# STENSTRÖM

# [ RINGS OF QUOTIENTS ]

PROPOSITION 1.3 Let the diagram

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, "\alpha"] \arrow[d, "\lambda"'] & M \arrow[r, "\beta"] \arrow[d, "\mu"'] & N \arrow[r] \arrow[d, "\eta"'] & 0 \\ 0 \arrow[r] & L' \arrow[r, "\alpha'"'] & M' \arrow[r, "\beta'"'] & N' \arrow[r] & 0 \end{tikzcd}$$

be commutative with exact rows. If $\lambda$ and $\eta$ are monomorphisms (resp. epimorphisms, isomorphisms), then so is $\mu$.

PROOF Suppose $\lambda$ and $\eta$ are monomorphisms, and let $x \in M$ with $\mu(x)=0$. Then $\eta\beta(x) = \beta'\mu(x) = 0$, and $\eta$ monic implies $x \in \operatorname{Ker}\beta = \operatorname{Im}\alpha$. If we put $x = \alpha(y)$, then $\alpha'\lambda(y) = \mu\alpha(y) = \mu(x) = 0$, and $\alpha'\lambda$ monic implies $y=0$.

Suppose $\lambda$ and $\eta$ are epimorphisms, and let $x \in M'$. Certainly $\operatorname{Im}\mu$ at least contains $\operatorname{Im}\alpha' = \operatorname{Ker}\beta'$, since $\lambda$ is epi. Hence suppose $\beta'(x) \neq 0$. Let $y \in N$ be s.t. $\eta(y) = \beta'(x) \in N'$, and $z \in M$ s.t. $\beta(z) = y$. Then

$$\beta'(\mu(z) - x) = 0$$

so that $\mu(z) = x + \alpha'(e') = x + \alpha'\lambda(e) = x + \mu\alpha(e)$. Hence $\mu(z - \alpha(e)) = x$. $\square$

PROPOSITION 2.3 If $A$ is a skew-field, then every right $A$-module is free.

EXAMPLES 1. Direct Summands and Idempotents: If $L$ is a submodule of a module $M$, then $L$ is a direct summand of $M$ if there exists a submodule $L'$ of $M$ such that $M = L \oplus L'$. The module $M$ is indecomposable if it has no direct summands $\neq 0, M$. The direct summands of $A$ as an $A$-module correspond to idempotent elements of $A$. For if $e^2 = e$, noticing that in Mod-$A$, $\operatorname{End}_A(A) = A$, as usual $A = \ker e \oplus \ker(1-e) = eA \oplus (1-e)A$. Conversely if $A = \mathfrak{a} \oplus \mathfrak{b}$ for right ideals $\mathfrak{a}, \mathfrak{b}$, then the idempotent $A \longrightarrow \mathfrak{a} \longrightarrow A$ of $A$ corresponds (by Yoneda) to an element of $A$, and this element is $e \in \mathfrak{a}$ s.t. $1 = e + f$, $f \in \mathfrak{b}$.

3. Group rings: Let $A$ be a commutative ring and $G$ a group. We define the group ring $A[G]$ to be the free $A$-module on the set $G$, with multiplication induced from the multiplication in the group $G$. Thus

$$A[G] = \left\{ \sum_{g \in G} a_g g \;\middle|\; a_g \in A \text{ and } a_g = 0 \text{ for almost all } g \right\}$$

and

$$\sum_g a_g g \cdot \sum_h b_h h = \sum_k c_k k \quad \text{where} \quad c_k = \sum_{gh=k} a_g b_h$$

## 3. Finitely Generated Modules and Noetherian Modules

A module is finitely generated if there exists a finite set of generators for $M$, or in other words, if there is an epimorphism $A^n \longrightarrow M$ for some $n$. In particular, $M$ is cyclic if there is an epimorphism $A \longrightarrow M$. It follows that $M$ is cyclic if and only if $M \cong A/\mathfrak{a}$ for a right ideal $\mathfrak{a}$ of $A$.

LEMMA 3.1 Let $L$ be a submodule of a module $M$. Then:

(i) If $M$ is finitely generated, so is $M/L$

(ii) If $L$ and $M/L$ are finitely generated, then so is $M$.

DEFINITION The module $M$ is *finitely presented* if there exists an exact sequence

$$A^m \longrightarrow A^n \longrightarrow M \longrightarrow 0$$

which means that not only is $M$ finitely generated, but also the module of "relations between the generators of $M$" is finitely generated.

PROPOSITION 3.2 If $M$ is finitely presented and $0 \to K \to L \to M \to 0$ is an exact sequence with $L$ finitely generated, then $K$ is finitely generated.

PROOF Since $M$ is finitely presented, there exists an exact sequence $A^m \to A^n \to M \to 0$. We get a commutative diagram

$$\begin{tikzcd}
 & A^m \arrow[r, "\eta"] \arrow[d, "\kappa"'] & A^n \arrow[r, "\mu"] \arrow[d, "\lambda"'] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & K \arrow[r, "\alpha"'] & L \arrow[r, "\beta"'] & M \arrow[r] & 0
\end{tikzcd}$$

where $\lambda$ is defined by projectivity and $\kappa$ by the usual argument. Now consider

$$\begin{tikzcd}
K \arrow[r, "\alpha"] \arrow[d] & L \arrow[d] \\
\operatorname{Coker} \kappa \arrow[r, dotted, "\bar{\alpha}"'] & \operatorname{Coker} \lambda
\end{tikzcd}$$

$$\operatorname{Coker} \kappa = K/\operatorname{Im} \kappa$$
$$\operatorname{Coker} \lambda = L/\operatorname{Im} \lambda$$

where $\bar{\alpha}$ is induced in the usual way. We claim that $\bar{\alpha}$ is an isomorphism. Suppose $\bar{\alpha}(\bar{x}) = 0$, $x \in K$. Then $\alpha(x) \in \operatorname{Im}\lambda$, so $\alpha(x) = \lambda(y)$ for some $y \in A^n$. But $\mu(y) = \beta\lambda(y) = \beta\alpha(x) = 0$, so $y \in \operatorname{Im}\eta$. Writing $y = \eta(z)$ for $z \in A^m$, we get $\alpha\kappa(z) = \lambda\eta(z) = \lambda(y) = \alpha(x)$ and hence $x = \kappa(z)$, since $\alpha$ is monic. It follows that $\bar{x} = 0$, so $\bar{\alpha}$ is a monomorphism. Next we show that $\bar{x} \in \operatorname{Im}\bar{\alpha}$ for every $x \in L$. We have $\beta(x) = \mu(y)$ for some $y \in A^n$. Then $x - \lambda(y) \in \operatorname{Ker}\beta$, so $x - \lambda(y) = \alpha(z)$ for some $z \in K$. But then $\bar{x} = \overline{\alpha(z)} = \bar{\alpha}(\bar{z})$.

The module $\operatorname{Coker}\lambda$ is finitely generated by 3.1(i), and hence so is $\operatorname{Coker}\kappa$, and so by 3.1 so is $K$, since $\operatorname{Im}\kappa$ is finitely generated. $\square$

It is of course not in general true that submodules of a finitely generated module are finitely generated. We call a submodule $M$ noethenan if every submodule of $M$ is finitely generated.

PROPOSITION 3.3 A module is noethenan if and only if every strictly ascending chain of submodules is finite.

PROOF. $\square$

PROPOSITION 3.4 Let $L$ be a submodule of $M$. Then $M$ is noethenan if and only if both $L$ and $M/L$ are noethenan.

PROOF $M$ noethenan obviously implies that $L$ is noethenan. It also implies that $M/L$ is noethenan, because the submodules of $M/L$ can be written as $M'/L$, where $L \subseteq M' \subseteq M$. Suppose conversely that $L$ and $M/L$ are noethenan. If $M'$ is a submodule of $M$, then $L \cap M'$ is finitely generated as a submodule of $L$, and

$$M'/_{L \cap M'} \cong L + M'/_L$$

is finitely generated as a submodule of $M/L$. It follows from Lemma 3.1(ii) that $M'$ is finitely generated. Hence $M$ is finitely generated. $\square$

DEFINITION A ring $A$ is *right noethenan* if $A$ is noethenan as a right $A$-module — i.e. every right ideal of $A$ is f.g.

PROPOSITION 3.5 If $A$ is right noetherian, then every finitely generated module is noetherian.

PROOF If $A$ is noetherian, then every finitely generated free module is noetherian by Prop. 3.4., and therefore every finitely generated module is a quotient of a noetherian module and hence noetherian. □

COROLLARY 3.6 If $A$ is right noetherian, then every finitely generated module is finitely presented.

PROPOSITION 3.7 If $A$ is a right noetherian ring, then the polynomial ring $A[x]$ is right noetherian.

PROOF Let $\mathfrak{a}$ be a right ideal in $A[x]$. For each $n$ we let $\mathfrak{a}_n$ be the set of $a \in A$ for which there exists a polynomial in $\mathfrak{a}$ with leading term $ax^n$. It is clear that $\mathfrak{a}_n$ is a right ideal of $A$, and that $\mathfrak{a}_n \subseteq \mathfrak{a}_{n+1}$. The ascending chain $\mathfrak{a}_0 \subseteq \mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \cdots$ becomes stationary at some $n_0$. Let $\{a_{ij}\}$ be a finite family of generators for the right ideal $\mathfrak{a}_i$ of $A$ $(i \le n_0)$, and let $f_{ij}$ be a polynomial in $\mathfrak{a}$ with leading term $a_{ij}x^i$. We assert that the polynomials $f_{ij}$ generate $\mathfrak{a}$. If there are polynomials in $\mathfrak{a}$ which cannot be written as linear combinations of $f_{ij}$'s then we choose one such polynomial of smallest degree, say $g(x) = bx^m + \cdots$, then $b \in \mathfrak{a}_m$, and if $m \le n_0$ then we can write $b = \sum_j a_{mj} b_j$. The polynomial

$$g(x) - \sum_j f_{mj}(x)\, b_j$$

in an element of $\mathfrak{a}$ of degree $< m$. It is therefore a linear combination of $f_{ij}$'s, and hence so is $g(x)$, a contradiction. If $m > n_0$, then we have $b = \sum_j a_{n_0 j} b_j$, and we look instead at the polynomial

$$g(x) - \sum_j f_{n_0 j}(x)\, x^{n-m_0}\, b_j$$

and argue as before. □

COROLLARY 3.8 If $K$ is a field, then the ring $K[x_1, \ldots, x_n]$ is noetherian.

If $A$ is right noetherian, and $\mathfrak{a}$ is a two-sided ideal of $A$, then the ring $A/\mathfrak{a}$ is also right noetherian.

EXAMPLES 1. Cyclic Submodules: Let $M$ be a module and $x \in M$. The element $x$ generates a cyclic submodule $xA$ of $M$. There is an epimorphism $\alpha : A \longrightarrow xA$ given by $a \mapsto x \cdot a$ and $\operatorname{Ker}\alpha = \{a \in A \mid x \cdot a = 0\} = \operatorname{Ann}(x)$, the annihilator of $x$. Hence $xA \cong A/\operatorname{Ann}(x)$.

2. Simple Modules: A module $M$ is simple (or irreducible) if $M \neq 0$ and the only submodules of $M$ are $0$ and $M$. Every simple module is clearly cyclic. It is clear that $M$ is simple iff. $M \cong A/\mathfrak{m}$ where $\mathfrak{m}$ is a maximal right ideal of $A$.

3. Artinian Modules and Rings: A module $M$ is called artinian if every strictly descending chain of submodules is finite. The ring $A$ is right artinian if it is artian as a right $A$-module

EXAMPLES Finite Presentation of Modules: If $M$ is an arbitrary finitely generated module, then there is an exact sequence

$$0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0$$

with $F$ finitely generated free. Since $K$ is the direct limit of its finitely generated submodules, and if these are labelled $K_i$, $F/K_i$ is also a direct system (using 0) of finitely presented objects whose limit is

$$\varinjlim F/K_i = F/\varinjlim K_i = F/K = M$$

since direct limits are exact. Hence every finitely generated module is a direct limit of finitely presented modules.

## 6. PROJECTIVE AND INJECTIVE MODULE

Let $A$ be a ring. A right $A$-module $P$ is projective if the functor $\operatorname{Hom}_A(P, -)$ is exact, which means that for every epimorphism $\beta : M \longrightarrow N$ and every $\gamma : P \longrightarrow N$ there exists $\gamma' : P \longrightarrow M$ such that $\beta\gamma' = \gamma$.

$$\begin{tikzcd} & P \arrow[dl, dotted, "\gamma'"'] \arrow[d, "\gamma"] & \\ M \arrow[r, "\beta"'] & N \arrow[r] & 0 \end{tikzcd}$$

Every free module $P$ is projective.

PROPOSITION 6.1 The following properties of a module $P$ are equivalent

(a) $P$ is projective

(b) $P$ is a direct summand of a free module

(c) every exact sequence $0 \to L \to M \to P \to 0$ splits.

The concept of "basis" for free modules can be extended to a weaker notion of "projective coordinates" for projective modules.

PROPOSITION 6.3 A module $P$ is projective if and only if there exist a family $(x_i)_I$ of elements of $P$ and homomorphisms $\varphi_i : P \to A$, such that for each $x \in P$ one has

$$x = \sum_i x_i \varphi_i(x)$$

where $\varphi_i(x) = 0$ for all but a finite number of $i \in I$.

PROOF Let $\beta : F \to P$ be an epimorphism of a free module $F$ onto $P$, let $(e_i)_I$ be a basis for $F$ and put $x_i = \beta(e_i)$, $i \in I$. If $P$ is projective, $\beta$ splits and there is $\varphi : P \to F$ with $\beta\varphi = 1_P$. $\varphi$ induces homomorphisms $\varphi_i : P \to A$ with the stated properties. Conversely, if the family $(\varphi_i, x_i)$ exists and one defines $\beta : F \to P$ as before, then the maps $\varphi_i$ induce $\varphi : P \to F$ with $\beta\varphi = 1_P$, and so $P$ is a direct summand of $F$. $\square$

When determining whether a module is injective, it suffices to consider extensions over a very restricted class of monomorphisms.

PROPOSITION 6.5 A module $E$ is injective if and only if for every right ideal $\mathfrak{a}$ of $A$ and homomorphism $\varphi : \mathfrak{a} \to E$ there exists $y \in E$ such that $\varphi(a) = ya$ for all $a \in \mathfrak{a}$.

PROOF The stated condition means simply that every $\varphi : \mathfrak{a} \to E$ can be extended to $\varphi' : A \to E$, so it is of course a necessary condition. Assume that $E$ satisfies the condition. Let $\alpha : L \to M$ be a monomorphism and $\varphi : L \to E$ an arbitrary morphism. Consider the set

$$\mathcal{M} = \{ \varphi' : L' \to E \mid L \le L' \le M \text{ and } \varphi' \text{ extends } \varphi \}$$

$\mathcal{M}$ can be partially ordered by declaring $\varphi' \le \varphi''$ if $\varphi''$ further extends $\varphi'$. If $\mathcal{T}$ is a totally ordered subset of $\mathcal{M}$ then we define $\bar{L}$ as the sum of all $L'$ in $\mathcal{T}$, and define $\bar{\varphi} : \bar{L} \to E$ so that it extends all $\varphi' \in \mathcal{T}$. Then $\bar{\varphi}$ is an upper bound for $\mathcal{T}$. The set $\mathcal{M}$ is thus inductive, and we can apply Zorn's Lemma to obtain a maximal $\varphi_0 : L_0 \to E$ in $\mathcal{M}$. We must show that $L_0 = M$. Suppose there exists $x \in M$ such that $x \notin L_0$. We will show that it is possible to extend $\varphi_0$ to $\psi : L_0 + xA \to E$, which will give the desired contradiction. Put

$$\mathfrak{a} = \{ a \in A \mid xa \in L_0 \}$$

which is a right ideal of $A$. There is a morphism $\alpha : \mathfrak{a} \to E$ given by $\alpha(a) = \varphi_0(xa)$, and by hypothesis there exists $y \in E$ s.t. $\alpha(a) = ya$ for all $a \in \mathfrak{a}$. Since $\varphi_0$ is a morphism, if $x$ were in $L_0$ we would have $\varphi_0(x) = y$ – hence our natural extension $\psi : L_0 + xA \to E$ is given by

$$\psi(z + xa) = \varphi_0(z) + ya \qquad z \in L_0,$$

If $z + xa = z' + xa'$, then $z - z' = x(a' - a)$ and $a' - a \in \mathfrak{a}$. Hence

$$\begin{aligned} \varphi_0(z) - \varphi_0(z') &= \varphi_0(x(a'-a)) \\ &= y(a'-a) \\ &= ya' - ya \end{aligned}$$

Hence $\varphi_0(z) + ya = \varphi_0(z') + ya'$. Furthermore $\psi$ is linear, and extends $\varphi_0$. $\square$

An injective module $E$ is an <u>injective cogenerator</u> if $\mathrm{Hom}_A(M, E) \neq 0$ for every non-zero module $M$; since $E$ is injective it actually suffices to require this for cyclic modules $M \neq 0$. The injective cogenerators are interesting because their existence (which will be established in §9) guarantees that every module can be embedded in an injective module.

PROPOSITION 6.7 An injective module $E$ is a cogenerator if and only if it contains an isomorphic copy of each simple module.

PROOF If $E$ is a cogenerator and $S$ is a simple module, there is a nonzero module morphism $S \to E$ which is monic since its kernel must be zero. To show that $E$ is a cogenerator, it suffices by a previous remark to prove that there is a nonzero morphism $C \to E$ for each cyclic module $C \neq 0$. But this is a consequence of the fact that $C$ has a simple quotient module, or somewhat more generally:

LEMMA 6.8 *Every non-zero finitely generated module $M$ has a maximal proper submodule.*

PROOF Let $\mathcal{M}$ be the set of all proper submodules of $M$, partially ordered under inclusion. If $\mathcal{T}$ is a totally ordered subset of $\mathcal{M}$, let $L$ be the sum of all $L \in \mathcal{T}$. If we can show that $L \neq M$, then $L$ will be an upper bound for $\mathcal{T}$ in $\mathcal{M}$, and Zorn's Lemma will apply to give a maximal proper submodule of $M$. If $L = M$, then $L$ contains a finite generating set $\{x_1, \dots, x_n\}$ for $M$. Each $x_i$ lies in some $L_i \in \mathcal{T}$ and then $\{x_1, \dots, x_n\} \subseteq L$ for some $L \in \mathcal{T}$ which implies $L = M$, a contradiction. □

This Lemma establishes in particular the existence of maximal right ideals in every ring $A$.

EXAMPLES

1. Non-free Projective Modules Let $A = \mathbb{Z}/6\mathbb{Z}$, which can be decomposed as $A = (2) \oplus (3)$. The ideals $(2)$ and $(3)$ are projective modules that are not free.

2. Hereditary Rings The ring $A$ is right hereditary if every right ideal of $A$ is a projective module. E.g. the ring $\mathbb{Z}$ is hereditary. More generally, every right principal domain, i.e. a ring without zero-divisors in which every right ideal is principal, is right hereditary. For a less trivial example of a hereditary ring, see Exercise 24.

3. Semi-Hereditary Rings The ring $A$ is right semi-hereditary if every finitely generated right ideal of $A$ is a projective module.

PROPOSITION 6.9 *If $A$ is right semi-hereditary, then every finitely generated submodule of a free module is isomorphic to a direct sum of finitely generated right ideals of $A$.*

PROOF Let $M$ be a finitely generated submodule of a free module $F$. We may assume that $F$ is finitely generated on a basis $x_1, \dots, x_n$. We proceed by induction on $n$. If $n = 1$, then $M$ is isomorphic to a finitely generated right ideal of $A$ and, by hypothesis, projective. Let $F'$ be the free submodule of $F$ generated by $x_1, \dots, x_{n-1}$, and let $\alpha : F \longrightarrow A$ be the projection $\alpha(\sum x_i a_i) = a_n$. The image of $M$ under $\alpha$ is a finitely generated right ideal of $\mathfrak{a}$, and if $M'$ is $\ker(M \to \mathfrak{a})$ we obtain

$$\begin{tikzcd} 0 \arrow[r] & M' \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & \mathfrak{a} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & F' \arrow[r] & F \arrow[r] & A \arrow[r] & 0 \end{tikzcd}$$

where the upper row splits since $\mathfrak{a}$ is projective. But then also $M'$ is finitely generated, and thus isomorphic to a direct sum of finitely generated right ideals by the induction hypothesis ($M' = M/\mathfrak{a}$ under splitting). Hence so is $M$, since $M = M' \oplus \mathfrak{a}$. □

COROLLARY *If $A$ is right semi-hereditary, every finitely generated submodule of a free module is projective.*

4. Divisible Modules. We call a module $M_A$ divisible if for each non-zero-divisor $s$ of $A$ and $x \in M$, there exists $y \in M$ such that $x = ys$. (technically $s$ only needs to not be a left zero divisor, so that $A \longrightarrow (s)$, $a \longmapsto sa$ has kernel $0$, so $(s)$ is free). An injective module $E$ must be divisible, for we can define $f : sA \longrightarrow E$ as $f(sa) = xa$ since $sA$ is free, and the definition of injectivity gives $y \in E$ s.t. $x = ys$. But divisible modules are in general not injective. (Ex. 25).

PROPOSITION 6.10 *If $A$ is a right principal domain, then a module is injective iff. it is divisible.*

PROOF We know that an injective module is divisible. Suppose $E$ is divisible. To show that $E$ is injective, it suffices by Prop 6.5 to consider a homomorphism $f : aA \longrightarrow E$. By divisibility there exists $y \in E$ s.t. $f(a) = ya$, and this is the element required by Prop 6.5. (N.B $A$ is domain, so no zero-divisors). □

The result applies in particular to the ring $A = \mathbb{Z}$. Hence $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are divisible, hence injective $\mathbb{Z}$-modules.

# 7. SEMI-SIMPLE RINGS AND MODULES



Recall that a nonzero module $S$ is simple if it has no submodules other than $0$ and $S$. More generally, a module is semi-simple if it is a sum of simple submodules. (Also the module $(0)$ is semi-simple, as an empty sum of simple modules) The following result shows that we can even take the sum to be direct:

PROPOSITION 7.1 Let $S$ be a sum of simple submodules $S_i$ ($i \in I$) and let $L$ be an arbitrary submodule of $S$. Then there exists $J \subseteq I$ s.t. $S = \bigoplus_J S_i \oplus L$.

PROOF The direct sum is here to be interpreted as an internal direct sum of submodules. By an easy application of Zorn's Lemma one finds a maximal subset $J$ of $I$ such that the sum $M = L + \sum_J S_i$ is direct. We must show that $M = S$, and to do this it suffices to show that $M$ contains every $S_i$. But if $S_{i_0} \not\subseteq M$, then $M \cap S_{i_0} = 0$ by the simplicity of $S_{i_0}$, so the sum $M + S_{i_0}$ is direct, which contradicts the maximality of $J$. □

As a consequence we get the following alternative descriptions of semi-simple modules:

PROPOSITION 7.2 The following properties of a module $S$ are equivalent:

(a) $S$ is semi-simple
(b) $S$ is a direct sum of simple modules
(c) Every submodule of $S$ is a direct summand.

PROOF (a)$\Leftrightarrow$(b) follows from Prop 7.1. (with $L = 0$), and also (a)$\Rightarrow$(c) is an immediate conseque of Prop. 7.1.
(c)$\Rightarrow$(a) The sum of all simple submodules of $S$ is a direct summand of $S$, and we show that the complementary summand is $0$. (Hence by Prop. 7.1. the sum is direct, i.e. any semi-simple module is the direct sum of all its simple submodules). To show the summand is zero, it suffices to show that every submodule $L$ contains a simple submodule. The submodule $L$ may as well be cyclic, so by Lemma 6.8. it contains a maximal proper submodule $M$. The submodule $M$ splits $S$ as $S = M \oplus K$, and then $L = M \oplus (K \cap L)$. It follows that $K \cap L \cong L/M$ is a simple submodule of $L$. □

COROLLARY If $S$ is semi-simple, it is the direct sum of all its simple submodules. (N.B so the decomposition may have isomorphic factors)

COROLLARY 7.3 Let $S$ be a sum of simple modules $S_i$ ($i \in I$). If $L$ is a submodule of $S$, then $L \cong \bigoplus_J S_j$ for some $J \subseteq I$.
PROOF We have $S = L \oplus K$ for some $K \subseteq S$, and $S = K \oplus (\bigoplus_J S_j)$ with $J \subseteq I$. Then $L \cong S/K \cong \bigoplus_J S_j$. □

COROLLARY 7.4 Every submodule and every quotient module of a semi-simple module is semi-simple.

Let $\Omega$ denote the set of isomorphism classes of simple right $A$-modules. For every $\omega \in \Omega$ and every semi-simple module $S$ we let $S_\omega$ denote the sum of all simple modules of $S$ of isomorphism class $\omega$ ($S_\omega$ is the $\omega$-isotypic component of $S$).

PROPOSITION 7.5 If $S$ is semi-simple, then $S = \bigoplus_{\Omega} S_\omega$

If $\alpha : S \longrightarrow S'$ is a morphism of semi-simple modules, then clearly $\alpha(S_\omega) \subseteq S'_\omega$ (notice the simple submodules of $S_\omega$ are all in $\omega$, and a quotient of a simple module is $0$ or the whole thing) The endomorphism ring of a finite sum of simple modules can be described rather nicely. First we note:

LEMMA 7.6 (Schur) If $S$ is a simple module, then $\mathrm{End}_A(S)$ is a skew-field.

