\mu^\mathcal{C}_{GX})"'] & \epsilon_X F(G\Omega_X G\Omega_X) \arrow[rrr, "1*F(G^2)"] & & & \epsilon_X FG(\Omega_X\Omega_X) \arrow[dddd, "1*FG(\mu^\mathcal{D}_X)"] \\
& \epsilon_X F\Omega_{GX}F\Omega_{GX} \arrow[ul, "1*F^2"'] & (5) & \Omega_X\Omega_X\epsilon_X \arrow[ur, "\epsilon_{\Omega_X\Omega_X}"'] \arrow[dd, "\mu^\mathcal{D}_X*1"] & \\
(1) & \epsilon_X\Omega_{FGX}\Omega_{FGX} \arrow[u, "1*k_{GX}*k_{GX}"] \arrow[r, "\Omega_{\epsilon_X}*1"'] \arrow[d, "1*\mu^\mathcal{D}_{FGX}"'] & \Omega_X\epsilon_X\Omega_{FGX} \arrow[ur, "1*\Omega_{\epsilon_X}"'] & & (3) \\
& \epsilon_X 1_{FGX} \arrow[r, "\rho"'] \arrow[dl, "1*F^0"'] & \epsilon_X \arrow[r, "\lambda^{-1}"'] & 1_X\epsilon_X \arrow[dr, "\epsilon_{1_X}"] & \\
\epsilon_X F1_{GX} \arrow[rrrr, "1*F(G^0)"'] & & (2) & & \epsilon_X FG1_X
\end{tikzcd}$$

Again we choose to omit bracketings and associators for clarity and conciseness. The sub-diagrams commute by the following logic:

1. The corresponding super coherence condition on F.
2. Lax unity for ϵ.
3. Naturality of the 2-cells of ϵ.
4. The modification condition for $\mu^\mathcal{D}$.
5. With ϵ_{Ω_X} defined as above and using the interchange law, it is straightforward to see that this sub-diagram commutes if and only if the lax naturality coherence condition of ϵ holds.

Using the usual trick, the second super coherence condition for G follows from commutativity of the outside diagram:

$$
\begin{tikzcd}
\epsilon_Y FG(f\Omega_X) \arrow[ddddd, "1*FG(\Omega_f)"'] & & & & \epsilon_Y F(Gf \circ \Omega_{GX}) \arrow[llll, "1*F(G^2)"'] \arrow[ddddd, "1*F(\Omega_{Gf})"] \\
& f\Omega_X\ \epsilon_X \arrow[ul, "\epsilon_{f\Omega_X}"] \arrow[r, "1*\epsilon_{\Omega_X}"] \arrow[dr, "1*\Omega^{-1}_{\epsilon_X}"'] \arrow[dd, "\Omega_f*1"'] & f\epsilon_X FG\Omega_X \arrow[r, "\epsilon_f*1"] \arrow[u, phantom, "(1)"] & \epsilon_Y FGfFG\Omega_X \arrow[ur, "1*F^2"'] & \\
& & f\epsilon_X \Omega_{FGX} \arrow[u, "1*1*k^F_{GX}"'] \arrow[r, "\epsilon_f*1"] \arrow[dl, "\Omega_{f\epsilon_X}"] \arrow[ur, phantom, "(5)"] & \epsilon_Y FGf\ \Omega_{FGX} \arrow[u, "1*1*k^F_{GX}"'] \arrow[dl, "\Omega_{\epsilon_Y FGf}"'] \arrow[d, "1*\Omega_{FGf}"] & \\
& \Omega_Y f\epsilon_X \arrow[r, "1*\epsilon_f"'] \arrow[dl, "\epsilon_{\Omega_Y f}"'] \arrow[l, phantom, "(2)"] \arrow[u, phantom, "(6)"] & \Omega_Y \epsilon_Y FGf \arrow[r, "\Omega^{-1}_{\epsilon_Y}*1"] \arrow[dr, "\epsilon_{\Omega_Y}*1"'] \arrow[u, phantom, "(8)"] & \epsilon_Y \Omega_{FGY} FGf \arrow[d, "1*k^F_{GY}*1"] \arrow[r, phantom, "(4)"] \arrow[ul, phantom, "(7)"] & \\
& & & \epsilon_Y FG\Omega_Y\ FGf \arrow[dr, "1*F^2"] \arrow[l, phantom, "(3)"] & \\
\epsilon_Y FG(\Omega_Y f) & & & & \epsilon_Y F(\Omega_{GY} Gf) \arrow[llll, "1*F(G^2)"]
\end{tikzcd}
$$

The labelled sub-diagrams commute due to the following logic:

1. Lax naturality of ϵ.
2. Naturality of the 2-cells of ϵ.
3. Lax naturality of ϵ.
4. The corresponding super coherence condition on F.
5. Interchange law.
6. Lax naturality for $\Omega^{\mathcal{C}}$.
7. Lax naturality for $\Omega^{\mathcal{C}}$.
8. Naturality of the 2-cells of $\Omega^{\mathcal{C}}$.

The remaining two triangles commute by our choice of ϵ_{Ω_X} and ϵ_{Ω_Y}.

We have shown now that $(G, \{k^G_X\}_{X \in \mathcal{C}_0})$ is a pseudo superfunctor and that ϵ is a strong supertransformation. We finally need to show that η is a strong supertransformation, so we want the following diagram to commute:

$$
\begin{tikzcd}
GF\Omega_X\ \eta_X \arrow[r, "\eta_{\Omega_X}"] & \eta_X \Omega_X \arrow[dd, "="] \\
G\Omega_{FX}\ \eta_X \arrow[u, "G(k^F_X)*1"] & \\
\Omega_{GFX}\ \eta_X \arrow[u, "="] & \eta_X \Omega_X \arrow[l, "\Omega_{\eta_X}"]
\end{tikzcd}
$$

Commutativity of the outside diagram below suffices:

$$
\begin{tikzcd}
\epsilon_{FX}F(GF\Omega_X\ \eta_X) \arrow[rrrr, equals] \arrow[dddddd, "1*F(\eta_{\Omega_X})" {pos=0.5}, phantom] & & & & \epsilon_{FX}F(GF\Omega_X\ \eta_X) \arrow[ddddddd, "1*F(\eta_{\Omega_X})"] \\
 & \epsilon_{FX}FGF\Omega_X\ F\eta_X \arrow[ul, "1*F^2"'] \arrow[rr, equals] & & \epsilon_{FX}FGF\Omega_X F\eta_X \arrow[ur, "1*F^2"] & \\
 & \arrow[rr, phantom, "(4)"] & & F\Omega_X\ \epsilon_{FX}F\eta_X \arrow[u, "\epsilon_{F\Omega_X}*1"'] & \\
\arrow[r, phantom, "(1)"] & \epsilon_{FX}FG\Omega_{FX}F\eta_X \arrow[uu, "1*FG(k^F_X)*1"] \arrow[ddddl, "1*F^2"] & \Omega_{FX}\epsilon_{FX}F\eta_X \arrow[l, "\epsilon_{\Omega_{FX}}*1"'] \arrow[ur, "k^F_X*1*1"] \arrow[dddl, "\Omega^{-1}_{\epsilon_{FX}}*1"] & F\Omega_X\ 1_{FX} \arrow[u, "1*\cong"'] & \\
 & \arrow[r, phantom, "(5)"] & \arrow[r, phantom, "(7)"] & F\Omega_X \arrow[u, "\rho^{-1}"'] \arrow[r, phantom, "(3)"] & \\
 & \arrow[r, phantom, "(6)"] & & 1_{FX}F\Omega_X \arrow[u, "\lambda"'] & \\
 & \epsilon_{FX}\ \Omega_{FGFX}F\eta_X \arrow[uuu, "1*k^F_{GFX}*1"'] \arrow[drr, phantom, "(2)"] & \epsilon_{FX}F\eta_X\ \Omega_{FX} \arrow[l, "1*\Omega_{F\eta_X}"] \arrow[r, "1*1*k^F_X"'] \arrow[uuu, "\Omega_{\epsilon_X F\eta_X}"'] & \epsilon_{FX}F\eta_X F\Omega_X \arrow[u, "\cong*1"'] \arrow[dr, "1*F^2"'] & \\
\epsilon_{FX}F(\Omega_{GFX}\ \eta_X) \arrow[uuuuuuu, "1*F(G(k^F_X)*1)"] & & & & \epsilon_{FX}F(\eta_X\Omega_X) \arrow[llll, "1*F(\Omega_{\eta_X})"]
\end{tikzcd}
$$

Each sub-diagram commutes by the following logic:

1. Naturality of F^2.
2. The second super functoriality condition on F.
3. η_{Ω_X} is the unique 2-cell making this diagram commute.
4. Naturality of the 2-cells of ϵ.
5. Follows from our definition of $\epsilon_{\Omega_{FX}}$.
6. Lax naturality of $\Omega^{\mathcal{C}}$.
7. Expand this sub-diagram as follows:

$$
\begin{tikzcd}
\epsilon_{FX}F\eta_X F\Omega_X \arrow[d, "\cong*1"'] & \epsilon_{FX}F\eta_X\Omega_{FX} \arrow[l, "1*1*k^F_X"'] \arrow[rr, "\Omega_{\epsilon_{FX}F\eta_X}"] \arrow[d, "\cong*1"] & & \Omega_{FX}\ \epsilon_{FX}F\eta_X \arrow[r, "k^F_X*1*1"] \arrow[d, "1*\cong"] & F\Omega_X\ \epsilon_{FX}F\eta_X \arrow[d, "1*\cong"] \\
1_{FX}F\Omega_X \arrow[ddrr, "\lambda"'] & 1_{FX}\Omega_{FX} \arrow[l, "1*k^F_X"'] \arrow[rr, "\Omega_{1_{FX}}"] \arrow[dr, "\lambda"'] & & \Omega_{FX}1_{FX} \arrow[r, "k^F_X*1"] & F\Omega_X 1_{FX} \\
 & & \Omega_{FX} \arrow[ur, "\rho^{-1}"'] \arrow[d, "k^F_X"'] & & \\
 & & F\Omega_X \arrow[uurr, "\rho^{-1}"'] & &
\end{tikzcd}
$$

The central square is naturality of the 2-cells of $\Omega^{\mathcal{C}}$, while the adjacent squares commute by the interchange law. The remaining squares are unitor naturality squares and the triangle is the lax unity coherence condition of $\Omega^{\mathcal{C}}$.

