Rings with Several Objects

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October 5, 2006

In this note we explore the idea that additive categories or *ringoids* are just "rings with several objects". This philosophy was developed in depth by Barry Mitchell [4], who showed that a substantial amount of noncommutative ring theory is still true in this generality. Ringoids and their modules provide a natural and transparent framework for discussing many algebraic constructions, including graded modules, chain complexes, sheaves of modules, and quiver representations.

Contents

1 Modules

In this note all rings are associative with identity. Throughout we will not distinguish between a ring R and the preadditive category with a single object whose endomorphism ring is R. Thus the ring R forms a full subcategory of $\mathbf{Mod}R$, where $a \in R$ is identified with the endomorphism $b \mapsto ab$. The endomorphism ring of R in $R\mathbf{Mod}$ is isomorphic to the *opposite* ring of R (partly explaining why we prefer right modules). In a similar spirit, if C is an object of a preadditive category $\mathcal{A}$, we may also refer to the endomorphism ring of this object by C. Recall our convention that if $\mathcal{A}, \mathcal{B}$ are additive categories and $\mathcal{A}$ is small, the category of additive functors $\mathcal{A} \longrightarrow \mathcal{B}$ is denoted by $(\mathcal{A}, \mathcal{B})$.

Recall that a left R-module is the same as a covariant additive functor $R \longrightarrow \mathbf{Ab}$, and that a right R-module is a contravariant additive functor $R \longrightarrow \mathbf{Ab}$. There are several obvious generalisations:

Definition 1. A *ringoid* is a small preadditive category $\mathcal{A}$. A morphism of ringoids $F : \mathcal{A} \longrightarrow \mathcal{A}'$ is an additive functor. A *left* (resp. *right*) $\mathcal{A}$-*module* is a covariant (resp. contravariant) additive functor $\mathcal{A} \longrightarrow \mathbf{Ab}$. These modules form two categories:

$$\begin{aligned}(\mathcal{A}, \mathbf{Ab}) &= \mathcal{A}\mathbf{Mod}\\ (\mathcal{A}^{\mathrm{op}}, \mathbf{Ab}) &= \mathbf{Mod}\mathcal{A}\end{aligned}$$

If $\mathcal{A}$ is a ringoid and $\mathcal{B}$ any preadditive category, then a *left* (resp. *right*) $\mathcal{A}$-*object* in $\mathcal{B}$ is a covariant (resp. contravariant) additive functor $\mathcal{A} \longrightarrow \mathcal{B}$. We denote the corresponding categories respectively by $(\mathcal{A}, \mathcal{B})$ and $(\mathcal{A}^{\mathrm{op}}, \mathcal{B})$. Following the notation for modules over a ring, the set of morphisms between two $\mathcal{A}$-modules F, G is denoted $Hom_{\mathcal{A}}(F, G)$ whenever a subscript is necessary. Observe that there is an *equality* of categories $\mathcal{A}\mathbf{Mod} = \mathbf{Mod}\mathcal{A}^{\mathrm{op}}$ so without loss of generality we can restrict our attention to right modules. Under this equality, H^C is identified with H_C for any object C of $\mathcal{A}$.

Recall that a *subobject* of a $\mathcal{A}$-module M is a monomorphism $\phi : N \longrightarrow M$. A *submodule* is a subobject ϕ with the property that for every object $A \in \mathcal{A}$ the map $\phi_A : N(A) \longrightarrow M(A)$ is the inclusion of a subset.

Remark 1. The empty category is a ringoid, called the *empty ringoid*. To avoid trivial cases, in this note all ringoids will be nonempty unless otherwise specified.

Since **Ab** is a complete, cocomplete abelian category which satisfies *Ab*5, the same is true of $\mathbf{Mod}\mathcal{A}$ and $\mathcal{A}\mathbf{Mod}$ for any small additive $\mathcal{A}$. Moreover, limits and colimits are computed pointwise and a sequence in $\mathbf{Mod}\mathcal{A}$ is exact iff. it is exact pointwise for each $A \in \mathcal{A}$.

Example 1. For any ring R let ${}_{\mathbb{Z}}R$ be the preadditive category with objects the elements of $\mathbb{Z}$, and morphisms specified as follows: there are only zero morphisms between distinct objects, but the endomorphism ring of each object is R. With composition defined in the obvious way, ${}_{\mathbb{Z}}R$ becomes a ringoid. The category of modules $\mathbf{Mod}_{\mathbb{Z}}R$ is precisely the category of $\mathbb{Z}$-graded R-modules.

We can define an additive functor $\sigma : \mathbf{Mod}_{\mathbb{Z}}R \longrightarrow \mathbf{Mod}_{\mathbb{Z}}R$ by "shifting". That is, $\sigma(F)(n) = F(n+1)$ and $\sigma(\phi)_n = \phi_{n+1}$. A well-known construction (see [7], IV Ex.11 or [2]) produces the category of chain complexes of R-modules as the "trivial extension" of $\mathbf{Mod}_{\mathbb{Z}}R$ by the functor σ.

For R-modules there is an isomorphism of groups $Hom_R(R, M) \cong M$. As a left (resp. right) module over itself, R is the additive covariant (resp. contravariant) functor $H^R : R \longrightarrow \mathbf{Ab}$ (resp. $H_R : R \longrightarrow \mathbf{Ab}$). We have the following generalisation (which also holds for nonadditive categories):

Lemma 1 (Yoneda). *Let $\mathcal{A}$ be any category, and consider a contravariant functor $T : \mathcal{A} \longrightarrow$ **Sets**. Then for any object $A \in \mathcal{A}$ we have a one-to-one correspondence*

$$Hom_{\mathcal{A}}(H_A, T) \longrightarrow T(A)$$

*which is natural in both A and T. If $\mathcal{A}$ and $T : \mathcal{A} \longrightarrow$ **Ab** are additive, this is an isomorphism of groups.*

Proof. A morphism $\phi : R \longrightarrow M$ of R-modules is uniquely determined by where $1 \in R$ goes, in the sense that if $1 \mapsto m \in M$, then $s \mapsto m \cdot s$. In this more general situation, a natural transformation $\phi : H_A \longrightarrow T$ is uniquely determined by where $1_A \in [A, A]$ goes, since naturality implies that for any A' and $\alpha : A' \longrightarrow A$,

$$\phi_{A'}(\alpha) = T(\alpha)(\phi_A(1_A))$$

from this point the proof is routine. □

Just as a ring R has a natural structure as a right R-module, we will see that the functors H_A for $A \in \mathcal{A}$ play a similar role for ringoids.

Lemma 2. *Let $\mathcal{A}$ be a ringoid. The covariant Yoneda embedding*

$$\mathbf{y} : \mathcal{A} \longrightarrow \mathbf{Mod}\mathcal{A}$$

is defined on objects $A \in \mathcal{A}$ by $\mathbf{y}(A) = H_A$ and on morphisms $\alpha : A \longrightarrow A'$ by

$$\mathbf{y}(\alpha) : H_A \longrightarrow H_{A'}$$
$$\mathbf{y}(\alpha)_D(f) = \alpha f$$

This functor is fully faithful, preserves and reflects limits and monics, and is an isomorphism of $\mathcal{A}$ with a full subcategory of $\mathbf{Mod}\mathcal{A}$.

In view of this Lemma, we will frequently confuse the functor H_A with the object A, and label morphisms $H_A \longrightarrow H_{A'}$ by the morphism α in $\mathcal{A}$ to which they correspond. For a module $F \in \mathbf{Mod}A$ we speak interchangeably of elements $x \in F(A)$ and morphisms $H_A \longrightarrow F$ (we will sometimes call these A-elements of F). Notice that we almost never have to think of F as a functor, since if $\alpha : A \longrightarrow A'$ is a morphism of $\mathcal{A}$, and $x : H_{A'} \longrightarrow F$ is an A'-element of F, then $F(\alpha)(x)$ corresponds to the morphism $H_A \longrightarrow F$ given by the composite

$$H_A \xrightarrow{\alpha} H_{A'} \xrightarrow{x} F$$

Rather than write $F(\alpha)(x)$, we will often write $x\alpha$.

Let F_i be a collection of right $\mathcal{A}$-modules. We know that the coproduct $\bigoplus_i F_i$ is defined pointwise, so that

$$\left(\bigoplus_i F_i\right)(A) = \bigoplus_i F_i(A)$$

Hence the value of $\bigoplus_i F_i$ on A is the set of all sequences $(x_i)_{i\in I}$ with $x_i \in F_i(A)$ and only finitely many x_i nonzero. If $\alpha : A' \longrightarrow A$ then $\bigoplus_i F_i(\alpha)$ takes a sequence $(x_i) \in \bigoplus_i F_i(A)$ to the sequence $(F_i(\alpha)(x_i)) = (x_i\alpha)$ in $\bigoplus_i F_i(A')$.

If $x : H_A \longrightarrow F$ is an A-element of F, then we denote the image of this morphism by (x) and call it the *submodule generated by* x. Explicitly, the submodule (x) of F is defined on objects by

$$(x)(A') = \{x\alpha \,|\, \alpha : A' \longrightarrow A\}$$

and on morphisms as the restriction of F's action. More generally, take any collection of elements $x_i : H_{A_i} \longrightarrow F$ of F. These induce a morphism $\phi : \bigoplus_i H_{A_i} \longrightarrow F$ out of the coproduct, which is defined by (all sums being finite)

$$\phi_A((\alpha_i)_i) = \sum_i x_i\alpha_i \qquad \alpha_i : A \longrightarrow A_i$$

The image $(x_i)_i$ of ϕ is the submodule of F defined pointwise by

$$(x_i)_i(A) = \{\sum_i x_i\alpha_i \,|\, \alpha_i : A \longrightarrow A_i \text{ for each } i\}$$

We say that the family $\{x_i\}_i$ is a *family of generators* for F if the morphism ϕ is an epimorphism, or equivalently if $(x_i)_i(A)$ is all of $F(A)$ for each $A \in \mathcal{A}$. Clearly any module F admits a family of generators consisting of every element $x \in F(A)$ as A ranges over all objects of $\mathcal{A}$. The $\{x_i\}_i$ are a *basis* if for each $A \in \mathcal{A}$, every $y \in F(A)$ can be written uniquely as

$$y = \sum_i x_i\alpha_i \qquad \alpha_i : A \longrightarrow A_i$$

Equivalently, the x_i are a basis if the corresponding morphism $\bigoplus_i H_{A_i} \longrightarrow F$ is an isomorphism. A module F is *free* if it has a basis.

In the case where F admits a finite family of generators, we say that F is *finitely generated*. This is clearly equivalent to F being a quotient of some finite coproduct $\bigoplus_i H_{A_i}$. If F is a quotient of a representable functor H_A, then F is *cyclic*.

The following proposition generalises the fact that R is a projective generator for its category of modules.

Proposition 3. *If $\mathcal{A}$ is a ringoid and $A \in \mathcal{A}$, then H_A is a small projective in* $\mathbf{Mod}\mathcal{A}$*. Moreover, the $\{H_A\}_{A\in\mathcal{A}}$ form a generating family for* $\mathbf{Mod}\mathcal{A}$*.*

Proof. We first show that H_A is projective. Let $F \longrightarrow F'$ be an epimorphism in $\mathbf{Mod}\mathcal{A}$. If $x : H_A \longrightarrow F'$ is a morphism in $\mathbf{Mod}\mathcal{A}$, then it corresponds to an element $x \in F'(A)$. Since an epimorphism in $\mathbf{Mod}\mathcal{A}$ is pointwise epi, there is $y \in F(A)$ mapping to x; the morphism $x : H_A \longrightarrow F$ factors as

$$H_A \xrightarrow{y} F \longrightarrow F'$$

This shows that H_A is projective. To see that H_A is small, consider a morphism $x : H_A \longrightarrow \bigoplus_i F_i$ of H_A into an arbitrary coproduct. This corresponds to an element $x \in \bigoplus_i F_i(A)$ which by definition belongs to some finite subcoproduct $\bigoplus_{j=1}^n F_j(A)$. It is then clear that $x : H_A \longrightarrow \bigoplus_i F_i$ factors through $\bigoplus_j F_j$.

The H_A form a generating family, since a nonzero morphism $\gamma : F \longrightarrow F'$ must satisfy $\gamma_A(x) \neq 0$ for some A and $x \in F(A)$, in which case the element $x : H_A \longrightarrow F$ is nonzero when composed with γ. □

Since the family $\{H_A\}_{A \in \mathcal{A}}$ forms a generating family of projectives for $\mathbf{Mod}\mathcal{A}$, their coproduct $\bigoplus_A H_A$ is a projective generator for $\mathbf{Mod}\mathcal{A}$. Hence for any ringoid $\mathcal{A}$, the category $\mathbf{Mod}\mathcal{A}$ is grothendieck abelian with exact products and enough projectives (AC,Lemma 47). At the end of the next section we will show directly that $\mathbf{Mod}\mathcal{A}$ admits an injective cogenerator (this also follows from the fact that $\mathbf{Mod}\mathcal{A}$ is grothendieck).

Notice that if $\mathcal{A}$ is a ringoid and $A, B \in \mathcal{A}$ then the morphism set $Hom(A, B)$ has a canonical structure as a B-A-bimodule (identifying A, B resp. with their endomorphism rings). Conversely, there is a way to construct ringoids from rings and bimodules, which we state in the following Proposition. The proof is a straightforward application of the definitions.

Proposition 4. *Suppose that I is an index set and we have the following data*

(i) A ring R_i for $i \in I$;

(ii) For each $i, j \in I$ an R_j-R_i-bimodule A_{ij} together with, for all triples $i, j, k \in I$, a morphism of R_k-R_i-bimodules

$$C_{ijk} : A_{jk} \otimes A_{ij} \longrightarrow A_{ik}$$

such that the following diagram always commutes

$$\begin{array}{ccc} A_{kl} \otimes A_{jk} \otimes A_{ij} & \xrightarrow{1 \otimes C_{ijk}} & A_{kl} \otimes A_{ik} \\ {\scriptstyle C_{jkl} \otimes 1} \downarrow & & \downarrow {\scriptstyle C_{ikl}} \\ A_{jl} \otimes A_{ij} & \xrightarrow[C_{ijl}]{} & A_{il} \end{array}$$

Then the category $\mathcal{A}$ with objects i, morphism sets $Hom(i, j) = A_{ij}$ for $i \neq j$ and $Hom(i, i) = R_i$ with the obvious composition is a small preadditive category.