PROOF If $\alpha : S \longrightarrow S$ is nonzero, $\mathrm{Ker}\,\alpha = 0$ and $\mathrm{Im}\,\alpha = S$, so $\alpha$ is iso. □

Let

$$S = S_{11} \oplus \cdots \oplus S_{1n_1} \oplus \cdots \oplus S_{k1} \oplus \cdots \oplus S_{kn_k}.$$

where $S_{ij}$ are simple modules s.t. $S_{ij} \cong S_{i'j'}$ iff. $i = i'$. The endomorphism ring of $S$ is the product of the endomorphism rings of each sum $S_{i1} \oplus \cdots \oplus S_{in_i}$,

$$\begin{aligned} \mathrm{End}_A(S) &= \mathrm{Hom}_A(S,S) = \mathrm{Hom}_A(S_{11} \oplus \cdots \oplus S_{1n_1} \oplus \cdots, S_{11} \oplus \cdots) \qquad \text{finite } \oplus = \times \\ &= \prod_{i,j,i',j'} \mathrm{Hom}_A(S_{ij}, S_{i'j'}) = \prod_j \prod_{n_j^2} \mathrm{End}_A(S_{j1}) \end{aligned}$$

If we put $D_i = \mathrm{End}_A(S_{i1})$, which is a skew-field, then

$$\mathrm{End}_A(S) = M_{n_1}(D_1) \times \cdots \times M_{n_k}(D_k)$$

In detail, if $S = S_{11} \oplus \cdots \oplus S_{1n_1} \oplus \cdots \oplus S_{k1} \oplus \cdots \oplus S_{kn_k}$ as on the previous page, we know that $\mathrm{End}_A(S)$ is isomorphic to the matrix ring $M$ with the usual setup:

$$
\begin{array}{cccccccc}
 & S_{11} & S_{12} & \cdots & S_{1n_1} & S_{21} & \cdots & S_{2n_2} \; \cdots \\
S_{11} & [S_{11}, S_{11}] & [S_{12}, S_{11}] & & [S_{1n_1}, S_{11}] & \cdots & & \\
S_{12} & [S_{11}, S_{12}] & [S_{12}, S_{12}] & & \vdots & \ddots & & \\
\vdots & & & & & & & \\
S_{21} & [S_{11}, S_{21}] & \vdots & & & & & \\
\vdots & \vdots & & & & & & \\
S_{2n_2} & [S_{11}, S_{2n_2}] & & & & & &
\end{array}
$$

where composition in $\mathrm{End}_A(S)$ becomes matrix multiplication in $M$. Notice, however, if $S \neq S'$ are two distinct simple modules that $[S, S'] = 0$. Hence $M$ consists of the product of diagonal blocks of the form

$$
\begin{array}{ccccc}
 & S_{i1} & \cdots & S_{in_i} \\
S_{i1} & [S_{i1}, S_{i1}] & \cdots & [S_{in_i}, S_{i1}] \\
\vdots & \vdots & \ddots & \vdots \\
S_{in_i} & [S_{i1}, S_{in_i}] & \cdots & [S_{in_i}, S_{in_i}]
\end{array}
$$

But each $S_{ij} \cong S_{ij'}$, so all these morphism sets are the same — $\mathrm{End}_A(S_{i1}) = D_i$. By the usual argument, we see that

$$\mathrm{End}_A(S) \cong M_{n_1}(D_1) \times \cdots \times M_{n_k}(D_k) \qquad (1)$$

DEFINITION The ring $A$ is semi-simple if $A$ is semi-simple as a right $A$-module.

A right ideal of $A$ which is simple as an $A$-module is called a minimal right ideal. A semi-simple ring is thus a direct sum of minimal right ideals, and every simple module is isomorphic to a minimal right ideal of $A$ (if $M = A/\mathfrak{m}$ is simple — hence quotient of $A$ by maximal right ideal $\mathfrak{m}$ — and if $A$ is semi-simple, hence by 7.2 (c) $A = \mathfrak{m} \oplus \mathfrak{a}$ as $A$-modules, $\mathfrak{a}$ a right ideal, then $M = \mathfrak{a}$, and $\mathfrak{a}$ is clearly minimal).

PROPOSITION 7.7 The following properties of a ring $A$ are equivalent

(a) $A$ is a semi-simple ring
(b) All right $A$-modules are semi-simple
(c) All right $A$-modules are projective
(d) All right $A$-modules are injective
(e) Every right ideal of $A$ is a direct summand of $A$.
(f) $A \cong M_{n_1}(D_1) \times \cdots \times M_{n_k}(D_k)$ for some skew-fields $D_1, \ldots, D_k$.

PROOF (a) $\Rightarrow$ (b) Every free module is semi-simple, and hence so is an arbitrary module by Prop 2.5. and Cor 7.4.
(a) $\Rightarrow$ (e) By Prop 7.2. and (e) $\Rightarrow$ (d) with the help of Prop 6.5.
(d) $\Rightarrow$ (c) If $0 \to L \to M \to N \to 0$ is exact, then $L$ injective implies the sequence splits. Hence by Prop 6.1, (d) $\Leftrightarrow$ every sequence splits $\Leftrightarrow$ (c), but the condition that every sequence splits is also equivalent to (b) by Prop 7.2.
Finally (b) $\Rightarrow$ (a) trivially, so it now remains to show the equivalence of (a) and (f). That (a) implies (f) is immediate by applying the formula (1) to the semi-simple module $A_A$, since $\mathrm{End}_A(A) = A$ as right $A$-modules, and rings.
(In fact, it is not quite immediate as we need the following proposition) (see over)

PROPOSITION Let $A$ be a semi-simple ring, then there are only finitely many isomorphism classes of simple modules, and only finitely many simple modules occur in any factorisation

$$A = \bigoplus_{i \in I} S_i \qquad (2)$$

PROOF As noted before, every simple $A$-module is isomorphic to some right ideal of $A$, and any semi-simple ring $A$ can, in particular, be written as the direct sum of all its simple submodules. Assume (2) is such a sum. Suppose

$$1 = s_1 + s_2 + \cdots + s_n \qquad s_i \in S_i$$

Then we claim that $S_1, \ldots, S_n$ are the only simple submodules of $A$. Let $S$ be another one, $x \in S \subseteq A$. Then

$$\begin{aligned} x = 1 \cdot x &= (s_1 + \cdots + s_n) \cdot x \\ &= s_1 x + \cdots + s_n x \end{aligned}$$

which is an element of $\sum_{i=1}^n S_i$. Hence $S$ is a subobject of $\bigoplus_{i=1}^n S_i$ – but by Corollary 7.3, it follows that $S$ is one of the $S_i$. □

Let us continue the proof of Proposition 7.7. (f) $\Rightarrow$ (a). Notice that the product $\prod A_i$ of semi-simple rings is semi-simple, since if $A_i = \bigoplus_j A_{i,j}$, take the ideals $A_{1,1} \times 0 \cdots \times 0, \ldots, 0 \times A_{2,1} \times 0 \cdots \times 0$, etc. Hence if we show $M_n(D)$ is semi-simple when $D$ is a skew-field, we will be done. We can always write $M_n(D)$ as a direct sum of right ideals $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$, where $\mathfrak{a}_i$ consists of matrices with only the $i$th row different from zero. This ideal is minimal since if $0 \neq x \in \mathfrak{a}_i$, and $y$ is any other element of $\mathfrak{a}_i$, say

$$\begin{aligned} x &= (x_1 \; x_2 \; \cdots \; x_n) \\ y &= (y_1 \; y_2 \; \cdots \; y_n) \end{aligned}$$

then consider the matrix

$$\phi = \begin{pmatrix} x_1^{-1}y_1 & x_1^{-1}y_2 & \cdots & x_1^{-1}y_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \qquad \begin{aligned} &\phi \in M_n(D) \\ &x\phi = y. \end{aligned}$$

(using cols)

Hence $\mathfrak{a}_i$ is simple. Notice this trick works on the left as well, so $M_n(D)$ is both left and right semisimple. □

Since the condition (f) is left-right symmetric, it follows that "right" and "left" semi-simplicity are equivalent. Consider a decomposition of the semi-simple ring $A$ as given in condition (f). Each factor $M_{n_i}(D_i)$ is a two-sided ideal in $A$. We assert that there are no other two-sided ideals in $A$ other than the products of these factors. Since each ideal in $A$ is of the form $\mathfrak{a}_1 \times \cdots \times \mathfrak{a}_k$ with $\mathfrak{a}_i$ an ideal in $M_{n_i}(D_i)$, this will follow from:

PROPOSITION 7.8 The following properties of a ring $A$ are equivalent:

(a) $A$ is a semi-simple ring and has no two-sided ideals except $0$ and $A$.
(b) $A$ is a semi-simple ring and there is only one isomorphism class of simple modules
(c) $A \cong M_n(D)$ for some skew-field $D$.

PROOF (a)$\Rightarrow$(b) The existence of two non-isomorphic simple modules would imply the existence of non-isomorphic minimal right ideals, and this would give rise to two-sided ideals in $A$ by formula ①. (ie $A \times 0$, $0 \times A$ style ideals)

(b) $\Rightarrow$ (c) Clear

(c) $\Rightarrow$ (a) Suppose $\mathfrak{a}$ is a two-sided ideal $\neq 0$ in $M_n(D)$. As a means to prove that $\mathfrak{a} = M_n(D)$, we show that $\mathfrak{a}$ contains the minimal right ideal $\mathfrak{a}_i$ consisting of matrices with only the $i$-th row different from $0$. Choose a matrix $\beta \in \mathfrak{a}$ with some nonzero entry $\beta_{st}$. Let $\alpha \in \mathfrak{a}_i$ be the matrix with $\alpha_{is} = 1$ and all other entries $0$. Then $0 \neq \alpha\beta \in \mathfrak{a}_i \mathfrak{a}$, so $0 \neq \mathfrak{a}_i \mathfrak{a} \subseteq \mathfrak{a}_i \cap \mathfrak{a}$ and $\mathfrak{a}_i$ minimal implies that $\mathfrak{a}_i \subseteq \mathfrak{a}$. □

A ring $A$ is simple as a right $A$-module iff. it contains no nonzero proper right ideals, but this is iff. it is a skew-field. Hence this is an uninteresting condition on a ring $A$. Hence we define

DEFINITION A ring $A$ is *simple* if it satisfies the equivalent conditions of Prop. 7.8.

NOTE If $A$ is semi-simple, so $A$ has only finitely many simple submodules, then note every right ideal of $A$ is a sum of a finite number of these. Further, if $M = A/\mathfrak{a}$ is cyclic, $\exists\, \mathfrak{b}$ s.t. $\mathfrak{a} \oplus \mathfrak{b} = A$, so $M = \mathfrak{b}$, so any cyclic $A$-module is a direct sum of finitely many simple modules.

## EXAMPLES

### 1. Commutative rings

**2. Socle** If $M$ is a module, the sum of all simple submodules of $M$ is called the socle of $M$ and is denoted by $s(M)$. If $x \in s(M)$, then $xA$ is a direct sum of a finite number of simple modules. From this it is easy to see that

$$s(M) = \{x \in M \mid \text{Ann}(x) \text{ is a finite intersection of maximal right ideals}\} \; ?$$

**3. Group Rings** The semi-simplicity of certain group rings is a basic result for the theory of group representations. It is based on the following "averaging" principle. Suppose $A$ is a commutative ring and $G$ is a finite group. Let $M$ and $N$ be $A[G]$-modules, and suppose $\alpha : M \longrightarrow N$ is an $A$-linear map. Define $\tilde{\alpha} : M \longrightarrow N$ by

$$\tilde{\alpha}(x) = \sum_{g \in G} \alpha(xg^{-1})g$$

NOTE Let $A$ be a ring. Clearly any simple $A$-module is noetherian, and hence any finite direct sum of simple modules is noetherian. It follows that as an $A$-module, any semi-simple ring $A$ (as the finite direct sum of minimal right ideals) is noetherian as a right (resp. left) $A$-module. Hence any semi-simple ring is both left and right noetherian.

# 8. Tensor Products

Let $A$ be a ring and let there be given modules $L_A$ and $_AM$. We shall define the tensor product of $L$ and $M$ as a kind of linearisation of the product $L\times M$. Let $G$ be an abelian group. A map $\varphi : L\times M \longrightarrow G$ will be called <u>bilinear</u> if

$$\begin{aligned}
\varphi(a+a',b) &= \varphi(a',b)+\varphi(a,b) && \forall a,a'\in L\ \forall b\in M\\
\varphi(a,b+b') &= \varphi(a,b)+\varphi(a,b') && \forall a\in L\ \forall b,b'\in M\\
\varphi(xa,y) &= \varphi(x,ay) && \forall x\in L,\ y\in M,\ a\in A
\end{aligned}$$

<u>DEFINITION</u> A <u>tensor product</u> of $L_A$ and $_AM$ is an abelian group $T$ together with a bilinear map $\tau : L\times M \longrightarrow T$ such that for every abelian group $G$ and bilinear map $\varphi : L\times M\longrightarrow G$ there exists a unique homomorphism $\alpha : T\longrightarrow G$ satisfying $\alpha\tau = \varphi$.

$$\begin{tikzcd}
L\times M \arrow[r, "\tau"] \arrow[dr, "\varphi"'] & T \arrow[d, "\alpha"] \\
 & G
\end{tikzcd}$$

<u>PROPOSITION 8.1</u> (Unicity) If $(T,\tau)$ and $(T',\tau')$ are tensor products of $L_A$ and $_AM$, then there exists an isomorphism $\alpha : T\longrightarrow T'$ s.t. $\alpha\tau = \tau'$.

<u>PROPOSITION 8.2</u> (Existence) A tensor product of $L_A$ and $_AM$ exists.

Because of its uniqueness (up to isomorphism), we will speak of <u>the</u> tensor product of $L$ and $M$. We denote it by $L\otimes_A M$, with generators $\tau(x,y) = x\otimes y$. Every element of $L\otimes_A M$ looks like $\sum n_i(x_i\otimes y_i)$ with finite summation, with relations

$$\begin{aligned}
(x+x')\otimes y &= x\otimes y + x'\otimes y\\
x\otimes(y+y') &= x\otimes y + x\otimes y'\\
xa\otimes y &= x\otimes ay
\end{aligned}$$

<u>PROPOSITION 8.3</u> $A\otimes_A M \cong M$ for every $_AM$

<u>PROOF</u> Define $\tau : A\times M \longrightarrow M$ as $\tau(a,x) = ax$. The map $\tau$ is bilinear, and every bilinear map $\varphi : A\times M\longrightarrow G$ can be uniquely factored over $\tau$ by $\alpha : M \longrightarrow G$, $\alpha(x) = \varphi(1,x)$. The couple $(M,\tau)$ is thus a tensor product of $A$ and $M$. $\square$

If $A\otimes_A M$ is an arbitrary tensor product of $A$ and $M$, then the isomorphism $A\otimes_A M\longrightarrow M$ is given by $a\otimes m\mapsto am$. The isomorphism is natural in the sense that there is a natural equivalence between the functor $A\otimes_A\cdot$ and the identity functor. In the future we will use the term "natural isomorphism" to indicate the existence of a natural equivalence between functors in the occurring variables.

<u>PROPOSITION 8.4</u> If $L_i$ $(i\in I)$ are right $A$-modules and $M$ is a left $A$-module, then there is a natural isomorphism

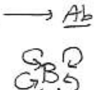
$$\Big(\bigoplus_I L_i\Big)\otimes_A M \cong \bigoplus_I\big(L_i\otimes_A M\big)$$

<u>PROOF</u> With the help of a suitable bilinear map one gets a homomorphism $\alpha : (\oplus_I L_i)\otimes M \longrightarrow \oplus_I(L_i\otimes M)$ such that $\alpha((x_i)_I\otimes y) = (x_i\otimes y)_I$. Similarly one obtains $\beta : \oplus_I(L_i\otimes M)\longrightarrow(\oplus_I L_i)\otimes M$ such that

$$\beta((x_i\otimes y_i)_I) = \sum_i(\lambda_i(x_i)\otimes y_i)$$

where $\lambda_i$ is the inclusion map. $\square$

Alternatively, notice that if $B$ is a right (resp. left) $A$-module – that is, a functor $A\longrightarrow \underline{Ab}$ which is contravariant (resp. covariant) then for any abelian group $C$, the two additive functors $H^C, H_C : \underline{Ab}\longrightarrow\underline{Ab}$ allow us to induce the following modules:



| <u>B left-module</u> | $C$ | | <u>B right-module</u> |
|---|---|---|---|
| $H^CB = [C,B]_{\underline{Ab}}$ | $H_CB = [B,C]_{\underline{Ab}}$ | $H^CB = [C,B]_{\underline{Ab}}$ | $H_CB = [B,C]_{\underline{Ab}}$ |
| left $A$-module | right $A$-module | right $A$-module | left $A$-module |
| $(a\cdot f)(c) = a\cdot f(c)$ | $(f\cdot a)(b) = f(a\cdot b)$ | $(f\cdot a)(c) \doteq f(c)\cdot a$ | $(a\cdot f)(b) = f(b\cdot a)$ |
| $a\cdot f = B(a)f$ | $f\cdot a = fB(a)$ | $f\cdot a = B(a)f$ | $a\cdot f = fB(a)$ |

Consider the bifunctor $Bil(L \times M, G) : \text{Mod-}A \times \underline{Ab} \longrightarrow \underline{Ab}$. This defines a functor, since if $\Theta : L' \times M \longrightarrow G$ is bilinear, $\alpha : L \longrightarrow L'$ a morphism of right $A$-modules, and $\beta : G \longrightarrow G'$ a morphism of abelian groups, then $\beta\Theta(\alpha\times 1)$ is bilinear.

PROPOSITION There is an isomorphism

$$[L \otimes_A M, G]_{\underline{Ab}} \cong Bil(L \times M, G)$$

natural in both $L$ and $G$.

PROOF There is no question of this isomorphism existing — we simply have to show it is natural. With $\alpha, \beta$ as above, this involves showing that the following diagrams commute

$$\begin{tikzcd}
{[L \otimes_A M, G]} \arrow[r, equal] \arrow[d, "{[1,\beta]}"'] & Bil(L \times M, G) \arrow[d, "{Bil(1,\beta)}"] \\
{[L \otimes_A M, G']} \arrow[r, equal] & Bil(L \times M, G')
\end{tikzcd}
\qquad
\begin{tikzcd}
{[L' \otimes_A M, G]} \arrow[r, equal] \arrow[d, "{[\alpha\otimes 1, 1]}"'] & Bil(L' \times M, G) \arrow[d, "{Bil(\alpha,1)}"] \\
{[L \otimes_A M, G]} \arrow[r, equal] & Bil(L \times M, G)
\end{tikzcd}$$

but these are both easy consequences of the definitions. □

By the previous page, $Hom_{\underline{Ab}}(M, G)$ has a canonical structure as a right $A$-module. Hence,

PROPOSITION There is an isomorphism

$$Bil(L \times M, G) \cong Hom_{\text{Mod-}A}(L, Hom_{\underline{Ab}}(M, G))$$

natural in both $L$ and $G$.

PROOF Of course, we identify both sides of this equation with subsets of $[L \times M, G]$, $[L, [M, G]]$, resp. satisfying certain conditions. Then using the adjunction

$$[L \times M, G] \cong [L, [M, G]]$$

and the fact that under this isomorphism, bilinear maps are identified precisely with those functions $\phi : L \longrightarrow [M, G]$ which preserve the action of $A$ and whose images are always morphisms of abelian groups, we have the result. □ (one should check the remaining details)

Hence the functor $- \otimes_A M : \text{Mod-}A \longrightarrow \underline{Ab}$ is left adjoint to the functor $Hom_{\underline{Ab}}(M, -)$ which takes an abelian group $G$ to the right $A$-module defined on $Hom_{\underline{Ab}}(M, G)$. Notice that the natural isomorphisms above are isomorphisms of groups, so that this is an adjunction of $\underline{Ab}$-valued bifunctors.

We show similarly that $L \otimes_A - : A\text{-}\underline{Mod} \longrightarrow \underline{Ab}$ is left adjoint to $Hom_{\underline{Ab}}(L, -) : \underline{Ab} \longrightarrow A\text{-Mod}$. Hence the tensor product is right exact in each of its variables, and preserves epis. More carefully,

PROPOSITION Let $L$ be a right $A$-module, $M$ a left $R$-module. Then there is an isomorphism of abelian groups

$$L \otimes_A M \cong M \otimes_{A^{op}} L$$

which is natural in $L, M$.

PROOF The map $T' : L \times M \longrightarrow M \otimes_{A^{op}} L$, $(\ell, m) \longmapsto m \otimes \ell$ is a tensor product of $L_A$ and ${}_AM$, for given $\varphi : L_A \times {}_AM \longrightarrow Q$ bilinear, induce bilinear $\bar{\varphi} : M_{A^{op}} \times {}_{A^{op}}L \longrightarrow Q$ and hence $\phi : M \otimes_{A^{op}} L \longrightarrow Q$ as required. Hence there is an isomorphism $\Theta_{L,M} : L \otimes_A M \longrightarrow M \otimes_{A^{op}} L$ unique making

$$\begin{tikzcd}
L \times M \arrow[r] \arrow[d, equal] & L \otimes_A M \arrow[d, equal, "\Theta_{L,M}"] \\
M \times L \arrow[r] & M \otimes_{A^{op}} L
\end{tikzcd}$$

commute. Then let $\alpha : L \longrightarrow L'$ and $\beta : M \longrightarrow M'$ (recall left $A$-morphisms are right $A^{op}$-morphisms). We must show

$$\begin{tikzcd} L'\otimes_A M \arrow[r, equal] \arrow[d, "\alpha\otimes_A\beta"'] & M\otimes_{A^{op}} L' \arrow[d, "\beta\otimes_{A^{op}}\alpha"] \\ L\otimes_A M' \arrow[r, equal] & M'\otimes_{A^{op}} L \end{tikzcd}$$

commutes. But this is trivial. $\square$

PROPOSITION 8.8 Let $(y_i)_I$ be a family of generators for ${}_AM$, and let $(x_i)_I$ be a family of elements of $L_A$ with almost all $x_i = 0$. Then $\sum_i x_i \otimes y_i = 0$ in $L \otimes_A M$ if and only if there exist a finite family $(u_j)_J$ of elements of elements of $L$ and a family $(a_{ji})_{J\times I}$ of elements of $A$ such that

(i) $a_{ji} = 0$ for almost all $(j,i)$

(ii) $\sum_I a_{ji} y_i = 0$ for each $j \in J$

(iii) $x_i = \sum_J u_j a_{ji}$ for each $i \in I$

PROOF It is clear that the given conditions are sufficient to make $\sum x_i \otimes y_i = 0$, because they give

$$\sum_i x_i \otimes y_i = \sum_{j,i} u_j a_{ji} \otimes y_i = \sum_j u_j \otimes \left(\sum_i a_{ji} y_i\right) = \sum_J u_j \otimes 0 = 0$$

Suppose on the other hand that $\sum_i x_i \otimes y_i = 0$. Since $y_i$ $(i \in I)$ generate $M$, there is an exact sequence

$$0 \longrightarrow K \xrightarrow{\alpha} A^{(I)} \xrightarrow{\beta} M \longrightarrow 0$$

where $\beta$ maps the canonical basis vector $e_i \in A^{(I)}$ to $y_i$. By tensoring with $L$ we get an exact sequence

$$L\otimes K \xrightarrow{1\otimes\alpha} L\otimes (A)^{(I)} \xrightarrow{1\otimes\beta} L\otimes M \longrightarrow 0$$

The hypothesis that $\sum x_i \otimes y_i = 0$ implies $\sum x_i \otimes e_i \in \mathrm{Ker}(1\otimes\beta) = \mathrm{Im}(1\otimes\alpha)$, so $\sum x_i \otimes e_i = \sum_j u_j \otimes \alpha(z_j)$ for some $u_j \in L$, $z_j \in K$. Each $\alpha(z_j) \in A^{(I)}$ can be expressed in the canonical basis as $\alpha(z_j) = \sum_i a_{ji} e_i$. Then $\beta\alpha = 0$ gives $0 = \beta(\sum_i a_{ji} e_i) = \sum_i a_{ji} y_i$, which is the desired condition (ii). We also have

$$\sum_i x_i \otimes e_i = \sum_j u_j \otimes \alpha(z_j) = \sum_{j,i} u_j \otimes a_{ji} e_i$$

in $L\otimes A^{(I)}$. Under the isomorphism $L\otimes A^{(I)} \cong L^{(I)}$, this gives $x_i = \sum_j u_j a_{ji}$ for each $i$ – condition (iii). $\square$

EXAMPLES

1. Tensoring with cyclic modules

Let $\mathfrak{a}$ be a right ideal of $A$ and $M$ a left $A$-module. From the exact sequence $0 \longrightarrow \mathfrak{a} \longrightarrow A \longrightarrow A/\mathfrak{a} \longrightarrow 0$ we get an exact sequence

$$\mathfrak{a}\otimes M \xrightarrow{\alpha} M \longrightarrow (A/\mathfrak{a})\otimes M \longrightarrow 0$$

where $\mathrm{Im}\,\alpha = \{\sum a_i x_i \mid a_i \in \mathfrak{a}, x_i \in M\} = \mathfrak{a}M$. It follows that $(A/\mathfrak{a})\otimes M \cong M/\mathfrak{a}M$. (as abelian groups – since $\mathfrak{a}$ may not be two-sided, $\mathfrak{a}M$ is not in general a submodule of $M$).

# 9. BIMODULES

Let $A$, $B$ and $C$ be rings. Recall that $M$ is a $B$-$A$-bimodule if $M$ is a left $B$-module and a right $A$-module, such that $(bx)a = b(xa)$ for $b \in B$, $x \in M$, $a \in A$. We write ${}_BM_A$ to indicate this situation. The $B$-module structure of a bimodule ${}_BM_A$ can be described as a ring homomorphism $\varphi : B \longrightarrow \mathrm{End}_A(M)$. For if ${}_BM_A$ is a bimodule, then we define $\varphi$ as $\varphi(b) : x \mapsto bx$ for $x \in M$; conversely, if $\varphi$ is given, then $M$ becomes a $B$-$A$-bimodule by putting $bx = \varphi(b)(x)$. Let $T : \underline{\mathrm{Mod}}-A \longrightarrow \underline{\mathrm{Mod}}-C$ be an additive functor. For a bimodule ${}_BM_A$ we get a map

$$B \longrightarrow \mathrm{End}_A(M) \longrightarrow \mathrm{End}_C(T(M)) \qquad (1)$$

where the second map, induced by $T$, is a ring homomorphism since $T$ is an additive functor. By the previous remark it follows that $T(M)$ is a $B$-$C$-bimodule. We now apply these considerations to the functors Hom and $\otimes$. Let $L$ be a right $A$-module, and consider the functor $\mathrm{Hom}_A(L, \cdot) : \underline{\mathrm{Mod}}-A \longrightarrow \underline{\mathrm{Mod}}-\mathbb{Z}$. If $M$ is a $B$-$A$-bimodule, then $\mathrm{Hom}_A(L,M)$ becomes a left $B$-module. When one examines what (1) means in this case, it turns out that the $B$-module structure is defined as follows: let $b \in B$ and $\varphi \in \mathrm{Hom}_A(L,M)$; then $b\varphi : L \longrightarrow M$ maps $x \in L$ to $b\varphi(x) \in M$.