By combining Theorem 3.4 just proven with the Whitehead theorem for bicategories, we get the following:

Corollary 3.5 (Whitehead theorem for superbicategories)**.** *A pseudo superfunctor between two k-superbicategories is a superbiequivalence if and only if the underlying pseudofunctor is*

- *Essentially surjective on adjoint equivalence classes of objects.*
- *Essentially full on* 1*-cells.*
- *Fully faithful on* 2*-cells.*

4 Strictification for superbicategories

Recall the k-superbicategory $\mathbf{sCat}_k$ of small k-supercategories, superfunctors and supertransformations as described in Proposition 2.14. We now construct the super-representable morphisms related to any k-superbicategory $(\mathcal{B}, \Omega^{\mathcal{B}}, \mu^{\mathcal{B}})$, analogous to the regular representable morphisms for bicategories as constructed in [JY20, Chapter 4]. This leads us to a super-Yoneda embedding of any k-superbicategory into the k-superbicategory of superfunctors

$$\tilde{\mathcal{Y}} : \mathcal{B} \to [\mathcal{B}^{op}, \mathbf{sCat}_k]^{sup}$$

The results of the previous section will allow us to conclude that any k-superbicategory is superbiequivalent to a super-2-category - the *strictification theorem for superbicategories.*

Representable superfunctors

Let $(\mathcal{B}, \Omega^{\mathcal{B}}, \mu^{\mathcal{B}})$ be a k-superbicategory and $X \in \mathcal{B}_0$. Define a *representable pseudo superfunctor* $\tilde{\mathcal{Y}}_X : \mathcal{B}^{op} \to \mathbf{sCat}_k$ as follows:

- The underlying pseudofunctor $\mathcal{B}^{op} \to \mathbf{sCat}_k$ sends $A \in \mathcal{B}_0$ to the local k-supercategory $(\mathcal{B}(A, X), \tilde{\Omega}_{A,X}, \tilde{\mu}_{A,X})$ as defined in Proposition 2.17.
- $\tilde{\mathcal{Y}}_X$ sends a 1-cell $f : A \to B$ to the superfunctor $f^* : \mathcal{B}(B, X) \to \mathcal{B}(A, X)$ as defined in Definition 2.19.
- $\tilde{\mathcal{Y}}_X$ sends a 2-cell $\beta : f \to g$ to the supertransformation β^* as defined in Definition 2.21.
- The functorial constraint $(\tilde{\mathcal{Y}}_X)^2_{g,f} : g^* f^* \to (fg)^*$ is the natural isomorphism given on h-component by
$$(h \circ f) \circ g \xrightarrow{\alpha} h \circ (f \circ g)$$
The reader may check that this defines a supertransformation between superfunctors.
- The unity constraint $(\tilde{\mathcal{Y}}_X)^0_A : 1_{\mathcal{B}(A,X)} \to 1^*_A$ is the natural isomorphism given on h-component by
$$h \xrightarrow{\rho^{-1}} h \circ 1_A$$
The reader may check that this defines a supertransformation between superfunctors. So far we have defined the data for a pseudofunctor $\mathcal{B}^{op} \to \mathbf{sCat}_k$. Naturality of the functorial constraints and α, ρ, λ-compatibility hold using the exact same logic as can be found for regular representable pseudofunctors, see for example [JY20, Chapter 4].

- To form a pseudo *super*functor, we pair the previously defined pseudofunctor with parity constraints (2-cells), being in this case the identity natural supertransformations

$$\begin{tikzcd}
\Omega_Y^* \arrow[r, "1"] \arrow[d, equal] & \Omega_Y^* \arrow[dd, equal] \\
\tilde{\Omega}_{Y,X} \arrow[d, equal] & \\
\tilde{\Omega}_{\tilde{\mathfrak{Y}}_X(Y)} \arrow[r, "k_Y^{\tilde{\mathfrak{Y}}_X}"'] & \tilde{\mathfrak{Y}}_X(\Omega_Y)
\end{tikzcd}$$

The data of a pseudo superfunctor is required to satisfy two super coherence conditions involving the parity constraints. The first is given on the left below, which is a diagram of supertransformations:

$$\begin{tikzcd}
\Omega_Y^* \circ \Omega_Y^* \arrow[r, "1"] \arrow[dd, "\tilde{\mu}_{Y,X}"'] & \Omega_Y^* \circ \Omega_Y^* \arrow[d, "\tilde{\mathfrak{Y}}_X^2"] \\
 & (\Omega_Y \circ \Omega_Y)^* \arrow[d, "(\mu_Y^{op})^*"] \\
1_{\mathcal{B}(Y,X)} \arrow[r, "\tilde{\mathfrak{Y}}_X^0"'] & 1_Y^*
\end{tikzcd}
\qquad
\begin{tikzcd}
(h\Omega_Y)\Omega_Y \arrow[r, "1"] \arrow[dd, "(\tilde{\mu}_{Y,X})_h"'] & (h\Omega_Y)\Omega_Y \arrow[d, "\alpha"] \\
 & h(\Omega_Y\Omega_Y) \arrow[d, "-1*\mu_Y"] \\
h \arrow[r, "\rho^{-1}"'] & h1_Y
\end{tikzcd}$$

The h-component of this diagram is given on the right - it commutes trivially after observing the definition of $(\tilde{\mu}_{Y,X})_h$ given in Proposition 2.17. To account for the sign, recall the definition of the opposite k-superbicategory in Definition 2.8. The second coherence condition asks that the left diagram below commutes:

$$\begin{tikzcd}
f^*\Omega_Y^* \arrow[r, "\tilde{\Omega}_{f^*}"] \arrow[d, "1"'] & \Omega_X^* f^* \arrow[d, "1"] \\
f^*\Omega_Y^* \arrow[d, "\tilde{\mathfrak{Y}}_X^2"'] & \Omega_X^* f^* \arrow[d, "\tilde{\mathfrak{Y}}_X^2"] \\
(\Omega_Y f)^* \arrow[r, "(\Omega_f^{-1})^*"'] & (f\Omega_X)^*
\end{tikzcd}
\qquad
\begin{tikzcd}
(h\Omega_Y)f \arrow[r, "(\tilde{\Omega}_{f^*})_h"] \arrow[d, "1"'] & (hf)\Omega_X \arrow[d, "1"] \\
(h\Omega_Y)f \arrow[d, "\alpha"'] & (hf)\Omega_X \arrow[d, "\alpha"] \\
h(\Omega_Y f) \arrow[r, "1*\Omega_f^{-1}"'] & h(f\Omega_X)
\end{tikzcd}$$

Again, the h-component diagram on the right commutes trivially if we expand the definition of $(\tilde{\Omega}_{f^*})_h$ given in Definition 2.19.

Representable supertransformations

Let $f : X \to Y$ be a 1-cell in $\mathcal{B}$. We define a strong supertransformation $\tilde{\mathfrak{Y}}_f : \tilde{\mathfrak{Y}}_X \to \tilde{\mathfrak{Y}}_Y$ between pseudo superfunctors to have 1-cells (superfunctors)

$$(\tilde{\mathfrak{Y}}_f)_A = f_* : \mathcal{B}(A, X) \to \mathcal{B}(A, Y)$$

The constraint naturality 2-cells (invertible supertransformations) are defined to be

$$(\tilde{\mathfrak{Y}}_f)_g : g^* f_* \to f_* g^*$$

with h-component

$$(f \circ h) \circ g \xrightarrow{\alpha} f \circ (h \circ g)$$

This is easily shown to be a supertransformation. The definition of $\tilde{\mathcal{Y}}_f$ is very similar to that of a regular representable transformation between representable pseudofunctors. The same small calculations as in that case show that this defines a strong transformation, see [JY20]. The super naturality coherence condition asks that the left diagram commute:

$$\begin{tikzcd}
\Omega^*_A\ f_* \arrow[r, "(\tilde{\mathcal{Y}}_f)_{\Omega_A}"] \arrow[d, equal] & f_*\ \Omega^*_A \arrow[d, equal] \\
\Omega^*_A\ f_* & f_*\ \Omega^*_A \arrow[l, "\tilde{\Omega}_{(\tilde{\mathcal{Y}}_f)_A}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
(fg)\Omega_A \arrow[r, bend left, "\alpha"] & f(g\Omega_A) \arrow[l, bend left, "\alpha^{-1}"]
\end{tikzcd}$$

The right diagram is the g-component, as we have used the definition of $\tilde{\Omega}_{(\tilde{\mathcal{Y}}_f)_A} = \tilde{\Omega}_{f_*}$ given in Definition 2.19. Therefore $\tilde{\mathcal{Y}}_f$ is a strong supertransformation.