Experts will recognise the case $|I| = 2$ of this Proposition as the definition of a *Morita context.*

We have already discussed how matrices enter the theory of categories in our Abelian Categories notes. For $A \in \mathcal{A}$ and nonzero integers m, n the morphism set

$$Hom_{\mathcal{A}}(H_A^m, H_A^n)$$

is the abelian group of all $n \times m$ matrices with entries in the ring A. When the coproduct involves terms H_A, H_B for different objects, the entries in the matrix become general morphisms of $\mathcal{A}$ and to write these matrices down, we have to agree to some ordering on the objects involved. For example, elements of $Hom(H_A \oplus H_B, H_C \oplus H_D)$ can be written as 2×2 matrices in the following way:

$$\begin{array}{cc} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} C \\ D \end{array} & \begin{pmatrix} a & b \\ c & d \end{pmatrix} \end{array}$$

As usual the rows correspond to projections, the columns to injections, so an element in the row labelled by C and the column labelled by B is a morphism $B \longrightarrow C$ in $\mathcal{A}$.

2 Tensor Products

Let R be a ring, M a right R-module and N a left R-module. Then the tensor product $M \otimes_R N$ is an abelian group. If N is in addition carries a right S-module structure compatible with the R-module structure, $M \otimes_R N$ becomes a right S-module. In Theorem 11, we will show how to take a right R-module M and a left R-object B in an arbitrary cocomplete abelian category $\mathcal{B}$, and define an object of $\mathcal{B}$ which behaves like a tensor product $M \otimes_R B$.

Even more generally, in Theorem 12 we take ringoid $\mathcal{A}$, a right $\mathcal{A}$-module M, and a left $\mathcal{A}$-object N in a cocomplete abelian category $\mathcal{B}$, and define an object $M \otimes_{\mathcal{A}} N$ in $\mathcal{B}$ in a functorial way.

As we saw in the last section, the representable modules H_A for a ringoid $\mathcal{A}$ form a family of small projective generators for $\mathbf{Mod}\mathcal{A}$. In Theorem 15 we show that this condition classifies modules over ringoids, in the sense that any cocomplete abelian category with such a family of small projective generators is equivalent to a module category over a ringoid (in particular, module categories over rings are those cocomplete abelian categories with a small projective generator). This explains the hypothesis of Theorem 9, from which all of the above results follow.

Theorem 9 is due to Mitchell [3], Chapter IV, and the proof is based on the one given there. However, due to the important role this Theorem plays in the sequel, we give considerably more detail and explain some subtle points. The reader desiring a less verbose proof is directed to [3].

Let us now recall the usual definition of the tensor product:

Definition 2. The *tensor product* of a right A-module M and a left A-module N is an abelian group G together with a bilinear map $\gamma : M \times N \longrightarrow G$, such that if H is another abelian group and $\eta : M \times N \longrightarrow H$ another bilinear map, there is a unique morphism of groups $\theta : G \longrightarrow H$ such that $\theta\gamma = \eta$.

The abelian group G and map γ are usually constructed by taking the free group on the set $M \times N$ and dividing out the subgroup generated by the elements

$$\begin{aligned} (m+m', n) - (m, n) - (m', n) \\ (m, n+n') - (m, n) - (m, n') \\ (ma, n) - (m, an) \end{aligned} \tag{1}$$

where $m, m' \in M$, $n, n' \in N$ and $a \in A$. This construction is fine when we want the tensor product to be an abelian group. However, when we come to construct tensor products in more general abelian categories, the group $\mathbb{Z}$ (and thus the free group $\bigoplus_{(m,n)\in M\times N} \mathbb{Z}$) will not be available.

An alternative construction which *will* generalise is to start with the abelian group ${}^M N = \bigoplus_{m\in M} N$, which consists of sequences $(n_m)_{m\in M}$ with only finitely many $n_m \neq 0$. Denoting the injections by $\widehat{u}_m$, we write (m, n) for $\widehat{u}_m(n)$. The pointwise operations mean that $(m, n+n') = (m, n)+(m, n')$, so we have included in our original group the second family of relations in Equation 1.

Denote by J the subgroup of ${}^M N$ generated by the elements $(m+m', n) - (m, n) - (m', n)$ and $(ma, n) - (m, an)$ for $m, m' \in M$, $n \in N$ and $a \in A$, and let $\mu : {}^M N \longrightarrow M \otimes_A N$ be the cokernel. The morphism of sets $M \times N \longrightarrow {}^M N$, $(m, n) \mapsto \widehat{u}_m(n) = (m, n)$ composes with μ to give a bilinear map. If $\eta : M \times N \longrightarrow H$ is another bilinear map, define $\theta : {}^M N \longrightarrow H$ out of the coproduct by $\theta\widehat{u}_m(n) = \eta(m, n)$. Then θ is zero on J, hence induces $\theta' : M \otimes_A N \longrightarrow H$ with the necessary properties, demonstrating that we have indeed constructed a tensor product.

We will need another characterisiation of the subgroup J. Every element $m \in M$ induces a morphism of right A-modules $m : A \longrightarrow M$, and thus a morphism $\pi : {}^M A \longrightarrow M$ with $\pi u_m = m$ (denoting the injections by u_m). The kernel K of this morphism consists of all those sequences $(k_m)_{m\in M}$ with $\sum_m m \cdot k_m = 0$. Again denoting $u_m(a)$ by (m, a), these sequences include

$$\begin{aligned} (m+m', 1) - (m, 1) - (m', 1) \\ (ma, 1) - (m, a) \end{aligned} \tag{2}$$

for $m, m' \in M$ and $a \in A$. Each sequence $k = (k_m)_m$ induces $k : A \longrightarrow {}^M A$ and collectively these morphisms induce $\lambda : {}^K A \longrightarrow {}^M A$ such that $\lambda v_k = k$, where v_k are the injections into ${}^K A$. We then have an exact sequence in $\mathbf{Mod}A$:

$$ {}^K A \xrightarrow{\lambda} {}^M A \xrightarrow{\pi} M \longrightarrow 0 $$

The map λ determines a morphism of abelian groups $\lambda' : {}^K N \longrightarrow {}^M N$ in $\mathbf{Ab}$. Denoting the injections into ${}^K N$ by $\widehat{v}_k$, we define λ' by

$$ \begin{aligned} \lambda' \widehat{v}_k : N &\longrightarrow {}^M N \\ n &\mapsto \sum_m (m, k_m \cdot n) \end{aligned} $$

Here the sum is implicitly over the finite number of $m \in M$ with $k_m \neq 0$. Using the elements k listed in Equation 2, we see that the image of λ' includes $(m + m', n) - (m, n) - (m', n)$ and $(ma, n) - (m, an)$ for all $m, m' \in M$, $n \in N$ and $a \in A$. Hence $J \subseteq Im\lambda'$. We show that the reverse inclusion holds by demonstrating that $\mu\lambda' = 0$, where $\mu : {}^M N \longrightarrow M \otimes_A N$ is the cokernel of J. For any $n \in N$,

$$ \begin{aligned} \mu\lambda' \widehat{v}_k(n) &= \mu\left(\sum_m (m, k_m \cdot n)\right) \\ &= \sum_m \mu(m, k_m \cdot n) = \sum_m \mu(m \cdot k_m, n) \\ &= \mu\left(\sum_m m \cdot k_m, n\right) = 0 \end{aligned} $$

since by assumption, $\sum_m m \cdot k_m = 0$. We have thus established

Proposition 5. *Let M be a right A-module and N a left A-module. Then there is an exact sequence of abelian groups*

$$ {}^K N \xrightarrow{\lambda'} {}^M N \xrightarrow{\mu} M \otimes_A N \longrightarrow 0 $$

Where λ' is defined by its composition with the injections $\widehat{v}_k$:

$$ \begin{aligned} \lambda' \widehat{v}_k : N &\longrightarrow {}^M N \\ n &\mapsto \sum_m (m, k_m \cdot n) \end{aligned} $$

In the next major Theorem we use a construction similar to that used in the formation of left derived functors. For the reader unfamiliar with derived functors in abelian categories, we review some basic results:

Definition 3. Let $\mathcal{A}$ be an abelian category with a generating set of small projectives $\mathcal{P}$. An object of $\mathcal{A}$ is *free* if it is a coproduct of objects in $\mathcal{P}$. A *presentation* of an object $A \in \mathcal{A}$ is an exact sequence

$$ F_1 \longrightarrow F_0 \longrightarrow A \longrightarrow 0 $$

where F_1, F_0 are free objects.

Lemma 6. *Consider the following diagram in an abelian category $\mathcal{A}$*

$$ \begin{array}{ccccccc} P_1 & \xrightarrow{d} & P_0 & \xrightarrow{\varepsilon} & A & & \\ & & & & \downarrow{\scriptstyle \alpha} & & \\ P_1' & \xrightarrow{d'} & P_0' & \xrightarrow{\varepsilon'} & A' & \longrightarrow & 0 \end{array} \tag{3} $$

Suppose that P_0, P_1 are projective, the bottom row exact, and that $\varepsilon d = 0$. Then there exist morphisms $f_0 : P_0 \longrightarrow P'_0$ and $f_1 : P_1 \longrightarrow P'_1$ such that (3) is commutative. Furthermore, let $T : \mathcal{A}' \longrightarrow \mathcal{B}$ be a covariant additive functor into an abelian category, where $\mathcal{A}'$ is a full subcategory of $\mathcal{A}$ containing P_0, P_1, P'_0 and P'_1. Then the induced morphism $Coker(T(d)) \longrightarrow Coker(T(d'))$ is independent of the choice of f_0 and f_1.

Proof. Using projectivity of P_0 we can find $f_0 : P_0 \longrightarrow P'_0$ such that $\alpha\varepsilon = \varepsilon' f_0$. Now $\varepsilon' f_0 d = \alpha\varepsilon d = 0$, and so the image of $f_0 d$ is a subobject of $Ker(\varepsilon') = Im(d')$. Hence by projectivity of P_1 we can find $f_1 : P_1 \longrightarrow P'_1$ such that $f_0 d = d' f_1$. Let $p : T(P_0) \longrightarrow F$ and $p' : T(P'_0) \longrightarrow F'$ be the cokernels of $T(d)$ and $T(d')$ respectively. Suppose that $g_0 : P_0 \longrightarrow P'_0$ and $g_1 : P_1 \longrightarrow P'_1$ are another pair of morphisms making (3) commutative. Let $\lambda, \mu : F \longrightarrow F'$ be the morphisms induced by f_0, f_1 and g_0, g_1 respectively. Now $\varepsilon'(f_0 - g_0) = 0$, and so again using projectivity of P_0 and exactness of the bottom row we obtain a morphism $h : P_0 \longrightarrow P'_1$ such that $d'h = f_0 - g_0$. Then

$$\begin{aligned}(\lambda - \mu)p &= \lambda p - \mu p = p'T(f_0) - p'T(g_0)\\ &= p'T(f_0 - g_0) = p'T(d'h)\\ &= p'T(d')T(h) = 0\end{aligned}$$

Therefore, since p is an epimorphism, $\lambda = \mu$. □

Lemma 7. *Let $\mathcal{A}$ be a cocomplete abelian category and suppose we have an exact sequence $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$. If there are projective objects F'_1, F'_0, F''_1, F''_0 and exact sequences*

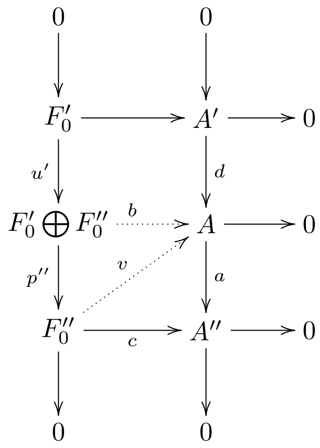
$$\begin{aligned}F'_1 \longrightarrow F'_0 \longrightarrow A' \longrightarrow 0\\ F''_1 \longrightarrow F''_0 \longrightarrow A'' \longrightarrow 0\end{aligned}$$

Then these three exact sequences fit into a commutative diagram in which all columns and rows are exact, and the left and middle columns are the canonical split exact sequences:

$$\begin{array}{ccccccc}
0 & & 0 & & 0 & & \\
\downarrow & & \downarrow & & \downarrow & & \\
F'_1 & \longrightarrow & F'_0 & \longrightarrow & A' & \longrightarrow & 0\\
\downarrow & & \downarrow & & \downarrow & & \\
F'_1 \bigoplus F''_1 & \longrightarrow & F'_0 \bigoplus F''_0 & \longrightarrow & A & \longrightarrow & 0\\
\downarrow & & \downarrow & & \downarrow & & \\
F''_1 & \longrightarrow & F''_0 & \longrightarrow & A'' & \longrightarrow & 0\\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & &
\end{array}$$

Proof. We start by considering the following diagram:

(4)

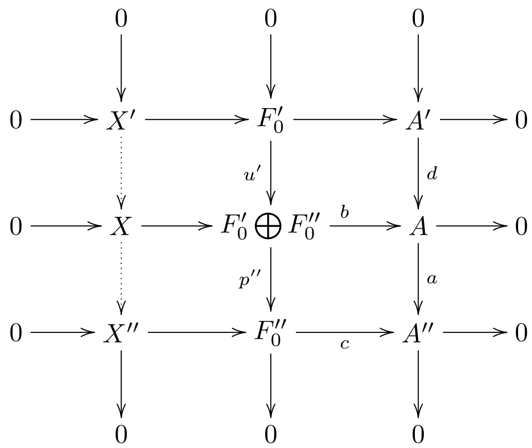


The left column is a split exact sequence with injections u', u'' and projections p', p''. Induce $v : F_0'' \longrightarrow A$ by projectivity of F_0'' and the fact that $A \longrightarrow A''$ is an epimorphism. Hence $av = c$. Let b be induced by the coproduct and the morphisms v and $F_0' \longrightarrow A' \longrightarrow A$. We claim that both squares commute.