Next we look at the contravariant functor $\mathrm{Hom}_A(\cdot, M) : \underline{\mathrm{Mod}}-A \longrightarrow \underline{\mathrm{Mod}}-\mathbb{Z}$. If $L$ is a $B$-$A$-bimodule, then $\mathrm{Hom}_A(L,M)$ is a right $B$-module. If $b \in B$ and $\varphi \in \mathrm{Hom}_A(L,M)$, then $\varphi b : L \longrightarrow M$ maps $x \in L$ to $\varphi(bx) \in M$. More generally, given two bimodules ${}_BL_A$ and ${}_CM_A$, one gets a $C$-$B$-bimodule $\mathrm{Hom}_A(L,M)$.

Finally, we consider the tensor product. Given bimodules ${}_BL_A$ and ${}_AM_C$, one gets a $B$-$C$-bimodule $L \otimes_A M$ with $b(x \otimes y)c = (bx) \otimes (yc)$ for $b \in B$, $x \in L$, $y \in M$ and $c \in C$. The associativity rule for tensor products can now be stated

PROPOSITION 9.1 Given modules $L_A$, ${}_AM_B$ and ${}_BN$, there is a natural isomorphism

$$(L \otimes_A M) \otimes_B N \cong L \otimes_A (M \otimes_B N)$$

PROOF We try to define the isomorphism on the generators of $(L \otimes_A M) \otimes_A N$ as $(x \otimes y) \otimes z \mapsto x \otimes (y \otimes z)$. The only problem is to show that this gives a well-defined homomorphism, for then by symmetry there is a similar map in the opposite direction, and the two maps are inverse of each other.

For each $z \in N$, we define $\alpha_z : M \longrightarrow M \otimes_B N$ as $\alpha_z(y) = y \otimes z$, and then get a homomorphism $\beta_z = 1 \otimes \alpha_z$: $L \otimes_A M \longrightarrow L \otimes_A (M \otimes_B N)$. We then define

$$\varphi : (L \otimes_A M) \times N \longrightarrow L \otimes_A (M \otimes_B N)$$

as $\varphi(u,z) = \beta_z(u)$. $\varphi$ is easily verified to be a bilinear map, since $\beta_{z+z'}(\ell \otimes m) = \ell \otimes \alpha_{z+z'}(m)$, which is $\ell \otimes (m \otimes z+z') = \ell \otimes (m \otimes z + m \otimes z') = \ell \otimes (m \otimes z) + \ell \otimes (m \otimes z')$. Then $\varphi$ induces the desired homomorphism. $\square$

The next result is a very useful formula relating the tensor product to the Hom-functor. Let $L_A$, ${}_AM_B$ and $N_B$ be given modules. We define $\mathrm{Bil}_B(L \times M, N)$ to consist of those bilinear maps $L \times M \longrightarrow N$ (bilinear in $A$) which also respect the $B$-structure, so

$$\varphi(xa, y) = \varphi(x, ay)$$
$$\varphi(x, yb) = \varphi(x,y)b.$$

We already know there is an isomorphism, natural in $L$ and $N$

$$\mathrm{Hom}(L \otimes_A M, N) = \mathrm{Bil}(L \times M, N) \cong \mathrm{Hom}_A(L, \mathrm{Hom}_{\underline{Ab}}(M,N))$$

and the bilinear maps on the left which correspond to those in $\mathrm{Bil}_B(L \times M, N)$ are precisely those whose images $\hat{\varphi} : L \longrightarrow \mathrm{Hom}_{\underline{Ab}}(M,N)$ are pointwise morphisms of $B$-modules. Hence

PROPOSITION 9.2 Given modules $L_A$, ${}_AM_B$ and $N_B$, there is a natural isomorphism

$$\mathrm{Hom}_B(L \otimes_A M, N) \cong \mathrm{Hom}_A(L, \mathrm{Hom}_B(M,N))$$

That is, the functor $- \otimes_A M : \underline{\mathrm{Mod}}-A \longrightarrow \underline{\mathrm{Mod}}-B$ is left adjoint to the functor $\mathrm{Hom}_B(M, -) : \underline{\mathrm{Mod}}\,B \longrightarrow \underline{\mathrm{Mod}}-A$.

PROPOSITION 9.3 The module $Hom_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$ is an injective cogenerator for Mod $-$ $A$.

PROOF Here we consider $A$ as an $A-\mathbb{Z}-$ bimodule. Then for every module $L_A$ we get

$$Hom_A(L_A, Hom_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})) \cong Hom_{\mathbb{Z}}(L \otimes_A A, \mathbb{Q}/\mathbb{Z}) \cong Hom_{\mathbb{Z}}(L, \mathbb{Q}/\mathbb{Z}). \square$$

PROPOSITION 9.5 The following properties of a ring $A$ are equivalent

(a) $A$ is right hereditary

(b) Every submodule of a projective module is projective

(c) Every quotient of an injective module is injective.

PROOF (a) $\Rightarrow$ (c). Let $E$ be an injective module, and consider an epimorphism $\alpha: E \longrightarrow M$. Suppose $\varphi: \mathfrak{a} \longrightarrow M$ is a homomorphism from a right ideal $\mathfrak{a}$. Since $\mathfrak{a}$ is a projective module, $\varphi$ can be lifted to a homomorphism $\varphi': \mathfrak{a} \longrightarrow E$. But since $E$ is injective, $\varphi'$ can be extended to $\psi: A \longrightarrow E$, and the composed map $\alpha\psi$ extends $\varphi$, as desired.

(c) $\Rightarrow$ (b) Let $L$ be a submodule of a projective module $P$. Suppose there is given an exact sequence $0 \rightarrow K \rightarrow M \rightarrow N \rightarrow 0$. and a homomorphism $\varphi: L \longrightarrow N$. If we consider $M$ as a submodule of an injective module $E$, we obtain a commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K \arrow[r] \arrow[d, equal] & M \arrow[r] \arrow[d, "\mu"'] & N \arrow[r] \arrow[d, "\eta"] & 0 \\
0 \arrow[r] & K \arrow[r] & E \arrow[r] \arrow[d, "\mu'"'] & E/K \arrow[r] \arrow[d, "\eta'"] & 0 \\
 & & E/M \arrow[r, equal] \arrow[d] & E/M & \\
 & & 0 & &
\end{tikzcd}$$

⌈ $\eta'$ induces since $\mu'\mu = 0$.
$\eta$ monic using M. notes p21a
LEMMA b, c. ⌋

since $E/K$ is an injective module, the map $\eta\varphi$ extends to $\varphi': P \longrightarrow E/K$. But since $P$ is projective, $\varphi'$ can be lifted to $\psi: P \longrightarrow E$. Then $\psi(L) \subseteq M$, because $\mu'\psi = \eta'\varphi'$ is zero on the submodule $L$. Then using the fact that $\eta$ is monic, we see that $\psi$ induces $L \longrightarrow M$ extending $\varphi$.

(b) $\Rightarrow$ (a) is trivial. $\square$

EXAMPLE (Extension of Scalars) Let $\varphi: A \longrightarrow B$ be a morphism of rings. If $M$ is a right $A$-module, then we may form $M \otimes_A B$, where $B$ is considered as a left $A$-module by restriction of scalars, and $M \otimes_A B$ is a right $B$-module. E.g if $M$ is a real vector space, then $M \otimes_{\mathbb{R}} \mathbb{C}$ is the "complexification" of $M$.

# 10. FLAT MODULES

DEFINITION A left $A$-module $F$ is *flat* if the functor $-\otimes_A F$ is exact.

Since the tensor product is always right exact, $F$ is flat iff. $\otimes_A F$ preserves monomorphisms of right $A$-modules.

PROPOSITION 10.1 If $(F_i)_I$ is a family of left $A$-modules, then $\oplus_I F_i$ is flat iff. each $F_i$ is flat.

PROOF Let $L \longrightarrow M$ be a monomorphism of right $A$-modules. By Prop 8.4 there is a commutative diagram

$$\begin{tikzcd} L\otimes_A(\oplus_I F_i) \arrow[r] \arrow[d, "\cong"] & M\otimes_A(\oplus_I F_i) \arrow[d, "\cong"] \\ \oplus_I(L\otimes_A F_i) \arrow[r] & \oplus_I(M\otimes_A F_i) \end{tikzcd}$$

The upper row is a monomorphism iff the bottom row is, and this happens iff. each $L\otimes_A F_i \longrightarrow M\otimes_A F_i$ is monic. □

COROLLARY 10.2 Every projective module is flat.

PROOF ${}_AA$ is flat since $L\otimes_A A \cong A$. Hence every free module is flat by Prop 10.1., and it follows from Prop 6.1 and 10.1 that projective modules are flat. □

From Prop 8.7 we obtain

PROPOSITION 10.3 Every direct limit of flat modules is flat.

The associativity relation between Hom and $\otimes$ establishes an intimate connection between flat modules and injective modules.

PROPOSITION 10.4 Let there be given modules ${}_AF_B$ and $E_B$, and assume that $E_B$ is an injective cogenerator. Then ${}_AF$ is flat if and only if $\mathrm{Hom}_B(F,E)$ is an injective right $A$-module.

PROOF Let $\alpha: M \longrightarrow N$ be an arbitrary monomorphism of right $A$-modules. It induces a commutative diagram

$$\begin{tikzcd} \mathrm{Hom}_A(N,\mathrm{Hom}_B(F,E)) \arrow[r, "\mathrm{Hom}(\alpha,1)"'] \arrow[d, "\cong"] & \mathrm{Hom}_A(M,\mathrm{Hom}_B(F,E)) \arrow[d, "\cong"] \\ \mathrm{Hom}_B(N\otimes_A F,E) \arrow[r, "\mathrm{Hom}(\alpha\otimes 1,1)"'] & \mathrm{Hom}_B(M\otimes_A F,E) \end{tikzcd}$$

with vertical isomorphisms furnished by Prop 9.2. If ${}_AF$ is flat, $\alpha\otimes 1: M\otimes_A F \longrightarrow L\otimes_A F$ is a monomorphism, and $E_B$ injective implies that $\mathrm{Hom}(\alpha\otimes 1,1)$ is an epimorphism, and hence $\mathrm{Hom}_B(F,E)$ is an injective $A$-module.

Assume now instead that $\mathrm{Hom}_B(F,E)$ is an injective $A$-module. Then as above, $\mathrm{Hom}(\alpha\otimes 1,1)$ is an epimorphism. Let $x$ be any nonzero element of $M\otimes_A F$. Since $E_B$ is an injective cogenerator, there is a $B$-linear map $\varphi: M\otimes_A F \longrightarrow E$ s.t. $\varphi(x) \neq 0$. Since $\mathrm{Hom}(\alpha\otimes 1,1)$ is an epimorphism, we have $\varphi = \psi(\alpha\otimes 1)$ for some $\psi: N\otimes_A F \longrightarrow E$. From $\varphi(x)\neq 0$ must then follow that $(\alpha\otimes 1)(x)\neq 0$, so $\alpha\otimes 1$ is a monomorphism. Hence ${}_AF$ is flat. □

If we take $B=\mathbb{Z}$ and $E=\mathbb{Q}/\mathbb{Z}$, we get in particular the remarkable

COROLLARY 10.5 A module ${}_AF$ is flat iff. $\mathrm{Hom}_{\mathbb{Z}}(F,\mathbb{Q}/\mathbb{Z})$ is an injective right $A$-module.

We can use this result to get an analogue of Prop 6.5 for flat modules. We denote $\hat{F} = \mathrm{Hom}_{\mathbb{Z}}(F,\mathbb{Q}/\mathbb{Z})$.

PROPOSITION 10.6 A module ${}_AF$ is flat if and only if the canonical map $\mathfrak{a}\otimes_A F \longrightarrow F$ is a monomorphism for every finitely generated right ideal $\mathfrak{a}$ of $A$.

PROOF Suppose $\mathfrak{a}\otimes F \longrightarrow F$ is a monomorphism for finitely generated right ideals $\mathfrak{a}$. Then it is so also for arbitrary right ideals $\mathfrak{a}$, for if $\sum a_i \otimes y_i \in \mathfrak{a}\otimes F$, then the $a_i$ lie in a finitely generated right ideal $\mathfrak{b}\subseteq \mathfrak{a}$, and the composed map $\mathfrak{b}\otimes F \longrightarrow \mathfrak{a}\otimes F \longrightarrow F$ is a monomorphism (hence so is $\mathfrak{b}\otimes F \longrightarrow \mathfrak{a}\otimes F$). So if $\sum a_i \otimes y_i$ gets zero in $F$, then $\sum a_i \otimes y_i$ is zero in $\mathfrak{b}\otimes F$ and hence in $\mathfrak{a}\otimes F$.

The inclusion map $\mathfrak{a} \longrightarrow A$ induces a monomorphism $\mathfrak{a}\otimes F \longrightarrow F$, and there results a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}(A, \hat{F}) \arrow[r] \arrow[d, "\cong"'] & \mathrm{Hom}(\mathfrak{a}, \hat{F}) \arrow[d, "\cong"] \\
\hat{F} \arrow[r] & \widehat{\mathfrak{a}\otimes F}
\end{tikzcd}$$

where the lower map is an epimorphism. So is then also the upper map, and therefore $\hat{F}$ is injective by Prop 6.5. But then $F$ is flat by Cor 10.5. □

EXAMPLES

1. Flat modules are torsion free Suppose ${}_AF$ is flat and let $s$ be a nonzero divisor of $A$. Define $\alpha : A \longrightarrow A$ as $\alpha(a) = sa$, which is a monomorphism of right $A$-modules. It gives rise to a commutative diagram

$$\begin{tikzcd}
A\otimes F \arrow[r, "\alpha\otimes 1"] \arrow[d, "\cong"'] & A\otimes F \arrow[d, "\cong"] \\
F \arrow[r, "\beta"'] & F
\end{tikzcd}$$

where $\beta(x) = sx$. Since $\alpha\otimes 1$ is a monomorphism, so also is $\beta$, and hence $x\neq 0$ implies $sx \neq 0$. The module $F$ is thus torsion-free, in the sense that $x\neq 0$ and $s$ not a zero divisor implies $sx\neq 0$.

PROPOSITION 10.7 A module ${}_AF$ is flat iff. it satisfies: $\sum_{i=1}^{n} b_i x_i = 0$ for $b_i\in A$, $x_i \in F$ then there exist $u_1,\dots,u_m$ in $F$ and $a_{ij}\in A$ $(i=1,\dots,n, j=1,\dots,m)$ such that

$$\sum_i b_i a_{ij} = 0$$

and

$$x_i = \sum_j a_{ij} u_j$$

PROOF Suppose ${}_AF$ is flat and $\sum b_i x_i = 0$ in $F$. Let $\mathfrak{a}$ be the right ideal generated by $b_1,\dots,b_n$. Then $\mathfrak{a}\otimes F \longrightarrow F$ is a monomorphism, so we must have $\sum b_i \otimes x_i = 0$ in $\mathfrak{a}\otimes F$, and we can apply Prop 8.8 to obtain the stated condition.

Suppose conversely that $F$ satisfies the condition in question. Let $\mathfrak{a}$ be an arbitrary right ideal and consider the map $\mathfrak{a}\otimes F \longrightarrow F$. If $\sum b_i \otimes x_i \in \mathfrak{a}\otimes F$ goes to zero in $F$, then $\sum b_i x_i = 0$, and there exist $u_j$ and $a_{ij}$ according to the condition. This gives immediately $\sum_i b_i \otimes x_i = \sum_{i,j} b_i \otimes a_{ij} u_j = \sum_{i,j} b_i a_{ij} \otimes u_j = \sum_j 0\otimes u_j = 0$. The map $\mathfrak{a}\otimes F \longrightarrow F$ is thus monic, and $F$ is flat by Prop 10.6. □

This characterises flat modules, and essentially states that a module $F$ is flat iff. all relations in $F$ are due to relations already existing in $A$.

## 11. PURE SUBMODULES

A short exact sequence $0 \to M' \to M \to M'' \to 0$ of left $A$-modules is called *pure* if $L \otimes_A M' \to L \otimes_A M$ is a monomorphism for every right $A$-module $L$. Since the tensor product commutes with direct limits, it suffices to require this for finitely generated modules $L$, and since every finitely generated module is a direct limit of finitely presented modules (Example 5.2), one may even assume $L$ finitely presented. The following result relates flatness to purity:

**PROPOSITION 11.1** The following properties of a module ${}_AF$ are equivalent:

(a) $F$ is flat

(b) Every exact sequence $0 \to M' \to M \to F \to 0$ is pure

(c) There is a pure exact sequence $0 \to M' \to M \to F \to 0$ where $M$ is a flat module.

**PROOF** (a)$\Rightarrow$(b) If $L$ is a right $A$-module, we choose an exact sequence $0 \to K \to H \to L \to 0$ with $H$ free. We get a commutative diagram

$$\begin{tikzcd}
 & K \otimes M' \arrow[r] \arrow[d] & K \otimes M \arrow[r] \arrow[d] & K \otimes F \arrow[r] \arrow[d, "\alpha"] & 0 \\
0 \arrow[r] & H \otimes M' \arrow[r, "\mu"] \arrow[d] & H \otimes M \arrow[r, "\varphi"] \arrow[d] & H \otimes F \arrow[r] \arrow[d] & 0 \\
 & L \otimes M' \arrow[r, "\beta"] \arrow[d] & L \otimes M \arrow[r] \arrow[d] & L \otimes F \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{1}$$

with exact rows and columns. By hypothesis we know that $\alpha$ is a monomorphism. We will show that it follows that $\beta$ is a monomorphism, by making a diagram chase. If $x \in \operatorname{Ker}\beta$, then $x$ can be lifted to $y \in H \otimes M'$, and $\mu(y) \in H \otimes M$ goes to $\beta(x) = 0$ in $L \otimes M$. Therefore $\mu(y)$ comes from some element $z \in K \otimes M$, and $z$ maps to an element $u$ in $K \otimes F$. But $\alpha(u) = \varphi\mu(y) = 0$, so $\alpha$ monic implies $u = 0$. The element $z \in K \otimes M$ must then be the image of some $v \in K \otimes M'$, and $v$ maps to $y$ in $H \otimes M'$ since $\mu$ is a monomorphism. It follows that $x \in \operatorname{Ker}\beta$ is zero, so $\beta$ is a monomorphism.

(b) $\Rightarrow$ (c) is obvious (write $F$ as the quotient of a free module $M$)

(c) $\Rightarrow$ (a) We let $0 \to M' \to M \to F \to 0$ be a pure exact sequence with $M$ flat, and consider any exact sequence $0 \to K \to H \to L \to 0$ of right $A$-modules. We then get a diagram (1), where now the middle column instead of the middle row is a short exact sequence. $\beta$ is a monomorphism, so we are in a situation completely symmetrical to the one in (a) $\Rightarrow$ (b), and it follows that $\alpha$ is a monomorphism. Hence $F$ is flat. $\square$

A submodule $M'$ of $M$ is a *pure* submodule if the exact sequence $0 \to M' \to M \to M/M' \to 0$ is pure. The following characterisation of pure submodules is an analogue of Prop 10.7 for flatness.

**PROPOSITION 11.2** A submodule $M'$ of $M$ is pure in $M$ if and only if it satisfies: if $y_1, \ldots, y_n$ are elements in $M'$, $(a_{ij})$ is an $m \times n$ matrix of elements in $A$ and the system of equations

$$\sum_{i=1}^{m} X_i a_{ij} = y_j \qquad (j = 1, \ldots, n)$$

has a solution $(x_1, \ldots, x_m)$ in $M$, then it has a solution in $M'$.

**PROOF** We assume the condition is satisfied, and want to show that for every left module $L$ we get a monomorphism $M' \otimes_A L \to M \otimes L$. Clearly we may assume $L$ finitely generated, and since every finitely generated module is a direct limit of finitely presented modules, we may even assume $L$ finitely presented. Thus there is an exact sequence

$$A^m \xrightarrow{\alpha} A^n \to L \to 0$$

where $\alpha$ is represented by a matrix $(a_{ij})$ by the rule $\alpha : (x_i) \mapsto (\sum_i a_{ij} x_i)$. There results a commutative diagram

$$\begin{tikzcd}
M'\otimes A^m \arrow[r] \arrow[d] & M'\otimes A^n \arrow[r] \arrow[d] & M'\otimes L \arrow[r] \arrow[d] & 0 \\
M\otimes A^m \arrow[r] & M\otimes A^n \arrow[r] & M\otimes L \arrow[r] & 0
\end{tikzcd} \tag{3}$$

where the two vertical arrows are monomorphisms. We need the following lemma

<u>LEMMA 11.3</u> Let the diagram

$$\begin{tikzcd}
M' \arrow[r, "\mu'"] \arrow[d, "\alpha'"] & M \arrow[r, "\mu"] \arrow[d, "\alpha"] & M'' \arrow[r] \arrow[d, "\alpha''"] & 0 \\
N' \arrow[r, "\eta'"'] & N \arrow[r, "\eta"'] & N'' \arrow[r] & 0
\end{tikzcd}$$

be commutative with exact rows. Suppose $\alpha$ is a monomorphism. Then $\alpha''$ is a monomorphism if and only if $\operatorname{Im}\eta' \cap \operatorname{Im}\alpha = \operatorname{Im}\alpha\mu'$.

<u>PROOF</u> Suppose $\alpha''$ is monic. Clearly $\operatorname{Im}\alpha\mu' \subseteq \operatorname{Im}\eta' \cap \operatorname{Im}\alpha$, so suppose $x \in N$ is s.t. $x = \alpha(y) = \eta'(z)$. Then $\alpha''\mu(y) = \eta\alpha(y) = \eta\eta'(z) = 0$, so that as $\alpha''$ is monic, $\mu(y) = 0$ and hence by exactness of the top row, $y = \mu'(q)$ some $q \in M'$. Hence $x = \alpha(y) = \alpha\mu'(q) \in \operatorname{Im}\alpha\mu'$, as required. Conversely, suppose $\operatorname{Im}\eta' \cap \operatorname{Im}\alpha = \operatorname{Im}\alpha\mu'$ and say $\alpha''(m) = 0$, $m \in M''$. Since $\mu$ is epi, let $n \in M$ be s.t. $m = \mu(n)$. Then $\alpha''\mu(n) = 0$, so $\eta\alpha(n) = 0$, hence $\alpha(n) \in \operatorname{Im}\eta' \cap \operatorname{Im}\alpha = \operatorname{Im}\alpha\mu'$ by exactness of the bottom row. Say $\alpha(n) = \alpha\mu'(x)$, $x \in M'$. Then $n = \mu'(x)$ since $\alpha$ is monic. Hence $m = \mu(n) = \mu\mu'(x) = 0$, so $\alpha''$ is monic. □

We will apply the lemma to diagram (3). An element $(x_1, \dots, x_m) \in M^m \cong M \otimes A^m$ maps to $u = \sum_i x_i \otimes (a_{ij}) \in M \otimes A^n$ which also is the image of $(y_1, \dots, y_n) \in M'^n$ just when $\sum_i x_i a_{ij} = y_j$. The existence of a solution $(z_1, \dots, z_m)$ in $M'$ of this system means that $u$ is the image of an element in $M' \otimes A^m = M'^m$, so the Lemma can be used to show that $M' \otimes L \longrightarrow M \otimes L$ is a monomorphism.

Conversely, assume that $M'$ is pure in $M$ and that the system (2) has a solution in $M$. The matrix $(a_{ij})$ determines a morphism $\alpha : A^m \longrightarrow A^n$, and one puts $L = A^n / \operatorname{Im}\alpha$. One can now follow the preceeding argument backwards to get the existence of a solution in $M'$. □

<u>COROLLARY 11.4</u> Let $M$ be a flat module and $0 \to K \to F \to M \to 0$ an exact sequence with $F$ free. For any finite family $u_1, \dots, u_m$ of elements of $K$ there exists a homomorphism $\varphi : F \longrightarrow K$ s.t. $\varphi(u_j) = u_j$, $j = 1, \dots, n$.

<u>PROOF</u> We can write

$$u_j = \sum_{i=1}^{m} x_i a_{ij} \quad j = 1, \dots, n \tag{4}$$

where $x_1, \dots, x_m$ is a basis (more precisely, <u>part</u> of a basis) for the free module $F$. Since $M$ is flat, the given exact sequence is pure (Prop 11.1), so the system (4) has a solution $z_1, \dots, z_m$ in $K$. We can define $\varphi : F \longrightarrow K$ so that $\varphi(x_i) = z_i$, and then get

$$\varphi(u_j) = \varphi\Big(\sum_i x_i a_{ij}\Big) = \sum_i z_i a_{ij} = u_j. \quad \square$$

A striking consequence of this Corollary is:

<u>COROLLARY 11.5</u> Every finitely presented flat module is projective.

<u>PROOF</u> Let $M$ be finitely presented and flat. Choose an exact sequence $0 \to K \to F \to M \to 0$ with $F$ finitely generated free and $K$ generated by $u_1, \dots, u_m$. If $\varphi : F \longrightarrow K$ maps $u_j$ to $u_j$, according to the corollary, then $\varphi$ splits the sequence, and hence $M$ is projective. □

Another very useful consequence of this corollary is:

<u>PROPOSITION 11.6</u> Let $A$ be a subring of a ring $B$. If $M$ is a finitely generated and flat right $A$-module, and $M \otimes_A B$ is a projective $B$-module, then $M$ is projective over $A$.