Representable modifications

Let $\beta : f \to g$ be a 2-cell between 1-cells $f, g : X \to Y$ in $\mathcal{B}$. Define a modification $\tilde{\mathcal{Y}}_\beta : \tilde{\mathcal{Y}}_f \to \tilde{\mathcal{Y}}_g$ to have component 2-cells (supertransformations)

$$(\mathcal{Y}_\beta)_A = \beta_* : f_* \to g_* \qquad : \mathcal{B}(A, X) \to \mathcal{B}(A, Y)$$

as given in Definition 2.21. This construction is highly analogous to regular representable modifications, see [JY20].

4.1 The Yoneda superfunctor

We now define a pseudo superfunctor $(\tilde{\mathcal{Y}}, \{k^{\tilde{\mathcal{Y}}}_X\}) : \mathcal{B} \to [\mathcal{B}^{op}, \mathbf{sCat}_k]^{sup}$ for any k-superbicategory $(\mathcal{B}, \Omega^{\mathcal{B}}, \mu^{\mathcal{B}})$, the underlying pseudofunctor is defined as follows:

- $\tilde{\mathcal{Y}}$ sends $X \in \mathcal{B}_0$ to the pseudo superfunctor $\tilde{\mathcal{Y}}_X$.
- A 1-cell $f : X \to Y$ maps to the strong supertransformation $\tilde{\mathcal{Y}}_f$.
- A 2-cell $\beta : f \to g$ maps to the modification $\tilde{\mathcal{Y}}_\beta$.
- The functorial constraint is a family of modifications

  $$(\tilde{\mathcal{Y}})^2_{f,g} : \tilde{\mathcal{Y}}_f \tilde{\mathcal{Y}}_g \to \tilde{\mathcal{Y}}_{fg}$$

  whose X-component $f_* g_* \to (fg)_*$ is the supertransformation with h-component

  $$f \circ (g \circ h) \xrightarrow{\alpha^{-1}} (f \circ g) \circ h$$

  Recall here f_*, g_* are the superfunctors as defined in Definition 2.19.
- The unital constraint is a family of modifications

  $$(\tilde{\mathcal{Y}})^0_X : 1_{\tilde{\mathcal{Y}}_X} \to \tilde{\mathcal{Y}}_{1_X}$$

  with Y-component $1_{\mathcal{B}(Y,X)} \to (1_X)_*$ a supertransformation with h-component

  $$h \xrightarrow{\lambda^{-1}} 1_X \circ h$$

This defines a pseudofunctor; all the relevant checks are identical to the Yoneda pseudofunctor case in [JY20]. The superstructure consists of a family of parity constraints (modifications)

$$k_X^{\tilde{\mathcal{Y}}} : \Omega^{[\mathcal{B}^{op},\mathbf{sCat}_k]}_{\tilde{\mathcal{Y}}_X} \to \tilde{\mathcal{Y}}_{\Omega^{\mathcal{B}}_X}$$

with Y-component a supertransformation $(\Omega_Y)^* \to (\Omega_X)_*$ with $(f : Y \to X)$-component

$$f \circ \Omega^{\mathcal{B}}_Y \xrightarrow{\Omega^{\mathcal{B}}_f} \Omega^{\mathcal{B}}_X \circ f$$

For this Y-component to define a supertransformation, we need the following diagram to commute:

$$\begin{tikzcd}
\Omega^*_Y\Omega^*_Y \arrow[r, "\tilde{\Omega}_{\Omega^*_Y}"] \arrow[d, "(k_X)_Y*1"'] & \Omega^*_Y\Omega^*_Y \arrow[d, "1*(k_X)_Y"] \\
(\Omega_X)_*\Omega^*_Y \arrow[r, "\tilde{\Omega}_{(\Omega_X)_*}"'] & \Omega^*_Y(\Omega_X)_*
\end{tikzcd}$$

On the f-component, this becomes the (f, Ω_Y)-component of lax naturality of $\Omega^{\mathcal{B}}$ as one may check. Recall that the natural transformations $\tilde{\Omega}_{g^*}, \tilde{\Omega}_{g_*}$ are defined as in Definition 2.19.

Remark 4.1. We will not omit the superscripts on $k_X^{\tilde{\mathcal{Y}}}$ as we usually would, to avoid confusion with the parity constraints $k_Y^{\tilde{\mathcal{Y}}_X}$ on the pseudo superfunctors $\tilde{\mathcal{Y}}_X$.

Proposition 4.2. *The data $(\tilde{\mathcal{Y}}, \{k_X^{\tilde{\mathcal{Y}}}\}_{X\in\mathcal{B}_0})$ defines a pseudo superfunctor $\mathcal{B} \to [\mathcal{B}^{op}, \boldsymbol{sCat}_k]^{sup}$, the **Yoneda pseudo superfunctor**.*

Proof. There are two super functoriality conditions to be checked for the data $(\tilde{\mathcal{Y}}, \{k_X^{\tilde{\mathcal{Y}}}\}_{X\in\mathcal{B}_0})$, the first is commutativity of the left diagram:

$$\begin{tikzcd}
\Omega_{\tilde{\mathcal{Y}}_X}\Omega_{\tilde{\mathcal{Y}}_X} \arrow[r, "k_X^{\tilde{\mathcal{Y}}}*k_X^{\tilde{\mathcal{Y}}}"] \arrow[dd, "\mu_{\tilde{\mathcal{Y}}_X}"'] & \tilde{\mathcal{Y}}_{\Omega_X}\tilde{\mathcal{Y}}_{\Omega_X} \arrow[d, "\tilde{\mathcal{Y}}^2"] \\
& \tilde{\mathcal{Y}}_{\Omega_X\Omega_X} \arrow[d, "\tilde{\mathcal{Y}}_{\mu_X}"] \\
1_{\tilde{\mathcal{Y}}_X} \arrow[r, "\tilde{\mathcal{Y}}^0"'] & \tilde{\mathcal{Y}}_{1_X}
\end{tikzcd}
\qquad
\begin{tikzcd}
\Omega^*_Y\Omega^*_Y \arrow[r, "(k_X^{\tilde{\mathcal{Y}}})_Y*(k_X^{\tilde{\mathcal{Y}}})_Y"] \arrow[dd, "\tilde{\mu}_{Y,X}"'] & (\Omega_X)_*(\Omega_X)_* \arrow[d, "(\tilde{\mathcal{Y}}^2)_Y"] \\
& (\Omega_X\Omega_X)_* \arrow[d, "(\mu_X)_*"] \\
1_{\mathcal{B}(Y,X)} \arrow[r, "(\tilde{\mathcal{Y}}^0)_Y"'] & (1_X)_*
\end{tikzcd}$$

This is a diagram of modifications, whose Y-component is the diagram of supertransformations on the right above. If we take the f-component, we get the outside diagram below:

$$\begin{tikzcd}
& & \Omega_X(f\Omega_Y) \arrow[drr, "1*\Omega_f"] & & \\
(f\Omega_Y)\Omega_Y \arrow[urr, "\Omega_{f\Omega_Y}"] \arrow[drr, "-\Omega_f*1"] \arrow[d, "\alpha"] & & & & \Omega_X(\Omega_X f) \arrow[dd, "\alpha^{-1}"] \\
f(\Omega_Y\Omega_Y) \arrow[d, "-1*\mu_Y"'] & & (\Omega_X f)\Omega_Y \arrow[uu, "\alpha"] & & \\
f1_Y \arrow[d, "\rho"] & & & & (\Omega_X\Omega_X)f \arrow[d, "\mu_X*1"] \\
f \arrow[rrrr, "\lambda^{-1}"'] & & & & 1_X f
\end{tikzcd}$$

The larger sub-diagram commutes, being the modification axiom for μ with two minus signs inserted (they cancel). The triangle commutes, as it simply the (f, Ω_Y)-component of the lax naturality condition on Ω where we have collapsed the following composition down into $-1_{(f\Omega_Y)\Omega_Y}$:

$$(f\Omega_Y)\Omega_Y \xrightarrow{\alpha} f(\Omega_Y\Omega_Y) \xrightarrow{1*\Omega_{\Omega_Y}} f\Omega_Y(\Omega_Y) \xrightarrow{\alpha^{-1}} (f\Omega_Y)\Omega_Y$$