The top square commutes by definition of b. For the bottom, notice that $abu' = F_0' \longrightarrow A' \longrightarrow A \longrightarrow A'' = 0 = cp''u'$, and $abu'' = av = c = cp''u''$. Hence $ab = cp''$. To see that b is an epimorphism, let $\alpha : A \longrightarrow X$ be its cokernel. Then $\alpha d = 0$, so there is $q : A' \longrightarrow X$ such that $qa = \alpha$. Since $\alpha v = 0$, we have $0 = qav = qc$ and hence $q = 0$. But $X = 0$ since q is an epimorphism, so b must be an epimorphism.

Now take kernels of the diagram (4):

(5)



The dashed morphisms are induced in the usual fashion, and form an exact sequence by ([6], 13.5.3). The sequences $F_1' \longrightarrow F_0' \longrightarrow A' \longrightarrow 0$ and $F_1'' \longrightarrow F_0'' \longrightarrow A'' \longrightarrow 0$ are exact, so there are epimorphisms $F_1' \longrightarrow X', F_1'' \longrightarrow X''$ and we can form a commutative diagram similar to (4) with F_0, F_0' replaced by F_1, F_1' and b replaced by $b' : F_1' \bigoplus F_1'' \longrightarrow X$. Finally, we have the commutative diagram

$$
\begin{array}{ccccccc}
0 & & 0 & & 0 & & \\
\downarrow & & \downarrow & & \downarrow & & \\
F_1' & \longrightarrow & F_0' & \longrightarrow & A' & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
F_1' \bigoplus F_1'' & \xrightarrow{d} & F_0' \bigoplus F_0'' & \longrightarrow & A & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
F_1'' & \longrightarrow & F_0'' & \longrightarrow & A'' & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & &
\end{array}
$$

Here d is the composite $F_1' \bigoplus F_1'' \longrightarrow X \longrightarrow F_0' \bigoplus F_0''$ and all columns and all rows are exact, as required. □

Lemma 8. *If $\mathcal{A} \cong \mathcal{B}$ are equivalent categories and $\mathcal{A}$ is abelian, so is $\mathcal{B}$, with both involved functors being additive.*

Proof. Suppose we have an equivalence of categories, $F : \mathcal{A} \longrightarrow \mathcal{B}$, with $G : \mathcal{B} \longrightarrow \mathcal{A}$ such that $FG \cong 1$ and $GF \cong 1$. Recall that F is both left and right adjoint to G, and that both functors are by definition fully faithful. It is easy to check that if $\mathcal{A}$ has has limits (colimits) for some diagram, so does $\mathcal{B}$. If $\mathcal{A}$ is normal or conormal, so is $\mathcal{B}$. If $\alpha \in \mathcal{B}$, and $G(\alpha)$ has an epi-mono factorsation in $\mathcal{A}$, then we can carry this back into an epi-mono factorisation of α. If $\mathcal{A}$ is abelian, then it has biproducts $A \oplus A$ for each A. Since F preserves limits and colimits, $\mathcal{B}$ has biproducts and is hence additive, since the above comments show that it is exact. With this structure, both F and G become additive functors ([3], I, 18.3). □

To clarify some notation used in the next result, if $\mathcal{P}$ is a subcategory of $\mathcal{A}$ and $T : \mathcal{P} \longrightarrow \mathcal{B}$ and $F : \mathcal{A} \longrightarrow \mathcal{B}$ are functors, then F *extends* T if the restriction of F to $\mathcal{P}$ is *equal* to T, not just naturally equivalent to T.

Theorem 9. *Let $\mathcal{P}$ be a full subcategory of a cocomplete abelian category $\mathcal{A}$, and suppose that the objects of $\mathcal{P}$ form a generating set of small projectives for $\mathcal{A}$. Let $T : \mathcal{P} \longrightarrow \mathcal{B}$ be an additive functor into a cocomplete abelian category. Then T can be extended uniquely (up to natural equivalence) to an additive colimit preserving functor $T'' : \mathcal{A} \longrightarrow \mathcal{B}$.*

Proof. An object of $\mathcal{A}$ is *free* if it is a coproduct in $\mathcal{A}$ of a set of objects in $\mathcal{P}$. First we extend T to a functor T' on the subcategory of $\mathcal{A}$ consisting of the free objects. Then we use presentations to extend this definition to all of $\mathcal{A}$. The main idea is simple but there are several tricky technical points.

Let F be a free object of $\mathcal{A}$: so there is a set $\{P_i\}_{i \in I}$ of objects in $\mathcal{P}$ and morphisms $u_i : P_i \longrightarrow F$ making F into a coproduct. We define

$$T'(F) = \bigoplus_{i \in I} T(P_i) \tag{6}$$

It is possible that F has the structure of a coproduct over several subsets of $\mathcal{P}$, so in order to define T' we use a strong axiom of choice to associate with every free object a specific subset of $\mathcal{P}$ and specific coproduct morphisms, with the understanding that the singleton $\{P\}$ is associated with $P \in \mathcal{P}$ and the empty set is associated with any initial object (which are coproducts over empty diagrams). Given F and the associated subset $\{P_i\}$ we then choose $T'(F)$ to be a specific coproduct over the objects $T(P_i)$ in $\mathcal{B}$.

Later we need to show that T' preserves coproducts. Suppose F is free and is associated with a subset $\{P_i\}_{i\in I}$ of $\mathcal{P}$. It is possible that there is another subset $\{Q_j\}_{j\in J}$ of $\mathcal{P}$ and morphisms $v_j : Q_j \longrightarrow F$ making F into a coproduct. If there is a free object F' whose associated set is $\{Q_j\}_{j\in J}$ then $F \cong F'$ and it is easy enough to check that the morphisms $T'(v_j) : T(Q_j) \longrightarrow T(F)$ form a coproduct (once we have defined T' on morphisms). In order to use this technique in showing that T' preserves coproducts over objects in $\mathcal{P}$, we reduce to the case where $\mathcal{A}$ has sufficiently many copies of every free object (this is already true if $\mathcal{A}$ is a category of modules over a ringoid).

We introduce a new category $\mathcal{A}'$. The objects of $\mathcal{A}'$ are the objects of $\mathcal{A}$ together with a distinct new object for each subset of $\mathcal{P}$ (including the empty set). If A, B are two objects of $\mathcal{A}$ then $Hom_{\mathcal{A}'}(A,B) = Hom_{\mathcal{A}}(A,B)$. Let $\{P_i\}_{i\in I}$ be a subset of $\mathcal{P}$ (possibly empty) and let F denote the corresponding new object of $\mathcal{A}'$. We select a specific coproduct $\bigoplus_{i\in I} P_i$ and define for each object C of $\mathcal{A}$:

$$Hom_{\mathcal{A}'}(F,C) = Hom_{\mathcal{A}}\left(\bigoplus_{i\in I} P, C\right)$$

$$Hom_{\mathcal{A}'}(C,F) = Hom_{\mathcal{A}}\left(C, \bigoplus_{i\in I} P\right)$$

For another new object G corresponding to a subset $\{Q_j\}_{j\in J}$, we define

$$Hom_{\mathcal{A}'}(F,G) = Hom_{\mathcal{A}}\left(\bigoplus_{i\in I} P_i, \bigoplus_{j\in J} Q_j\right)$$

$$Hom_{\mathcal{A}'}(G,F) = Hom_{\mathcal{A}}\left(\bigoplus_{j\in J} Q_j, \bigoplus_{i\in I} P_i\right)$$

Composition is defined in the obvious way. In the category $\mathcal{A}'$ each new object F is isomorphic to its selected coproduct $\bigoplus_{i\in I} P_i$. It follows that the canonical inclusion of $\mathcal{A}$ in $\mathcal{A}'$ is an equivalence, so $\mathcal{A}'$ is a cocomplete abelian category. The objects of $\mathcal{P}$ form a generating family of small projectives in $\mathcal{A}'$, and to prove the theorem for $\mathcal{A}$ it suffices to prove it for $\mathcal{A}'$. Hence we can reduce to the case where it is possible to assign subsets of $\mathcal{P}$ to free objects in such a way that every subset is assigned to some free object.

With the notation refreshed, let $\mathcal{F}$ denote the full subcategory of free objects in $\mathcal{A}$, and define the functor $T' : \mathcal{F} \longrightarrow \mathcal{B}$ on objects as above. In our choices of coproducts in $\mathcal{B}$, we can arrange for T' to agree with T on objects and for T' to map any initial object to a specific zero object 0 in $\mathcal{B}$.

To define T' on morphisms, write free objects as coproducts over the selected index sets and consider a morphism α of $\mathcal{F}$:

$$\alpha : \bigoplus_{i\in I} P_i \longrightarrow \bigoplus_{j\in J} Q_j$$

Let α_i denote the composition of α with the ith injection into $\bigoplus_{i\in I} P_i$. By ([3] II, 16.1) since P_i is small we can write $\alpha_i = \sum_{j\in J_i} u_j p_j \alpha_i$ where u_j and p_j are the jth injection and projection for the coproduct $\bigoplus_{j\in J} Q_j$, and J_i is the set of all $j \in J$ with $p_j\alpha_i \neq 0$. We define $T'(\alpha)$ to be the morphism $\bigoplus_{i\in I} T(P_i) \longrightarrow \bigoplus_{j\in J} T(Q_j)$ which gives $\sum_{j\in J_i} u'_j T(p_j\alpha_i)$ when composed with the ith injection into $\bigoplus_{i\in I} T(Q_i)$. Here u'_j is the jth injection into the coproduct $\bigoplus_{j\in J} T(Q_j)$. In a more compact form

$$T'(\alpha) = T'((\alpha_i)_i) = \left(\sum_{j\in J_i} u'_j T(p_j\alpha_i)\right)_i$$

With the convention that summing over the empty set gives zero, the above gives $T'(0) = 0$ for any zero morphism of $\mathcal{F}$. If $\{F_i\}_{i\in I}$ is a set of free objects and the morphisms $u_i : F_i \longrightarrow C$ are a coproduct in $\mathcal{A}$, then clearly C is free and since every subset of $\mathcal{P}$ is associated with some free object it is not difficult to check that the morphisms $T'(u_i) : T'(F_i) \longrightarrow T'(C)$ are a coproduct in $\mathcal{B}$.

Next we check that T' is an additive functor. It is easy to see that $T'(1_A) = 1_{T'(A)}$, and checking $T'(\alpha\beta) = T'(\alpha)T'(\beta)$ is straightforward. It is also worth noticing that, in the above notation, $T'(u_j)$ is the injection u'_j into the coproduct in $\mathcal{B}$.

To verify additivity, let $\alpha, \beta : \bigoplus_{i\in I} P_i \longrightarrow \bigoplus_{j\in J} Q_j$ be two morphisms of $\mathcal{F}$. Let s_i, t_i and u_j, p_j denote the injections and projections for the two coproducts, respectively, and s'_i, t'_i, u'_j, p'_j the corresponding morphisms for the coproducts in $\mathcal{B}$. It would suffice to show that for each $i \in I$

$$T'(\alpha+\beta)s'_i = \left(T'(\alpha) + T'(\beta)\right) s'_i \tag{7}$$

Fix $i \in I$ and suppose that

$$\alpha_i = \alpha s_i = \sum_{j\in J_\alpha} u_j p_j \alpha_i, \qquad \beta_i = \beta s_i = \sum_{j\in J_\beta} u_j p_j \beta_i$$

and

$$(\alpha+\beta)_i = \sum_{j\in J_{\alpha+\beta}} u_j p_j (\alpha+\beta)_i$$

Clearly $(\alpha+\beta)_i = \alpha_i + \beta_i$ It is not difficult to see that $J_{\alpha+\beta} = \{j \in J \mid p_j\alpha_i \neq -p_j\beta_i\}$. Hence $J_{\alpha+\beta} \subseteq J_\alpha \bigcup J_\beta$, and by definition

$$\sum_{j\in J_\alpha \bigcup J_\beta} u_j p_j(\alpha_i + \beta_i) = \sum_{j\in J_{\alpha+\beta}} u_j p_j(\alpha_i+\beta_i)$$

since all the other summands are zero. For $k \in (J_\alpha \bigcup J_\beta) \setminus J_{\alpha+\beta}$ we have $p_k\alpha_i = -p_k\beta_i$. Hence $T(p_k(\alpha_i+\beta_i)) = T(0) = 0$. It follows that

$$\begin{aligned}
\left(T'(\alpha) + T'(\beta)\right) s'_i &= \sum_{j\in J_\alpha} u'_j T(p_j\alpha_i) + \sum_{j\in J_\beta} u'_j T(p_j\beta_i) \\
&= \sum_{j\in J_\alpha \bigcup J_\beta} u'_j T(p_j\alpha_i) + \sum_{j\in J_\alpha \bigcup J_\beta} u'_j T(p_j\beta_i) \\
&= \sum_{j\in J_\alpha \bigcup J_\beta} u'_j T(p_j\{\alpha_i+\beta_i\}) \\
&= \sum_{j\in J_{\alpha+\beta}} u'_j T(p_j\{\alpha+\beta\}_i) = T'(\alpha+\beta)s'_i
\end{aligned}$$

This completes the proof that T' is an additive functor.