PROOF Choose an exact sequence $0 \to K \to F \to M \to 0$ with $F$ finitely generated free. Since $M$ is flat, tensoring with $B$ gives an exact sequence

$$0 \to K \otimes B \to F \otimes B \to M \otimes B \to 0$$

by Prop 11.1. This sequence splits, because $M \otimes B$ was assumed to be $B$-projective. It follows that $K \otimes B$ is finitely generated over $B$, say with generators $k_1 \otimes 1, \ldots, k_n \otimes 1$. By Prop 11.4 there exists a homomorphism $\varphi : F \to K$ such that $\varphi(h_j) = k_j$ for $j = 1, \ldots, n$. We will show that $\varphi(k) = k$ holds for all $k \in K$, so that $\varphi$ splits and $M$ is projective. If $k \in K$, we may write $k \otimes 1 = \sum (k_i \otimes 1) b_i$ with $b_i \in B$. Then

$$\begin{aligned} \varphi(k) \otimes 1 &= (\varphi \otimes 1)(k \otimes 1) \\ &= \sum (\varphi \otimes 1)(k_i \otimes b_i) \\ &= \sum \varphi(k_i) \otimes b_i \\ &= \sum k_i \otimes b_i \\ &= k \otimes 1 \end{aligned}$$

in $K \otimes B$. But since $F$ and $M$ are flat, also $K$ is flat (Exercise 36), and therefore $K \cong K \otimes A \to K \otimes B$ is a monomorphism. Hence $\varphi(k) = k$ for each $k \in K$. $\square$

EXAMPLE Integral Domains If $A$ is a commutative integral domain with field of fractions $K$, then every finitely generated flat-module is projective by Prop 11.6 since $M \otimes_A K$ is a vector space over $K$.

EXERCISES Ch I Stenström.

**Q36** Suppose $0 \to L \to M \to N \to 0$ is exact and that $N$ is flat. We claim then that $L$ is flat iff. $M$ is flat. Let $C \to D$ be a monomorphism of right $A$-modules, so that we have a commutative diagram with exact rows:

$$\begin{tikzcd} C\otimes L \arrow[r, "\theta"] \arrow[d, "\alpha'"] & C\otimes M \arrow[r, "\eta"] \arrow[d, "\alpha"] & C\otimes N \arrow[r] \arrow[d, "\alpha''"] & 0 \\ D\otimes L \arrow[r, "\theta'"'] & D\otimes M \arrow[r, "\eta'"'] & D\otimes N \arrow[r] & 0 \end{tikzcd}$$

since $N$ is flat, by Prop 11.1 $0 \to L \to M \to N \to 0$ is pure, so that $\theta, \theta'$ are both monic. So suppose $L$ is flat. Then $\alpha'$ is monic, so if $\alpha(x) = 0$, $\eta'\alpha(x) = 0$, so $\alpha''\eta(x) = 0$, hence since $N$ is flat $\alpha''$ is monic, and $\eta(x) = 0$. This implies $x = \theta(y)$ for some $y \in C\otimes L$. Hence $\theta'\alpha'(y) = \alpha\theta(y) = \alpha(x) = 0$, so $\alpha'(y) = 0$ and hence $y = 0$. Thus $\alpha$ is monic. Conversely if $M$ is flat, so $\alpha$ is monic, suppose $\alpha'(y) = 0$. Then $\alpha\theta(y) = \theta'\alpha'(y) = 0$, so since $\alpha\theta$ is monic, $y = 0$. Since $C \to D$ was arbitrary, we have shown the result. □

**Q18** Let $A$ be a ring and $1 : \underline{\text{Mod}}\,A \to \underline{\text{Mod}}\,A$ the identity functor. For $a \in A$ in the center of $A$ we define a natural transformation $1 \to 1$ by $\varphi(a)_M : x \mapsto xa$ for $x \in M$. This gives an isomorphism of rings

$$\varphi : \text{cen}A \longrightarrow [1,1]$$

since $\varphi(a)_A(1) = a$, so $\varphi$ is injective, and $\varphi(ab)_M(x) = x\cdot ab = (x\cdot b)\cdot a = \varphi(b)_M(x)\cdot a = \varphi(a)_M(\varphi(b)_M(x))$, which is $(\varphi(a)\varphi(b))_M(x)$, and $\varphi(a+b)_M(x) = x\cdot(a+b) = x\cdot a + x\cdot b = \varphi(a)_M(x) + \varphi(b)_M(x) = (\varphi(a)+\varphi(b))_M(x)$. Moreover, $\varphi(0)_M(x) = 0 = 0_M(x)$, and $\varphi(1)_M(x) = x = 1_M(x)$, so $\varphi$ is a morphism of rings. It is surjective since if $\phi : 1 \to 1$, let $a = \phi_A(1)$. Then for $M \in \underline{\text{Mod}}\,A$ and $x \in M$, let $\beta : A \to M$ be $1 \mapsto x$. Then by naturality of $\phi$,

$$\begin{aligned} \phi_M(x) &= \phi_M(\beta(1)) \\ &= \beta(\phi_A(1)) = \beta(a) = x\cdot a. \end{aligned}$$

and so $\phi = \varphi(a)$. Hence $\text{cen}A \cong [1,1]$.

**Q9** Let $M$ be an $A$-$A$-bimodule. Define a multiplication on the abelian group $A\times M$ as

$$(a,m)(b,m') = (ab,\ am' + mb)$$

then (i) $A\times M$ is a ring: If $(a,m),(b,m'),(c,m'') \in A\times M$ then

$$\begin{aligned} (a,m)\big((b,m')\cdot(c,m'')\big) &= (a,m)(bc,\ bm'' + m'c) \\ &= (a(bc),\ a(bm''+m'c) + m(bc)) \\ &= ((ab)c,\ (ab)m'' + (am')c + (mb)c) \end{aligned}$$

and

$$\begin{aligned} \big((a,m)(b,m')\big)(c,m'') &= (ab,\ am' + mb)(c,m'') \\ &= ((ab)c,\ (ab)m'' + (am'+mb)c) \\ &= ((ab)c,\ (ab)m'' + (am')c + (mb)c) \end{aligned}$$

so $A\times M$ is associative. It is also distributive since

$$\begin{aligned} (a,m)\big((b,m')+(c,m'')\big) &= (a,m)(b+c,\ m'+m'') = (a(b+c),\ a(m'+m'') + m(b+c)) \\ &= (ab+ac,\ am' + am'' + mb + mc) \\ &= (ab,\ am'+mb) + (ac,\ am''+mc) = (a,m)(b,m') + (a,m)(c,m'') \end{aligned}$$

right distributivity follows similarly. Finally, $(a,m)(1,0) = (a,\ a\cdot 0 + m\cdot 1) = (a,m)$, so $(1,0)$ is the identity.

we define a ring morphism $\alpha: A \longrightarrow A\times M$ by $\alpha(a) = (a,0)$. Then $\alpha(1) = (1,0) = 1$, $\alpha(0) = 0$ and

$$\alpha(a+b) = (a+b,0) = \alpha(a)+\alpha(b)$$
$$\alpha(ab) = (ab,0) = (a,0)(b,0)$$

also, the projection $\varphi: A\times M \longrightarrow A$, $(a,m)\longmapsto a$ is a ring morphism, by def$^N$. Clearly $\varphi\alpha = 1_A$. (In particular, $\alpha$ is clearly monic).

(ii) the subset $0\times M \subseteq A\times M$ is an additive subgroup, and is also a two sided ideal, since

$$(0,m)(a,m') = (0,\ 0\cdot m' + ma) = (0,ma) \in 0\times M$$
$$(a,m)(0,m) = (0,\ am + m\cdot 0) = (0,am) \in 0\times M$$

(this also shows that any sub-bimodule of $M$ produces a two sided ideal.), if $(0,m)\in 0\times M$ then $(0,m)(0,m) = (0,0)$, so $(0\times M)^2 = 0$.

$\boxed{Q10}$ Let $A, B$ be rings and ${}_AM_B$ a bimodule. Consider $M$ as a bimodule over the ring $A\times B$ by

$$(a,b)x = ax, \qquad x(a,b) = xb$$

(i) We claim the trivial extension $(A\times B)\times M$ is isomorphic to the generalised matrix ring $\begin{pmatrix} A & M \\ 0 & B\end{pmatrix}$ consisting of

$$\begin{pmatrix} a & m \\ 0 & b\end{pmatrix} \qquad a\in A,\ b\in B,\ m\in M$$

with the obvious addition and multiplication

$$\begin{pmatrix} a & m \\ 0 & b\end{pmatrix}\begin{pmatrix} a' & m' \\ 0 & b'\end{pmatrix} = \begin{pmatrix} aa' & am'+mb' \\ 0 & bb'\end{pmatrix}$$

in fact, this is trivially obvious when we identify $\begin{pmatrix} a & m \\ 0 & b\end{pmatrix}$ with $((a,b),m)\in (A\times B)\times M$.

(ii) The right ideals of $\begin{pmatrix} A & M \\ 0 & B\end{pmatrix}$ are

$$R = \left\{ \begin{pmatrix} a & m \\ 0 & b\end{pmatrix} \,\middle|\, a\in\mathfrak{a},\ (m,b)\in L\right\}$$

where $\mathfrak{a}$ is a right ideal of $A$, and $L$ is a $B$-submodule of $M\oplus B$ s.t. $\mathfrak{a}M\subseteq L$.

<u>PROOF</u> Obviously any such collection is closed under addition. If $\begin{pmatrix} a & m \\ 0 & b\end{pmatrix}\in R$ and $\begin{pmatrix} a' & m' \\ 0 & b'\end{pmatrix}\in\begin{pmatrix} A & M \\ 0 & B\end{pmatrix}$, then

$$\begin{pmatrix} a & m \\ 0 & b\end{pmatrix}\begin{pmatrix} a' & m' \\ 0 & b'\end{pmatrix} = \begin{pmatrix} aa' & am'+mb' \\ 0 & bb'\end{pmatrix}$$

which is in $R$ since $aa'\in\mathfrak{a}$, and if $(m,b)\in L\leq M\oplus B$, then $(am'+mb', bb') = (am', 0) + (m,b)b'$, which is in $L$ since $\mathfrak{a}M\subseteq L$ (that is, $(am,0)\in L$, $a\in\mathfrak{a}$ and $m\in M$). Conversely, let $R$ be any right ideal of $\begin{pmatrix} A & M \\ 0 & B\end{pmatrix}$. Let $\mathfrak{a} = \{a\in A \mid \begin{pmatrix} a & m \\ 0 & b\end{pmatrix}\in R$ for some $m\in M$, $b\in B\}$. Then clearly $\begin{pmatrix} 0 & 0 \\ 0 & 0\end{pmatrix}\in\mathfrak{a}$ and $\mathfrak{a}$ is closed under addition. If $a\in\mathfrak{a}$ and $a'\in A$ then let $m, b$ be s.t. $\begin{pmatrix} a & m \\ 0 & b\end{pmatrix}\in R$. Then $\begin{pmatrix} aa' & 0 \\ 0 & 0\end{pmatrix} = \begin{pmatrix} a & m \\ 0 & b\end{pmatrix}\begin{pmatrix} a' & 0 \\ 0 & 0\end{pmatrix}\in R$, since $R$ is a right ideal, hence $aa'\in\mathfrak{a}$, so $\mathfrak{a}$ is a right ideal.

Now let $L = \{(m,b)\in M\oplus B \mid \exists a\in\mathfrak{a}$ (equiv $a\in A$) s.t. $\begin{pmatrix} a & m \\ 0 & b\end{pmatrix}\in R\}$. Then $L$ is an additive subgroup of $M\oplus B$ and if $b'\in B$, $(m,b)\in L$, say $\begin{pmatrix} a & m \\ 0 & b\end{pmatrix}\in R$, then $\begin{pmatrix} 0 & mb' \\ 0 & bb'\end{pmatrix} = \begin{pmatrix} a & m \\ 0 & b\end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & b'\end{pmatrix}\in R$, since $R$ is a right ideal. Finally, if $a\in A$ then say $\begin{pmatrix} a & m \\ 0 & b\end{pmatrix}\in R$ — then for any $m'\in M$,

$$\begin{pmatrix} 0 & am' \\ 0 & 0\end{pmatrix} = \begin{pmatrix} a & m \\ 0 & b\end{pmatrix}\begin{pmatrix} 0 & m' \\ 0 & 0\end{pmatrix}\in R$$

hence $(am', 0)\in L$ — that is, $\mathfrak{a}M\subseteq L$. Hence $R$ has the required form — if $a\in\mathfrak{a}$ and $(m,b)\in L$, then

$$\begin{pmatrix} a & m' \\ 0 & b'\end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0\end{pmatrix} + \begin{pmatrix} 0 & m' \\ 0 & b'\end{pmatrix} = \begin{pmatrix} a & m \\ 0 & b\end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0\end{pmatrix} + \begin{pmatrix} a' & m' \\ 0 & b'\end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1\end{pmatrix}\in R$$

and similarly the left ideals of $\begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ are

$$L = \left\{ \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \middle| \; b \in \mathfrak{b}, \; (m,a) \in Q \right\}$$

where $\mathfrak{b}$ is a left ideal of $B$ and $Q$ is an $A$-submodule of $M \oplus A$ s.t. $M\mathfrak{b} \subseteq L$.

$\boxed{Q15}$ We claim the ring

$$\begin{pmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix}$$

is right noetherian, but not left noetherian. By Ex. 10 (considering $\mathbb{Q}$ as a $\mathbb{Z}-\mathbb{Q}$ bimodule) this matrix ring is s.t. every right ideal is

$$R = \left\{ \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \middle| \; a \in \mathfrak{a}, \; (m,b) \in L \right\}$$

where $\mathfrak{a}$ is an ideal of $\mathbb{Z}$, and $L$ is a $\mathbb{Q}$-submodule of $\mathbb{Q} \oplus \mathbb{Q}$, s.t. $\mathfrak{a}\mathbb{Q} \subseteq L$. Since $\mathfrak{a} = (n)$, some $n \in \mathbb{Z}$, $\mathfrak{a}\mathbb{Q} \subseteq L$ is equivalent to $(nq, 0) \in L$, $\forall q \in \mathbb{Q}$, which is the same as $\mathbb{Q} \oplus 0 \subseteq L$. Then either $L = \mathbb{Q} \oplus 0$, or $\exists\, 0 \neq q' \in \mathbb{Q}$ and $q \in \mathbb{Q}$ s.t. $(q, q') \in L$. Then $(q/q', 1) \in L$, and since $(1/q' q, 0) \in \mathbb{Q} \oplus 0 \subseteq L$, this implies $(0,1) \in L$, and hence that $L = \mathbb{Q} \oplus \mathbb{Q}$. Hence there are two classes of right ideal

$$M_n = \begin{pmatrix} (n) & \mathbb{Q} \\ 0 & 0 \end{pmatrix} \qquad P_n = \begin{pmatrix} (n) & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix} \qquad n \geqslant 0$$

the first is generated by $\begin{pmatrix} n & 1 \\ 0 & 0 \end{pmatrix}$ and the second by $\begin{pmatrix} n & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix}$, so every right ideal is finitely generated, and consequently the ring is right noetherian.

To see that $\begin{pmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix}$ is not left noetherian, let $Q_n = \left\{ \frac{m}{2^n} \mid m \in \mathbb{Z} \right\} \subseteq \mathbb{Q}$ be the subgroup of $\mathbb{Q}$ generated by $1/2^n$. Then $Q_0 = \mathbb{Z}$ and $Q_n \subseteq Q_{n+1}$. The sets

$$M_n = \begin{pmatrix} \mathbb{Z} & Q_n \\ 0 & 0 \end{pmatrix}$$

are left ideals, and form an infinite ascending sequence $M_n \subset M_{n+1} \subset \cdots$. Hence this ring is <u>not</u> left noetherian.

$\boxed{Q24}$ We claim $\begin{pmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix}$ is right hereditary. We use the characterisation of the right ideals achieved in Q15. Consider an epimorphism $\phi : M \longrightarrow N$ and $\varphi : M_n \longrightarrow N$

$$\begin{tikzcd} & M_n \arrow[d, "\varphi"] \arrow[dl, dotted, "\lambda"'] \\ M \arrow[r, twoheadrightarrow, "\phi"'] & N \end{tikzcd}$$

suppose that $\varphi\begin{pmatrix} n & 1 \\ 0 & 0 \end{pmatrix} = a \in N$, and let $x \in M$ be s.t. $\phi(x) = a$. Define

$$\lambda\begin{pmatrix} mn & q \\ 0 & 0 \end{pmatrix} = x\begin{pmatrix} m & q/n \\ 0 & 0 \end{pmatrix}. \quad \text{if } n \neq 0$$
$$= x\begin{pmatrix} 0 & 0 \\ 0 & q \end{pmatrix} \quad \text{if } n = 0$$

In both cases $\lambda$ is a morphism of right $\begin{pmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix}$ - modules, and $\phi\lambda = \rho$. Hence $M_n$ is projective.

We handle the ideals $P_n$ for $n=0$ and $n \neq 0$. seperately. The ideal

$$P_0 = \begin{pmatrix} 0 & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix}$$

is projective since in a diagram

$$\begin{tikzcd} & P_0 \arrow[dl, dashed, "\lambda"'] \arrow[d, "\rho"] \\ N \arrow[r, two heads, "\phi"'] & M \end{tikzcd} \qquad (2)$$

we let $a = \rho\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $b = \rho\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Let $x, y \in N$ be s.t. $\phi(x) = a$, $\phi(y) = b$, and define $\lambda$ by

$$\lambda\begin{pmatrix} 0 & q \\ 0 & q' \end{pmatrix} = x\begin{pmatrix} 0 & 0 \\ 0 & q \end{pmatrix} + y\begin{pmatrix} 0 & 0 \\ 0 & q' \end{pmatrix}$$

then $\lambda$ is a module morphism and $\phi\lambda = \rho$ since they agree on generators. Hence $P_0$ is projective.

The right ideals $P_n$ have single generators $(n \neq 0)$, which are $\begin{pmatrix} n & 1 \\ 0 & 1 \end{pmatrix}$ since

$$\begin{pmatrix} mn & q \\ 0 & q' \end{pmatrix} = \begin{pmatrix} n & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} m & \frac{q-q'}{n} \\ 0 & q' \end{pmatrix}$$

hence in the situation of (2) we define $\lambda\begin{pmatrix} mn & q \\ 0 & q' \end{pmatrix} = x\begin{pmatrix} m & \frac{q-q'}{n} \\ 0 & q' \end{pmatrix}$ where $\rho\begin{pmatrix} n & 1 \\ 0 & 1 \end{pmatrix} = a$ and $\phi(x) = a$. The map $\lambda$ is then a morphism of modules and $\phi\lambda = \phi$, so $P_n$ is projective. Hence $\begin{pmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix}$ is right hereditary.

$\boxed{Q34}$ Let $A \times M$ be the trivial extension of $A$ by a bimodule $M$. Let $X'$ be a right $A \times M$ module. Using the morphism $\alpha : A \longrightarrow A \times M$, $a \longmapsto (a, 0)$, $X'$ becomes a right $A$-module which we denote by $X$. Define the bilinear map

$$\lambda : X \times M \longrightarrow X \qquad (x, m) \longmapsto x \cdot (0, m)$$

then $\lambda$ is easily seen to be bilinear, hence induces $\hat{\lambda} : X \otimes_A M \longrightarrow X$, $x \otimes m \longmapsto x \cdot (0, m)$. This is even a morphism of right $A$-modules, since

$$\begin{aligned} \hat{\lambda}\Big(\big(\textstyle\sum_i x_i \otimes m_i\big) \cdot a\Big) &= \hat{\lambda}\big(\textstyle\sum_i x_i \otimes m_i \cdot a\big) \\ &= \textstyle\sum_i x_i \cdot (0, m_i \cdot a) \\ &= \textstyle\sum_i x_i \cdot \{(0, m_i)(a, 0)\} \\ &= \big(\textstyle\sum_i x_i (0, m_i)\big)(a, 0) = \hat{\lambda}\big(\textstyle\sum x_i \otimes m_i\big) \cdot a \end{aligned}$$

the composition

$$X \otimes_A M \otimes_A M \xrightarrow{\lambda \otimes 1} X \otimes_A M \xrightarrow{\lambda} X$$

is zero, since $\lambda(\lambda \otimes 1)(x \otimes m \otimes m') = \lambda\big(\lambda(x \otimes m) \otimes m'\big) = \lambda\big(x \cdot (0, m) \otimes m'\big) = \big(x \cdot (0, m)\big) \cdot (0, m')$
$= x \cdot (0, 0) = 0.$

Now suppose we are given a right $A$-module $X$ and $\lambda : X \otimes_A M \longrightarrow X$ $A$-linear s.t. $\lambda(\lambda \otimes 1) = 0$. We give $X$ the structure of an $A \times M$ module by defining

$$x \cdot (a,m) = xa + \lambda(x \otimes m)$$

this is valid since

$$\begin{aligned} x \cdot ((a,m) + (a',m')) &= x \cdot (a+a', m+m') = x(a+a') + \lambda(x \otimes m+m') \\ &= xa + xa' + \lambda(x \otimes m) + \lambda(x \otimes m') \\ &= x(a,m) + x(a',m') \end{aligned}$$

similarly $(x+y)\cdot(a,m) = x(a,m) + y(a,m)$

more interesting is
$$\begin{aligned} (x\cdot(a,m))\cdot(a',m') &= \{xa + \lambda(x\otimes m)\}(a',m') \\ &= \{xa + \lambda(x \otimes m)\}a' + \lambda(\{xa + \lambda(x\otimes m)\}\otimes m') \\ &= x(aa') + \lambda(x \otimes ma') + \lambda(x \otimes am') \\ &\qquad \text{since } \lambda(\lambda(x\otimes m)\otimes m') = 0. \\ &= x(aa') + \lambda(x \otimes (ma' + am')) \\ &= x \cdot ((a,m)(a',m')) \end{aligned}$$

$x \cdot (1,0) = x$, trivially.

Hence $X$ is a right $A \times M$ module. Since $x \cdot (0,m) = \lambda(x \otimes m)$, the map $X \otimes_A M \longrightarrow X$ induced as in the first part of this question coincides with $\lambda$. Since $x \cdot (a,0) = xa$, the $A$-module structures also coincide. Conversely, beginning with an $A \times M$-module $X$, producing $\lambda$, the structure defined above is the original $A \times M$-module structure, since

$$\begin{aligned} x \cdot' (a,m) &= xa + \lambda(x \otimes m) \\ &= x\cdot(a,0) + x \cdot (0,m) = x \cdot (a,m) \end{aligned}$$

so the two processes are inverse and we have the desired correspondence. (We write $(X,\lambda)$ for these modules)

Let $(X,\alpha)$ and $(Y,\beta)$ be $A\times M$-modules. A morphism of $A \times M$ modules $\varphi : (X,\alpha) \longrightarrow (Y,\beta)$ is a function $\varphi : X \longrightarrow Y$ s.t. $\varphi(x \cdot (a,m)) = \varphi(x)\cdot(a,m)$, but since $(a,m) = (a,0) + (0,m)$, it is equivalent that $\varphi$ satisfy

$$\begin{aligned} \varphi(x\cdot(a,0)) &= \varphi(x)\cdot(a,0) \\ \varphi(x \cdot (0,m)) &= \varphi(x)\cdot(0,m) \end{aligned}$$

the first equation is precisely "$\varphi$ is $A$-linear" and the second is precisely commutativity of the diagram

$$\begin{tikzcd} X \otimes_A M \arrow[r, "\alpha"] \arrow[d, "\varphi\otimes 1"'] & X \arrow[d, "\varphi"] \\ Y \otimes_A M \arrow[r, "\beta"'] & Y \end{tikzcd}$$

$\boxed{Q35}$ Let $A, B$ be rings and $M$ an $A-B$-bimodule. Consider $M$ as an $A\times B$-bimodule as in Q10

(i) We claim $M \otimes_{A\times B} M = 0$. This is obvious since $m = m\cdot(0,1)$, so

$$m \otimes n = m(0,1) \otimes n = m \otimes (0,1)n = m \otimes 0 = 0$$

NOTE Products of Rings and their Modules.

Let $A_1, \dots, A_n$ be rings, and $A_1 \times \cdots \times A_n$ their product. We claim that there is an equivalence

$$\underline{Mod}-A_1 \times \cdots \times \underline{Mod}-A_n \longrightarrow \underline{Mod}-(A_1 \times \cdots \times A_n)$$

given by $(M_1, \dots, M_n) \longmapsto M_1 \times \cdots \times M_n$. Given $(\varphi_1, \dots, \varphi_n) : (M_1, \dots, M_n) \longrightarrow (N_1, \dots, N_n)$ we define

$$\varphi : M_1 \times \cdots \times M_n \longrightarrow N_1 \times \cdots \times N_n$$
$$(m_1, \dots, m_n) \longmapsto (\varphi_1(m_1), \dots, \varphi_n(m_n))$$

Thus defined, the functor is certainly faithful. It is full since if $\phi : M_1 \times \cdots \times M_n \longrightarrow N_1 \times \cdots \times N_n$ and we let $\phi_i : M_i \longrightarrow N_i$ be functions s.t. $\phi(m_1, \dots, m_n) \longrightarrow (\phi_i(m_1), \dots, \phi_n(m_n))$, then

$$\begin{aligned} \phi_i(m_1 + m_1') &= \phi(m_1 + m_1', 0, \dots, 0), \\ &= \phi(m_1, 0, \dots, 0), + \phi(m_1', 0, \dots, 0), \\ &= \phi_i(m_1) + \phi(m_1') \end{aligned}$$

and similarly $\phi_i(m a_i) = \phi_i(m) a_i$, so $\phi_i$ is a morphism of $A_i$-modules, and moreover $\phi$ is clearly the image of $(\phi_1, \dots, \phi_n)$ under our functor, which is thus full.