The second and final super functoriality condition on $\tilde{\mathcal{Y}}$ is commutativity of the left diagram below for $f : X \to Y$ a 1-cell in $\mathcal{B}$:

$$\begin{tikzcd}
\tilde{\mathcal{Y}}_f \circ \Omega_{\tilde{\mathcal{Y}}_X} \arrow[r, "\Omega_{\tilde{\mathcal{Y}}_f}"] \arrow[d, "1*k^{\tilde{\mathcal{Y}}}_X"'] & \Omega_{\tilde{\mathcal{Y}}_Y} \circ \tilde{\mathcal{Y}}_f \arrow[d, "k^{\tilde{\mathcal{Y}}}_X*1"] \\
\tilde{\mathcal{Y}}_f \circ \tilde{\mathcal{Y}}_{\Omega_X} \arrow[d, "\tilde{\mathcal{Y}}^2"'] & \tilde{\mathcal{Y}}_{\Omega_Y} \circ \tilde{\mathcal{Y}}_f \arrow[d, "\tilde{\mathcal{Y}}^2"] \\
\tilde{\mathcal{Y}}_{f\Omega_X} \arrow[r, "\tilde{\mathcal{Y}}_{\Omega_f}"'] & \tilde{\mathcal{Y}}_{\Omega_Y f}
\end{tikzcd}
\qquad
\begin{tikzcd}
f_*\Omega^*_Y \arrow[r, "\tilde{\Omega}_{f_*}"] \arrow[d, "1*(k^{\tilde{\mathcal{Y}}}_X)_Y"'] & \Omega^*_Y f_* \arrow[d, "(k^{\tilde{\mathcal{Y}}}_X)_Y*1"] \\
f_*(\Omega_X)_* \arrow[d, "(\tilde{\mathcal{Y}}^2)_Y"'] & (\Omega_X)_* f_* \arrow[d, "(\tilde{\mathcal{Y}}^2)_Y"] \\
(f\Omega_X)_* \arrow[r, "(\Omega_f)_*"'] & (\Omega_X f)_*
\end{tikzcd}$$

This is again a diagram of modifications, the Z-component is the diagram of supertransformations on the right. The g-component of this diagram is the (f, g)-component of the lax naturality condition on Ω.

□

Given a k-superbicategory $(\mathcal{B}, \Omega^{\mathcal{B}}, \mu^{\mathcal{B}})$, Proposition 2.34 defines a natural k-superbicategory structure on $[\mathcal{B}^{op}, \mathbf{sCat}_k]^{sup}$. Further, Proposition 2.17 defines a natural k-supercategory structure on each local category $[\mathcal{B}^{op}, \mathbf{sCat}_k]^{sup}(F, G)$, which we shall denote $\mathrm{sStr}(F, G)$ for the rest of this section. This is the k-supercategory of strong supertransformations $F \to G$ between pseudo superfunctors, with modifications as morphisms.

Definition 4.3. For each pseudo superfunctor $(F, \{k^F_X\}_{X \in \mathcal{B}_0}) : \mathcal{B}^{op} \to \mathbf{sCat}_k$, define the data of a superfunctor, called *evaluation*

$$e_A : \mathrm{sStr}(\tilde{\mathcal{Y}}_A, F) \to (FA, \tilde{\Omega}_{FA}, \tilde{\mu}_{FA})$$

sending a supertransformation $\beta : \tilde{\mathcal{Y}}_A \to F$ to $\beta_A(1_A)$ and a modification $\Gamma : \beta \to \beta'$ to $(\Gamma_A)_{1_A}$. Additionally, we need a natural isomorphism

$$e_A \circ \Omega^{\mathrm{sStr}(F,G)} \cong \tilde{\Omega}_{FA} \circ e_A$$

We define the β-component as

$$e_A(\beta \circ \Omega_{\tilde{\mathcal{Y}}_A}) = \beta_A(\tilde{\Omega}_{\mathcal{B}(A,A)}(1_A)) \xrightarrow{(\tilde{\Omega}_{\beta_A})_{1_A}} \tilde{\Omega}_{FA}(\beta_A(1_A))$$

Here $\beta_A : \mathcal{B}(A, A) \to FA$ is a superfunctor, coming equipped with a natural isomorphism $\tilde{\Omega}_{\beta_A} : \beta_A \circ \tilde{\Omega}_{\mathcal{B}(A,A)} \to \tilde{\Omega}_{FA} \circ \beta_A$, of which we have taken the 1_A-component above.

The data of a superfunctor must satisfy a coherence condition consisting of a commuting diagram of natural transformations. The β-component of this unwritten coherence diagram for e_A is simply the 1_A-component of the same condition on the superfunctor β_A, so the data above defines a superfunctor e_A.

Lemma 4.4 (Objectwise super Yoneda lemma)**.** *The superfunctor e_A is a superequivalence of k-supercategories.*

Proof. By Proposition 2.16 it suffices to show that the underlying functor is an equivalence of categories, for which it suffices to that show the underlying functor is essentially surjective and fully faithful. For the former property, let $D \in FA$ be given. We shall construct a strong supertransformation $\overline{D} : \tilde{\mathcal{Y}}_A \to F$ such that $e_A(\overline{D}) \cong D$.

- Define $\overline{D}_X : (\mathcal{B}(X, A), \tilde{\Omega}_{X,A}, \tilde{\mu}_{X,A}) \to (FX, \tilde{\Omega}_{FX}, \tilde{\mu}_{FX})$ to be the superfunctor with underlying functor

$$f \mapsto (Ff)(D)$$
$$(\beta : f \to f') \mapsto (F\gamma)_D$$

Here $Ff : FA \to FX$ is a superfunctor between k-supercategories and $F\gamma : Ff \to Ff'$ is a supertransformation. The parity constraint for $\overline{D}_X$ is a natural transformation

$$\tilde{\Omega}_{\overline{D}_X} : \overline{D}_X \circ \Omega_X^* \to \tilde{\Omega}_{FX} \circ \overline{D}_X$$

Define the f-component to be the composition

$$(F(f \circ \Omega_X))(D) \xrightarrow{(F^2)_D^{-1}} F(\Omega_X)((Ff)(D)) \xrightarrow{(k_X)^{-1}_{(Ff)(D)}} \Omega^{FX}((Ff)(D))$$

Instead of bloating this proof, we delegate the proof that this data defines a superfunctor $\overline{D}_X$ to the appendix, see Proposition A.10.

- Define $\overline{D}_g : Fg \circ \overline{D}_X \to \overline{D}_Y \circ \tilde{\mathcal{Y}}_A(f)$ to be the supertransformation with h-component

$$Fg(Fh(D)) \xrightarrow{(F^2)_D} F(hg)(D)$$

We prove that $\overline{D}_g$ is a supertransformation in the appendix, see Proposition A.11.

We need the left super naturality diagram to commute:

$$\begin{tikzcd}
F\Omega_X \circ \overline{D}_X \arrow[r, "\overline{D}_{\Omega_X}"] & \overline{D}_X \circ \tilde{\mathcal{Y}}_A(\Omega_X) \\
\tilde{\Omega}_{FX} \circ \overline{D}_X \arrow[u, "k_X^F * 1"] & \overline{D}_X \circ \tilde{\Omega}_{\tilde{\mathcal{Y}}_A(X)} \arrow[l, "\tilde{\Omega}_{\overline{D}_X}"] \arrow[u, "1 * k_X^{\tilde{\mathcal{Y}}_A}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
(F\Omega_X)((Ff)D) \arrow[rr, "(F^2)_D"] & & (F(f \circ \Omega_X))(D) \\
\tilde{\Omega}_{FX}((Ff)D) \arrow[u, "(k_X^F)_{(Ff)D}"] & F\Omega_X((Ff)D) \arrow[l, "(k_X^F)^{-1}_{(Ff)D}"] & (F(f \circ \Omega_X))(D) \arrow[l, "(F^2)_D^{-1}"] \arrow[u, "1"']
\end{tikzcd}$$

The f-component is on the right, which clearly commutes. Thus $\overline{D}$ is a strong supertransformation. We have

$$e_A(\overline{D}) = \overline{D}_A(1_A) = (F1_A)(D) \xrightarrow{(F^0)_D^{-1}} 1_{FA}(D) = D$$

So e_A is essentially surjective. Let $\Gamma : \beta \to \beta'$ be a modification between two strong transformations $\tilde{\mathcal{Y}}_A \to F$. Thus the following diagram commutes for $f : X \to A$:

$$\begin{tikzcd}
Ff\beta_A \arrow[r, "\beta_f"] \arrow[d, "1 * \Gamma_A"'] & \beta_X f^* \arrow[d, "\Gamma_X * 1"] \\
Ff\beta'_A \arrow[r, "\beta'_f"'] & \beta'_X f^*
\end{tikzcd}$$