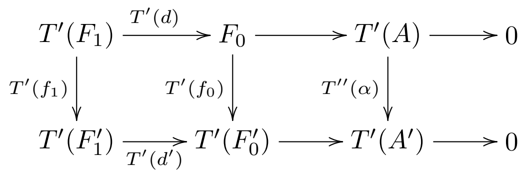
Next we extend T' to a functor $T'' : \mathcal{A} \longrightarrow \mathcal{B}$, by taking presentations. For any A in $\mathcal{A}$, there are objects $F_1, F_0 \in \mathcal{F}$ and an exact sequence

$$F_1 \xrightarrow{\ d\ } F_0 \longrightarrow A \longrightarrow 0 \tag{8}$$

Select a presentation for each object of $\mathcal{A}$, with the convention that for a free object F the selected presentation is $0 \longrightarrow F \longrightarrow F \longrightarrow 0$. Define the functor $T'' : \mathcal{A} \longrightarrow \mathcal{B}$ as follows: For an object A, choose a cokernel of $T'(d)$ and set $T''(A)$ equal to this object. Let $\alpha : A \longrightarrow A'$ be a morphism of $\mathcal{A}$, and let $F'_1 \longrightarrow F'_0 \longrightarrow A' \longrightarrow 0$ be the selected presentation for A'. Then by Lemma 6 we can find morphisms f_1 and f_0 making the following diagram commutative:

$$\begin{array}{ccccccc}
F_1 & \xrightarrow{\ d\ } & F_0 & \longrightarrow & A & \longrightarrow & 0 \\
\downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle \alpha} & & \\
F'_1 & \xrightarrow[\ d'\ ]{} & F'_0 & \longrightarrow & A' & \longrightarrow & 0
\end{array}$$

Define $T''(\alpha)$ to be the morphism making the following diagram commute:

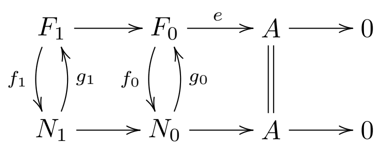


By Lemma 6 the morphism $T''(\alpha)$ is independent of the choice of f_0 and f_1.

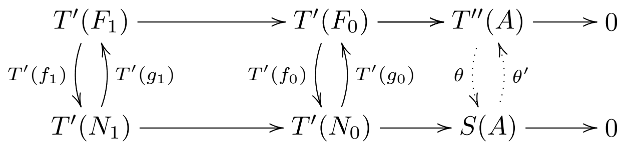
For $\alpha = 1_A : A \longrightarrow A$, choosing $f_1 = 1_{F_1}$ and $f_0 = 1_{F_0}$ gives $T'(1_A) = 1_{T'(A)}$. It is similarly obvious that $T''(\alpha\beta) = T''(\alpha)T''(\beta)$. By choosing our cokernels properly, we can arrange for T'' to extend T' on both the objects and morphisms of $\mathcal{F}$. Since T'' is easily seen to be additive, it only remains to show that T'' preserves colimits and is unique. First we show that (up to natural equivalence) T'' is independent of the choices of presentations and cokernels made above.

Suppose that for each object of $\mathcal{A}$ we have a second presentation, and that we use these presentations to define a functor $S : \mathcal{A} \longrightarrow \mathcal{B}$ (we allow arbitrary presentations for the free objects, so S may not agree with T'). We claim that S is naturally equivalent to T''.

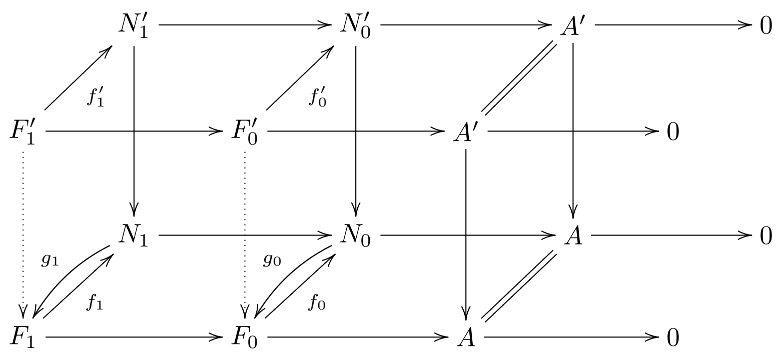
For $A \in \mathcal{A}$, let the presentation used in the definition of T'' be $F_1 \longrightarrow F_0 \longrightarrow A \longrightarrow 0$ and let $N_1 \longrightarrow N_0 \longrightarrow A \longrightarrow 0$ be the new presentation. Using Lemma 6 we produce a commutative diagram



Applying T' to the left-hand square and taking cokernels gives a commutative diagram



In defining $1_{T''(A)}$ and $1_{S(A)}$ we could have used g_0f_0, g_1f_1 and f_0g_0, f_1g_1 respectively, so it follows from Lemma 6 that $\theta\theta' = 1$ and $\theta'\theta = 1$. To show these isomorphisms are natural, let a morphism $\alpha : A' \longrightarrow A$ be given and let $F_1' \longrightarrow F_0' \longrightarrow A' \longrightarrow 0$ and $N_1' \longrightarrow N_0' \longrightarrow A' \longrightarrow 0$ be the respective presentations of A'. Consider the diagram



The solid morphisms are chosen so that all solid squares commute. In defining the morphism $T''(\alpha)$ we need to select morphisms $F_0' \longrightarrow F_0$ and $F_1' \longrightarrow F_1$ making two of these squares commute, and we may choose these morphisms to be $F_i' \longrightarrow N_i' \longrightarrow N_i \longrightarrow F_i$ for $i = 0, 1$. When we map the left-hand cube into $\mathcal{B}$ using T', take cokernels, and induce the isomorphisms $\theta_A : T''(A) \longrightarrow S(A)$ and $\theta_{A'} : T''(A') \longrightarrow S(A')$ it is not to difficult to check that $S(\alpha)\theta_{A'} = \theta_A T''(\alpha)$. Hence T'' is naturally equivalent to S.

Next we show that T'' preserves cokernels. Let $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ be an exact sequence in $\mathcal{A}$ and let $F_1' \longrightarrow F_0' \longrightarrow A' \longrightarrow 0$ and $F_1'' \longrightarrow F_0'' \longrightarrow A \longrightarrow 0$ be the presentations used in the definition of T''. Apply the functor T'' to the two left hand columns of the commutative diagram produced by Lemma 7. Since T'' is additive these columns remain split exact in $\mathcal{B}$.

If we consider "taking cokernels" as a functor defined on diagrams of the form $B \longleftarrow A \longrightarrow 0$, morphisms between which are just commutative squares, then a sequence of such diagrams is exact iff. it is pointwise exact. Hence we have a short exact sequence of such diagrams in $\mathcal{B}$. Applying the cokernel functor, which is right exact, we obtain an exact sequence

$$T''(A') \longrightarrow S(A) \longrightarrow T''(A'') \longrightarrow 0$$

Where S is the functor defined by choosing the same presentations as T'' for every object except A, for which $F_1' \oplus F_1'' \longrightarrow F_0' \oplus F_0'' \longrightarrow A \longrightarrow 0$ is the selected presentation. Since T'' is naturally equivalent to S it follows that the sequence $T''(A') \longrightarrow T''(A) \longrightarrow T''(A'') \longrightarrow 0$ is exact, and T'' preserves cokernels.

To show that T'' preserves colimits, it suffices to show that T'' preserves coproducts. Given a family $\{A_i\}_{i \in I}$ of objects in $\mathcal{A}$, let the chosen presentation for each $i \in I$ be

$$F_{1,i} \xrightarrow{d_i} F_{0,i} \xrightarrow{e_i} A_i \longrightarrow 0$$

Using the fact that T'' preserves coproducts of free objects (since T'' agrees with T' on $\mathcal{F}$, and T' preserves such coproducts) and the fact that coproducts are cokernel preserving ([3], II 12.2), we produce an exact sequence

$$T''(\bigoplus F_{1,i}) \xrightarrow{T''(\bigoplus d_i)} T''(\bigoplus F_{0,i}) \longrightarrow \bigoplus T''(A_i) \longrightarrow 0$$

But $\bigoplus F_{1,i} \longrightarrow \bigoplus F_{0,i} \longrightarrow \bigoplus A_i \longrightarrow 0$ is exact in $\mathcal{A}$, so we can define a functor S using this presentation for $\bigoplus A_i$. Since T'' is naturally equivalent to S it follows that T'' maps the coproduct $A_i \longrightarrow \bigoplus A_i$ to a coproduct in $\mathcal{B}$.

We have produced an additive, colimit preserving extension $T'' : \mathcal{A} \longrightarrow \mathcal{B}$ of the original functor $T : \mathcal{P} \longrightarrow \mathcal{B}$. Let $S : \mathcal{A} \longrightarrow \mathcal{B}$ be any colimit preserving functor extending T. Any colimit preserving functor between abelian categories is additive, and it is not difficult to check that the restriction of S to $\mathcal{F}$ is naturally equivalent to T'. It is now easy to adapt earlier arguments to show that S is naturally equivalent to T''.

□

The uniqueness part of the previous Theorem shows that if S is colimit preserving and restricts to give T then S is naturally equivalent to T''. This conclusion is also true if we only assume that S is naturally equivalent to T on $\mathcal{P}$, but the proof is subtler and requires more conditions on the category $\mathcal{A}$.

Theorem 10. *Le $\mathcal{A}$ be a grothendieck abelian category with a full subcategory $\mathcal{P}$ whose objects form a generating set of projectives for $\mathcal{A}$. Let $T, S : \mathcal{A} \longrightarrow \mathcal{B}$ be additive, colimit preserving functors where $\mathcal{B}$ is an abelian category. Denote the restrictions of T and S to $\mathcal{P}$ by $T|_{\mathcal{P}}$ and $S|_{\mathcal{P}}$. Then any natural transformation $\varphi : T|_{\mathcal{P}} \longrightarrow S|_{\mathcal{P}}$ can be extended uniquely to a natural transformation $\widetilde{\varphi} : T \longrightarrow S$, such that if φ is a natural equivalence, so is $\widetilde{\varphi}$.*

Proof. See ([3], IV, 5.4). □

In the case where $\mathcal{A}$ and $\mathcal{B}$ are module categories, Theorem 9 simply defines the tensor product. To be precise, if R, S are rings and $\mathcal{A} = \mathbf{Mod}R$ and $\mathcal{B} = \mathbf{Mod}S$ then giving a right S-module C the structure of an R-S-bimodule is equivalent to giving an additive covariant functor $R \longrightarrow \mathbf{Mod}S$. If we denote this functor by the same symbol C, then it is defined by $C(r)(x) = x \cdot r$ and $C(R) = C$.

Identifying R as a full subcategory of $\mathbf{Mod}R$ (the objects of which clearly form a generating family of small projectives) we are in a position to apply Theorem 9. This produces a unique, additive, colimit preserving functor $Q : \mathbf{Mod}R \longrightarrow \mathbf{Mod}S$ which extends T. Examining the proof of Theorem 9 we see that in this case the construction is just the one given prior to Proposition

5. Hence $Q = - \otimes_R C$. Alternatively, tensoring with C is already known to be additive, colimit preserving and agrees with C on the subcategory R, so the uniqueness part of Theorem 9 will imply $Q = - \otimes_R C$.

Example 2. As an amusing aside, let us examine what Q produces when it is applied to the morphisms between free objects in **Mod**R. Consider a ring R as a right module over itself. We have already noticed that the endomorphism ring of the n-fold biproduct R^n in **Mod**R is the matrix ring $M_n(R)$. If $m, n \geq 1$ are integers, then the abelian group $Hom(R^m, R^n)$ can be naturally identified with the abelian group of all $n \times m$ matrices over R. Rows correspond to projections, and columns to injections. If $\alpha : R^m \longrightarrow R^n$ then we denote the composition of α with the ith injection by $\alpha_i : R \longrightarrow R^n$, and think of α_i as the ith column of the matrix α. Similarly the composition of α with the jth projection is the jth row of α. We write α_{ji} for the element of R obtained by composing α with the ith injection and jth projection (so it is the element of the matrix at position (j, i)).

In the above notation, an $n \times m$ matrix over R takes the form of a morphism:

$$\alpha : R^m \longrightarrow R^n$$

So $Q(\alpha)$ is a morphism $C^m \longrightarrow C^n$ in **Mod**S. We write elements of C^m as column vectors with m rows, and we claim that

$$Q(\alpha)\begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1m} \\ \alpha_{21} & & & \vdots \\ \vdots & & & \vdots \\ \alpha_{n1} & \dots\dots\dots & & \alpha_{nm} \end{pmatrix}\begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}$$

where we interpret the matrix product in the obvious way.

For $1 \leq j \leq n$, let $u'_j : C \longrightarrow C^n$ be the jth injection into C^n, and $p_j : R^n \longrightarrow R$ the jth projection from R^n. Considering the proof of Theorem 9, the composition of $Q(\alpha)$ with the ith injection into C^m is given by the sum

$$\sum_{j=1}^{n} u'_j T(p_j \alpha_i) = \sum_{j=1}^{n} u'_j T(\alpha_{ji})$$

and so for $x \in C$,

$$Q(\alpha)_i(x) = \sum_{j=1}^{n} u'_j(x \cdot \alpha_{ji}) = \begin{pmatrix} x \cdot \alpha_{1i} \\ x \cdot \alpha_{2i} \\ \vdots \\ x \cdot \alpha_{ni} \end{pmatrix}$$

but by definition of the coproduct,

$$Q(\alpha)\begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = \sum_{i=1}^{m} Q(\alpha)_i(x_i) = \begin{pmatrix} \sum_i x_i \cdot \alpha_{1i} \\ \vdots \\ \sum_i x_i \cdot \alpha_{ni} \end{pmatrix}$$

which is precisely how we define the action of the matrix α.

We now define the tensor product of right R-modules with left R-objects in a cocomplete abelian category. Recall that for a left R-module N, tensoring with N defines a functor $- \otimes_R N$: **Mod**$R \longrightarrow$ **Ab**. This functor has a right adjoint $Hom(N, -)$: **Ab** $\longrightarrow$ **Mod**R which for any abelian group A gives the morphism set $Hom(N, A)$ its canonical structure as a right R-module.