To see that it is representative, let $X$ be a $A_1 \times \cdots \times A_n$ module and $\alpha_i =$ the endomorphism of $X$ corresponding to $(0, \dots, \underset{i}{1}, \dots 0)$ in $A_1 \times \cdots \times A_n$. Then $\alpha_i X$ is an abelian subgroup of $X$, if $y \in X$ then

$$y = y \cdot (1, \dots, 1) = \sum_{i=1}^{n} \alpha_i(y)$$

and since $\alpha_i \alpha_j = 0$ if $i \neq j$ and $\alpha_i \alpha_i = \alpha_i$, if $\alpha_i X \cap \sum_{j \neq i} \alpha_j X$ is nonzero, say $\alpha_i(x) \neq 0$ and

$$\alpha_i(x) = \sum_{j \neq i} \alpha_j(y_j)$$

then on applying $\alpha_i$ we have $\alpha_i(x) = 0$, a contradiction.

Notice that $\alpha_i X$ is closed under the action of $(0, \dots, a_i, \dots, 0)$ $a_i \in A_i$, since

$$\begin{aligned} &\alpha_i(x) \cdot (0, \dots, a_i, \dots, 0) \\ &= ( \cdot (0, \dots, 1, 0, \dots 0)) \cdot (0, \dots, a_i, \dots, 0) \\ &= (x \cdot (0, \dots, a_i, 0, \dots 0)) \cdot (0, \dots, 1, \dots, 0) \\ &= \alpha_i ( x(0, \dots, a_i, \dots, 0)) \end{aligned}$$

and in this way $\alpha_i X$ becomes an $A_i$-module. We define

$$\phi : X \longrightarrow \alpha_1 X \times \cdots \times \alpha_n X$$
$$\phi(x) = (\alpha_1(x), \dots, \alpha_n(x))$$

which we already know is an isomorphism of groups. (It is onto since $\phi(\alpha_1(x_1) + \cdots + \alpha_n(x_n)) = (\alpha_1(x_1), \dots, \alpha_n(x_n))$). It is a morphism of $A_1 \times \cdots \times A_n$ modules since

$$\begin{aligned} \phi(x(a_1, \dots, a_n)) &= (\alpha_i(x(a_1, \dots, a_n))) \\ &= ( x(a_1, \dots, a_n)(0, \dots, \overset{i}{1}, \dots, 0)) \\ &= ( x(0, \dots, a_i, \dots, 0)) \\ &= ( \alpha_i(x) a_i ) = \phi(x)(a_1, \dots, a_n) \end{aligned}$$

(ii) Let $X$ be a right $A$-module, $Y$ a right $B$-module and $\alpha : X \otimes_A M \longrightarrow Y$ $B$-linear. Then $X \oplus Y$ becomes a right $\begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$-module by

$$(x,y)\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} = (xa, \alpha(x \otimes m) + yb)$$

since

$$\begin{aligned}(x,y)\begin{pmatrix} a+a' & m+m' \\ 0 & b+b' \end{pmatrix} &= (xa + xa', \alpha(x \otimes m+m') + yb+yb') \\ &= (xa, \alpha(x \otimes m) + yb) + (xa', \alpha(x \otimes m') + yb') \\ &= (x,y)\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} + (x,y)\begin{pmatrix} a' & m' \\ 0 & b' \end{pmatrix}\end{aligned}$$

$$\begin{aligned}((x,y)+(x',y'))\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} = (x+x', y+y')\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} &= (xa+x'a, \alpha(x+x' \otimes m) + yb+y'b) \\ &= (xa, \alpha(x \otimes m) + yb) + (x'a, \alpha(x' \otimes m) + y'b) \\ &= (x,y)\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} + (x',y')\begin{pmatrix} a & m \\ 0 & b \end{pmatrix}\end{aligned}$$

$$\begin{aligned}\left\{(x,y)\cdot\begin{pmatrix} a & m \\ 0 & b \end{pmatrix}\right\}\begin{pmatrix} a' & m' \\ 0 & b' \end{pmatrix} &= \{(xa, \alpha(x \otimes m) + yb)\}\cdot\begin{pmatrix} a' & m' \\ 0 & b' \end{pmatrix} \\ &= ((xa)a', \alpha(\quad xa \quad \otimes m') + \alpha(x \otimes m)b' + (yb)b') \\ &= (x(aa'), \alpha(x \otimes am') + \alpha(x \otimes mb') + y(bb')) \\ &= (x(aa'), \alpha(x \otimes (am' + mb')) + y(bb')) \\ &= (x,y)\left(\begin{pmatrix} a & m \\ 0 & b \end{pmatrix}\begin{pmatrix} a' & m' \\ 0 & b' \end{pmatrix}\right)\end{aligned}$$

$$(x,y)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = (x, \alpha(x \otimes 0) + y) = (x,y)$$

we write $(X\ Y)_\alpha$ for this module.

(iii) We claim every right module over $\begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ is $(X\ Y)_\alpha$ for some ${}_AX$, ${}_BY$ and $\alpha : X \otimes_A M \longrightarrow Y$ $B$-linear. Recall that $\begin{pmatrix} A & M \\ 0 & B \end{pmatrix} \cong (A \times B) \times M$, the trivial extension of $A \times B$ by the $A\times B$-bimodule $M$. By Q34, any right $(A \times B) \times M$ module comes from a right $A\times B$-module together with an $A\times B$-linear map $X \otimes_{A\times B} M \longrightarrow X$ such that

$$X \otimes_{A\times B} M \otimes_{A\times B} M \longrightarrow X \otimes_{A \times B} M \longrightarrow X$$

is zero. Since $M \otimes_{A\times B} M = 0$, this is trivially satisfied. One can continue in this fashion, but thus motivated it is probably clearer to do it directly:

Let $X$ be a right $\begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$-module, and define

$$X_1 = \left\{ x\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \,\middle|\, x \in X \right\} \quad X_2 = \left\{ x\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \,\middle|\, x \in X \right\}$$

these are subgroups, $X_1 \cap X_2 = 0$ since if $x\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = y\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, on multiplying by $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ we have $x\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 0$.
Also, for $x \in X$,

$$x = x\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = x\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + x\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

so as abelian groups $X = X_1 \oplus X_2$. $X_1$ is a right $A$-module (resp. $X_2$ is a right $B$-module) under the actions

$$x_1 \cdot a = x_1\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \qquad x_2 \cdot b = x_2\begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}$$

we define a function $\tilde{\lambda}: X_1 \times M \longrightarrow X_2$ by $\tilde{\lambda}(x_1, m) = x_1 \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}$. Since for $x \in X$

$$x \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} = x \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} = x \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in X_2$$

this is well-defined. It is bilinear – for ex.

$$\tilde{\lambda}(x_1 a, m) = \left(x_1 \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\right)\begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} = x_1 \begin{pmatrix} 0 & am \\ 0 & 0 \end{pmatrix} = \tilde{\lambda}(x_1, am)$$

and hence induces $\lambda: X_1 \otimes_A M \longrightarrow X_2$, which is $B$-linear since

$$\begin{aligned} \lambda(\{x_1 \otimes m\} b) &= \lambda(x_1 \otimes mb) \\ &= x_1 \begin{pmatrix} 0 & mb \\ 0 & 0 \end{pmatrix} = \left(x_1 \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}\right)\begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} \\ &= \lambda(x_1 \otimes m) b \end{aligned}$$

then we have the $\begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ module $(X_1\ X_2)_\lambda$, which we claim is isomorphic to $X$ via

$$\begin{aligned} \Phi : X &\longrightarrow (X_1\ X_2)_\lambda \\ x &\longmapsto \Big(\underbrace{x\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{x_1}, \underbrace{x\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{x_2}\Big) \end{aligned}$$

we need only show it is $\begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$-linear. But

$$\begin{aligned} \Phi\left( x\begin{pmatrix} a & m \\ 0 & b \end{pmatrix}\right) &= \left( x\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, x\begin{pmatrix} 0 & m \\ 0 & b \end{pmatrix} \right) \\ &= \left( x_1 a,\ x_1 \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} \right) \\ &= (x_1 a,\ \lambda(x_1 \otimes m) + x_2 b) \\ &= \Phi(x) \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \end{aligned}$$

(iv) Let $\phi : (X\ Y)_\alpha \longrightarrow (X'\ Y')_\beta$. Notice that since

$$\phi(x,0) = \phi\left( (x,0)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right) = \phi(x,0)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

we see that there is $x' \in X'$ s.t. $\phi(x,0) = (x',0)$. Denote this assignment by $\varphi: X \longrightarrow X'$, so

$$\varphi(x) = \left\{\phi(x,0)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right\}_1 \qquad \phi(x,0) = (\varphi(x), 0)$$

similarly we define $\psi: Y \longrightarrow Y'$ s.t.

$$\psi(y) = \left\{\phi(0,y)\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}_2 \qquad \phi(0,y) = (0, \psi(y))$$

both $\varphi, \psi$ are clearly group morphisms and

$$\varphi(xa) = \left\{ \phi(xa,0)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right\}_1 = \left\{ \phi\left( (x,0)\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\right)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right\}_1 = \left\{\phi(x,0)\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\right\}_1 = \varphi(x) a$$

similarly $\psi(yb) = \psi(y)b$. Since $\phi(x,y) = \phi(x,0) + \phi(0,y) = (\varphi(x), \psi(y))$, we have $\phi = (\varphi, \psi)$, so it only remains to show that

$$\begin{tikzcd} X \otimes_A M \arrow[r, "\alpha"] \arrow[d, "\varphi \otimes 1"'] & Y \arrow[d, "\psi"] \\ X' \otimes_A M \arrow[r, "\beta"'] & Y' \end{tikzcd}$$

commutes.

This diagram commutes, since consider the fact that $\phi$ is $\begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$-linear, so

$$\phi\left((x,y)\begin{pmatrix} a & m \\ 0 & b \end{pmatrix}\right) = \phi(x,y)\begin{pmatrix} a & m \\ 0 & b \end{pmatrix}$$
$$\phi\left(xa, \alpha(x\otimes m)+yb\right) = (\varphi(x), \psi(y))\begin{pmatrix} a & m \\ 0 & b \end{pmatrix}$$
$$\left(\varphi(xa), \psi(\alpha(x\otimes m))+\psi(yb)\right) = \left(\varphi(xa), \beta(\varphi(x)\otimes m)+\psi(yb)\right)$$

Hence $\psi(\alpha(x\otimes m)) = \beta(\varphi(x)\otimes m)$, as required.

# CHAPTER II : Rings of Fractions

## 1. The Ring of Fractions

Let $A$ be a ring and let $S$ be a multiplicatively closed subset of $A$, i.e., $s, t \in S$ implies $st \in S$ and $1 \in S$. We define a right ring of fractions of $A$ with respect to $S$ as a ring $A[S^{-1}]$ together with a ring homomorphism $\varphi: A \longrightarrow A[S^{-1}]$ satisfying

F1. $\varphi(s)$ is invertible for every $s \in S$

F2. Every element in $A[S^{-1}]$ has the form $\varphi(a)\varphi(s)^{-1}$ with $s \in S$

F3. $\varphi(a) = 0$ iff. $as = 0$ for some $s \in S$.

Similarly one defines a left ring of fractions $[S^{-1}]A$ of $A$ with respect to $S$ (F2', F3'). It is not immediately clear that the axioms F1–3 determine $A[S^{-1}]$ uniquely, but that so is the case follows from the fact that $A[S^{-1}]$ is a solution of a universal problem.

PROPOSITION 1.1 When $A[S^{-1}]$ exists, it has the following universal property: for every ring homomorphism $\psi: A \longrightarrow B$ such that $\psi(s)$ is invertible in $B$ for every $s \in S$, there exists a unique homomorphism $\sigma: A[S^{-1}] \longrightarrow B$ s.t. $\sigma\varphi = \psi$.

PROOF $\sigma$ is defined as $\sigma(\varphi(a)\varphi(s)^{-1}) = \psi(a)\psi(s)^{-1}$. We have to verify that this is well-defined. So suppose

$$\varphi(a)\varphi(s)^{-1} = \varphi(b)\varphi(t)^{-1}$$

Then $\varphi(a) = \varphi(b)\varphi(t)^{-1}\varphi(s) = \varphi(b)\varphi(c)\varphi(u)^{-1}$ for some $c \in A$ and $u \in S$, by F(2). This gives

$$\varphi(a)\varphi(u) = \varphi(b)\varphi(c)$$
$$\varphi(s)\varphi(u) = \varphi(t)\varphi(c)$$

from which we obtain with the use of F3 that $auv = bcv$ and $suv' = tcv'$ for some $v$ and $v'$ in $S$. Since $\psi(v)$ and $\psi(v')$ are invertible, this implies

$$\psi(a)\psi(u) = \psi(b)\psi(c)$$
$$\psi(s)\psi(u) = \psi(t)\psi(c)$$

and we can go backwards to find that $\psi(a)\psi(s)^{-1} = \psi(b)\psi(t)^{-1}$. It remains to show that $\sigma$ is a morphism of rings, but it is clear that $\sigma\varphi = \psi$, and uniqueness is clear. Hence, let

$$a = \varphi(a_1)\varphi(a_2)^{-1} \quad b = \varphi(b_1)\varphi(b_2)^{-1} \quad a+b = \varphi(c_1)\varphi(c_2)^{-1}$$

Then

$$\varphi(c_1)\varphi(c_2)^{-1} = \varphi(a_1)\varphi(a_2)^{-1} + \varphi(b_1)\varphi(b_2)^{-1}$$

We get rid of inverses using the above technique — let $\varphi(a_2)^{-1}\varphi(c_2) = \varphi(d_1)\varphi(d_2)^{-1}$, $\varphi(b_2)^{-1}\varphi(c_2) = \varphi(e_1)\varphi(e_2)^{-1}$. Then let $\varphi(e_2)^{-1}\varphi(d_2) = \varphi(f_1)\varphi(f_2)^{-1}$. Multiplying everything out, we get

$$c_2 d_2 v = a_2 d_1 v$$
$$c_2 e_2 v' = b_2 e_1 v'$$
$$e_2 f_1 v'' = d_2 f_2 v''$$

$$c_1 d_2 f_2 v''' = a_1 d_1 f_2 v''' + b_1 e_1 f_1 v'''$$

hence applying $\psi$ we go backward to find that $\psi(c_1)\psi(c_2)^{-1} = \psi(a_1)\psi(a_2)^{-1} + \psi(b_1)\psi(b_2)^{-1}$ that is, $\sigma(a+b) = \sigma(a) + \sigma(b)$. Similarly one verifies that $\sigma(ab) = \sigma(a)\sigma(b)$ $(\sigma(1) = \sigma(\varphi(1)\varphi(1)^{-1}) = \psi(1)\psi(1)^{-1} = 1)$. $\square$

COROLLARY 1.2 $A[S^{-1}]$ is unique up to isomorphism.

COROLLARY 1.3 If both $A[S^{-1}]$ and $[S^{-1}]A$ exist, then they are canonically isomorphic.

That is, both $\{F_1, F_2, F_3\}$ and $\{F_1, F_2', F_3'\}$ imply the universal condition.

**DEFINITION** A right denominator set is a multiplicatively closed subset $S$ satisfying

S1. If $s \in S$ and $a \in A$, then there exist $t \in S$ and $b \in A$ s.t.
$$sb = at$$

S2. If $sa = 0$ with $s \in S$, then $at = 0$ for some $t \in S$.

Clearly these conditions are satisfied by any multiplicatively closed subset of a commutative ring. The axioms S1, S2 are properties of any $\varphi : A \longrightarrow A[S^{-1}]$ satisfying F1, F2, F3, and introduce just enough commutativity to make things work.

**PROPOSITION 1.4** Let $S$ be a multiplicatively closed subset of $A$. Then $A[S^{-1}]$ exists if and only if $S$ is a right denominator set. When $A[S^{-1}]$ exists, it has the form
$$A[S^{-1}] = A \times S/\sim$$
where $\sim$ is the equivalence relation $(a,s) \sim (b,t)$ iff. $\exists\, c, d \in A$ s.t. $ac = bd$ and $sc = td \in S$.

**PROOF** Suppose $A[S^{-1}]$ exists and let $\varphi : A \longrightarrow A[S^{-1}]$ be the canonical map. We first verify S1. If $a \in A$ and $s \in S$, then F2 gives $\varphi(s)^{-1}\varphi(a) = \varphi(b)\varphi(t)^{-1}$ for some $b \in A$, $t \in S$. Hence $\varphi(at) = \varphi(sb)$ and by F3 this means that $atu = sbu$ for some $u \in S$, as required. If $sa = 0$ with $s \in S$, then $\varphi(a) = 0$ by F1, and hence there is $t \in S$ s.t. $at = 0$ by F3. This gives us S2.

Let us now explain the equivalence relation $\sim$. Given as a morphism $\varphi : A \longrightarrow A[S^{-1}]$, a localisation realises a fraction $(a,s)$ by $\varphi(a)\varphi(s)^{-1}$. Hence
$$\varphi(a)\varphi(s)^{-1} = \varphi(b)\varphi(t)^{-1}$$

OR

$$\varphi(a)\varphi(s)^{-1}\varphi(t) = \varphi(b)$$
let $\varphi(s)^{-1}\varphi(t) = \varphi(a_1)\varphi(a_2)^{-1}$ $a_2 \in S$
- $\varphi(t)\varphi(a_2) = \varphi(s)\varphi(a_1)$
- $\varphi(b)\varphi(a_2) = \varphi(a)\varphi(a_1)$

$$\varphi(a) = \varphi(b)\varphi(t)^{-1}\varphi(s)$$
let $\varphi(t)^{-1}\varphi(s) = \varphi(c_1)\varphi(c_2)^{-1}$ $c_2 \in S$
- $\varphi(t)\varphi(c_1) = \varphi(s)\varphi(c_2)$
- $\varphi(b)\varphi(c_1) = \varphi(a)\varphi(c_2)$

and this motivates the definition of $\sim$. (Notice that if $ac = bd$, $sc = td \in S$ then $\varphi(td)$ a unit $\Rightarrow \varphi(t), \varphi(d)$ units, so $\varphi(a)\varphi(s)^{-1} = \varphi(b)\varphi(t)^{-1}$)
This relation reduces to the canonical one in the case of a commutative ring, since if $t'(at - sb) = 0$, let $c = t't$ and $d = t's$. Conversely, if $ac = bd$, $sc = td \in S$ we let $t' = cd$. Let us now show that $\sim$ is an equivalence relation. It is clearly reflexive and symmetric — so let $(a,s) \sim (b,t)$ and $(b,t) \sim (p,q)$ via
$$ac = bd \qquad be = pf$$
$$sc = td \qquad te = qf$$

Using S1, let $m \in A$, $m' \in S$ be s.t. $tdm = tem'$. Then by S2, $dmn = em'n$ some $n \in S$. Then
$$a(cmn) = bdmn = bem'n = pfm'n = p(fm'n)$$
$$s(cmn) = tdmn = tem'n = qfm'n = q(fm'n)$$

and $qf \in S$, $m' \in S$, $n \in S$ implies $qfm'n \in S$, as required. Hence we form $A \times S/\sim$ and define addition and multiplication in the following way.

$$(a,s) + (b,t) = (ac + bd, u) \qquad \text{where } u = sc = td \in S$$
$$(a,s)(b,t) = (ac, tu) \qquad \text{where } sc = bu \text{ and } tu \in S$$

obviously can find $u \in S$, so trivially $tu \in S$. But we need it to work if $u \notin S$ and just $tu \in S$, for something later

This is motivated by finding a common denominator in the first case, since if $sc = td$ then $\frac{ac}{sc} + \frac{bd}{td}$ can be written as above. In the second case $ac/sc \cdot bu/tu$ will cancel $sc$, $bu$ to give $ac/tu$. We first verify that, once a representative of the involved classes is chosen, the sum does not depend on which $c, d$ (resp. $c, u$) we choose. (S1 will provide at least one pair). Suppose
$$sc = td \in S$$
$$sc' = td' \in S$$

we can reduce this to the case $d = d'$ by letting $e \in A$, $e' \in S$ be s.t. $tde = td'e'$, and using S2 to find $m \in S$ s.t.

$dem = d'e'm$. Then

$$\begin{aligned} s(cem) &= tdem \\ &= td'e'm \\ &= s(c'e'm) \end{aligned}$$

where $sc' \in S$, $e' \in S$, $m \in S$ implies all these elements are in $S$. And clearly $(a,s) \sim (ac, sc)$ any $c \in A$, provided $sc \in S$, so we assume that $d = d'$, so $sc = td = td' = sc'$ and hence by $S2$ there is $n \in S$ s.t. $cn = c'n$. Then

$$\begin{aligned} (ac+bd)n &= (ac'+bd')n \\ scn &= sc'n \in S \end{aligned}$$

and hence $(ac+bd, sc) \sim (ac'+bd', sc')$. Likewise, if $sc = bu$ and $sc' = bu'$ with $tu, tu' \in S$, let $e \in A$, $e' \in S$ be s.t. $tue = tu'e'$. Then $scem = buem = bu'e'm = sc'e'm$. Hence by $S2$ there is $q \in S$ s.t. $ceq = c'e'q$. Then
$\therefore$ let $m \in S$ $uem = u'e'm$

$$\begin{aligned} ac(eq) &= ac'e'q = ac'(e'q) \\ tu(eq) &= tu'e'q = tu'(e'q) \in S \end{aligned}$$

and hence the product is independent of these choices as well. We now show that the sum and product are well-defined on equivalence classes. First, suppose that $(a,s) \sim (a_1, s_1)$ via $ac = a_1 d$, $sc = s_1 d \in S$. Let $(b,t)$ be another pair, and suppose $sq = tr$. Then let $e \in A$, $e' \in S$ be s.t. $(sc)e = (sq)e'$. Then by $S2$ there is $m \in S$ s.t. $cem = qe'm$. Hence

$$s_1(dem) = scem = sqe'm = t(re'm) \in S$$

and hence

$$\begin{aligned} (a_1,s_1) + (b,t) = (a_1dem + bre'm, s_1dem) &\sim (acem + bre'm, scem) \\ &\sim (aqe'm + bre'm, sqe'm) \\ &\sim (aq + br, sq) = (a,s) + (b,t) \end{aligned}$$

a similar method works when we fix $(a,s)$ and consider $(b,t) \sim (b_1, t_1)$. Hence if $(a,s) \sim (a_1,s_1)$ and $(b,t) \sim (b_1,t_1)$

$$(a,s) + (b,t) \sim (a_1,s_1) + (b,t) \sim (a_1,s_1) + (b_1,t_1)$$

as required. Now suppose $(a,s) \sim (a_1,s_1)$ and $(b,t)$ are pairs, and $sq = bu$, $tu \in S$, $ac = a_1d$, $sc = s_1d \in S$. Let $e \in A$, $e' \in S$ be s.t. $(sc)e = (sq)e'$. Then $cem = qe'm$ some $m \in S$. Then $s_1(dem) = scem = sqe'm = bue'm$, and $tue'm \in S$.
Hence

$$\begin{aligned} (a_1,s_1)(b,t) &= (a_1dem, tue'm) \\ &\sim (aqe'm, tue'm) \\ &\sim (aq, tu) \\ &= (a,s)(b,t) \end{aligned}$$

and so $(a_1,s_1)(b_1,t_1) \sim (a,s)(b_1,t_1) \sim (a,s)(b,t)$, since if $(b_1,t_1) \sim (b,t)$ via $bc = b_1d$, $tc = t_1d \in S$, let $e \in S$, $e' \in A$ be s.t. $ce = ue'$. Then if $sq = bu$, we have $b_1(de) = bce = bue' = sqe'$, and $t_1de \in S$. Hence

$$\begin{aligned} (a,s)(b_1,t_1) &= (aqe', t_1de) \\ &\sim (aqe', tce) \\ &\sim (aqe', tue') \\ &\sim (aq, tu) = (a,s)(b,t). \end{aligned}$$

And hence addition and multiplication are both well-defined. Addition is obviously commutative, associativity is not difficult to check, the class of $(0,1)$ serves as an identity, and $(a,s) + (-a,s) = (0,s)$. Associativity of multiplication follows since:

$$\begin{array}{ccc} (a,s) & (b,t) & (c,u) \\ & sn = bn' & tm = cm' \\ & n' \in S & m' \in S \end{array}$$

Let $n'p = mp'$ with $p' \in S$. Then $tn'p = cm'p'$ and $snp = bmp'$, so

$$((a,s)\cdot(b,t))\cdot(c,u) = (an, tn')\cdot(c,u) = (anp, um'p') = (a,s)\cdot(bm, um') = (a,s)\cdot((b,t)\cdot(c,u))$$

To show left distributivity, consider fractions $(a,s)$, $(b,t)$, $(c,u)$. Suppose

$$sx = by \qquad y \in S$$
$$sx' = cy' \qquad y' \in S$$

and that $(ty)d = (uy')d'$ with $d' \in S$. Then $(a,s)(b,t) + (a,s)(c,u) = (ax, ty) + (ax', uy') = (axd' + ax'd', uy'd')$.
And

$$\begin{aligned}(a,s)\big((b,t)+(c,u)\big) &= (a,s)\big((by,ty)+(cy',uy')\big)\\ &= (a,s)(byd + cy'd', uy'd')\\ &= (a,s)(sxd + sx'd', uy'd')\\ &= (a,s)(s(xd+x'd'), uy'd')\\ &= (a(xd+x'd'), uy'd)\\ &= (axd + ax'd', uy'd)\\ &= (a,s)(b,t) + (a,s)(c,u)\end{aligned}$$

$$(a,s)(sb,y) = (ab,y)$$

right distributivity is similarly verified. Hence $A \times S/\sim$ is a ring, with $(1/1)$ obviously being an identity. Moreover, $\varphi : A \to A\times S/\sim$ defined by $\varphi(a) = [(a,1)]$ is a morphism of rings, since for $a, b \in A$ and $s, t \in S$

$$(a+b, s) = (a,s) + (b,s)$$
$$(a,1)(b,1) = (ab,1)$$

For any $s \in S$, $(s/1)(1/s) = (1/1)$, since $(a,a) = 1$ for any $a \in S$, and

$$\begin{aligned}\varphi(a)\varphi(s)^{-1} &= (a,1)(1,s)\\ &= (a,s)\end{aligned}$$

also, notice that $(a,1) = 0 = (0,1)$ implies $ac = 0d$ and $1 \cdot c = 1 \cdot d \in S$, some $c, d \in A$. Hence $ac = 0$, $c \in S$. Hence $A \times S/\sim = A[S^{-1}]$. Moreover, uniqueness means any localisation has this form. □

We will now show that S1 implies S2 under certain circumstances. A right ideal of the form $r(S) = \{a \in A \mid Sa = 0\}$, for some subset $S$ of $A$, is called a right annihilator.