Evaluating this diagram at 1_A gives commutativity of the left diagram below, the right commutes due to Γ_X being a natural transformation:

$$\begin{tikzcd}
Ff(\beta_A(1_A)) \arrow[r, "(\beta_f)_{1_A}"] \arrow[d, "Ff((\Gamma_A)_{1_A})"'] & \beta_X(1_A f) \arrow[r, "\beta_X(\rho)"] \arrow[d, "(\Gamma_X)_{1_A f}"] & \beta_X(f) \arrow[d, "(\Gamma_X)_f"] \\
Ff(\beta'_A(1_A)) \arrow[r, "(\beta'_f)_{1_A}"'] & \beta'_X(1_A f) \arrow[r, "\beta'_X(\rho)"'] & \beta'_X(f)
\end{tikzcd}$$

Explicitly, $(\Gamma_X)_f$ is equal to the following composite

$$\beta_X(f) \xrightarrow{\beta_X(\rho)^{-1}} \beta_X(1_A f) \xrightarrow{(\beta_f)^{-1}_{1_A}} Ff(\beta_A(1_A)) \xrightarrow{Ff((\Gamma_A)_{1_A})} Ff(\beta'_A(1_A)) \xrightarrow{(\beta'_f)_{1_A}} \beta'_X(1_A f) \xrightarrow{\beta'_X(\rho)} \beta'_X(f)$$

so that Γ is completely determined by $(\Gamma_A)_{1_A}$. Thus e_A is fully faithful. □

Remark 4.5. The previous lemma was super-adapted from the *objectwise Yoneda lemma* in [JY20]. In particular, the proof strategy and details can be seen to be very similar. The contribution by our work is the construction of the various parity constraints and checks of super-coherence conditions. The strategy for the proof of the strictification result Theorem 4.6 coming up is adapted in a similar way from [JY20, Lemma 8.3.12, Yoneda Embedding].

4.2 Strictification for superbicategories

Theorem 4.6. *Any k-superbicategory is superbiequivalent to a strict k-superbicategory.*

Proof. Let $(\mathcal{B}, \Omega^{\mathcal{B}}, \mu^{\mathcal{B}})$ be a k-superbicategory. Define $\tilde{\mathcal{B}}$ to be the essential image of the Yoneda pseudo superfunctor $\tilde{\mathcal{Y}} : \mathcal{B} \to [\mathcal{B}^{op}, \mathbf{sCat}_k]$. That is, objects of $\tilde{\mathcal{B}}$ consist of those pseudo superfunctors that are (adjoint) equivalent to the representables. The 1-cells and 2-cells are all of those from the ambient k-superbicategory relevant to the included objects. Give $\tilde{\mathcal{B}}$ the obvious k-superbicategory structure as a sub-bicategory. It is clear that $\tilde{\mathcal{B}}$ with this structure is a strict k-superbicategory

Since $\tilde{\mathcal{Y}} : \mathcal{B} \to \tilde{\mathcal{B}}$ is clearly essentially surjective on adjoint equivalence classes, it suffices to show that the underlying pseudofunctor is a local equivalence. By the Whitehead theorem this will say that $\tilde{\mathcal{Y}}$ is a biequivalence, then Theorem 3.4 will say that $\tilde{\mathcal{Y}}$ is a superbiequivalence. Let us show then that the local super-Yoneda functors

$$\tilde{\mathcal{Y}} : \mathcal{B}(X, Y) \to \mathrm{sStr}(\tilde{\mathcal{Y}}_X, \tilde{\mathcal{Y}}_Y)$$

are equivalences of categories. We claim that the evaluation (super)functor

$$e_X : \mathrm{sStr}(\tilde{\mathcal{Y}}_X, \tilde{\mathcal{Y}}_Y) \to \mathcal{B}(X, Y)$$

provides an inverse. Observe that

$$\begin{aligned}(e_X\tilde{\mathcal{Y}})(f) &= e_X(\tilde{\mathcal{Y}}_f)\\ &= (\tilde{\mathcal{Y}}_f)_X(1_X)\\ &= f_*(1_X)\\ &= f \circ 1_X\end{aligned}$$

The left unitor provides a natural isomorphism

$$e_X \circ \tilde{\mathcal{Y}} \to 1_{\mathcal{B}(X,Y)}$$

Next we need a natural isomorphism

$$\tilde{\mathcal{Y}} \circ e_X \to 1_{\mathrm{sStr}(\tilde{Y}_A, \tilde{Y}_B)}$$

The β-component of which will be an isomorphism (invertible modification) in $\mathrm{sStr}(\mathcal{Y}_A, \mathcal{Y}_B)$, natural in β:

$$\tilde{\mathcal{Y}}_{\beta_X(1_X)} \to \beta \tag{16}$$

Such a modification will have as A-component a supertransformation between superfunctors

$$(\beta_X(1_X))_* \to \beta_A \tag{17}$$

The $(h : A \to X)$-component isomorphism of which is defined as

$$\beta_X(1_X) \circ h \xrightarrow{(\beta_h)_{1_X}} \beta_A(1_X \circ h) \xrightarrow{\beta_A(\lambda)} \beta_A(h)$$

We leave naturality in h to the reader. To show that (17) defines a *super*transformation requires commutativity of the following diagram:

$$\begin{tikzcd}
(\beta_X(1_X))_* \circ \tilde{\Omega}_{A,X} \arrow[r, "\tilde{\Omega}_{(\beta_X(1_X))_*}"] \arrow[d, "\cong * 1"'] & \tilde{\Omega}_{A,Y} \circ (\beta_X(1_X))_* \arrow[d, "1 * \cong"] \\
\beta_A \circ \tilde{\Omega}_{A,X} \arrow[r, "\tilde{\Omega}_{\beta_A}"'] & \tilde{\Omega}_{A,Y} \circ \beta_A
\end{tikzcd}$$

The h-component of the above diagram is the outside diagram below:

$$\begin{tikzcd}
\beta_X(1_X) \circ (h \circ \Omega_A) \arrow[rr, "\alpha^{-1}"] \arrow[d, "(\beta_{h \circ \Omega_A})_{1_X}"'] \arrow[drr, phantom, "(1)"] & & (\beta_X(1_X) \circ h) \circ \Omega_A \arrow[d, "(\beta_h)_{1_X} * 1"] \\
\beta_A(1_X \circ (h \circ \Omega_A)) \arrow[d, "\beta_A(\lambda)"'] \arrow[dr, phantom, "(2)"] & \beta_A((1_X \circ h) \circ \Omega_A) \arrow[l, "\beta_A(\alpha)"'] \arrow[dl, "\beta_A(\lambda * 1)"] \arrow[dr, phantom, "(3)"] & \beta_A(1_X \circ h) \circ \Omega_A \arrow[l, "(\beta_{\Omega_A})_{1_X \circ h}"'] \arrow[d, "\beta_A(\lambda) * 1"] \\
\beta_A(h \circ \Omega_A) \arrow[rr, bend right, "(\tilde{\Omega}_{\beta_A})_h"'] \arrow[rr, phantom, "(4)", shift right=3] & & \beta_A(h) \circ \Omega_A \arrow[ll, "(\beta_{\Omega_A})_h"']
\end{tikzcd}$$

Each labelled sub-diagram commutes due to the respective logic below:

1. This is the 1_X-component of the lax naturality coherence condition on the underlying strong transformation of β.
2. The image under the functor β_A of a diagram that commutes in any bicategory, see the diagrams on page 6.
3. Naturality of the natural transformation β_{Ω_A}.
4. This is the h-component of the super naturality coherence condition on β. It is particularly simple since the parity constraints on $\tilde{\mathcal{Y}}_X, \tilde{\mathcal{Y}}_Y$ are identities.

Therefore (17) is a supertransformation. We leave it to the reader to show that (16) is a modification, natural in β. □

Corollary 4.7. *The local super-Yoneda functors*

$$\tilde{\mathcal{Y}} : \mathcal{B}(X,Y) \to sStr(\tilde{\mathcal{Y}}_X, \tilde{\mathcal{Y}}_Y)$$

given the structure of a superfunctor by Remark 2.33, are superequivalences.

Proof. By Proposition 2.16 it suffices to show that the underlying functor is an equivalence of categories, which was done in the proof of Theorem 4.6. □

A Appendix

A.1 Pasting diagrams

The aim of this section is to introduce, and justify the usage of, pasting and string diagrams in a bicategory. We essentially give a light-on-details overview of [JY20, Chapter 3].

Definition A.1. A connected plane graph G is *anchored* if each face is anchored, according to the following definitions:

- An interior face F of G is *anchored* if its boundary contains two vertices s_F, t_F, called the source and sink of F respectively, together with two directed[4] paths $\mathrm{dom}_F, \mathrm{cod}_F : s_F \to t_F$ such that the interior of F is always to the right while traversing the boundary in the order $s_F \xrightarrow{\mathrm{dom}_F} t_F \xrightarrow{\mathrm{cod}^*_F} s_F$. Here cod^*_F refers to the path cod_F in reversed order.
- The exterior face $F = \mathrm{ext}_G$ of G is *anchored* if there are two vertices s_F, t_F together with directed paths $\mathrm{dom}_F, \mathrm{cod}_F : s_F \to t_F$ such that the interior of F is always to the left while traversing the boundary in the order $s_F \xrightarrow{\mathrm{dom}_F} t_F \xrightarrow{\mathrm{cod}^*_F} s_F$.