A left R-object in a cocomplete abelian category $\mathcal{B}$ is a covariant additive functor $C : R \longrightarrow \mathcal{B}$ (we tend to name the functor by the object of $\mathcal{B}$ that R gets mapped to). By Theorem 9 the functor C extends to an additive colimit preserving functor $-\otimes_R C : \mathbf{Mod}R \longrightarrow \mathcal{B}$. This tensor product has a right adjoint, given by the functor $H^C : \mathcal{B} \longrightarrow \mathbf{Mod}R$

$$H^C(B) = Hom_{\mathcal{B}}(C, B)$$

The morphism set $Hom_{\mathcal{B}}(C, B)$ is an R-module, where the action of $r \in R$ on a morphism $\phi : B \longrightarrow C$ is given by composing on the right by the image of r under the functor C. The adjunction $-\otimes_R C \dashv H^C$ is established in the next theorem, whose proof is based on the one given in [3], Chapter IV §3.

Theorem 11. *Let $\mathcal{B}$ be a cocomplete abelian category and R a ring. If C is a left R-object in $\mathcal{B}$, then the functor $H^C : \mathcal{B} \longrightarrow \mathbf{Mod}R$ has a left adjoint, which we denote by $-\otimes_A C : \mathbf{Mod}R \longrightarrow \mathcal{B}$.*

Proof. A left R-object C is just an additive functor T into $\mathcal{B}$ from the full subcategory of $\mathbf{Mod}R$ consisting of the single object R. Applying Theorem 9, we get a unique, additive colimit preserving functor $\mathbf{Mod}R \longrightarrow \mathcal{B}$ which extends T, whose value on the right R-module M we denote by $M \otimes_R C$. In the discussion following the proof of Theorem 9, we observed that in the case where $\mathcal{B} = \mathbf{Mod}S$ and C is an R-S-bimodule this functor is $-\otimes_R C$, and in that case we already know that $-\otimes_R C \dashv H^C$. It now remains to prove that this adjunction holds in the general case.

Explicitly, for a right R-module M, the object $M \otimes_R C$ is defined as follows: take an exact sequence of R-modules

$${}^K R \xrightarrow{\lambda} {}^M R \xrightarrow{\pi} M \longrightarrow 0$$

where the mth coordinate of π is the morphism $R \longrightarrow M$ corresponding to $m \in M$, K is the set of elements of ${}^M R$ in the kernel of π, and λ has kth coordinate $R \longrightarrow {}^M R$ corresponding to this element $k \in {}^M R$. Then $M \otimes_R C$ is the cokernel of the induced morphism ${}^K C \longrightarrow {}^M C$ in $\mathcal{B}$. If $\mu : M \longrightarrow M'$ is a morphism of modules then we have the morphism ${}^M R \longrightarrow {}^{M'} R$ which is such that composition with the mth injection into ${}^M R$ gives the $\mu(m)$th injection into ${}^{M'} R$. The morphism $\mu \otimes_R C$ is then unique making the diagram

$$\begin{array}{ccc} {}^M C & \longrightarrow & M \otimes_R C \\ \downarrow & & \downarrow{\scriptstyle \mu\otimes_R C} \\ {}^{M'} C & \longrightarrow & M' \otimes_R C \end{array}$$

commute. We will continue using the notation introduced in the previous section, so that the element of ${}^M R$ which is zero except for an r in the mth place will be denoted by (m, r).

We now define an isomorphism

$$\theta_{M,A} : Hom_R(M, Hom_{\mathcal{B}}(C, A)) \longrightarrow Hom_{\mathcal{B}}(M \otimes_R C, A)$$

natural in M and A, which will complete the proof of the Theorem.

A right R-module morphism $f : M \longrightarrow Hom_{\mathcal{B}}(C, A)$ determines a morphism $\widehat{\theta}(f) : {}^M C \longrightarrow A$ whose mth coordinate is $f(m)$, and the composite ${}^K C \longrightarrow {}^M C \longrightarrow A$ is zero, since if we compose $\widehat{\theta}(f)$ with the kth coordinate of $\lambda \otimes_R C$, and use the definition of the functor $-\otimes_R C$ given in Theorem 9, we have (denoting the mth injection and projection from ${}^M C$ and ${}^M R$ resp. by u'_m, p_m)

$$\begin{aligned} \widehat{\theta}(f)(\lambda \otimes_R C)_k &= \widehat{\theta}(f) \sum_{m \in S_k} u'_m T(p_m \lambda_k) \\ &= \sum_m f(m) T(k_m) = f(\sum_m m \cdot k_m) = 0 \end{aligned}$$

hence we have an induced morphism $\theta(f) : M \otimes_R C \longrightarrow A$.

On the other hand, given $g : M \otimes_R C \longrightarrow A$ we can compose g with ${}^M C \longrightarrow M \otimes_R C$ to get a morphism $\widehat{g} : {}^M C \longrightarrow A$. Then $\widehat{g}$ determines the function

$$\theta'(g) : M \longrightarrow Hom_{\mathcal{B}}(C, A)$$

such that $\theta'(g)(m)$ is the mth coordinate of $\widehat{g}$. We have to show that $\theta'(g)$ is a morphism of right R-modules. Firstly, note that for $m \in M$ the element $(m, 1) \in {}^M R$, considered as a morphism $R \longrightarrow {}^M R$, is precisely the mth injection into the coproduct. Hence $(m, 1) \otimes_R C : C \longrightarrow {}^M C$ is the mth injection u'_m into the coproduct ${}^M C$. Also, $(m, r) \in {}^M R$ is just the mth injection $R \longrightarrow {}^M R$ preceeded by the endomorphism $r : R \longrightarrow R$. Hence

$$(m, r) \otimes_R C = ((m, 1) \otimes_R C)T(r)$$

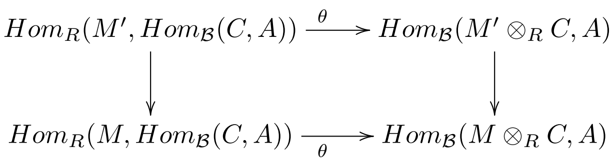
and since $(m, r) - (mr, 1) \in K \subseteq {}^M R$, we have

$$\widehat{g}((m, r) \otimes_R C - (mr, 1) \otimes_R C) = 0$$

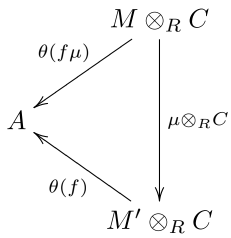
so finally,

$$\begin{aligned} \theta'(g)(m) \cdot r = \theta'(g)(m)T(r) &= \widehat{g}u'_m T(r) \\ &= \widehat{g}((m, 1) \otimes_R C)T(r) = \widehat{g}((m, r) \otimes_R C) \\ &= \widehat{g}u'_{m \cdot r} = \theta'(g)(m \cdot r) \end{aligned}$$

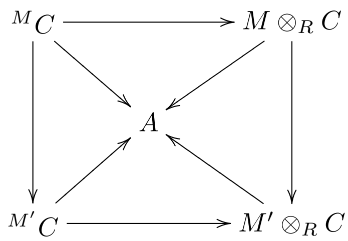
It is easy to see that θ and θ' are mutually inverse, so it only remains to show that θ is natural in M and A. If $\mu : M \longrightarrow M'$ is a morphism of right R-modules, we have to show that the diagram



commutes. This amounts to showing that for an R-module morphism $f : M' \longrightarrow Hom_{\mathcal{B}}(C, A)$ the diagram



commutes, which follows from the fact that in the diagram



The square and each triangle, save possibly the righthand one, commute. Since ${}^M C \longrightarrow M \otimes_R C$ is an epimorphism it follows that this triangle commutes also, which is what we wanted. This establishes naturality of θ in M, and naturality in A is easy, so this concludes the proof of the Theorem. □

We have now defined the tensor product $M \otimes_R C$ of a right R-module M and a left R-object C. The next task is to replace modules over a ring by modules over a ringoid.

Let $\mathcal{A}$ be a ringoid, and $\mathcal{B}$ a cocomplete abelian category. A left $\mathcal{A}$-object in $\mathcal{B}$ is an additive functor $Q : \mathcal{A} \longrightarrow \mathcal{B}$,and just as in the single object case, we can define a functor $H^Q : \mathcal{B} \longrightarrow$

Mod$\mathcal{A}$ which corresponds to "morphisms out of Q". Of course, the data encapsulated in Q now consists of a whole famliy of objects $\{Q(A)\}_{A\in\mathcal{A}}$ in $\mathcal{B}$, together with various morphisms between them. An object $B \in \mathcal{B}$ then determines a family of morphism sets, $Hom_{\mathcal{B}}(Q(A), B)$, and this family is really a right $\mathcal{A}$-module, which we call $H^Q(B)$:

$$H^Q(B) : \mathcal{A}^{\text{op}} \longrightarrow \mathbf{Ab}, \quad H^Q(B)(A) = Hom_{\mathcal{B}}(Q(A), B)$$

For a morphism $\alpha : A \longrightarrow A'$ of $\mathcal{A}$, the action on the module is defined by mapping to $\mathcal{B}$ with Q and then composing:

$$\begin{aligned} H^Q(B)(\alpha) : Hom_{\mathcal{B}}(Q(A'), B) &\longrightarrow Hom_{\mathcal{B}}(Q(A), B) \\ \psi \cdot \alpha &= \psi Q(\alpha) \end{aligned}$$

This defines the right $\mathcal{A}$-module $H^Q(B)$. We define H^Q on morphisms by composition, so for $\phi : B \longrightarrow B'$ in $\mathcal{B}$,

$$\begin{aligned} H^Q(\phi) : H^Q(B) &\longrightarrow H^Q(B') \\ H^Q(\phi)_A : Hom_{\mathcal{B}}(Q(A), B) &\longrightarrow Hom_{\mathcal{B}}(Q(A), B') \\ \psi &\mapsto \phi\psi \end{aligned}$$

As before, we apply Theorem 9 to the additive functor Q, and obtain a unique additive, colimit preserving functor $- \otimes_{\mathcal{A}} Q : \mathbf{Mod}\mathcal{A} \longrightarrow \mathcal{B}$. We now show that H^Q is right adjoint to $- \otimes_{\mathcal{A}} Q$.

Theorem 12. *Let $\mathcal{B}$ be a cocomplete abelian category and $\mathcal{A}$ a ringoid. If Q is a left $\mathcal{A}$-object in $\mathcal{B}$, then the functor $H^Q : \mathcal{B} \longrightarrow \mathbf{Mod}\mathcal{A}$ has a left adjoint, which we denote by $-\otimes_{\mathcal{A}} Q : \mathbf{Mod}\mathcal{A} \longrightarrow \mathcal{B}$.*

Proof. Let $F \in \mathbf{Mod}\mathcal{A}$ be given. For each $A \in \mathcal{A}$ and $x \in F(A)$ there is a morphism $x : H_A \longrightarrow F$, and the collection of these morphisms induces $\pi : \bigoplus_{A\in\mathcal{A}}{}^{F(A)} H_A \longrightarrow F$. We label the injections $u_{A,x}$ and projections $p_{A,x}$ by the object A and the element $x \in F(A)$, so

$$\pi u_{A,x} : H_A \longrightarrow F$$

is the element x. For $E \in \mathcal{A}$ the component of π looks like

$$\bigoplus_A{}^{F(A)} Hom_{\mathcal{A}}(E, A) \xrightarrow{\pi_E} F(E)$$

An element on the left is a sequence $(f_{A,x})_{A\in\mathcal{A},x\in F(A)}$ with $f_{A,x} : E \longrightarrow A$ and only finitely many terms nonzero. Since $\pi u_{A,x}$ is x, when we compose π_E with the Eth component of the injection $u_{A,x}$ we get $Hom_{\mathcal{A}}(E, A) \longrightarrow F(E)$, $g \mapsto F(g)(x)$. Hence $\pi_E((f_{A,x})) = \sum F(f_{A,x})(x)$.

The tensor product $F \otimes_{\mathcal{A}} Q$ is determined by the process given in the proof of Theorem 9. We first take the kernel K of π, and induce $\lambda : \bigoplus_A{}^{K(A)} H_A \longrightarrow \bigoplus_A{}^{F(A)} H_A$ fitting into the exact sequence

$$\bigoplus_A{}^{K(A)} H_A \xrightarrow{\lambda} \bigoplus_A{}^{F(A)} H_A \xrightarrow{\pi} F \longrightarrow 0$$

We denote the injections into $\bigoplus_A{}^{K(A)} H_A$ by $v_{E,y}$. For $y \in K(E)$ let $(g_{A,x}) \in \bigoplus{}^{F(A)} Hom_{\mathcal{A}}(E, A)$ be the corresponding sequence. Then $g_{A,x} = p_{A,x}\lambda v_{E,y}$, and since $\pi_E(y) = 0$ we have

$$\sum_{A\in\mathcal{A},x\in F(A)} F(p_{A,x}\lambda v_{E,y})(x) = 0 \qquad \forall k \in K(E) \tag{9}$$

The construction is completed by extending Q to the free objects, then using this extension to map λ into $\mathcal{B}$:

$$\bigoplus_A{}^{K(A)} Q(A) \xrightarrow{\lambda\otimes Q} \bigoplus_A{}^{F(A)} Q(A)$$

and taking the cokernel.

If $\mu : F \longrightarrow F'$ is a morphism of $\mathcal{A}$-modules then there is a morphism

$$\theta : \bigoplus_A {}^{F(A)} H_A \longrightarrow \bigoplus_A {}^{F'(A)} H_A$$
$$\theta u_{A,x} = u_{A,\mu_A(x)}$$

Note that $\pi'\theta = \mu\pi$ (where π' denotes the morphism $\pi' : \bigoplus_A {}^{F'(A)} H_A \longrightarrow F'$). Then $\mu \otimes Q$ is the unique morphism in $\mathcal{B}$ making the following diagram commute:

$$\begin{array}{ccc} \bigoplus_A {}^{F(A)} Q(A) & \longrightarrow & F \otimes_{\mathcal{A}} Q \\ \downarrow{\scriptstyle \theta\otimes Q} & & \downarrow{\scriptstyle \mu\otimes Q} \\ \bigoplus_A {}^{F'(A)} Q(A) & \longrightarrow & F' \otimes_{\mathcal{A}} Q \end{array}$$

With these preliminaries out of the way, we define an isomorphism

$$\Phi_{F,A} : Hom_{\mathcal{A}}(F, H^Q(B)) \longrightarrow Hom_{\mathcal{B}}(F \otimes_{\mathcal{A}} Q, B)$$

which is natural in F and $B \in \mathcal{B}$. In what follows we refer to the injections and projections for the coproducts in $\mathcal{B}$ by putting hats $\widehat{u}_{A,x}$ above the corresponding morphism in $\mathbf{Mod}\mathcal{A}$.