PROPOSITION 1.5 Assume $A$ satisfies ACC on right annihilators. If $S$ is multiplicatively closed and satisfies S1, then $S$ is a right denominator set.

PROOF Suppose $sa = 0$ for some $s \in S$. There exists an integer $n$ s.t. $r(s^k) = r(s^n)$ for all $k \geq n$. By S1 one can find $t \in S$ and $b \in A$ such that $s^n b = at$. Then $s^{n+1} b = sat = 0$, so $b \in r(s^{n+1}) = r(s^n)$. Hence $at = s^n b = 0$, which establishes S2.

EXAMPLES 1. Commutative Rings When $A$ is a commutative ring, both S1 and S2 are automatically satisfied, so every multiplicatively closed set $S$ is a denominator set.

2. Ore Rings The most important example of a multiplicatively closed set is the set $S_{reg}$ of all regular elements (i.e. non-zero-divisors) of $A$. The ring of fractions $A[S_{reg}^{-1}]$ is usually called the classical right ring of quotients. (or sometimes the total right ring of fractions). We will denote it by $Q_{cl}^r(A)$, or more often simply by $Q_{cl}$. The corresponding left-hand notation is $Q_{cl}^{\ell}(A)$. Recall that when both $Q_{cl}^r(A)$ and $Q_{cl}^{l}(A)$ exist, they coincide. Since condition S2 is automatically satisfied, we have:

PROPOSITION 1.6 $Q_{cl}^r(A)$ exists if and only if $A$ satisfies the right Ore condition: i.e. for $a$ and $s$ in $A$ with $s$ regular, there exist $b$ and $t$ in $A$ with $t$ regular, such that $at = sb$.

3. Ore Domains If $A$ has no zero-divisors, the right Ore condition states that $aA \cap sA \neq 0$ for all non-zero elements $a, s$ of $A$, and this is the same as saying that $\mathfrak{a} \cap \mathfrak{b} \neq 0$ for all non-zero right ideals $\mathfrak{a}$ and $\mathfrak{b}$. A ring without zero-divisors and satisfying the right Ore condition is called a right Ore domain, and its classical right ring of quotients is a skew-field. Numerous examples of Ore domains are obtained from:

PROPOSITION 1.7 Every right noetherian ring without zero-divisors is a right Ore domain.

PROOF Let $a$ and $b$ be nonzero elements of $A$. Consider the right ideals $\mathfrak{a}_n = bA + abA + \cdots + a^n bA$. Since $A$ is right noetherian, there exists a smallest $n$ for which $\mathfrak{a}_n = \mathfrak{a}_{n+1}$, and then $a^{n+1}b = bc_0 + \cdots + a^n bc_n$. This gives

$$bc_0 = a(a^n b - bc_1 - \cdots - a^{n-1}bc_n) \neq 0$$

where the minimality of $n$ implies that the bracket is $\neq 0$. Thus $bA \cap aA \neq 0$, which is the right Ore condition. □

4. Bezout Domains A right Bezout domain is a ring without zero divisors in which every finitely generated right ideal is principal.

PROPOSITION 1.8 Every right Bezout domain is a right Ore domain.

PROOF If $A$ does not satisfy the Ore condition, then there exist non-zero elements $a$ and $b$ of $A$ such that $aA \cap bA = 0$. Put $aA + bA = cA$, and write $b = cd$. Then

$$aA = aA/aA \cap bA \cong aA + bA/bA \cong cA/bA \cong A/dA.$$

since $d \neq 0$, this is a contradiction. □

6. Rings of Quotients A ring $A$ is called a ring of quotients if every non-zero-divisor of $A$ is invertible, i.e., $A$ is its own classical right and left ring of quotients. E.g every regular ring is a ring of quotients, as one easily sees. Also rings with minimum condition on right ideals are rings of quotients, in fact one has more generally

PROPOSITION 1.9 If $A$ satisfies DCC on principal right ideals, then $A$ is a ring of quotients.

PROOF See the next Lemma. □

Here DCC stands for "descending chain condition", ie every strictly descending chain (of principal right ideals) is finite.

LEMMA 1.10 Suppose $A$ satisfies DCC on principal right ideals. The following properties of an element $a$ are equivalent

(a) $a$ is invertible

(b) $a$ is regular

(c) $r(a) = 0$

PROOF Obviously (a) $\Rightarrow$ (b) $\Rightarrow$ (c). Assume the element $a$ satisfies the condition (c). Consider the descending chain $aA \supseteq a^2A \supseteq \cdots$. By hypothesis $a^nA = a^{n+1}A$ for some $n$. Thus $a^n = a^{n+1}b$ for some $b \in A$, and then $a^n(1-ab) = 0$, and (c) implies $1 = ab$. We then also have $aba = a$, and (c) implies $ba = 1$. Hence (c) $\Rightarrow$ (a). □

Given a ring $Q$ of quotients, one may be interested in investigating those subrings of $Q$ for which $Q$ is the classical right ring of quotients. Such subrings are called right orders in $Q$.

DEFINITION Let $Q$ be a ring of quotients, and $R$ a subring of $Q$. $R$ is called a right order in $Q$ if the injection $R \hookrightarrow Q$ is a classical right ring of quotients for $R$.

Notice that $A \longrightarrow A[S_{reg}^{-1}]$ is always injective, hence $A$ is always canonically identified with a right order of $A[S_{reg}^{-1}]$. So saying "$A$ is a right order" in a semi-simple ring" is equivalent to "$A[S_{reg}^{-1}]$ is semi-simple", since any two right rings of quotients are canonically isomorphic.

# 2. ORDERS IN A SEMISIMPLE RING

In this section we prove a theorem by Goldie which characterises those rings which have a semisimple classical ring of quotients. Before stating the theorem we have to make some definitions.

DEFINITION A ring $A$ has finite rank if there are no infinite direct sums of non-zero right ideals in $A$, i.e., $A$ does not contain any right ideals of the form $\mathfrak{a} = \mathfrak{a}_1 \oplus \mathfrak{a}_2 \oplus \cdots$ with $\mathfrak{a}_i \neq 0$, $i \in \mathbb{N}$. Every right noetherian ring has of course finite right rank.

DEFINITION A right ideal $\mathfrak{a}$ is nilpotent if $\mathfrak{a}^n = 0$ for some $n$, and is nil if every element of $\mathfrak{a}$ is nilpotent. The ring $A$ is prime if there are no non-zero two-sided ideals $\mathfrak{a}$ and $\mathfrak{b}$ such that $\mathfrak{a}\mathfrak{b} = 0$, and $A$ is semi-prime if it has no nonzero nilpotent ideals. Finally, the right ideal $\mathfrak{a}$ is essential if $\mathfrak{a} \cap \mathfrak{b} \neq 0$ holds for every right ideal $\mathfrak{b} \neq 0$.

LEMMA 2.1 The ring $A$ is semi-simple if and only if it has no proper essential right ideals.

PROOF If $A$ is semi-simple, then every proper right ideal of $A$ is a direct summand and cannot therefore be essential. Conversely, suppose there are no essential right ideals $\neq A$. If $\mathfrak{a}$ is any right ideal of $A$, one can use Zorn's lemma to find a right ideal $\mathfrak{b}$ s.t. $\mathfrak{b}$ is maximal with the property that $\mathfrak{a} \cap \mathfrak{b} = 0$. But then $\mathfrak{a} + \mathfrak{b}$ is an essential right ideal of $A$, for if $\mathfrak{c}$ is a right ideal s.t. $(\mathfrak{a} + \mathfrak{b}) \cap \mathfrak{c} = 0$, then one has $\mathfrak{a} \cap (\mathfrak{b} + \mathfrak{c}) = 0$, which implies that $\mathfrak{c} = 0$ by the maximality of $\mathfrak{b}$. It follows that $\mathfrak{a} \oplus \mathfrak{b} = A$, so every right ideal of $A$ is a direct summand. Hence $A$ is semi-simple. □

THEOREM 2.2 (Goldie) The following properties of the ring $A$ are equivalent:

(a) $A$ is a right order in a semi-simple ring.

(b) $A$ has finite right rank, satisfies ACC on right annihilators and is a semi-prime ring

(c) A right ideal of $A$ is essential if and only if it contains a regular element.

PROOF (a) $\Rightarrow$ (b) Let $Q$ be the semi-simple right ring of quotients of $A$. If $\mathfrak{a}$ and $\mathfrak{b}$ are right ideals of $A$ such that $\mathfrak{a} \cap \mathfrak{b} = 0$, then it easy to see that also $\mathfrak{a}Q \cap \mathfrak{b}Q = 0$. Now $Q$ has finite right rank (any semi-simple ring does) since if $\mathfrak{a} = \mathfrak{a}_1 \oplus \cdots$ is an infinite direct sum of nonzero ideals in $Q$, let $\mathfrak{b}$ be a right ideal s.t. $\mathfrak{a} \oplus \mathfrak{b} = Q$. By earlier results, $\mathfrak{b} \oplus \mathfrak{a}_1 \oplus \cdots = Q$ implies that only finitely many of the $\mathfrak{a}_i$ are nonzero. The above two observations imply that $A$ has finite right rank. $A$ satisfies ACC on right annihilators because $r_A(S) = r_Q(S) \cap A$ for any $S \subseteq A$, where the subscripts indicate in which ring the annihilator is taken. Suppose $r_A(S_1) \subseteq r_A(S_2) \subseteq \cdots$ is an ascending chain of right annihilators. If $x \in r_Q(S_i)$, then let $x = as^{-1}$ $a \in A$, $s \in S$ (that is, $s$ regular). Then $s_i x = 0$ $\forall s_i \in S_i$, or $0 = s_i a s^{-1} \Leftrightarrow 0 = s_i a$ and so $a \in r_A(S_i) \subseteq r_A(S_j)$ which implies $x \in r_Q(S_j)$. Hence $r_Q(S_1) \subseteq r_Q(S_2) \subseteq \cdots$ is also an ascending chain of right annihilators. Notice that as the coproduct of a finite number of simple modules (which must be Noetherian) any semi-simple ring is right and left noetherian. Hence for some $i$, $\forall n \geq i$ $r_Q(S_n) = r_Q(S_i)$. But then $\forall n \geq i$, $r_A(S_n) = r_Q(S_n) \cap A = r_Q(S_i) \cap A = r_A(S_i)$, so that $A$ has ACC on right annihilators.

Finally, let $\mathfrak{a}$ be a nilpotent two-sided ideal in $A$. $\ell(\mathfrak{a})$ is essential as a right ideal of $A$, because if $\mathfrak{b}$ is any non-zero right ideal of $A$, then there is a largest integer such that $\mathfrak{b}\mathfrak{a}^n \neq 0$, and clearly $\mathfrak{b} \cap \ell(\mathfrak{a}) \supseteq \mathfrak{b}\mathfrak{a}^n \neq 0$. But if $\ell(\mathfrak{a})$ is essential in $A$, then $\ell(\mathfrak{a})Q = \{\sum a_i s_i^{-1} \mid a_i \in \ell(\mathfrak{a}),\ s_i \text{ in } A \text{ regular}\}$ is an essential right ideal in $Q$. This follows since if $\mathfrak{b}$ is a nonzero right ideal of $Q$, then $\tilde{\mathfrak{b}} = \mathfrak{b} \cap A$ is also nonzero (else $0 \neq as^{-1} \in \mathfrak{b} \Rightarrow (as^{-1})s \in \mathfrak{b} \Rightarrow a \in \mathfrak{b} \cap A = 0$) and is a right ideal of $A$. Hence $\ell(\mathfrak{a}) \cap \tilde{\mathfrak{b}} \neq 0$, say $m$ belongs to this intersection. Then $m \in \ell(\mathfrak{a})Q$, and $m \in \tilde{\mathfrak{b}} = \mathfrak{b} \cap A$, so $m \in \mathfrak{b} \cap \ell(\mathfrak{a})Q$ which is thus nonempty. Hence $\ell(\mathfrak{a})Q$ is essential. But $Q$ is semi-simple, and hence $\ell(\mathfrak{a})Q = Q$, and we may write $1 = \sum a_i s_i^{-1}$ with $a_i \in \ell(\mathfrak{a})$ and $s_i \in A$ regular. There exists a regular element $s \in A$ s.t. all $s_i^{-1}s \in A$. (see EX. 2). Then $s\mathfrak{a} = \sum a_i s_i^{-1} s \mathfrak{a} \subseteq \sum a_i \mathfrak{a} = 0$, and it follows that $\mathfrak{a} = 0$.

(b) $\Rightarrow$ (c) Every non-zero divisor generates an essential right ideal:

LEMMA 2.3 If $A$ has finite right rank and $r(s) = 0$, then $sA$ is an essential right ideal.

PROOF If $\mathfrak{a}$ is a non-zero right ideal such that $sA \cap \mathfrak{a} = 0$, then clearly the right ideals $\mathfrak{a}, s\mathfrak{a}, s^2\mathfrak{a}, \ldots$ are all nonempty, and suppose that $x \in s^{i_0}\mathfrak{a} \cap (s^{j_1}\mathfrak{a} + \cdots + s^{j_n}\mathfrak{a})$ for some $i_0, j_1, \ldots, j_n$ all distinct non-negative integers. Say $s^{i_0}a = x = s^{j_1}a_1 + \cdots + s^{j_n}a_n$. Let $k = \min\{i_0, j_1, \ldots, j_n\} \geq 0$. We have

$$s^{i_0 - k}(-a) + s^{j_1 - k}a_1 + \cdots + s^{j_n - k}a_n = 0$$

where some term is $a_i \in \mathfrak{a}$ and the rest (since $i_0, j_1, \ldots, j_n$ all distinct) are terms of the form $s^m a'$, $m \geq 1$. Hence since $sA \cap \mathfrak{a} = 0$, $a_i = 0$. By induction we may as well assume all $a_j \neq 0$, hence contradiction unless $x = 0$. Hence the sum $\mathfrak{a} \oplus s\mathfrak{a} \oplus s^2\mathfrak{a} \oplus \cdots \oplus s^n\mathfrak{a} \oplus \cdots$ is infinite direct, contradicting the fact that $A$ has finite right rank. Hence $sA \cap \mathfrak{a} \neq 0$ and $sA$ is essential. □

It requires more work to show that every essential right ideal contains a regular element. Here we follow the proof given by Goldie in [7]. We need:

LEMMA 2.4 If $A$ is semi-prime and satisfies ACC on right annihilator ideals, then $A$ has no left or right nil ideals $\neq 0$.

PROOF If $Aa$ is nil and $b \in A$, $ba$ nilpotent implies $\exists n \geq 1$ s.t. $(ba)^n = bababa \cdots ba = 0$. Hence $a(ba)^n b = 0$, that is, $(ab)^{n+1} = 0$. Hence $Aa$ nil implies $aA$ nil and conversely, so it is enough to consider a left nil ideal $Aa$. Assume $Aa \neq 0$. Among the nonzero elements of $Aa$, choose one $b \in Aa$ with maximal right annihilator. For each $c \in A$, let $k$ be the smallest integer s.t. $(cb)^k = 0$. Since $r(b) \subseteq r((cb)^{k-1})$ if $k > 1$, and by the choice of $b$ we have then $r(b) = r((cb)^{k-1})$. If $k=1$, clearly $cb \in r(b)$, and likewise if $k > 1$, $r(b) = r((cb)^{k-1})$ implies $cb \in r(b)$, and hence $bAb = 0$ by the arbitrary choice of $c$. From this we obtain $(AbA)^2 = 0$, but every nilpotent ideal is zero, so $AbA = 0$. Hence $b = 0$, which is a contradiction, and we must have $Aa = 0$. □

Now let $\mathfrak{a}$ be an essential right ideal of $A$. We have to find a regular element in $A$. Since $A$ has ACC on right annihilators and $\mathfrak{a}$ is not a nil ideal (by the preceeding lemma) there exists $a_1 \neq 0$ in $\mathfrak{a}$ s.t. $r(a_1) = r(a_1{}^2)$ (pick any non-nilpotent $a \in \mathfrak{a}$, use ACC on r.a. to see $r(a) \subseteq r(a^2) \subseteq \cdots$ terminates, say $r(a^k) = r(a^n)$, $\forall k \geq n$. Let $a_1 = a^n$). If $\mathfrak{a} \cap r(a_1) \neq 0$, we continue and choose $a_2 \in \mathfrak{a} \cap r(a_1)$ such that $a_2 \neq 0$ and $r(a_2) = r(a_2{}^2)$. If then $\mathfrak{a} \cap r(a_1) \cap r(a_2) \neq 0$, choose $a_3 \in \mathfrak{a} \cap r(a_1) \cap r(a_2)$, and so on. At each step we obtain a direct sum $a_1 A \oplus \cdots \oplus a_k A$. This is proved by induction: suppose $a_1 A \oplus \cdots \oplus a_{k-1} A$ is direct and $a_k b = a_1 b_1 + \cdots + a_{k-1} b_{k-1}$; since for each $i < k$ we have $a_i a_k = 0$, we get $b_i \in r(a_i{}^2) = r(a_i)$, and hence $\sum_{i=1}^{k-1} a_i b_i = 0$. But $A$ has finite right rank, so the process must stop at some stage, where we have

$$\mathfrak{a} \cap r(a_1) \cap \cdots \cap r(a_k) = 0$$

Then $r(a_1) \cap \cdots \cap r(a_k) = 0$ since $\mathfrak{a}$ is essential, so if $c = a_1 + \cdots + a_k \in \mathfrak{a}$, then $r(c) = 0$, since if $cb = 0$,

$$a_1 b + \cdots + a_k b = 0$$

and hence $a_1{}^2 b + a_1 a_2 b + \cdots + a_1 a_k b = 0$, so $a_1{}^2 b = 0 \Rightarrow b \in r(a_1{}^2) = r(a_1)$, so $a_1 b = 0$. In this fashion, we show $b \in r(a_i)$, $1 \leq i \leq k$, and hence $b = 0$.

$c$ is thus the kind of element we are looking for, but it remains to show that also $\ell(c) = 0$. Note that $cA$ is an essential right ideal by Lemma 2.3. Define the <u>right singular ideal</u> of $A$ as

$$Z = \{ a \in A \mid r(a) \text{ is an essential right ideal of } A \}$$

This a right ideal, since $r(0)$ is certainly essential, and if $a, b \in Z$, let $\mathfrak{b}$ be a nonzero ideal of $A$. Then $\mathfrak{b} \cap r(b) \neq 0$, hence $r(a) \cap r(b) \cap \mathfrak{b} \neq 0$, say it contains $x$. Then $(a+b)x = ax + bx = 0$, so that $x \in \mathfrak{b} \cap r(a+b)$, which is thus nonempty. Hence $a + b \in Z$. If $a \in Z$ and $c \in A$, then $r(a)$ is essential and $r(ca) \supseteq r(a)$, so $r(ca)$ must be essential, and hence $ca \in Z$. To show $Z$ is a right ideal, use $I = r(a)$ in the following Lemma, since $r(ac) = \{x \mid acx = 0\} = \{x \mid cx \in r(a)\} = (r(a) : c)$ (Recall for an ideal {right} of $A$ and $x \in A$, $(I : x) = \{ y \mid xy \in I \}$ is a right ideal)

LEMMA If $I$ is an essential right ideal of an arbitrary ring $R$, and $x \in R$, then $(I : x)$ is also essential.

PROOF Let $\mathfrak{b}$ be a nonzero ideal, and consider the right ideal $x\mathfrak{b}$. If $x\mathfrak{b} = 0$, then if $0 \neq b \in \mathfrak{b}$, $xb = 0 \in I$, so $b \in (I : x)$ and hence $\mathfrak{b} \cap (I : x) \neq 0$. Otherwise $x\mathfrak{b} \neq 0$ and since $I$ is essential, $I \cap x\mathfrak{b} \neq 0$, say $i \in I$ and $i = xb$. Then clearly $b \in \mathfrak{b} \cap (I : x)$, and $b \neq 0$ since we assume $i \neq 0$. Hence $(I : x)$ is essential. □

{Notice that $(I : x)$ is the pullback of the sieve $I$ along $x : R \longrightarrow R$, so the above lemma is a stability result}. Hence $Z$ is a two-sided ideal, and $\ell(c) \subseteq Z$ since if $xc = 0$, $r(x) \supseteq cA$, which is essential since $r(c) = 0$. Since $A$ contains no nilpotent ideals $\neq 0$, we may conclude that $\ell(c) = 0$ using:

LEMMA 2.5 If $A$ satisfies ACC on right annihilators, then the right singular ideal is nilpotent.

PROOF We will show that the ascending chain $r(Z) \subseteq r(Z^2) \subseteq r(Z^3) \subseteq \cdots$ would be strictly ascending if $Z$ were not nilpotent. If $Z^n \neq 0$, choose an element $a \in Z$ with $Z^{n-1} a \neq 0$ and largest possible right annihilator. For each $b \in Z$ we have $r(b) \cap aA \neq 0$, since $r(b)$ is essential in $A$. So there exists $c \in A$ s.t. $ac \neq 0$, but $bac = 0$. This means that $r(ba)$ is strictly bigger than $r(a)$, and by the choice of $a$ we must therefore have $Z^{n-1} ba = 0$. Since $b \in Z$ is arbitrary, we get $Z^n a = 0$, and hence $r(Z^{n-1})$ is strictly contained in $r(Z^n)$. □

(c) $\Rightarrow$ (a) We first verify the right Ore condition. Let $s$ be a regular element and $a$ arbitrary in $A$. Since $sA$ is essential, by the Lemma $(sA:a)$ is also essential, and it therefore contains a regular element $t$. Thus $at = sb$ for some $b \in A$.

To prove that $Q = Q_{cl}(A)$ is semi-simple, it suffices by Lemma 2.1 to show that $Q$ has no proper essential right ideals. If $\mathfrak{a}$ is any right ideal of $Q$, then $(\mathfrak{a} \cap A)Q = \mathfrak{a}$ since every element $q \in \mathfrak{a}$ can be written as $q = as^{-1}$ with $a = qs \in \mathfrak{a} \cap A$. If furthermore $\mathfrak{a}$ is an essential right ideal of $Q$, then $\mathfrak{a} \cap A$ is an essential right ideal of $A$ and thus contains a regular element $t$. ($\mathfrak{a} \cap A$ is essential since if $0 \neq \mathfrak{b}$ in $A$, $0 \neq \mathfrak{b}Q$ in $Q$, so $0 \neq \mathfrak{a} \cap \mathfrak{b}Q$, say $x = as^{-1}$ in $\mathfrak{a}$, with $a \in \mathfrak{a} \cap A$ and $s$ regular, and $x = \sum_i b_i s_i^{-1}$ in $\mathfrak{b}Q$ with $b_i \in \mathfrak{b}$, $s_i$ regular. Find $q$ regular s.t. $s^{-1}q$, $s_i^{-1}q$ are all in $A$. Then $xq = as^{-1}q = \sum b_i s_i^{-1} q \in \mathfrak{a} \cap A \cap \mathfrak{b}$, and $xq \neq 0$ since $q$ regular). Since $t$ is invertible in $Q$, $\mathfrak{a} = (\mathfrak{a} \cap A)Q = Q$. This concludes the proof of the Theorem. □

As a special case of Goldie's Theorem we obtain a characterisation of those rings $A$ for which $Q_{cl}(A)$ is a simple ring.

PROPOSITION 2.6 The following properties of a ring $A$ are equivalent:

(a) $A$ is a right order in a simple ring.

(b) $A$ has finite right rank, satisfies ACC on right annihilators and is a prime ring.

PROOF (a) $\Rightarrow$ (b) It only remains to show that $A$ is a prime ring. Let $\mathfrak{a}$ and $\mathfrak{b}$ be two sided ideals of $A$ with $\mathfrak{a}\mathfrak{b} = 0$. Then $\mathfrak{b}Q\mathfrak{a} \cap A$ is a two sided ideal of $A$ and $(\mathfrak{b}Q\mathfrak{a} \cap A)^2 = 0$. Since $A$ has no non-zero nilpotent ideals by the Theorem, we have $\mathfrak{b}Q\mathfrak{a} \cap A = 0$. But this clearly implies that $\mathfrak{b}Q\mathfrak{a} = 0$.