An *atomic graph* is an anchored graph with a unique interior face F.

Necessarily all atomic graphs have the form

$$\begin{tikzcd} \bullet \arrow[r, "P"] & \bullet \arrow[r, bend left, "R"] \arrow[r, bend right, "S"'] \arrow[r, phantom, "F"] & \bullet \arrow[r, "Q"] & \bullet \end{tikzcd}$$

where P, Q, R, S are all directed paths, with $R = \mathrm{dom}_F, S = \mathrm{cod}_F$. Atomic graphs are used to represent a whiskering of a 2-cell on either side or both by 1-cells in a 2-category. Suppose that we have two anchored graphs G, H such that

- $s_G = s_H, t_G = t_H$.
- $\mathrm{cod}_G = \mathrm{dom}_H$.

Then there is an associative operation, suggestively called *vertical composition*, that returns a new anchored graph HG by gluing G and H along $\mathrm{cod}_G \sim \mathrm{dom}_H$ with the obvious anchoring. Below is an example of the vertical compostion of two atomic graphs:

G

$$\begin{tikzcd} \bullet \arrow[r, "P"] & \bullet \arrow[r, bend left, "R"] \arrow[r, bend right, "S"'] & \bullet \end{tikzcd}$$

H

$$\begin{tikzcd} \bullet \arrow[r, bend left, "P"] \arrow[r, bend right, "Q"'] & \bullet \arrow[r, "S"] & \bullet \end{tikzcd}$$

$$\longmapsto$$

HG

$$\begin{tikzcd} \bullet \arrow[r, bend left, "P"] \arrow[r, bend right, "Q"'] & \bullet \arrow[r, bend left, "R"] \arrow[r, bend right, "S"'] & \bullet \end{tikzcd}$$

[4]By a directed path we mean a path such that each edge is traversed from tail to head.

Definition A.2. A *pasting scheme* is an anchored graph G together with a decomposition

$$G = G_n \dots G_1$$

into atomic graphs. The decomposition itself is called a *pasting scheme presentation* for the anchored graph G.

Suppose that G is an anchored graph, a G-*diagram* is a representation of G in some 2-category $\mathcal{A}$:

G
$$\begin{tikzcd} & \bullet \arrow[rr] \arrow[dr] & & \bullet \\ \bullet \arrow[ur] \arrow[rr] & & \bullet \arrow[ur] & \end{tikzcd}
\qquad
\begin{tikzcd} & Z \arrow[rr, "h"] \arrow[dr, "j"] \arrow[d, phantom, "\beta\Downarrow"] & \arrow[d, phantom, "\Downarrow\gamma"] & W \\ X \arrow[ur, "g"] \arrow[rr, "f"'] & {} & Y \arrow[ur, "i"'] & \end{tikzcd}$$

Note that the domain and codomain of G is fixed by considering orientations, as are those for each interior face for the same reason. By convention we associate a 2-cell to each interior face F whose direction is governed by the domain and codomain of F. For example, the domain of the left-most interior face above is associated with the 1-cell $j \circ g$ in the G-diagram, which serves as the domain for the 2-cell β. The anchored graph G above has a unique pasting scheme presentation $G = G_2 G_1$ with

G_1
$$\begin{tikzcd} & \bullet \arrow[rr] \arrow[dr] & & \bullet \\ \bullet \arrow[ur] & & \bullet \arrow[ur] & \end{tikzcd}$$

G_2
$$\begin{tikzcd} & \bullet \arrow[dr] & & \bullet \\ \bullet \arrow[ur] \arrow[rr] & & \bullet \arrow[ur] & \end{tikzcd}$$

If G has a pasting scheme presentation then we call any G-diagram a *pasting diagram.* To any pasting diagram we thus have a set of instructions for how to construct a 2-cell from the domain to the codomain 1-cells: each atomic graph corresponds to a whiskering of a 2-cell β in one the following forms:

- $1 * \dots * \beta * \dots * 1$
- $\beta * \dots * 1$
- $1 * \dots * \beta$

which we then compose vertically in the order given by the pasting presentation. The G-diagram above has composite given by

$$(1 * \beta) \circ (\gamma * 1) : hg \xrightarrow{\gamma * 1} ijg \xrightarrow{1 * \beta} if$$

The 2-categorical pasting lemma says that the composite of a pasting diagram is independent of the choice of pasting scheme presentation used. As a contrived but illuminating example, consider the following two pasting scheme presentation for G below:

G
$$\begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] & \bullet \arrow[r, bend left] \arrow[r, bend right] & \bullet \end{tikzcd}$$

G_1
$$\begin{tikzcd} \bullet \arrow[r, bend left] & \bullet \arrow[r, bend left] \arrow[r, bend right] & \bullet \end{tikzcd}$$

G_2
$$\begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] & \bullet \arrow[r, bend right] & \bullet \end{tikzcd}$$

G_1'
$$\begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] & \bullet \arrow[r, bend left] & \bullet \end{tikzcd}$$

G_2'
$$\begin{tikzcd} \bullet \arrow[r, bend right] & \bullet \arrow[r, bend left] \arrow[r, bend right] & \bullet \end{tikzcd}$$

We thus have two different ways to form a composite of the pasting diagram

$$\begin{tikzcd}
\bullet \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow\beta"] & \bullet \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\gamma\Downarrow"] & \bullet
\end{tikzcd}$$

Both ways give the same composite 2-cell, precisely due to the interchange law. The situation is understandably more complicated in a bicategory, to which we turn to now:

Definition A.3. A *bracketing* of an anchored graph G is simply a choice of bracketing for the paths dom_G, cod_G and dom_F, cod_F for each interior face F. A G-diagram in a bicategory $\mathcal{B}$ for a bracketed graph G is a representation of G in $\mathcal{B}$, similar to the 2-categorical case, except now the bracketing on dom_F, cod_F governs the bracketing on the source and target 1-cells of the 2-cell corresponding to the face F.

Example A.4. The following constitutes the data of a bracketing of G:

$$\begin{tikzcd}
 & \bullet \arrow[r, "b"] & \bullet \arrow[r, "c"] \arrow[d, "g"] & \bullet \\
G & \bullet \arrow[u, "a"] \arrow[r, "d"'] & \bullet \arrow[r, "e"'] & \bullet \arrow[u, "f"']
\end{tikzcd}$$

$$\begin{tikzcd}
\bullet \arrow[r, "b)"] \arrow[dr, phantom, "F_1"] & \bullet \arrow[r, "c"] \arrow[d, "g"] \arrow[dr, phantom, "F_2"] & \bullet \\
\bullet \arrow[u, "(a"] \arrow[r, "d"'] & \bullet \arrow[r, "(e"'] & \bullet \arrow[u, "f)"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\bullet \arrow[r, "c"] \arrow[d, "g"'] \arrow[dr, phantom, "F_2"] & \bullet \\
\bullet \arrow[r, "(e"'] & \bullet \arrow[u, "f)"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\bullet \arrow[r, "(b"] \arrow[dr, phantom, "F_1"] & \bullet \arrow[d, "g)"] \\
\bullet \arrow[u, "a"] \arrow[r, "d"'] & \bullet
\end{tikzcd}$$

$\mathrm{dom}_G{=}c(ba)$, $\mathrm{cod}_G{=}(fe)d$, $\mathrm{dom}_{F_1}{=}(gb)a$, $\mathrm{cod}_{F_1}{=}d$ $\mathrm{dom}_{F_2}{=}c$, $\mathrm{cod}_{F_2}{=}(fe)g$

A G-diagram in some bicategory $\mathcal{B}$ would look like the following:

$$\begin{tikzcd}
A \arrow[r, "b)"] \arrow[dr, phantom, "\Downarrow\beta"] & B \arrow[r, "c"] \arrow[d, "g"] \arrow[dr, phantom, "\Downarrow\gamma"] & C \\
D \arrow[u, "(a"] \arrow[r, "d"'] & E \arrow[r, "(e"'] & F \arrow[u, "f)"']
\end{tikzcd}$$

However, this presentation alone is potentially confusing, since due to the bracketings specified on the faces of G we actually have $\beta : (gb)a \to d, \gamma : c \to (fe)g$. Conversely, knowing the source and target of all the 2-cells in the diagram above, we know exactly the bracketing on each interior face. Thus to specify a G-diagram in a bicategory, it suffices to only make explicit the outside bracketing, leaving the bracketing on each interior face implicitly defined by the associated 2-cell.

Definition A.5. Given two bracketed graphs G, H satisfying the following:

- $s_G = s_H, t_G = t_H$.
- $(\mathrm{cod}_G) = (\mathrm{dom}_H)$ as bracketed paths.

there is an associative operation yielding a new bracketed graph HG, called the vertical composition of G and H, where HG is given all the obvious bracketings. Essentially this is just gluing the codomain of G onto the domain of H as in the case for anchored graphs. A *composition scheme* is a bracketed graph G together with a decomposition

$$G = G_n \dots G_1$$

into *consistent graphs*[5].