A morphism $f : F \longrightarrow H^Q(B)$ in $\mathbf{Mod}\mathcal{A}$ consists of a natural collection $f_C : F(C) \longrightarrow Hom_{\mathcal{B}}(Q(C), B)$, $C \in \mathcal{A}$. We define $\dot{\Phi}(f) : \bigoplus_A {}^{F(A)} Q(A) \longrightarrow B$ by $\dot{\Phi}\widehat{u}_{A,x} = f_A(x)$. For $y \in K(E)$ we have the collection

$$J_{E,y} = \{(A,x) \,|\, p_{A,x}\lambda v_{E,y} \neq 0\}$$

Then by definition

$$\begin{aligned} \dot{\Phi}(\lambda \otimes Q)\widehat{v}_{E,y} &= \dot{\Phi}(f) \sum_{(A,x)\in J_{E,y}} \widehat{u}_{A,x} Q(p_{A,x}\lambda v_{E,y}) \\ &= \sum_{(A,x)\in J_{E,y}} f_A(x) Q(p_{A,x}\lambda v_{E,y}) \end{aligned}$$

But $f_A(x)Q(p_{A,x}\lambda v_{E,y}) = H^Q(B)(p_{A,x}\lambda v_{E,y})(f_A(x)) = f_E(F(p_{A,x}\lambda v_{E,y})(x))$, and hence by (9)

$$\dot{\Phi}(\lambda \otimes Q)\widehat{v}_{E,y} = f_E\left(\sum_{(A,x)\in J_{E,y}} F(p_{A,x}\lambda v_{E,y})(x)\right) = 0$$

By definition of cokernel this induces a unique morphism $\Phi(f) : F \otimes Q \longrightarrow B$.

In the other direction, let $g : F \otimes Q \longrightarrow B$ be given. Compose with $\bigoplus_A {}^{F(A)} Q(A) \longrightarrow F \otimes Q$ to get a morphism $g' : \bigoplus_A {}^{F(A)} Q(A) \longrightarrow B$. Then g' determines the map

$$\Phi'(g) : F \longrightarrow H^Q(B)$$
$$\Phi'(g)_A : F(A) \longrightarrow Hom(Q(A), B)$$

given by $\Phi'(g)_A(x) = g'\widehat{u}_{A,x}$. We have to show that $\Phi'(g)$ is a morphism in $\mathbf{Mod}\mathcal{A}$. This will follow from the fact that $g'(\lambda \otimes Q) = 0$. First we have to show that $\Phi'(g)_E$ is a morphism of groups. This will follow from the fact that for $x, x' \in F(E)$ we have $g'\widehat{u}_{E,x+x'} = g'\widehat{u}_{E,x} + g'\widehat{u}_{E,x'}$. To prove this, consider

$$u_{E,x+x'} - u_{E,x} - u_{E,x'} : H_E \longrightarrow \bigoplus_A {}^{F(A)} H_A$$

This factors through λ since

$$\pi_E(u_{A,x+x'} - u_{A,x} - u_{A,x'})(1_E) = x + x' - x - x' = 0$$

Hence this morphism defines an element $y \in K(E)$, with $\lambda v_{E,y} = u_{E,x+x'} - u_{E,x} - u_{E,x'}$. Then $J_{E,y} = \{(E, x+x'), (E,x), (E,x')\}$, so

$$\begin{aligned} 0 = g'(\lambda \otimes Q)\widehat{v}_{E,y} &= \sum_{(D,x)\in J_{E,y}} g'\widehat{u}_{D,x}(p_{D,x}\lambda v_{E,y}) \\ &= g'\widehat{u}_{E,x+x'}Q(1 - p_{E,x+x'}u_{E,x} - p_{E,x+x'}u_{E,x'}) \\ &+ g'\widehat{u}_{E,x}Q(p_{E,x}u_{E,x+x'} - 1 - p_{E,x}u_{E,x'}) \\ &+ g'\widehat{u}_{E,x'}Q(p_{E,x'}u_{E,x+x'} - p_{E,x'}u_{E,x} - 1) \end{aligned}$$

If $x \neq x'$ and neither are 0, the results follows immediately. The other cases are trivial. To see that $\Phi'(g)$ is natural, let $\alpha : E \longrightarrow E'$ in $\mathcal{A}$. Denoting the induced morphism $H_E \longrightarrow H_{E'}$ also by α, we have for $x \in F(E')$

$$u_{E',x}\alpha - u_{E,F(\alpha)(x)} : H_E \longrightarrow \bigoplus_A{}^{F(A)} H_A$$

Again this defines an element $y \in K(E)$ such that $\lambda v_{E,y} = u_{E',x}\alpha - u_{E,F(\alpha)(x)}$. Then $J_{(E,y)} = \{(E',x), (E, F(\alpha)(x))\}$, and

$$\begin{aligned} 0 = g'(\lambda \otimes Q)\widehat{v}_{E,y} &= \sum_{(D,x)\in J_{E,y}} g'\widehat{u}_{D,x}Q(p_{D,x}\lambda v_{E,y}) \\ &= g'\widehat{u}_{E',x}Q(\alpha - p_{E',x}u_{E,F(\alpha)(x)}) \\ &+ g'\widehat{u}_{E,F(\alpha)(x)}Q(p_{E,F(\alpha)(x)}u_{E',x}\alpha - 1) \end{aligned}$$

If $E \neq E'$ then $J_{C,y}$ has two entries and we see that $g'\widehat{u}_{E',x}Q(\alpha) = g'\widehat{u}_{E,F(\alpha)(x)}$, as required. If $E = E'$ but $F(\alpha)(x) \neq x$ the same holds. If $E = E'$ and $F(\alpha)(x) = x$ we get naturality trivially if $\alpha = 1$ and if $\alpha \neq 1$, $J_{C,y} = \{(E',x)\}$ and $0 = g'\widehat{u}_{E',x}Q(\alpha - 1)$, as required. Hence $\Phi'(g) : F \longrightarrow H^Q(B)$ is natural.

It is easy to see that the operations Φ, Φ' are mutually inverse, so it only remains to show that Φ is natural. This follows from an argument similar to the one given in the proof of Theorem 11. □

In the case where the cocomplete abelian category $\mathcal{B}$ is the category of modules over a ringoid, we can give an explicit description of the functor $- \otimes_{\mathcal{A}} Q$. In this case it makes sense to call an additive covariant functor $\mathcal{A} \longrightarrow \mathbf{Mod}\mathcal{B}$ a $\mathcal{A} - \mathcal{B}$-bimodule.

Theorem 13. *Let $\mathcal{A}, \mathcal{B}$ be ringoids and let Q be a $\mathcal{A}$-$\mathcal{B}$-bimodule. Let F be a right $\mathcal{A}$-module and $B \in \mathcal{B}$. Then the abelian group $(F \otimes_{\mathcal{A}} Q)(B)$ is the free abelian group on the symbols*

$$x \otimes f \qquad A \in \mathcal{A}, x \in F(A), f \in Q(A)(B)$$

subject to the relations

$$\begin{aligned} (x + x') \otimes f &= x \otimes f + x \otimes f & & x, x' \in F(A), f \in Q(A)(B) \\ x \otimes (f + f') &= x \otimes f + x \otimes f' & & x \in F(A), f, f' \in Q(A)(B) \\ (x \cdot \alpha) \otimes f &= x \otimes Q(\alpha)_B(f) & & \alpha : A' \longrightarrow A, x \in F(A), f \in Q(A')(B) \end{aligned}$$

For $\beta : B \longrightarrow B'$ and $x \otimes f \in (F \otimes_{\mathcal{A}} Q)(B')$ we have $(x \otimes f) \cdot \beta = x \otimes (\beta \cdot f) = x \otimes Q(A)(\beta)(f)$. For a morphism of $\mathcal{A}$-modules $\phi : F \longrightarrow F'$ and $B \in \mathcal{B}, A \in \mathcal{A}, x \in F(A), f \in Q(A)(B)$ we have

$$(\phi \otimes_{\mathcal{A}} Q)_B(x \otimes f) = \phi_A(x) \otimes f$$

Proof. Given the covariant functor $Q : \mathcal{A} \longrightarrow \mathbf{Mod}\mathcal{B}$, our definition of $- \otimes_{\mathcal{A}} Q$ follows from an application of Theorem 9. By unrolling this definition, we will eventually arrive at the characterisation of $- \otimes_{\mathcal{A}} Q$ described in the Theorem.

First let us establish some notation. We fix a right $\mathcal{A}$-module F and form the coproduct $\bigoplus_{A\in\mathcal{A}}{}^{F(A)} H_A$, the injections into which we denote by $u_{A,x}$. Define a morphism $\pi : \bigoplus_{A\in\mathcal{A}}{}^{F(A)} H_A \longrightarrow F$ by

$$\pi u_{A,x} = x : H_A \longrightarrow F$$

For $A, C \in \mathcal{A}, x \in F(A)$ and $f : C \longrightarrow A$ we denote $(u_{A,x})_C(f)$ by (x, f). In this notation,

$$\pi_C\left(\sum_x (x, f_x)\right) = \sum_x x \cdot f_x \in F(C)$$

Let K be the kernel of π. For $C \in \mathcal{A}$ the group $K(C)$ is easily seen to contain the following families of elements:

$$\begin{array}{rl} (x + x', f) - (x, f) - (x', f) & A \in \mathcal{A}, x, x' \in F(A), f : C \longrightarrow A \\ (x \cdot \alpha, f) - (x, \alpha f) & \alpha : A' \longrightarrow A, x \in F(A), f : C \longrightarrow A' \end{array}$$

For $k \in K(C) \subseteq \bigoplus_A{}^{F(A)} Hom(C, A)$ we let $k_x : C \longrightarrow A$ denote the xth component of k for $x \in F(A)$. Denoting the injections into the coproduct $\bigoplus_A{}^{K(A)} H_A$ by $v_{C,k}$, we define the morphism λ as follows:

$$\bigoplus_A{}^{K(A)} H_A \xrightarrow{\lambda} \bigoplus_A{}^{F(A)} H_A$$
$$\lambda v_{C,k} = k : H_C \longrightarrow \bigoplus_A{}^{F(A)} H_A$$

The next step in the construction is to introduce the morphism λ' of $\mathbf{Mod}\mathcal{B}$, which is defined by:

$$\bigoplus_A{}^{K(A)} Q(A) \xrightarrow{\lambda'} \bigoplus_A{}^{F(A)} Q(A)$$
$$\lambda' \widehat{v}_{C,k} = \sum_x \widehat{u}_{A,x} Q(k_x)$$

Explicitly if $g \in Q(C)(B)$ and we denote $(\widehat{v}_{C,k})_B(g)$ by (k, g), then

$$\lambda'_B(k, g) = \sum_x (x, Q(k_x)_B(g))$$

Notice that $\bigoplus_A{}^{F(A)} Q(A)(B)$ is the free abelian group on the set $\{(x, f) \,|\, x \in F(A), f \in Q(A)(B)\}$ modulo the subgroup generated by elements of the form $(x, f + f') - (x, f) - (x, f')$. Since $(F \otimes_\mathcal{A} Q)(B)$ is defined to be the quotient of $\bigoplus_A{}^{F(A)} Q(A)(B)$ by $Im(\lambda'_B)$, to show that $(F \otimes_\mathcal{A} Q)(B)$ has the form claimed in the Theorem it suffices to show that $Im(\lambda'_B)$ is the subgroup generated by all elements of the form

$$\begin{array}{rl} (x + x', f) - (x, f) - (x', f) & A \in \mathcal{A}, x \in F(A), f, f' \in Q(A)(B) \\ (x \cdot \alpha, f) - (x, Q(\alpha)_B(f)) & \alpha : A' \longrightarrow A, x \in F(A), f \in Q(A')(B) \end{array}$$

Call this subgroup Z. First we show that $Z \subseteq Im(\lambda'_B)$. For $A \in \mathcal{A}$, elements $x, x' \in F(A)$ and $g \in Q(A)(B)$ consider the element:

$$k = (x + x', 1_A) - (x, 1_A) - (x', 1_A) \in K(A)$$

By applying λ'_B to (k, g) we see that $(x + x', g) - (x, g) - (x', g)$ belongs to $Im(\lambda'_B)$. Similarly if $x \in F(A), g \in Q(A')(B), \alpha : A' \longrightarrow A$, consider

$$k' = (x \cdot \alpha, 1_{A'}) - (x, \alpha) \in K(A')$$

Applying λ'_B to (k', g) we see that $(x \cdot \alpha, g) - (x, Q(\alpha)_B(g))$ also belongs to $Im(\lambda'_B)$.