Recall for two rings $A, B$ and an $A$–$B$-bimodule $M$, the functor $- \otimes_A M : \underline{Mod}\text{-}A \longrightarrow \underline{Mod}\text{-}B$ is left adjoint to $Hom_B(M,-) : \underline{Mod}\text{-}B \longrightarrow \underline{Mod}\text{-}A$. Now, let $\varphi : A \longrightarrow B$ be a morphism of rings, and define functors

$$\begin{tikzcd} \underline{Mod}\text{-}A \arrow[r, shift left=2, "\varphi^*"] \arrow[r, leftarrow, "\varphi_*" description] \arrow[r, shift right=2, "\varphi^!"'] & \underline{Mod}\text{-}B \end{tikzcd}$$

$\varphi^*(M) = M \otimes_A B$ (extension of scalars)

$\varphi_*(N) = N$ (restriction of scalars)

$\varphi^!(M) = Hom_A(B, M)$

Notice that $Hom_B(B,-) : \underline{Mod}\text{-}B \longrightarrow \underline{Mod}\text{-}A$ is exactly what we're calling $\varphi_*$. (that is, there is a natural equivalence of $\varphi_*$ with $Hom_B(B,-)$). Hence $\varphi^* \dashv \varphi_*$. We claim that $\varphi_* \dashv \varphi^!$. That is, we consider $B$ as a $B$–$A$-bimodule, and give $Hom_A(B,M)$ the canonical structure as a right $B$-module : $(\phi \cdot b)(b') = \phi(bb')$. We need to show that there is an isomorphism natural in $N, M$

$$Hom_A(\varphi_*(N), M) \cong Hom_B(N, \varphi^!(M))$$
$$\theta : Hom_A(N, M) \cong Hom_B(N, Hom_A(B, M))$$

This is defined by $\theta(\phi)(n)(b) = \phi(n \cdot b)$, $\theta^{-1}(p)(n) = p(n)(1)$. To check naturality, let $\alpha : N \longrightarrow N'$ in Mod–$B$, $\beta : M \longrightarrow M'$ in Mod–$A$ and check that

$$\begin{tikzcd} Hom_A(N', M) \arrow[r, equal] \arrow[d] & Hom_B(N', Hom_A(B,M)) \arrow[d] \\ Hom_A(N, M) \arrow[r, equal] & Hom_B(N, Hom_A(B,M)) \end{tikzcd} \qquad \begin{tikzcd} Hom_A(N, M) \arrow[r, equal] \arrow[d] & Hom_B(N, Hom_A(B,M)) \arrow[d] \\ Hom_A(N, M') \arrow[r, equal] & Hom_B(N, Hom_A(B,M')) \end{tikzcd}$$

commute, which is not difficult. □

For the first adjunction the natural isomorphism is

$$\psi : Hom_B(M \otimes_A B, N) \cong Hom_A(M, \varphi_* N)$$

defined by $\psi(\alpha)(m) = \alpha(m \otimes 1)$ and $\psi^{-1}(\beta)(m \otimes b) = \beta(m \cdot b)$. Hence the unit and counit are defined as follows.

unit: $\eta : 1 \longrightarrow \varphi_* \varphi^*$
$\eta_M : M \longrightarrow M \otimes_A B$
$m \longmapsto m \otimes 1$

counit: $\epsilon : \varphi^* \varphi_* \longrightarrow 1$
$\epsilon_N : N \otimes_A B \longrightarrow N$
$n \otimes b \longmapsto n \cdot b$

## 3. MODULES OF FRACTIONS

Let $S$ be a right denominator set in the ring $A$. For each right $A$-module $M$ we define its module of fractions with respect to $S$ as $M[S^{-1}] = M \otimes_A A[S^{-1}]$ with its canonical structure as a right $A[S^{-1}]$ module. This is a reasonable definition in view of the following two results:

PROPOSITION 3.1 The canonical $A$-linear map $\mu_M : M \longrightarrow M[S^{-1}]$ has the following universal property: for each right $A[S^{-1}]$ module $N$ and $A$-linear map $\alpha : M \longrightarrow N$ there exists a unique $A[S^{-1}]$-linear map $\delta : M[S^{-1}] \longrightarrow N$ such that $\delta\mu_M = \alpha$.

PROOF We have

$$\mathrm{Hom}_{A[S^{-1}]}(M \otimes_A A[S^{-1}], N) \cong \mathrm{Hom}_A(M, \mathrm{Hom}_{A[S^{-1}]}(A[S^{-1}], N)) \cong \mathrm{Hom}_A(M, N). \square$$

PROPOSITION 3.2 $M[S^{-1}] \cong M \times S/\sim$ where $\sim$ is the equivalence relation defined as $(x,s) \sim (y,t)$ if there is $c, d \in A$ such that $xc = yd$ and $sc = td \in S$.

PROOF The earlier proof that $\sim$ is an equivalence relation applies muta mutandis. The operations in $M \times S/\sim$ are defined as

$$(x,s)+(y,t) = (xc+yd, u) \qquad u = sc = td \in S \quad (\text{using } S1)$$
$$(x,s)\cdot(b,t) = (xc, tu) \qquad \text{for } (b,t) \in A[S^{-1}] \text{ where } sc = bu \text{ and } u \in S.$$

The earlier proofs also show these operations to be independent of the $c,d$ (resp. $c,u$) chosen, and also that all operations are independent of any representative choices (including $(b,t) \in A[S^{-1}]$). Addition is commutative and associative, with identity $(0,1)$, and $(a,s) + (-a,s) = (0,1)$. Similarly all the module axioms are already taken care of. Hence $M \times S/\sim$ is a right $A[S^{-1}]$-module. Define $\hat{\mu}_M : M \longrightarrow M \times S/\sim$ by $\hat{\mu}_M(m) = (m,1)$. Then for $a \in A$, $\hat{\mu}_M(m \cdot a) = (m \cdot a, 1) = (m,1)(a,1)$ so that $\hat{\mu}_M$ is a morphism of $A$-modules. Let $N$ be a right $A[S^{-1}]$-module and $\alpha : M \longrightarrow N$ an $A$-linear map.

$$\begin{tikzcd} M \arrow[r, "\hat{\mu}_M"] \arrow[dr, "\alpha"'] & M \times S/\sim \arrow[d, dotted, "\tilde{\alpha}"] \\ & N \end{tikzcd}$$

Then we define $\tilde{\alpha} : M \times S/\sim \longrightarrow N$ by $\tilde{\alpha}(m,s) = \alpha(m).(1,s)$. Then $\tilde{\alpha}$ is clearly a morphism of $A[S^{-1}]$-modules unique s.t. $\tilde{\alpha}\hat{\mu}_M = \alpha$. Hence $M[S^{-1}] \cong M \times S/\sim$. $\square$

COROLLARY 3.3 The kernel of $\mu_M : M \longrightarrow M[S^{-1}]$ consists of those $x \in M$ for which there exist $s \in S$ with $xs = 0$.

From this corollary it follows that $t(M) = \{x \in M \mid xs = 0 \text{ for some } s \in S\}$ is a submodule of $M$, called the S-torsion submodule. $M$ is an S-torsion module if $M = t(M)$, ie., $M[S^{-1}] = 0$, and is S-torsion free if $t(M) = 0$.

LEMMA 3.4 $M/t(M)$ is $S$-torsion-free.

PROOF If $x \in M$ and $\bar{x} \in t(M/t(M))$ then $xs \in t(M)$ for some $s \in S$, and this means $x(st) = 0$ for $t \in S$. Hence $x \in t(M)$ and $\bar{x} = 0$. $\square$

Now, let $\varphi : A \longrightarrow A[S^{-1}]$ be canonical. We know this induces functors $- \otimes_A A[S^{-1}] : \text{Mod-}A \longrightarrow \text{Mod-}A[S^{-1}]$ and $\varphi_* : \text{Mod-}A[S^{-1}] \longrightarrow \text{Mod-}A$, with $- \otimes_A A[S^{-1}] \dashv \varphi_*$. (This is shown generally elsewhere, and 3.1 establishes it in this particular case). In particular, since $- \otimes_A A[S^{-1}]$ has a left adjoint (see our notes on Ring extensions), $A[S^{-1}]$ is flat as a left $A$-module. Alternatively:

PROPOSITION 3.5 $A[S^{-1}]$ is flat as a left $A$-module

PROOF Let $\alpha : L \longrightarrow M$ be a monomorphism in Mod$-A$. We must show that the induced morphism $L[S^{-1}] \longrightarrow M[S^{-1}]$ is a monomorphism. Suppose $xs^{-1}$ (with $x \in L$, $s \in S$) has the property that $\alpha(x)s^{-1} = 0$ in $M[S^{-1}]$. By Prop 3.2. this means that $\alpha(x)c = 0$ in $M$ for some $c \in A$ s.t. $sc \in S$. Since $\alpha$ is a monomorphism, this implies $xc = 0$ in $L$, and then $xs^{-1} = 0$ in $L[S^{-1}]$. $\square$

We want to have an intrinsic characterisation of those $A$-modules which are modules of fractions, i.e., may be considered as $A[S^{-1}]$-modules.

DEFINITIONS (i) An $A$-module $M$ is <u>$S$-injective</u> if for each right ideal $\mathfrak{a}$ of $A$ such that $\mathfrak{a} \cap S \neq \emptyset$ and each morphism $\alpha : \mathfrak{a} \longrightarrow M$, there exists $m \in M$ such that $\alpha(a) = xa$ for all $a \in \mathfrak{a}$.

(ii) $M$ is <u>$S$-divisible</u> if $M = Ms$ for each $s \in S$.

PROPOSITION 3.7 Assume $S$ is a right denominator set in $A$. The following properties of a right $A$-module $M$ are equivalent:

(a) $M \cong M[S^{-1}]$ (via the canonical map $\mu_M$)

(b) $M$ is $S$-torsion-free and $S$-injective

(c) $M$ is $S$-torsion-free and $S$-divisible

PROOF (a) $\Rightarrow$ (c) The condition (a) means that $M$ is an $A[S^{-1}]$-module. For suppose $\tilde{M}$ denotes $M$ with an $A[S^{-1}]$-module structure extending the $A$-module structure of $M$. Then $1_M : M \longrightarrow \tilde{M}$ induces $A[S^{-1}]$-linear map $\varphi : M[S^{-1}] \longrightarrow \tilde{M}$ s.t. $\varphi(m,1) = m$ and $\varphi(m,s) = m\cdot(1,s)$ in $\tilde{M}$. Since $M = \tilde{M}$ as $A$-modules, this map $\varphi$ is surjective. If $\varphi(m,s) = \varphi(n,t)$, then $m\cdot(1,s) = n\cdot(1,t)$. Let $b, b'$ be s.t. $sb = tb'$ with $t' \in S$. Then

$$(m\cdot(1,s))\cdot(sb,1) = (n\cdot(1,t))\cdot(tb',1)$$
$$m\cdot(b,1) = n\cdot(b',1)$$
$$m\cdot b = n\cdot b'$$

Hence by def$^N$ in $M[S^{-1}]$, $(m,s) = (n,t)$. Thus $\varphi$ is an isomorphism of $A[S^{-1}]$-modules, whence $\mu_M$ is an isomorphism. Now, if $x \in M$ and $xs = 0$ for some $s \in S$, then $x = xs\cdot s^{-1} = 0$, so $M$ is $S$-torsion-free. For each $x \in M$ and $s \in S$ we can write $x = xs^{-1}\cdot s \in Ms$, so $M$ is $S$-divisible.

(c) $\Rightarrow$ (a) $M$ $S$-torsion-free implies that $M$ is an $A$-submodule of $M[S^{-1}]$. For each $x \otimes s^{-1} \in M[S^{-1}]$ we can write $x = ys$ for some $y \in M$ by $S$-divisibility. Then $x \otimes s^{-1} = y \otimes ss^{-1} = y \otimes 1$, and so the map $M \longrightarrow M[S^{-1}]$ is an epimorphism also.

(c) $\Rightarrow$ (b) Suppose we are given $\alpha : \mathfrak{a} \longrightarrow M$, where $\mathfrak{a}$ is a right ideal and $s \in \mathfrak{a} \cap S$. Write $\alpha(s) = xs$ for some $x \in M$. We want to show that $\alpha(a) = xa$ holds for all $a \in \mathfrak{a}$. By the condition S1 we may write $at = sb$ for some $t \in S$ and $b \in A$. Then $\alpha(a)t = \alpha(s)b = xsb = xat$, and $M$ $S$-torsion-free implies that $\alpha(a) = xa$.

(b) $\Rightarrow$ (c) Given $x \in M$ and $s \in S$, we define a homomorphism $\alpha : sA \longrightarrow M$ by $\alpha(sa) = xa$. $\alpha$ is well-defined because if $sa = sb$ then $at = bt$ for some $t \in S$ by S2 and $xat = xbt$ implies $xa = xb$ since $M$ is $S$-torsion-free. By hypothesis there exists $y \in M$ such that $\alpha(s) = ys$, and thus $x = ys$. This shows that $M = Ms$. $\square$

Considering the adjoints $-\otimes_A A[S^{-1}] \longrightarrow \varphi_*$, $\varphi : A \longrightarrow A[S^{-1}]$, $-\otimes_A A[S^{-1}] : \underline{\text{Mod}}\text{-}A \longrightarrow \underline{\text{Mod}}\text{-}A[S^{-1}]$, $\varphi_* : \underline{\text{Mod}}\text{-}A[S^{-1}] \to \underline{\text{Mod}}\text{-}A$ with unit $\mu : 1 \longrightarrow \varphi_*(-\otimes_A A[S^{-1}])$, $\mu_M : M \longrightarrow M[S^{-1}]$, the above Proposition classifies those modules $M \in \underline{\text{Mod}}-A$ for which $\mu$ is an isomorphism. Now the counit $\varepsilon_N : N \otimes_A A[S^{-1}] \longrightarrow N$ for $A[S^{-1}]$-modules $N$ is defined by $n \otimes (a,s) \mapsto n\cdot(a,s)$. Recall that there is a canonical isomorphism of $A[S^{-1}]$-modules $N \otimes_A A[S^{-1}] \cong N[S^{-1}] = N \times S/\sim$. Here $\varepsilon_N : N[S^{-1}] \longrightarrow N$ becomes $(n,s) \mapsto n\cdot s^{-1}$. Hence $\varepsilon_N$ is always an isomorphism, since it is certainly always onto, and if $n\cdot s^{-1} = m\cdot t^{-1}$, say $sa = ta'$, $a \in S$, so $n = mt^{-1}s = ma'a^{-1}$, so $na = ma'$. Hence by def$^N$ $(n,s) \sim (m,t)$, so $\varepsilon_N$ is also injective. Hence Lemma 4, Ch II §6 of SAL gives (noticing that (a) $\Rightarrow$ (c) above shows that for any $A[S^{-1}]$-module $N$, $\mu_{\varphi_* N}$ is an isomorphism) an equivalence of categories

$$\underline{\text{Mod}}\text{-}A[S^{-1}] \cong \mathcal{A}$$

where $\mathcal{A}$ is the full subcategory of $\underline{\text{Mod}}-A$ consisting of $S$-torsion-free and $S$-divisible (equiv. $S$-injective) modules. This follows since (a) $\Rightarrow$ (c) above can just as well show that if any $A$-module $M$ is isomorphic as an $A$-module to any $A[S^{-1}]$-module $\tilde{M}$, then both $M, \tilde{M}$ are isomorphic (as $A$ and $A[S^{-1}]$-modules resp.) to $M[S^{-1}]$. Hence $M \in \underline{\text{Mod}}\text{-}A$ is isomorphic to $\varphi_* N$ for some $N$ iff. it is canonically isomorphic to $M[S^{-1}]$, which is iff. (b) (or (c)) above hold. Moreover, by the lemma, $\mathcal{A}$ is a reflective subcategory of $\underline{\text{Mod}}-A$.

Note that by showing $\varepsilon$ above is an isomorphism, we showed $\varphi_*$ is full and faithful, which is all we really needed. It is not hard to see, then, that two $A[S^{-1}]$-modules are isomorphic iff. they are isomorphic as $A$-modules, and hence that $\varphi_*$ preserves distinctness of objects (i.e. is an embedding).

EXAMPLES

1. Sreg-torsion The definition of Sreg-torsion elements was first proposed by Levy. Assume $S_{reg}$ is a right denominator set in $A$. Instead of "Sreg-torsion" and "Sreg-divisible" one uses the shorter terms "torsion" and "divisible". Note that every Sreg-injective module is divisible, because the homomorphism $\alpha : sA \longrightarrow M$, given by $\alpha(sa) = xa$, is allways well-defined when $s$ is regular, and can be extended to $A$. Also, every flat module is torsion-free (by the argument used in Example I.10.1)

2. Orders in a semisimple ring Suppose $A$ has a classical right ring of quotients $Q$. Then $Q$ is semi-simple if and only if Sreg-injective modules are injective:

PROPOSITION 3.8 The following properties of a ring $A$ with a classical right ring of quotients $Q$ are equivalent

(a) Every torsion-free and divisible right $A$-module is injective
(b) Every right $Q$-module is injective over $A$
(c) Every Sreg-injective right $A$-module is injective
(d) $Q$ is a semi-simple ring.

PROOF (a) $\Leftrightarrow$ (b) by Prop 3.7, and also (c) $\Rightarrow$ (b) follows from this proposition.

(b) $\Rightarrow$ (d) Let $q$ be any right ideal in $Q$. Since $q$ is injective over $A$, $Q$ splits as an $A$-module as $Q = q \oplus K$. Now $K$ is torsion-free and divisible, and is therefore a $Q$-module. (Canonically, $(a+q)s^{-1} = (as^{-1}+q)$) consider

$$\begin{tikzcd} 0 \arrow[r] & q \arrow[r, "i"'] & Q \arrow[l, bend right, "\alpha"'] \arrow[r] & Q/q \arrow[l, bend right, "\varphi"'] \arrow[r] & 0 \end{tikzcd}$$

$Q$ is the direct sum of $q$ and $\varphi$ as $A$-modules, so to show $Q$ is the direct sum of $q$ and some other right ideal, it suffices to show $\varphi(Q/q)$ is an $Q$-submodule of $Q$. But $\operatorname{Ker}\alpha = \varphi$, so it suffices to show $\alpha$ is a morphism of $Q$-modules (it is a morphism of $A$-modules by construction). Let $q' \in q$ be s.t. $q' = \alpha(1)$, and let $q \in Q$, say $q = as^{-1}$. Then, since $q$ is a right ideal $\alpha(q)s = \alpha(as^{-1})s = \alpha(as^{-1}s) = \alpha(a) = q'a$. Hence $\alpha(q) = q'as^{-1}$ in $q$, i.e. $\alpha(q) = q'q$. Hence if $p \in Q$, $\alpha(q \cdot p) = \alpha(qp) = q'qp = (q'q)p = \alpha(q)p$, so $\alpha$ is a morphism of $Q$-modules. Hence $\varphi(Q/q)$ is closed under $Q$, and so $Q$ is semi-simple.

(d) $\Rightarrow$ (c) Let $M$ be an Sreg-injective module. Consider any morphism $\alpha : \mathfrak{a} \longrightarrow M$ from a right ideal $\mathfrak{a}$ of $A$. Let $\mathfrak{b}$ be a right ideal which is maximal w.r.t. $\mathfrak{a} \cap \mathfrak{b} = 0$. $\alpha$ can trivially be extended to $\alpha' : \mathfrak{a} + \mathfrak{b} \longrightarrow M$. But $\mathfrak{a}+\mathfrak{b}$ is an essential right ideal of $A$ (see the proof of Lemma 2.1) and therefore contains a regular element $s$ (Theorem 2.2). Since $M$ is Sreg-injective, $\alpha'$ can be extended to the whole of $A$. $\square$

## 4. INVERTIBLE IDEALS AND HEREDITARY ORDERS

Let $A$ be a right order in the ring $Q$. As usual we let $S_{reg}$ denote the set of regular elements of $A$. For each right $A$-submodule $I$ of $Q$ we put $I^* = \{q \in Q \mid qI \subseteq A\}$.

PROPOSITION 4.1 If $I$ is a right $A$-submodule of $Q$, with $I \cap S_{reg} \neq \emptyset$, then $I^* \cong \operatorname{Hom}_A(I, A)$.

PROOF If $q \in I^*$ then $x \longmapsto qx$ defines a homomorphism $I \longrightarrow A$. Conversely, suppose $\varphi : I \longrightarrow A$ is a morphism. Let $s \in I \cap S_{reg}$. For each $x \in I$ there exists $t \in S_{reg}$ with $xt \in A$, and by the Ore condition there exists $b \in A$ and $u \in S_{reg}$ such that $sb = xtu$. Then

$$\varphi(x)tu = \varphi(xtu) = \varphi(sb) = \varphi(s)b = \varphi(s)s^{-1}sb = \varphi(s)s^{-1}xtu$$

which implies $\varphi(x) = \varphi(s)s^{-1}x$ with $\varphi(s)s^{-1} \in I^*$. Notice that given $q \in I^*$, the induced morphism $x \longmapsto qx$ induces the element $qss^{-1}$ for some regular $s$, which is $q$. Likewise given $\varphi : I \longrightarrow A$, $q = \varphi(s)s^{-1}$, and clearly $x \longmapsto qx$ is $\varphi$. $\square$

DEFINITION A right $A$-submodule $I$ of $Q$ is right invertible if there exists $a_1, \ldots, a_n \in I$ and $q_1, \ldots, q_n \in I^*$ such that $1 = \sum a_i q_i$.

Note also that under these circumstances, we have for each $x \in I$, $x = \sum a_i q_i x$ with $q_i x \in A$, so $a_1, \ldots, a_n$ generate the module $I$. Thus

PROPOSITION 4.2 Every right invertible $A$-submodule of $Q$ is finitely generated

PROPOSITION 4.3 A right $A$-submodule $I$ of $Q$ is right invertible iff. $I \cap S_{reg} \neq \emptyset$ and $I$ is a projective module.

PROOF Suppose $I$ is right invertible, i.e. $1 = \sum a_i q_i$ with $a_i \in I$ and $q_i I \subseteq A$. Define morphisms $\varphi_i : I \longrightarrow A$ as $\varphi_i(x) = q_i x$. The family $(\varphi_i, a_i)$ clearly satisfies the requirements of Prop. I.6.3, and therefore makes $I$ into a finitely generated projective module. Choose $s \in S_{reg}$ such that all $q_i s \in A$. Then $s = \sum a_i q_i s \in I \cap S_{reg}$.
Assume conversely that $I$ is a projective $A$-module with $I \cap S_{reg} \neq \emptyset$. Let $(\varphi_j, a_j)_J$ be a family of projective coordinates for $I$. Each $\varphi_j : I \to A$ is of the form $\varphi_j(x) = q_j x$ for some $q_j \in I^*$. If $s \in I \cap S_{reg}$, then $s = \sum a_j \varphi_j(s)$ $= \sum a_j q_j s$, and $s$ regular implies $1 = \sum a_j q_j$. Hence $I$ is right invertible. □

A divisible module has the following injectivity property with respect to invertible right ideals:

PROPOSITION 4.4 Suppose $A$ is a two-sided order in $Q$. Let $\mathfrak{a}$ be a right invertible right ideal of $A$ and let $M$ be a divisible module. For each morphism $\alpha : \mathfrak{a} \longrightarrow M$ there exists $x \in M$ such that $\alpha(a) = xa$ for all $a \in \mathfrak{a}$.

PROOF Let $a_1, \dots, a_n \in \mathfrak{a}$ and $q_1, \dots, q_n \in \mathfrak{a}^*$ with $\sum a_i q_i = 1$. Since $A$ is assumed to be also a left order in $Q$, there exists $s \in S_{reg}$ such that all $s q_i \in A$. The divisibility of $M$ implies that we can write $\alpha(a_i) = x_i s$ with $x_1, \dots, x_n \in M$. For every $a \in \mathfrak{a}$ we obtain

$$\alpha(a) = \alpha\left(\sum a_i q_i a\right) = \sum \alpha(a_i) q_i a = \sum x_i s q_i a = xa$$

with $x = \sum x_i s q_i \in M$. □

This result is used to prove the following characterisation of hereditary orders.

PROPOSITION 4.5 Suppose $A$ is a twosided order in a semi-simple ring. The following properties of $A$ are equivalent.

(a) $A$ is right hereditary
(b) Every right ideal containing a regular element is invertible
(c) Every divisible module is injective.

PROOF (a) ⟹ (b) is a consequence of Prop 4.3
(b) ⟹ (c) Every divisible module is $S_{reg}$-injective by Prop 4.4, and is therefore injective by Prop 3.8.
(c) ⟹ (a) Every quotient module of an injective module is divisible, and is therefore injective by (c). Hence $A$ is right hereditary by Prop. I.9.5. □

We already knew that for PIDs, divisible ⟺ injective, and the above theorem tells us that provided $A$ is a two-sided order in a semi-simple ring, hereditary ⟺ (divisible ⟺ injective). The following is another important property of orders in semi-simple rings.

PROPOSITION 4.6 If $A$ is a two-sided order in a semi-simple ring, then every finitely generated torsion free module is a submodule of a free module.

PROOF Let $M$ be a torsion free module with generators $x_1, \dots, x_n$. By Cor 3.3. there is a monomorphism $M \longrightarrow M \otimes_A Q$ where $Q$ is the classical right ring of quotients for $A$. Now $M \otimes_A Q$ is a finitely generated $Q$-module, and since $Q$ is semi-simple, this implies that $M \otimes_A Q$ is a submodule of a finitely generated free $Q$-module $F$. Let $v_1, \dots, v_n$ be a basis for $F$ over $Q$. Then $x_j = \sum_i v_i q_{ij}$ for certain $q_{ij} \in Q$. Choose $s \in S_{reg}$ such that all $s q_{ij} \in A$. Then $x_j = \sum_i v_i s^{-1} s q_{ij}$, so $M$ is imbedded in the free $A$-submodule of $F$ generated by $v_1 s^{-1}, \dots, v_n s^{-1}$. □

COROLLARY 4.7 Let $A$ be a two-sided order in a semi-simple ring. The following properties of $A$ are equivalent

(a) $A$ is right semi-hereditary
(b) Every finitely generated torsion-free module is projective
(c) Every torsion-free module is flat
(d) Every right ideal is flat

PROOF (a) ⟹ (b) Every finitely generated torsion free module is a submodule of a free module (Prop 4.6), and is therefore projective (Prop I.6.9).

(b) ⇒ (c) Every module is the direct limit of its finitely generated submodules, and the direct limit of projective modules is flat. (Prop I.10.3)

(c) ⇒ (d) is trivial.

(d) ⇒ (a) follows from Prop I.11.6. □

EXAMPLES 1. Dedekind Domains When $A$ is a commutative integral domain, Prop 4.5 gives

PROPOSITION 4.8 The following properties of a commutative integral domain $A$ are equivalent

(a) $A$ is hereditary

(b) Every non-zero ideal is invertible

(c) Every divisible module is injective.

A ring satisfying these conditions is called a Dedekind domain. Every Dedekind domain is a noetherian ring (by Prop 4.2), and its invertible modules form an abelian group under multiplication.