Example A.6. Composition schemes for bracketed graphs are the analogue to pasting schemes for anchored graphs. Just as a pasting scheme allows us to define the composite of any G-diagram when G is an anchored graph, a composition scheme allows us to define the composite of any G-diagram in a bicategory when G is a bracketed graph. As an example, the composite of the following diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "f"'] \arrow[rr, bend left=50, "1_X", "\Downarrow\eta"'] & Y \arrow[r, "g"] \arrow[d, phantom, "\Downarrow\alpha^{-1}"] & X \arrow[d, "f"] \\
Y \arrow[r, "g"'] \arrow[rr, bend right=50, "1_Y"', "\Downarrow\epsilon"] & X \arrow[r, "f"'] & Y
\end{tikzcd}$$

is

$$f1_X \xrightarrow{1*\epsilon} f(gf) \xrightarrow{\alpha^{-1}} (fg)f \xrightarrow{\eta*1} 1_Y f$$

Notice that we left out the bracketing of all three inner faces, as per the discussion in Example A.4. The domain and codomain of the entire diagram both are the composition of only two 1-cells, so no brackets are needed. The (unique) composition scheme that we used was the following:

$$\begin{tikzcd}
\bullet \arrow[r, "(-"] \arrow[rr, bend left=50] & \bullet \arrow[r, "-)"] & \bullet \arrow[d, "-"] \\
& & \bullet
\end{tikzcd}
\qquad
\begin{tikzcd}
\bullet \arrow[r, "(-"] \arrow[d, "-"'] & \bullet \arrow[r, "-)"] & \bullet \arrow[d, "-"] \\
\bullet \arrow[r, "(-"'] & \bullet \arrow[r, "-)"'] & \bullet
\end{tikzcd}
\qquad
\begin{tikzcd}
\bullet \arrow[d, "-"'] & & \\
\bullet \arrow[r, "(-"'] \arrow[rr, bend right=50] & \bullet \arrow[r, "-)"'] & \bullet
\end{tikzcd}$$

Example A.7. For the bracketed graph in Example A.4, a candidate for a potential composition scheme presentation $G = G_2G_1$ is the following data:

$$G_1 \qquad \begin{tikzcd}
\bullet \arrow[r, "b)"] & \bullet \arrow[r, "c"] \arrow[d, "(g"'] & \bullet \\
\bullet \arrow[u, "(a"] & \bullet \arrow[r, "(e"'] & \bullet \arrow[u, "f))"']
\end{tikzcd} \qquad \mathrm{dom}_{G_1} = c(ba),\ \mathrm{cod}_{G_1} = ((fe)g)(ba)$$

$$G_2 \qquad \begin{tikzcd}
\bullet \arrow[r, "(b"] & \bullet \arrow[d, "g))"] & \bullet \\
\bullet \arrow[u, "(a"] \arrow[r, "d"'] & \bullet \arrow[r, "(e"'] & \bullet \arrow[u, "f)"']
\end{tikzcd} \qquad \mathrm{dom}_{G_2} = (fe)((gb)a),\ \mathrm{cod}_{G_2} = (fe)d$$

There is however one glaring problem, the codomain of G_1 is not the same bracketed path as the domain of G_2. This obstacle prevents G from having *any* composition scheme presentation, in fact.

[5]See [JY20] for the definition, but essentially a consistent graph is an atomic graph with a boundary bracketing that is consistent on the intersection of domain with codomain.

Definition A.8. Let G be a bracketed graph. If the underlying anchored graph of G has a pasting scheme presentation, then any G-diagram in a bicategory is called a *pasting diagram.*

This definition at first may seem odd - as Example A.4 and Example A.7 show, a pasting diagram in a bicategory may completely fail to have a composition scheme presentation, the problem being that certain domains and codomains may have different bracketings. Yet we know how to rebracket 1-cells in a bicategory, in fact there is a canonical isomorphim between any two bracketings of a sequence of 1-cells using associators due to the pentagon axiom. It is a fact proven in [JY20] that any bracketed graph admitting a pasting scheme presentation (for the underlying anchored graph) admits a composition scheme presentation after extending it to include extra faces, corresponding to those rebracketings. The G-diagram in Example A.4 has such an extension, as the underlying anchored graph of G has a pasting scheme presentation. A particular extension may be chosen so as to obtain the composite

$$c(ba) \xrightarrow{\gamma*1} ((fe)g)(ba) \xrightarrow{\alpha} (fe)(g(ba)) \xrightarrow{1*\alpha^{-1}} (fe)((gb)a) \xrightarrow{1*\beta} (fe)d$$

The bicategorical pasting theorem says that every pasting diagram has a composite obtained by extension in this way, independent of which extension is used. A different extension above could lead instead to the composite

$$c(ba) \xrightarrow{\gamma*1} ((fe)g)(ba) \xrightarrow{\alpha^{-1}} (((fe)g)b)a \xrightarrow{\alpha*1} ((fe)(gb))a \xrightarrow{\alpha} (fe)((gb)a) \xrightarrow{1*\beta} (fe)d$$

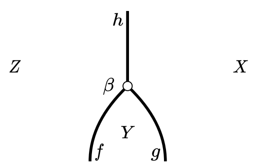
The pentagon identity in any bicategory shows that this is in fact the same composite, showcasing the uniqueness statement in the bicategorical pasting theorem.

In summary, a pasting diagram in a bicategory can be represented as a diagram (with a specific structure - the underlying graph is anchored and has a pasting scheme representation) with a specified bracketing on the domain and codomain. The inner faces have an implicit bracketing from the 2-cells that fill them, and a unique composite can always be defined by expanding the diagram using associators. As a final note, when expressing an equality of 2-cells in the form "pasting diagram A = pasting diagram B", one need not specify the bracketings of the common domain and codomain because equality in one choice of bracketings will imply equality in any other. Again the argument here is that there is a canonical isomorphism between any two bracketings.

A.2 String diagrams

String diagrams are another notational tool for expressing vertical compositions of whiskered 2-cells, their clarity in expressing the data of *adjunctions* and *mates* in a bicategory prove extremely useful in this thesis. String diagrams for monoidal categories are discussed and justified in [JS91]. A brief introduction to pasting and string diagrams, as well as their relationship with adjunctions in a bicategory, is given in [SP14, Appendix A.4].

Objects in a bicategory are now represented as 2D regions, 1-cells as 'strings' on the boundary of two regions, and 2-cells as nodes or junctions where several string meet. The following is a basic example expressing the data of a 2-cell $\beta : f \circ g \to h$, where $g : X \to Y, f : Y \to Z, h : X \to Z$:



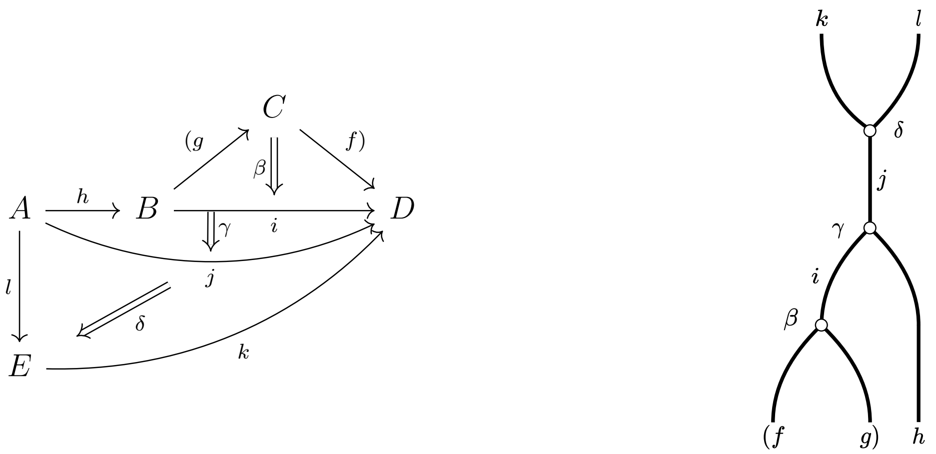
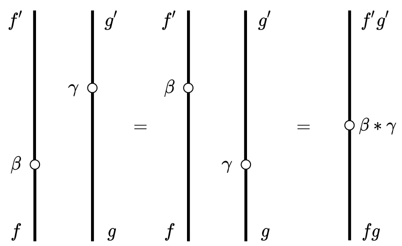


Figure 16: A pasting diagram and a string diagram respectively, both representing the composite 2-cell $(fg)h \xrightarrow{\beta * 1} ih \xrightarrow{\gamma} j \xrightarrow{\delta} kl$ in some bicategory. Labellings for objects in the string diagram are omitted here.

String diagrams are to be read from bottom to top, right to left. The strings that enter a junction from below constitute the source of the 2-cell, while the strings that exit from above constitute the target. Usually we omit labellings for 2D regions. String diagrams and pasting diagrams are two aesthetically different representations of the same types of composition in a bicategory, see Figure 16 for a particular translation. As such, we only need to bracket those sequences of 1-cells comprising the domain and codomain of the string diagram - we are assured of a unique composite 2-cell regardless of how we decide to insert associators. Lastly, when dealing with an equality of the form "string diagram A = string diagram B" we can omit the bracketing on the domain and codomain.