Finally, we show that $Im(\lambda'_B) \subseteq Z$. Let $\mu : \bigoplus_A {}^{F(A)} Q(A)(B) \longrightarrow T$ be the quotient of Z. Then it would be enough to show that $\mu\lambda'_B = 0$. But since λ'_B is a morphism out of a coproduct, it would suffice to show that $\mu\lambda'_B(\widehat{v}_{C,k})_B = 0$ for all $C \in \mathcal{A}$ and $k \in K(C)$. But for $g \in Q(C)(B)$

$$\begin{aligned}\mu\lambda'_B(k,g) &= \mu\left(\sum_x (x, Q(k_x)_B(g))\right)\\ &= \sum_x \mu(x, Q(k_x)_B(g))\\ &= \sum_x \mu(x \cdot k_x, g)\\ &= \mu\left(\sum_x x \cdot k_x, g\right)\\ &= \mu(0,g) = 0\end{aligned}$$

as required. Hence $Im(\lambda'_B) = Z$. If we denote the coset of (x,f) in $(F \otimes_{\mathcal{A}} Q)(B)$ by $x \otimes f$, then we have established the first claim of the Theorem. The other properties of $F \otimes_{\mathcal{A}} \mathcal{B}$ are easy to verify from the construction given in Theorem 12. □

Lemma 14. *Let $\mathcal{A}, \mathcal{B}$ be ringoids and Q a $\mathcal{A}$-$\mathcal{B}$-bimodule. We know that the functor $- \otimes_{\mathcal{A}} Q$ is left adjoint to H^Q:*

$$\mathbf{Mod}\mathcal{A} \underset{H^Q}{\overset{-\otimes_{\mathcal{A}} Q}{\rightleftarrows}} \mathbf{Mod}\mathcal{B} \qquad - \otimes_{\mathcal{A}} Q \dashv H^Q$$

The adjunction maps are defined explicitly as follows. For a right $\mathcal{A}$-module F, a right $\mathcal{B}$-module G, and elements $x \in F(A), f \in Q(A)(B)$ we have:

$$\begin{gathered}\Phi : Hom_{\mathcal{B}}(F \otimes_{\mathcal{A}} Q, G) \longrightarrow Hom_{\mathcal{A}}(F, H^Q(G))\\ \Phi(g)_A(x)_B(y) = g_B(x \otimes f)\\ \Phi^{-1}(h)_B(x \otimes f) = h_A(x)_B(f)\end{gathered}$$

Proof. This is a straightforward exercise using the explicit adjunction formulae given in Theorem 12. □

Example 3. For a ringoid $\mathcal{A}$, let $N : \mathcal{A} \longrightarrow \mathbf{Ab}$ be a left $\mathcal{A}$-module. This induces adjoint functors

$$- \otimes_{\mathcal{A}} N \dashv H^N$$

$$\mathbf{Mod}\mathcal{A} \underset{H^N}{\overset{-\otimes_{\mathcal{A}} N}{\rightleftarrows}} \mathbf{Ab}$$

The functor $- \otimes_{\mathcal{A}} N$ is additive and colimit preserving, and $H^N : \mathbf{Ab} \longrightarrow \mathbf{Mod}\mathcal{A}$ is the multi-object version of $Hom(N, -)$ defined by $H^N(Z)(A) = Hom(N(A), Z)$. Applying Theorem 13 with $\mathcal{B} = \mathbb{Z}$ we see that for a right $\mathcal{A}$-module F the abelian group $F \otimes_{\mathcal{A}} N$ is the free abelian group on the symbols

$$x \otimes y \qquad x \in F(A), y \in N(A)$$

subject to the relations

$$\begin{aligned}(x + x') \otimes y &= x \otimes y + x' \otimes y & x, x' \in F(A), y \in N(A)\\ x \otimes (y + y') &= x \otimes y + x \otimes y' & x \in F(A), y, y' \in N(A)\\ (x \cdot \alpha) \otimes y &= x \otimes (\alpha \cdot y) & \alpha : A' \longrightarrow A, x \in F(A), y \in N(A')\end{aligned}$$

If $\phi : F \longrightarrow F'$ is a morphism of modules, then $\phi \otimes_{\mathcal{A}} N$ is defined by

$$(\phi \otimes_{\mathcal{A}} N)(x \otimes y) = \phi_A(x) \otimes y$$

We use Lemma 14 to give the adjunction map:

$$\Phi : Hom_{\mathcal{B}}(F \otimes_{\mathcal{A}} N, Z) \longrightarrow Hom_{\mathcal{A}}(F, H^N(Z))$$
$$\Phi(g)_A(x)(y) = g(x \otimes f)$$
$$\Phi^{-1}(h)(x \otimes y) = h_A(x)(y)$$

It is not difficult to check that a natural transformation $F \longrightarrow H^N(Z)$ consists precisely of the following data: for each $A \in \mathcal{A}$ a map $m_A : F(A) \times N(A) \longrightarrow Z$, additive in each variable, such that for any $\alpha : A' \longrightarrow A$ and $x \in F(A), y \in N(A')$

$$m_A(x \cdot \alpha, y) = m_A(x, \alpha \cdot y)$$

Hence morphisms $F \otimes_{\mathcal{A}} N \longrightarrow Z$ correspond to natural families of bilinear maps $F(A) \times N(A) \longrightarrow Z$.

To define a functor which takes the tensor product with a fixed right $\mathcal{A}$-module F, consider F as a covariant functor $\mathcal{A}^{\text{op}} \longrightarrow \mathbf{Ab}$. This induces an additive, colimit preserving functor $- \otimes_{\mathcal{A}} F : \mathbf{Mod}\mathcal{A}^{\text{op}} \longrightarrow \mathbf{Ab}$. Of course, $\mathbf{Mod}\mathcal{A}^{\text{op}}$ is $A\mathbf{Mod}$, and we prefer to write this functor as $F \otimes_{\mathcal{A}} - : A\mathbf{Mod} \longrightarrow \mathbf{Ab}$.

For each $A \in \mathcal{A}$ we have the left $\mathcal{A}$-module $H^A : \mathcal{A} \longrightarrow \mathbf{Ab}$, which corresponds in the single-object case to considering R as a left module over itself. What can we say about the functor $- \otimes_{\mathcal{A}} H^A : \mathbf{Mod}\mathcal{A} \longrightarrow \mathbf{Ab}$? Recall that the covariant "evaluation" functor $E_A : \mathbf{Mod}\mathcal{A} \longrightarrow \mathbf{Ab}$ is defined by taking a module T to $T(A)$ and a morphism μ to μ_A. We think of E_A as being the forgetful functor at A. The evaluation functor is additive, and it is a fundamental fact about functor categories that it is also colimit preserving. Since it obviously extends H^A, the uniqueness part of Theorem 9 implies that the functor $- \otimes_{\mathcal{A}} H^A$ is just E_A.

By definition for any left $\mathcal{A}$-module N, $H_A \otimes_{\mathcal{A}} N = N(A)$. We have now shown that for any right $\mathcal{A}$-module M,

$$M \otimes_{\mathcal{A}} H^A = M(A)$$

We also observe that the right adjoint $H^{H^A} : \mathbf{Ab} \longrightarrow \mathbf{Mod}\mathcal{A}$ is defined by

$$H^{H^A}(Z)(A') = Hom_{\mathbf{Ab}}(H^A(A'), Z) = Hom_{\mathbf{Ab}}(Hom_{\mathcal{A}}(A, A'), Z)$$

for any abelian group Z.

The following Theorem gives necessary and sufficient conditions on a category for it to be equivalent to the category of modules over a ringoid.

Theorem 15. *A category $\mathcal{B}$ is equivalent to a module category* $\mathbf{Mod}\mathcal{A}$ *for some ringoid $\mathcal{A}$ if and only if $\mathcal{B}$ is cocomplete abelian with a generating set of small projectives.*

Proof. We have already seen that $\mathbf{Mod}\mathcal{A}$ is cocomplete abelian, and that the family $\{H_A\}_{A \in \mathcal{A}}$ form a generating set of small projectives. Conversely, suppose that $\mathcal{B}$ is cocomplete abelian and that $\mathcal{P}$ is a full subcategory of $\mathcal{B}$, the objects of which form a generating set of small projectives. Then $\mathcal{B}$ and $\mathbf{Mod}\mathcal{P}$ are cocomplete abelian categories each with a full subcategory whose objects form a generating set of small projectives, and furthermore both these subcategories are isomorphic to $\mathcal{P}$. Theorem 9 then gives us functors $T : \mathcal{B} \longrightarrow \mathbf{Mod}\mathcal{P}$ and $S : \mathbf{Mod}\mathcal{P} \longrightarrow \mathcal{B}$ extending this isomorphism, and moreover the uniqueness part of the theorem shows that ST and TS are naturally equivalent to the identity functors on $\mathcal{B}$ and $\mathbf{Mod}\mathcal{P}$, respectively. Hence the two categories are equivalent. ∎

Corollary 16. *A cocomplete abelian category $\mathcal{B}$ is equivalent to a module category over a ring if and only if it has a small projective generator X. If U is the endomorphism ring of X, then $\mathcal{B}$ is equivalent to* $\mathbf{Mod}U$ *via the functor*

$$H^X : \mathcal{B} \longrightarrow \mathbf{Mod}U$$
$$B \mapsto Hom_{\mathcal{B}}(X, B)$$

Proof. In the notation of the previous Theorem, $S : \mathbf{Mod}U \longrightarrow \mathcal{B}$ is the unique extension of the functor embedding X into $\mathcal{B}$. By Theorem 11 this functor S has a right adjoint $H^X : \mathcal{B} \longrightarrow \mathbf{Mod}U$ defined by $B \mapsto Hom_{\mathcal{B}}(X, B)$. Since T is also right adjoint to S, it follows that $H^X \cong T$ and so H^X is an equivalence, as desired. □

3 Injective Cogenerators

We can now show that for any ringoid $\mathcal{A}$ the category $\mathbf{Mod}\mathcal{A}$ has an injective cogenerator (and hence enough injectives). Just as the family $\{H_A\}_{A\in\mathcal{A}}$ forms a generating family of projectives, we will produce a family of right $\mathcal{A}$-modules Q_A which form a cogenerating family of injectives. Taking the product of all these objects, we will obtain the required injective cogenerator. The following argument imitates the original proof given by Eckmann for modules [1].

Let E be an injective cogenerator for $\mathbf{Ab}$ (for example, $\mathbb{Q}/\mathbb{Z}$). As observed in the previous section, for each $A \in \mathcal{A}$ there is a functor $H^{H^A} : \mathbf{Ab} \longrightarrow \mathbf{Mod}\mathcal{A}$ right adjoint to $- \otimes_{\mathcal{A}} H^A$. Define Q_A by using this functor to lift E up into $\mathbf{Mod}\mathcal{A}$:

$$Q_A = H^{H^A}(E)$$

More explicitly, $Q_A : \mathcal{A}^{\mathrm{op}} \longrightarrow \mathbf{Ab}$ is defined on objects by

$$Q_A(A') = Hom_{\mathbf{Ab}}(Hom_{\mathcal{A}}(A, A'), E)$$

Since the evaluation functor is exact, its right adjoint H^{H^A} must preserve injectives (AC,Proposition 25). Hence each Q_A is injective. Alternatively notice that for a right $\mathcal{A}$-module F, we can use the adjunction $E_A \dashv H^{H^A}$ to see that there is an isomorphism

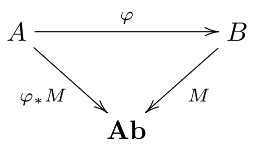
$$\begin{aligned} Hom_{\mathcal{A}}(F, Q_A) &= Hom_{\mathcal{A}}(F, H^{H^A}(E)) \\ &= Hom_{\mathbf{Ab}}(F(A), E) \end{aligned} \tag{10}$$

which is natural in both F and E. Naturality of F immediately implies that Q_A is injective. To see that $\{Q_A\}_{A\in\mathcal{A}}$ forms a cogenerating family, suppose that $\phi : F \longrightarrow G$ is a nonzero morphism. Then some component $\phi_A : F(A) \longrightarrow G(A)$ is nonzero, and thus there is a morphism of groups $\rho : G(A) \longrightarrow E$ such that $\rho\phi_A$ is nonzero. Since the isomorphism in Equation 10 is natural in F the induced morphism $\rho' : G \longrightarrow Q_A$ satisfies $\rho'\phi \neq 0$ as required.

Taking the product over $A \in \mathcal{A}$, we see that $\prod_{A\in\mathcal{A}} Q_A$ is an injective cogenerator for $\mathbf{Mod}\mathcal{A}$.

4 Adjoint Triples from Ringoid Morphisms

Let $\varphi : A \longrightarrow B$ be a morphism of rings. The action $a \cdot b = \varphi(a)b$ gives B the structure of an A-B-bimodule. Equivalently, think of φ as an additive functor from A into $\mathbf{Mod}B$. As usual, this extends to the tensor product $- \otimes_A B : \mathbf{Mod}A \longrightarrow \mathbf{Mod}B$ with right adjoint $Hom_{\mathcal{B}}(B, -) : \mathbf{Mod}B \longrightarrow \mathbf{Mod}A$. It is easy to see that this second functor is naturally equivalent to the "restriction of scalars" functor φ_*, which uses φ to turn a right B-module M into a right A-module via $m \cdot a = m \cdot \varphi(a)$. Looking at φ as an additive functor between ringoids and M as an additive functor $\mathcal{A}^{\mathrm{op}} \longrightarrow \mathbf{Ab}$, the functor φ_* acts by composition with φ, as illustrated in the following diagram



Of course, φ also makes B into a right A-module via $b \cdot a = b\varphi(a)$. This is clearly a B-A-bimodule, and so induces an additive covariant functor $B \longrightarrow \mathbf{Mod}A$. We thus have another pair $- \otimes_B B : \mathbf{Mod}B \longrightarrow \mathbf{Mod}A$ with right adjoint $Hom_A(B, -) : \mathbf{Mod}A \longrightarrow \mathbf{Mod}B$.

Since φ_* extends the functor $B \longrightarrow \mathbf{Mod}A$, is additive and preserve colimits (a cocone in $\mathbf{Mod}B$ or $\mathbf{Mod}A$ is a colimit iff. it is a colimit for abelian groups), the uniqueness part of Theorem 9 shows that $-\otimes_B B$ is naturally equivalent to φ_*. We call $Hom_A(B,-)$ the *coextension functor*.

Proposition 17. *For any morphism of rings $\varphi : A \longrightarrow B$ there is a triple of adjoints*

$$\varphi^* \dashv \varphi_* \dashv \varphi^!$$

where $\varphi^* = -\otimes_A B$ is "extension of scalars", φ_* is "restriction of scalars" and $\varphi^! = Hom_A(B,-)$ is "coextension of scalars".

More generally let $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ be a morphism of ringoids. Thinking of φ as a left $\mathcal{A}$-object and applying Theorem 9 we obtain the "extension" functor or tensor product $-\otimes_{\mathcal{A}} \mathcal{B} : \mathbf{Mod}\mathcal{A} \longrightarrow \mathbf{Mod}\mathcal{B}$ (it has been our convention to denote this functor $-\otimes_{\mathcal{A}} \varphi$, but we make an exception here)

$$-\otimes_{\mathcal{A}} \mathcal{B} \dashv H^{\varphi}$$

The "restriction" functor $\varphi_* : \mathbf{Mod}\mathcal{B} \longrightarrow \mathbf{Mod}\mathcal{A}$ is defined by

$$\varphi_*(F) = F\varphi, \quad \varphi_*(\phi) = \phi\varphi$$

That is, $\varphi_*(\phi)_A = \phi_{\varphi(A)}$. The functor φ_* is easily seen to be additive and colimit preserving. Moreover, there is an obvious natural equivalence of H^{φ} with φ_*. The restriction of φ_* to the full subcategory $\mathcal{B} \subseteq \mathbf{Mod}\mathcal{B}$ defines a left $\mathcal{B}$-object in $\mathbf{Mod}\mathcal{A}$. By the uniqueness part of Theorem 9 we see that φ_* has a right adjoint $\varphi^! = H^{\mathcal{B}} : \mathbf{Mod}\mathcal{A} \longrightarrow \mathbf{Mod}\mathcal{B}$ defined by

$$\begin{aligned} H^{\mathcal{B}}(F) : \mathcal{B}^{\mathrm{op}} &\longrightarrow \mathbf{Ab} \\ B &\mapsto Hom_{\mathcal{A}}(\varphi_* H_B, F) \end{aligned}$$

Again, we call this the *coextension functor*.

Theorem 18. *For any morphism of ringoids $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ there is a triple of adjoints*

$$\mathbf{Mod}\mathcal{A} \; \begin{array}{c} \xrightarrow{\varphi^*} \\ \xleftarrow{\varphi_*} \\ \xrightarrow[\varphi^!]{} \end{array} \; \mathbf{Mod}\mathcal{B} \qquad \varphi^* \dashv \varphi_* \dashv \varphi^!$$

Here $\varphi^* = -\otimes_{\mathcal{A}} \mathcal{B}$ is "extension of scalars", φ_* is "restriction of scalars" and $\varphi^! = H^{\mathcal{B}} = \mathscr{H}om_{\mathcal{A}}(\mathcal{B},-)$ is "coextension of scalars". These functors are defined explicitly as follows:

Extension *For a right $\mathcal{A}$-module F and $B \in \mathcal{B}$, the abelian group $(F \otimes_{\mathcal{A}} \mathcal{B})(B)$ is the free abelian group on the symbols*

$$x \otimes f \qquad A \in \mathcal{A}, x \in F(A), f : B \longrightarrow \varphi(A)$$

subject to the relations

$$\begin{aligned} (x + x') \otimes f &= x \otimes f + x' \otimes f \\ x \otimes (f + f') &= x \otimes f + x \otimes f' \\ (x \cdot \alpha) \otimes f &= x \otimes (\varphi(\alpha) f) \end{aligned}$$

For $\beta : B \longrightarrow B'$ and $x \otimes f \in (F \otimes_{\mathcal{A}} \mathcal{B})(B')$ we have $(x \otimes f) \cdot \beta = x \otimes (f\beta)$. For a morphism of $\mathcal{A}$-modules $\phi : F \longrightarrow F'$ and $B \in \mathcal{B}, A \in \mathcal{A}, x \in F(A), f : B \longrightarrow \varphi(A)$ we have

$$(\phi \otimes_{\mathcal{A}} \mathcal{B})_B(x \otimes f) = \phi_A(x) \otimes f$$

Restriction *For a right $\mathcal{B}$-module G and $A \in \mathcal{A}$, we have*

$$\varphi_* G(A) = G(\varphi(A))$$

For $\alpha : A \longrightarrow A'$ and $x \in \varphi_* G(A') = G(\varphi(A'))$ we have $x \cdot \alpha = x \cdot \varphi(\alpha)$. For a morphism of $\mathcal{B}$-modules $\psi : G \longrightarrow G'$ and $A \in \mathcal{A}, x \in \varphi_* G(A)$ we have

$$(\varphi_* \psi)_A(x) = \psi_{\varphi(A)}(x)$$

Coextension *For a right $\mathcal{A}$-module F and $B \in \mathcal{B}$ we have*

$$\mathscr{H}om_{\mathcal{A}}(\mathcal{B}, F)(B) = Hom_{\mathcal{A}}(\varphi_* H_B, F)$$

For $\beta : B \longrightarrow B'$ and $f \in \mathscr{H}om_{\mathcal{A}}(\mathcal{B}, F)(B')$ we have $f \cdot \beta = f \varphi_*(H_\beta)$. That is, for $A \in \mathcal{A}$ and $x \in (\varphi_* H_B)(A) = Hom_{\mathcal{B}}(\varphi(A), B)$,

$$(f \cdot \beta)_A(x) = f_A(\beta x)$$

For a morphism of $\mathcal{A}$-modules $\phi : F \longrightarrow F'$ and $B \in \mathcal{B}, f \in \mathscr{H}om_{\mathcal{A}}(\mathcal{B}, F)(B)$ we have

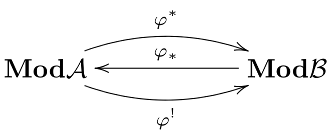
$$\mathscr{H}om_{\mathcal{A}}(\mathcal{B}, \phi)_B(f) = \phi f$$

Proof. It follows immediately from the definitions that the restriction and coextension functors have the stated form. To establish the results about the extension functor, we apply Theorem 13 to the case where the covariant functor $Q : \mathcal{A} \longrightarrow \mathbf{Mod}\mathcal{B}$ is the functor φ followed by the Yoneda embedding $\mathcal{B} \longrightarrow \mathbf{Mod}\mathcal{B}$. □

Intuitively, the extension functor works in the following way: given a $\mathcal{A}$-module F, the obvious way to assign an abelian group to the object $\varphi(A) \in \mathcal{B}$ is to choose $F(A)$. But we must also define the action of the morphisms of $\mathcal{B}$ on these elements. So we take any $x \in F(A)$ (thought of as sitting at $\varphi(A)$), a morphism $f : B \longrightarrow \varphi(A)$ and introduce the element $x \otimes f$ belonging at B. Once we have "spread out" the element of F in this fashion, we arrive at the $\mathcal{B}$-module $F \otimes_{\mathcal{A}} \mathcal{B}$.

Let us now write down explicitly the adjunction maps established in the previous Theorem.

Proposition 19. *Let $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ be a morphism of ringoids. Then we have the following adjunctions:*



$$\varphi^* \dashv \varphi_* \dashv \varphi^!$$

The adjunction maps are defined explicitly as follows. For a right $\mathcal{A}$-module F, a right $\mathcal{B}$-module G, and elements $x \in F(A), f : B \longrightarrow \varphi(A)$ we have:

$$\Phi : Hom_{\mathcal{B}}(F \otimes_{\mathcal{A}} \mathcal{B}, G) \longrightarrow Hom_{\mathcal{A}}(F, \varphi_* G)$$
$$\Phi(g)_A(x) = g_{\varphi(A)}(x \otimes 1_{\varphi(A)})$$
$$\Phi^{-1}(h)_B(x \otimes f) = h_A(x) \cdot f$$

For $y \in G(B), f : \varphi(A) \longrightarrow B, z \in G(\varphi(A))$ we have

$$\Psi : Hom_{\mathcal{A}}(\varphi_* G, F) \longrightarrow Hom_{\mathcal{B}}(G, \mathscr{H}om_{\mathcal{A}}(\mathcal{B}, F))$$
$$\Psi(g)_B(y)_A(f) = g_A(y \cdot f)$$
$$\Psi^{-1}(h)_A(z) = h_{\varphi(A)}(z)_A(1_{\varphi(A)})$$

Proof. This is a straightforward exercise using the explicit adjunction formulae given in Theorem 12. □

Example 4. Let $\mathcal{A}$ be a ringoid and $A \in \mathcal{A}$. As usual we also write A for the endomorphism ring of this object. Then there is the obvious inclusion $\varphi : A \longrightarrow \mathcal{A}$, which is a morphism of ringoids. What do the above adjoints look like in this special case?

Extension Let M be a right A-module. Then $\varphi^*(M) = M \otimes_A \mathcal{A}$ is the $\mathcal{A}$-module defined by

$$\begin{aligned} (M \otimes_A \mathcal{A})(C) &= M \otimes_A Hom_{\mathcal{A}}(C, A) \\ (m \otimes f) \cdot \alpha &= m \otimes (f\alpha) \end{aligned} \tag{11}$$

In particular for $m \in M$ and $f : C \longrightarrow A$ we have $m \cdot f = m \otimes f$. Thus to extend an A-module to a $\mathcal{A}$-module, we "spread out" the elements at A across all the other objects of the ringoid.

Restriction of Scalars The functor φ_* takes a right $\mathcal{A}$-module F and maps it to the A-module $F(A)$. Similarly, a morphism of $\mathcal{A}$-modules ϕ is mapped to the morphism of A-modules ϕ_A.

Coextension For a right A-module M, the $\mathcal{A}$-module $\varphi^! M$ is defined by

$$\varphi^! M(B) = Hom_A(Hom_{\mathcal{A}}(A, B), M)$$

5 Identifying the Additive Functors

Let $\mathcal{A}$ be a ringoid, and consider the category $\mathbf{Ab}^{\mathcal{A}^{\mathrm{op}}}$ of all contravariant functors $F : \mathcal{A} \longrightarrow \mathbf{Ab}$. Some of these functors are additive, and some are not. We show in this section that the additive functors form a coreflective subcategory of $\mathbf{Ab}^{\mathcal{A}^{\mathrm{op}}}$.

Definition 4. For a covariant functor $F : \mathcal{A}^{\mathrm{op}} \longrightarrow \mathbf{Ab}$ and $C \in \mathcal{A}$, an element $x \in F(C)$ is *additive* if for all parallel pairs of arrows $f, h : D \longrightarrow C$ in $\mathcal{A}$, we have

$$F(f + h)(x) = F(f)(x) + F(h)(x)$$

Obviously a functor F is additive iff. for each $C \in \mathcal{C}$, every element of $F(C)$ is additive. For $C \in \mathcal{C}$, we have the additive functor $H_C : \mathcal{A}^{\mathrm{op}} \longrightarrow \mathbf{Ab}$ given by $H_C(A) = Hom(A, C)$. For a set valued functor $G : \mathcal{A}^{\mathrm{op}} \longrightarrow \mathbf{Sets}$ and the set-valued functor $H_C : \mathcal{A}^{\mathrm{op}} \longrightarrow \mathbf{Sets}$, a morphism of functors $H_C \longrightarrow G$ is a pointwise morphism of sets. By the Yoneda lemma, such morphisms are in bijective correspondence with elements of $G(C)$. A morphism $H_C \longrightarrow F$ of functors in $\mathbf{Ab}^{\mathcal{A}^{\mathrm{op}}}$ is a pointwise morphism of groups - but again, if F is additive then such morphisms correspond to elements of $F(C)$. The next lemma considers what happens when F is *not* additive:

Lemma 20. *For a functor $F \in \mathbf{Ab}^{\mathcal{A}^{op}}$ and $C \in \mathcal{A}$ there is a bijection between morphisms $H_C \longrightarrow F$ of functors in $\mathbf{Ab}^{\mathcal{A}^{op}}$ and additive elements of $F(C)$.*

Proof. Corresponding to $\phi : H_C \longrightarrow F$ is $x = \phi_C(1_C)$. It is easy to verify that this is an additive element of $F(C)$, using naturality of ϕ and the fact that ϕ is a pointwise morphism of abelian groups. Conversely, given an additive element $x \in F(C)$, define $\phi_D : Hom(D, C) \longrightarrow F(D)$ by $\phi_D(f) = F(f)(x) \in F(D)$. The fact that x is additive ensures that ϕ_D is a morphism of groups. The correspondence is clearly bijective. ☐

Given a functor $F : \mathcal{A}^{\mathrm{op}} \longrightarrow \mathbf{Ab}$ we define a new functor $F^+ : \mathcal{A}^{\mathrm{op}} \longrightarrow \mathbf{Ab}$ by

$$F^+(C) = \{x \in F(C) \,|\, x \text{ is additive}\} = Hom(H_C, F)$$

It is not hard to see that $F^+(C)$ is a subfunctor of F, since the restriction of an additive element is additive. Further, an element $y \in F^+(C)$ is additive iff. it is additive as an element of $F(C)$, so F^+ is an additive functor.

Let $\mathbf{i} : (\mathcal{A}^{\mathrm{op}}, \mathbf{Ab}) \longrightarrow \mathbf{Ab}^{\mathcal{A}^{\mathrm{op}}}$ denote the inclusion of the additive functors in $\mathbf{Ab}^{\mathcal{A}^{\mathrm{op}}}$ and define

$$\begin{gathered} \mathbf{e} : \mathbf{Ab}^{\mathcal{A}^{\mathrm{op}}} \longrightarrow (\mathcal{A}^{\mathrm{op}}, \mathbf{Ab}) \\ \mathbf{e}(F) = F^+ \\ \mathbf{e}(\phi)_C(x) = \phi_C(x) \end{gathered}$$

Proposition 21. *The functor* $\mathbf{e}$ *is right adjoint to* $\mathbf{i}$*, and hence* $(\mathcal{A}^{op}, \mathbf{Ab})$ *is a coreflective subcategory of* $\mathbf{Ab}^{\mathcal{A}^{op}}$.

Proof. For $F \in \mathbf{Ab}^{\mathcal{A}^{\mathrm{op}}}$ let $\epsilon_F : F^+ \longrightarrow F$ denote the pointwise inclusion of the additive elements. If G is another additive functor and $\phi : G \longrightarrow F$, then it is easy to see that for $x \in G(C)$, $\phi_C(x)$ is an additive element of $F(C)$, and thus belongs to $F^+(C)$. Hence ϕ factors uniquely through ϵ_F by a morphism $G \longrightarrow F^+$ in $(\mathcal{A}^{\mathrm{op}}, \mathbf{Ab})$. Since any functor naturally equivalent to an additive functor is additive, this completes the proof. □

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