The interchange law says (in particular) that we always have the following equality of string diagrams:



This is due to the calculation

$$\begin{aligned}(1 * \gamma) \circ (\beta * 1) &= (1 \circ \beta) * (\gamma \circ 1)\\ &= \beta * \gamma\\ &= (\beta \circ 1) * (1 \circ \gamma)\\ &= (\beta * 1) \circ (1 * \gamma)\end{aligned}$$

Whenever we invoke "the interchange law" to rearrange 2-cells in string diagrams, we mean so in this fashion.

Remark A.9. Identity 1-cells are often depicted as dotted strings in string diagrams, only when they are not omitted entirely. Consider 2-cells

$$\eta : 1_X \to g \circ f \qquad \epsilon : f \circ g \to 1_Y$$

which we may depict follows

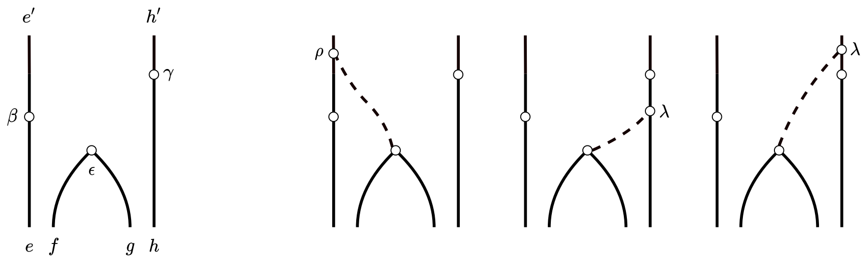
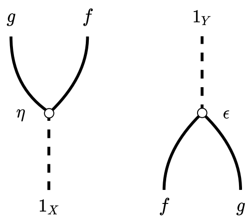


Figure 17: Here $\epsilon : f \circ g \to 1_Y$ is a 2-cell. Any (sensible) choice of drawing in the string for 1_Y is equivalent.



Very often we also remove the junctions corresponding to the unitors ρ, λ and their inverses. *A priori* this leads to ambiguity, for instance, when/where does the invisible string emanating from the ϵ node in the left-most diagram of Figure 17 ultimately reach another junction?

There is no factor of 1_Y in the sequence of 1-cells comprising the codomain of the string diagram, so it must eventually extingiush itself at a unitor-junction with some other string in the diagram. There are many options, some shown in each of the other subfigures. Fortunately all such choices are equivalent, as can be shown using basic bicategory identities. For example, naturality of the unitor λ allows us to conclude that the right-most diagrams yield the same composite.

A.3 Miscellaneous calculations

Proposition A.10. *The data of $\overline{D}_X$ on page 54 defines a superfunctor.*

Proof. We need the following diagram of transformations to commute:

$$\begin{tikzcd}
\overline{D}_X \Omega^*_X \Omega^*_X \arrow[r, "\tilde{\Omega}_{\overline{D}_X} * 1"] \arrow[d, "1 * \tilde{\mu}_{X,A}"'] & \tilde{\Omega}_{FX} \overline{D}_X \Omega^*_X \arrow[d, "1 * \tilde{\Omega}_{\overline{D}_X}"] \\
\overline{D}_X & \tilde{\Omega}_{FX} \tilde{\Omega}_{FX} \overline{D}_X \arrow[l, "\tilde{\mu}_{FX} * 1"]
\end{tikzcd}$$

The expansion of the g-component of this diagram is the outside diagram below:

$$
\begin{tikzcd}
F((g\Omega_X)\Omega_X)(D) \arrow[rr, "(F^2)^{-1}_D"] \arrow[d, "F(\alpha)_D"'] \arrow[drr, phantom, "(1)"] & & F\Omega_X F(g\Omega_X)(D) \arrow[r, "(k_X)^{-1}_{F(g\Omega_X)(D)}"] \arrow[d, "1*(F^2)^{-1}_D"] \arrow[dr, phantom, "(4)"] & \tilde{\Omega}_{FX} F(g\Omega_X)(D) \arrow[d, "1*(F^2)^{-1}_D"] \\
F(g(\Omega_X\Omega_X))(D) \arrow[d, "F(-1*\mu_X)_D"'] \arrow[dr, phantom, "(2)"] & F(\Omega_X\Omega_X)Fg(D) \arrow[l, "F^2_D"] \arrow[d, "F(-\mu_X)_{Fg(D)}"] & F\Omega_X F\Omega_X Fg(D) \arrow[l, "F^2_{Fg(D)}"] \arrow[r, "(k_X)^{-1}_{F\Omega_X Fg(D)}"] & \tilde{\Omega}_{FX} F\Omega_X Fg(D) \arrow[ddd, "1*(k_X)^{-1}_{Fg(D)}"] \\
F(g1_X)(D) \arrow[dd, "F(\rho)_D"'] \arrow[dr, phantom, "(3)"] & F(1_X)Fg(D) \arrow[l, "F^2_D"'] \arrow[dr, phantom, "(5)"] & & \\
& 1_{FX}Fg(D) \arrow[u, "F^0_{Fg(D)}"'] \arrow[dl, dash, "="] & & \\
Fg(D) & & & \tilde{\Omega}_{FX}\tilde{\Omega}_{FX} Fg(D) \arrow[lll, "(\tilde{\mu}_{FX})_{Fg(D)}"] \arrow[uuul, "(k_X*k_X)_{Fg(D)}"']
\end{tikzcd}
$$

Each labelled sub-diagram commutes due to the following respective logic:

1. α-compatibility of F.
2. Naturality of the 2-cells F^2.
3. ρ-compatibility of F.
4. Interchange law.
5. The first super functoriality condition on F. Recall from Definition 2.8 that $\mu_X^{op} = -\mu_X$.

The unlabelled triangle commutes simply by definition of horizontal composition of natural transformations. ☐

Proposition A.11. *The data of $\overline{D}_g$ on page 54 defines a supertransformation.*

Proof. We need to show that the outside diagram below commutes:

$$
\begin{tikzcd}
Fg\overline{D}_X\Omega_X^* \arrow[r, "1*\tilde{\Omega}_{\overline{D}_X}"] \arrow[rr, bend left, "\tilde{\Omega}_{Fg\overline{D}_X}"] \arrow[d, "\overline{D}_g*1"'] & Fg\tilde{\Omega}_{FX}\overline{D}_X \arrow[r, "\tilde{\Omega}_{Fg}*1"] \arrow[u, phantom, "||"] & \tilde{\Omega}_{FY}Fg\overline{D}_X \arrow[d, "1*\overline{D}_g"] \\
\overline{D}_Y g^*\Omega_X^* \arrow[r, "1*\tilde{\Omega}_{g^*}"'] \arrow[rr, bend right, "\tilde{\Omega}_{\overline{D}_Y g^*}"'] & \overline{D}_Y\Omega_Y^* g^* \arrow[r, "\tilde{\Omega}_{\overline{D}_Y}*1"'] \arrow[d, phantom, "||"] & \tilde{\Omega}_{FY}\overline{D}_Y g^*
\end{tikzcd}
$$

The h-component expanded is the outside diagram below:

$$
\begin{tikzcd}
FgF(h\Omega_X)(D) \arrow[r, "1*(F^2)^{-1}_D"] \arrow[d, "(F^2)_D"'] \arrow[dr, phantom, "(1)"] & FgF\Omega_X Fh(D) \arrow[r, "1*(k_X)^{-1}_{Fh(D)}"] \arrow[d, "F^2_{Fh(D)}"] \arrow[ddr, phantom, "(4)"] & Fg\tilde{\Omega}_{FX}Fh(D) \arrow[r, "(\tilde{\Omega}_{Fg})_{Fh(D)}"] & \tilde{\Omega}_{FY}FgFh(D) \arrow[ddd, "1*F^2_D"] \arrow[ddl, "(k_Y)_{FgFh(D)}"] \\
F((h\Omega_X)g)(D) \arrow[d, "F(\alpha)_D"'] \arrow[dr, phantom, "(2)"] & F(\Omega_X g)Fh(D) \arrow[d, "F(\Omega_g^{-1})_{Fh(D)}"] \arrow[dl, "F^2_D"] & & \\
F(h(\Omega_X g))(D) \arrow[d, "F(1*\Omega_g^{-1})_D"'] & F(g\Omega_Y)Fh(D) \arrow[dl, "F^2_D"] \arrow[dr, phantom, "(3)"] & F\Omega_Y FgFh(D) \arrow[l, "F^2_{Fh(D)}"] \arrow[d, "1*F^2_D"] \arrow[r, phantom, "(5)"] & {} \\
F(h(g\Omega_Y))(D) \arrow[r, "F(\alpha^{-1})_D"'] & F((hg)\Omega_Y)(D) \arrow[r, "(F^2)^{-1}_D"'] & F\Omega_Y F(hg)(D) \arrow[r, "(k_Y)^{-1}_{F(hg)(D)}"'] & \tilde{\Omega}_{FY}F(hg)(D)
\end{tikzcd}
$$

1. α-compatibility of F.
2. Naturality of F^2.
3. α-compatibility of F.
4. The second super functoriality condition on F.
5. Interchange law.

□

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