2. Prüfer Domains A semi-hereditary commutative integral domain is called a Prüfer domain.

# EXERCISES Ch II Stenstrom.

**Q1** $S$ a right denominator set, $\varphi : A \longrightarrow A[S^{-1}]$ canonical.

a) $\varphi$ epi in $\underline{Rng}$. Suppose $f, g : A[S^{-1}] \longrightarrow Q$ s.t. $f\varphi = g\varphi$. Then $(f\varphi)(s) = g\varphi(s)$ is a unit $\forall s \in S$, so that by the universal property of $A[S^{-1}]$ there is a unique morphism $h : A[S^{-1}] \longrightarrow Q$ s.t. $h\varphi = f\varphi = g\varphi$. Hence $f = h = g$.

b) Let $q_1, \dots, q_n \in A[S^{-1}]$, say $q_i = (a_i, s_i)$. Then $q_i(s_i, 1) = (a_i, 1) \in \varphi(A)$. Let $s_1 u = s_2 u'$ with $u' \in S$. Then $q_1(s_1 u, 1) = q_1(s_1, 1)(u, 1) = (a_1 u, 1) \in \varphi(A)$, and $q_2(s_2 u', 1) = (a_2 u', 1) \in \varphi(A)$, where $(s_1 u, 1) = (s_2 u', 1) \in \varphi(S)$. We proceed by induction to the general case.

c) Let $A$ be a right Ore domain. We claim $A[x]$ is also a right Ore domain. We know that since $A$ has no zero-divisors, neither does $A[x]$ – represent $f, g \in A[x]$ as functions $\mathbb{N} \longrightarrow A$, so that $f = f(0) + f(1)x + \cdots + f(\deg f)x^{\deg f}$, where $m' = \deg f$ is largest s.t. $f(m') \neq 0$, resp. $m = \deg g$. Here $(fg)(s) = \sum_{x+y=s} f(x)g(y)$. Thus

$$(fg)(m'+m) = f(m')g(m)$$

Hence $fg = 0$ implies $f(m')g(m) = 0$, in $A$ which implies $f(m') = 0$ or $g(m) = 0$. Either is a contradiction.

Let $Q(A)$ denote the right ring of quotients of $A$ – it is a skew-field, and is hence right noetherian. It follows from the Hilbert basis theorem that $Q(A)[x]$ is likewise noetherian, lacks zero-divisors, and is hence a right Ore ring (domain). We have monics

$$A \longrightarrow Q(A) \longrightarrow Q(A)[x]$$

(Notice that for a general $\varphi : A \longrightarrow B$ and $b \in B$ we cannot induce $\tilde{\varphi} : A[x] \longrightarrow B$, $x \longmapsto b$, because without commutativity there is no gaurantee $\tilde{\varphi}$ is a morphism of rings). $A[x]$ can be made isomorphic to a certain subring of $Q(A)[x]$, so make this identification and suppose $a, s \in A[x]$, $s \neq 0$. If $a = 0$, then $a \cdot 1 = s \cdot 0$, so suppose $a \neq 0$. Then since $Q(A)[x]$ is a right Ore domain, we have $\tilde{t}, \tilde{b} \in Q(A)[x]$ s.t. $a\tilde{t} = s\tilde{b}$ in $Q(A)[x]$, with say $\tilde{t} \neq 0$, and hence $\tilde{b} \neq 0$. Let

$$\begin{aligned} \tilde{t} &= a_0 s_0^{-1} + a_1 s_1^{-1} x + \cdots + a_n s_n^{-1} x^n \\ &= q_0 + q_1 x + \cdots + q_n x^n \qquad\qquad q_i \in Q(A). \end{aligned}$$

Then by b) let $u \in S$ be s.t. $q_i u \in A$, $0 \le i \le n$. (We may identify $A$ with a subring of $Q(A)$). Then $\tilde{t}u$ is a polynomial of $Q(A)[x]$ contained in the subring $A[x]$. Do the same with $\tilde{b}$, say $v \in S$ s.t. $\tilde{b}v \in A[x]$, and let $w \in A$, $w' \in S$ be s.t. $uw = vw'$ ($A$ is a right Ore domain). Then $\tilde{t}uw, \tilde{b}vw' \in A[x]$ and

$$a\tilde{t}uw = s\tilde{b}vw'$$

Since $\tilde{t} \neq 0$, $\tilde{b} \neq 0$, both $\tilde{t}uw$, $\tilde{b}vw'$ are nonzero elements of $A[x]$. Hence $A[x]$ is a right Ore domain.

d) Let $S$ be a right denominator set in $A$. Consider the ring $M_n(A)$, and identify $A$ with the usual subring of diagonal matrices. Then $S$ is identified with a right denominator set of $M_n(A)$, since if $s \in S$ and $a = (a_{ij}) \in M_n(A)$ are given, then let $b_{ij}$ be s.t. $sb_{ij} = a_{ij}t_{ij}$ $t_{ij} \in S$. We can modify the $b_{ij}$s to get all the $t_{ij}$'s the same – for example let $t_{11}v = t_{12}v'$, then $sb_{11}v = a_{11}t_{11}v$ and $sb_{12}v' = a_{12}t_{12}v'$. In this way produce $b' = (b_{ij}')$ s.t. $sb_{ij}' = a_{ij}t$ for some $t \in S$. That is, $sb' = at$, with $t \in S$ as required. To check S2, if $sa = 0$, $(sa_{ij}) = 0$, so each $sa_{ij} = 0$, in $A$. Find a common $t$ s.t. $a_{ij}t = 0$ as above.

(ii) Let $\varphi : A \longrightarrow A[S^{-1}]$ be canonical. This induces $\tilde{\varphi} : M_n(A) \longrightarrow M_n(A[S^{-1}])$. This itself is interesting. Let $\phi : A \longrightarrow B$ be any morphism of rings – this induces $\phi^* : \text{Mod-}A \longrightarrow \text{Mod-}B$, $N \longmapsto N \otimes_A B$. Hence $\phi^*(A) = A \otimes_A B = B$. Since $\phi^*$ is right exact, $\phi^*(\oplus_{i=1}^n A) = \oplus_{i=1}^n B$, so $\phi^*$ (as a functor) induces a morphism of the endomorphism rings $M_n(A) = \text{End}_A(\oplus_{i=1}^n A)$ to $M_n(B) = \text{End}_B(\oplus_{i=1}^n B)$. Suppose $m = (m_{ij})$ is an endomorphism of $\oplus_{i=1}^n A$. Then by general yoga, $\phi^*(m) = (\phi^*(m_{ij}))$, so that in terms of matrices, $\phi : A \longrightarrow B$ induces $M_n(A) \longrightarrow M_n(B)$, $(a_{ij}) \longmapsto (\phi(a_{ij}))$. In any case, we have

$$\begin{aligned} \tilde{\varphi} : M_n(A) &\longrightarrow M_n(A[S^{-1}]) \\ (a_{ij}) &\longmapsto ((a_{ij}, 1)) \end{aligned}$$

Clearly for $s \in S$, $\tilde{\varphi}(s)$ is a unit, and if $\tilde{\varphi}((a_{ij})) = ((a_{ij}, 1)) = 0$ then each $(a_{ij}, 1) = 0$, that is, $\varphi(a_{ij}) = 0$ so there is $t_{ij} \in S$ s.t. $a_{ij}t_{ij} = 0$ in $A$. We have $n^2$ such equations, and we can keep piling on elements of $S$ until $t_{ij} = t_{ke}$ $\forall i,j,k,e$. That is, until we have $t \in S$ s.t. $(a_{ij})t = 0$. Now suppose $q = ((a_{ij}, s_{ij})) \in M_n(A[S^{-1}])$. As above, let $u_{ij} \in A$ be s.t. $s = s_{ij}u_{ij}$ $\forall i,j$, $s \in S$. Then $q = ((a_{ij}u_{ij}, 1))s^{-1} = \tilde{\varphi}((a_{ij}u_{ij}))\tilde{\varphi}(s)^{-1}$, as required. Hence

$$M_n[A][S^{-1}] = M_n(A[S^{-1}])$$

(iii) Now notice that in (i), given $s \in S$ and $a = (a_{ij}) \in UT_n(A)$, we can clearly find $b'$ also upper triangular, since $sb' = at$. Hence $S$ is a right denominator set in $UT_n(A)$, also. The morphism $\tilde{\varphi}$ above restricts to $UT_n(A) \longrightarrow UT_n(A[S^{-1}])$, and the rest of the proof carries through unchanged to show

$$UT_n(A)[S^{-1}] = UT_n(A[S^{-1}])$$

(Q5) Suppose $A$ is injective as a right and left $A$-module. Let $a \in A$ be a nonzerodivisor. The corresponding morphism of left (right) $A$-modules $A \longrightarrow A$ defined by $b \longmapsto b \cdot a$ ($b \longmapsto a \cdot b$) is thus monic, since $a\varphi = a\phi \Leftrightarrow a\varphi(x) = a\phi(x)\ \forall x \Rightarrow$ $(\varphi(x) - \phi(x))a = 0$ ($a(\varphi(x) - \phi(x)) = 0$) and hence $\varphi(x) = \phi(x)$, $\varphi = \phi$. But then by injectivity there is an $x$ s.t.

$$\begin{tikzcd} & A \\ A \arrow[r, "a"'] \arrow[ur] & A \arrow[u, "x"'] \end{tikzcd}$$

where $x$ corresponds to an element $x(1)$ of $A$, and $(xa)(1) = 1 \Leftrightarrow x(a(1)) = 1 \Leftrightarrow ax = 1$ ($xa = 1$). Hence $a$ is a unit.

Q6 Let $S$ and $T$ be subsets of $A$. Then

(i) $r(\ell(r(S))) = r(S)$

$\ulcorner$ $x \in r(\ell(r(S))) \Leftrightarrow \forall y \in \ell(r(S))\ yx = 0$

$\Leftrightarrow$ whenever $yu = 0\ \forall u \in r(S)$, $yx = 0$

$\Leftrightarrow$ if for any $q$ s.t. $sq = 0$, $\forall s \in S$ we have $yq = 0$, then $yx = 0$

If $x \in r(S)$, and $y \in \ell(r(S))$, then in particular $yx = 0$, so $x \in r(\ell(r(S)))$. Now clearly $S \subseteq \ell(r(S))$, so that if $x \in r(\ell(r(S)))$, then in particular $x \in r(S)$ $\lrcorner$

(ii) $r(S \cup T) = r(S) \cap r(T)$

$\ulcorner$ By defn $\lrcorner$

(iii) $S \subseteq T$ implies $r(S) \supseteq r(T)$

$\ulcorner$ By defN $\lrcorner$

Q7 Notice that also $\ell(r(\ell(S))) = \ell(S)$ and that $S \subseteq T \Rightarrow \ell(S) \supseteq \ell(T)$. If $A$ has ACC on right annihilators, and

$$\ell(S_1) \supseteq \ell(S_2) \supseteq \cdots$$

is a descending chain of left annihilators, then $r(\ell(S_1)) \subseteq r(\ell(S_2)) \subseteq \cdots$ terminates, say $r(\ell(S_k)) = r(\ell(S_n))\ \forall k \geq n$. Then $\ell(r(\ell(S_k))) = \ell(r(\ell(S_n)))\ \forall k \geq n$; that is, $\ell(S_k) = \ell(S_n)\ \forall k \geq n$, as required. The converse is similar.

Q8 Let $A$ be a prime ring, $\mathfrak{a}$ a two-sided ideal other than $0$. To see $\mathfrak{a}$ is essential as a right ideal, suppose $\mathfrak{b}$ is a right ideal with $\mathfrak{a} \cap \mathfrak{b} = 0$. Hence $\mathfrak{b}\mathfrak{a} \subseteq \mathfrak{a} \cap \mathfrak{b}$ implies $\mathfrak{b}\mathfrak{a} = 0$. Let $A\mathfrak{b}$ denote the two-sided ideal $\{\sum a_i b_i \mid a_i \in A, b_i \in \mathfrak{b}\}$ containing $\mathfrak{b}$. Then $A\mathfrak{b}\mathfrak{a} = 0$ implies $A\mathfrak{b} = 0$, since $A$ is prime. Hence $\mathfrak{b} = 0$, and $\mathfrak{a}$ is essential as a right ideal. Alternatively, if $\mathfrak{b}$ is a left ideal, $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{a} \cap \mathfrak{b} = 0$ and so $\mathfrak{b}A = 0$, so $\mathfrak{b} = 0$ again. Hence every nonzero two-sided ideal is both right and left essential. It is also clearly essential w.r.t. two sided ideals, by the same argument.

Q11 Let $S$ be a right denominator set in $A$, and $\mu_M : M \longrightarrow M[S^{-1}]$ canonical. A submodule $L$ of $M$ is called <u>$S$-saturated</u> if it satisfies: if $x \in M$ and $xs \in L$ for some $s \in S$, then $x \in L$.

(i) Since $A[S^{-1}]$ is flat as a left $A$-module, a subobject $L \rightarrowtail M$ induces $L[S^{-1}] \rightarrowtail M[S^{-1}]$, $\ell s^{-1} \longmapsto \ell s^{-1}$. Conversely, any $A[S^{-1}]$-submodule $N$ of $M[S^{-1}]$ is in particular an $A$-submodule of the right $A$-module $M[S^{-1}]$, whence $\mu_M^{-1}N$ is a right $A$-submodule of $M$. Clearly such $\mu^{-1}N$ is $S$-saturated, since if $x \in M$ and $xs \in \mu^{-1}N$, so $\mu(xs) = (xs, 1)$ is in $N$, then $(x,1) = (xs,1)(1,s)$ is in $N$, so that $x \in \mu^{-1}N$. Also, $(\mu^{-1}N)[S^{-1}]$ and $N$ are the same subobject, since $(\mu^{-1}N)[S^{-1}] \longrightarrow M[S^{-1}]$ is the inclusion of all $(n,s)$ where $n \in \mu^{-1}N$, or equivalently $(n,1) \in N$. But if $(n,1) \in N$ so does $(n,s)$, and conversely if $x = (m,t) \in N$, $m \in M$, $t \in S$, then $(m,1) \in N \Rightarrow m \in \mu^{-1}N$ so that $(m,t) \in \mu^{-1}N[S^{-1}]$.

Going the other way, certainly $L \subseteq \mu^{-1}L[S^{-1}]$, and the reverse inclusion holds iff. whenever $(x,1) \in L[S^{-1}]$, $x \in L$. But $(x,1) \in L[S^{-1}] \Leftrightarrow \exists y \in L, t \in S$ s.t. $(x,1) = (y,t) \Leftrightarrow \exists y \in L, t, t', s \in S$ s.t. $xs = yt'$, $s = tt' \Leftrightarrow \exists s \in S$ s.t. $xs \in L$. Hence $\mu^{-1}L[S^{-1}] \subseteq L$ iff. $L$ is $S$-saturated. Thus we have a bijection

$$\begin{array}{ccc} \mathrm{Sub}\,M & & \mathrm{Sub}\,M[S^{-1}] \\ L & \longmapsto & L[S^{-1}] \\ \mu^{-1}N & \longleftarrow & N \end{array}$$

between the $S$-saturated submodules of $M$ and all submodules of $M[S^{-1}]$ (this is an isomorphism of lattices). Notice that when $A = M$, a prime right ideal $\mathfrak{a}$ is $S$-saturated iff. it avoids $S$, and we get the usual correspondence.

(ii) Suppose that $M$ is $S$-torsion-free, and suppose that $xas = 0$ for $x \in M$, $a \in A$ and $s \in S$. Then $xa \in M$ since $M$ is an $A$-module, so since $M$ is $S$-torsion free, $xa = 0$. Hence $\forall x \in M$, $\mathrm{Ann}(x)$ is $S$-saturated. Conversely, suppose each $\mathrm{Ann}(x)$ is $S$-saturated, and suppose $xs = 0$, $x \in M$, $s \in S$. Then $0 = x \cdot (1s)$, whence $1s \in \mathrm{Ann}(x)$, so since $\mathrm{Ann}(x)$ is $S$-saturated, $1 \in \mathrm{Ann}(x)$. Hence $x = 0$ and $M$ is $S$-torsion-free.

(iii) Notice the bijection in (i) is inclusion preserving, so with $A = M$ we have an inclusion preserving correspondence between right ideals of $A[S^{-1}]$ and $S$-saturated right ideals of $A$. If $A$ is right noetherian and

$$N_0 \leq N_1 \leq \cdots$$

is a chain of right ideals in $A[S^{-1}]$, then $\mu^{-1}N_0 \leq \mu^{-1}N_1 \leq \cdots$ in $A$, so $\mu^{-1}N_k = \mu^{-1}N_m$ all $k \geq m$ some $m$. Hence $N_k = (\mu^{-1}N_k)[S^{-1}] = (\mu^{-1}N_m)[S^{-1}] = N_m$ all $k \geq m$, so $A[S^{-1}]$ is right noetherian.

$\boxed{Q12}$ This is trivial since $M[S^{-1}] = M \otimes_A A[S^{-1}]$, and we already know $- \otimes_A A[S^{-1}] \dashv \varphi_*$ where $\varphi : A \longrightarrow A[S^{-1}]$ is canonical. Hence

$$\begin{aligned} \mathrm{Hom}_{A[S^{-1}]}(M[S^{-1}], N) &= \mathrm{Hom}_{A[S^{-1}]}(M \otimes_A A[S^{-1}], N) \\ &\cong \mathrm{Hom}_A(M, \varphi_* N) \\ &= \mathrm{Hom}_A(M, N) \end{aligned}$$

(ii) The functor $\varphi_* : \underline{\mathrm{Mod}}\text{-}A[S^{-1}] \longrightarrow \underline{\mathrm{Mod}}\text{-}A$ has a left adjoint which preserves monics ($A[S^{-1}]$ is flat) and hence by the usual yoga $\varphi_*$ preserves injectives. Thus if a right $A[S^{-1}]$-module $N$ is injective, then it is also injective as a right $A$-module. The converse is easy, since if $N \in \underline{\mathrm{Mod}}\text{-}A[S^{-1}]$ is injective as an $A$-module, and

$$\begin{tikzcd} & N \\ M \arrow[r, tail] \arrow[ur] & M' \arrow[u, dashed, "\alpha"'] \end{tikzcd} \qquad (1)$$

is a diagram in $\underline{\mathrm{Mod}}\text{-}A[S^{-1}]$, it is also a diagram in $\underline{\mathrm{Mod}}\text{-}A$. (monics preserved since reflective). Let $\alpha$ make (1) commute. $\underline{\mathrm{Mod}}\text{-}A[S^{-1}]$ is full in $\underline{\mathrm{Mod}}\text{-}A$, so $\alpha$ is a morphism of $A[S^{-1}]$-modules, so $N$ is injective in $\underline{\mathrm{Mod}}\text{-}A[S^{-1}]$.

$\boxed{Q14}$ (i) Let $\mathfrak{b} = (q_1, \ldots, q_n)$ be a finitely generated right ideal of $Q$, with $q_i = a_i s_i^{-1}$. Then clearly $a_i \in \mathfrak{b}$, $1 \leq i \leq n$, and hence if $\mathfrak{a} = (a_1, \ldots, a_n)$ is the right ideal of $A$ generated by the $a_i$, $\mathfrak{a}Q \leq \mathfrak{b}$. Conversely, all the $q_i$ belong to $\mathfrak{a}Q$, hence $\mathfrak{b} \leq \mathfrak{a}Q$ — so $\mathfrak{b} = \mathfrak{a}Q$, as required.

(ii) Suppose $A$ is right semi-hereditary. Notice that if $\mathfrak{a} \to A$ is a right ideal of $A$, $\mathfrak{a}Q$ is the submodule $\mathfrak{a} \otimes_A Q \longrightarrow A \otimes_A Q = Q$, so in particular, (i) states that any finitely generated submodule of $Q$ is an object of this form. Let $\mathfrak{b} = \mathfrak{a}Q$ be finitely generated. Then $\mathfrak{a}$ is projective, since $A$ is semi-hereditary. Hence by I, 6.3 we have $(x_i)_I$, $x_i \in \mathfrak{a}$ and $\varphi_i : \mathfrak{a} \longrightarrow A$ such that $\forall x \in \mathfrak{a}$,

$$x = \sum_i x_i \varphi_i(x)$$

Consider the morphisms of $Q$-modules $\hat{\varphi}_i = \varphi_i \otimes_A Q : \mathfrak{a} \otimes_A Q \longrightarrow A \otimes_A Q$, that is, $\hat{\varphi}_i : \mathfrak{a}Q \longrightarrow Q$. Let $y = \sum_j a_j q_j$ be an element of $\mathfrak{a}Q$, $a_j \in \mathfrak{a}$ and $q_j \in Q$. Then with $x_i \in \mathfrak{a} \leq \mathfrak{a}Q$ as before,

$$\begin{aligned} \sum_i x_i \hat{\varphi}_i(y) &= \sum_i x_i \hat{\varphi}_i\Big(\sum_j a_j q_j\Big) \\ &= \sum_{i,j} x_i \hat{\varphi}_i(a_j q_j) \\ &= \sum_{i,j} x_i \varphi_i(a_j) q_j \\ &= \sum_j \Big(\sum_i x_i \varphi_i(a_j)\Big) q_j \\ &= \sum_j a_j q_j = y \end{aligned}$$

where the sums are all finite since $\hat{\varphi}_i(y) = \sum_j \varphi_i(a_j) q_j$, and by assumption $\varphi_i(a_j) = 0$ for all but finite $i \in I$. Hence $(x_i), \hat{\varphi}_i$ form a system of projective coordinates for $\mathfrak{a}Q$, so $\mathfrak{a}Q$ is projective. Hence $Q$ is semi-hereditary.

$\boxed{Q15}$ Let $A$ be a right order in $Q$. If every divisible right $A$-module is injective, then $A$ is right hereditary by I, Prop 9.5 since injective $\Rightarrow$ divisible, and every quotient of a divisible module is divisible, and hence by hypothesis, injective. $Q$ is then semi-simple by II, Prop 3.8.

# CHAPTER III : MODULAR LATTICES

## 1. LATTICES

Let $P$ be a partially ordered set, with the partial ordering denoted by $\leq$, and $S$ a subset of $P$. An upper bound for $S$ in $P$ is an element $x \in P$ such that $s \leq x$ for all $s \in S$. An element $s_0 \in S$ is a greatest element in $S$ if $s \leq s_0$ for all $s \in S$. There can be at most one greatest element in $S$. Similarly one defines lower bound and least element. In particular, a least upper bound for $S$ is a least element in the set of upper bounds for $S$, and similarly for greatest lower bound for $S$.

A lattice is a partially ordered set in which every couple of elements $x, y$ has a least upper bound (called the join of $x, y$ and written $x \vee y$), and a greatest lower bound (called the meet of $x$ and $y$, written $x \wedge y$). It follows then by induction that every non-empty finite set of elements has a join and a meet.

Suppose $L$ is a lattice. If one reverses the partial ordering in $L$, then one obtains a new lattice, called the oppposite (or dual) lattice of $L$ and denoted $L^{op}$. As in the case of categories, this leads to a duality principle, by which every concept or result for lattices carries with it a dual concept or result.

If $L$ and $L'$ are lattices, then a morphism $\alpha : L \longrightarrow L'$ is a map from $L$ to $L'$ satisfying

$$\alpha(x \vee y) = \alpha(x) \vee \alpha(y) \qquad \alpha(x \wedge y) = \alpha(x) \wedge \alpha(y) \qquad \forall x, y \in L$$

Notice that $a \leq b$ in $L$ iff. $a \vee b = a$, so that if $\alpha : L \longrightarrow L'$ is a morphism and $a \leq b$, $\alpha(a) \leq \alpha(b)$. Hence a morphism is a functor preserving finite limits and colimits. In this way we obtain the category $\underline{Lat}$ of lattices. It is a subcategory of the category of partially ordered sets, where the morphisms are the order preserving maps (we call them order morphisms). Since the lattice operations are defined in terms of the ordering, one obtains:

PROPOSITION 1.1 If $L$ and $L'$ are lattices, then every order isomorphism $L \longrightarrow L'$ is a lattice isomorphism.

A lattice morphism $L^{op} \longrightarrow L'$ is often said to be an anti-morphism from $L$ to $L'$. If $L$ is a lattice, then a sublattice of $L$ is a subset $L'$ of $L$ such that $x, y \in L'$ implies $x \vee y \in L'$ and $x \wedge y \in L'$; $L'$ is then itself a lattice. A lattice $L$ is complete if every subset $S$ of $L$ has a least upper bound, written $\sup S$ or $\bigvee_{s \in S} s$ and called the join of $S$, and a greatest lower bound, written $\inf S$ or $\bigwedge_{s \in S} s$ and called the meet of $S$. In a complete lattice there exists a greatest element $\sup L$, denoted by $1$, and a smallest element $\inf L$, denoted by $0$. Note that we also have $1 = \inf \emptyset$ and $0 = \sup \emptyset$.

PROPOSITION 1.2 If $L$ is a partially ordered set and every subset of $L$ has a least upper bound in $L$, then $L$ is a complete lattice.

PROOF For each subset $S$ of $L$ consider the set $B$ of all lower bounds of $S$ in $L$. ($B$ is nonempty because $0 = \sup \emptyset \in L$) Let $x$ be the least upper bound of $B$ in $L$. If $s \in S$ then for each $b \in B$ we have $b \leq s$, so $s \geq \sup B = x$. If $y$ is any element s.t. $s \geq y$ $\forall s \in S$, then $y \in B$ and hence $y \leq x$. It follows that $x$ is a greatest lower bound for $S$. □

If $L$ is a complete lattice, then a complete sublattice of $L$ is a sublattice $L'$ of $L$ such that $S \subseteq L'$ implies that $\sup S$ and $\inf S$ (taken in $L$) belong to $L'$; it is then itself a complete lattice.

EXAMPLES

1. Intervals Let $L$ be a lattice and $a, b$ elements of $L$ with $a \leq b$. Then $[a,b] = \{x \in L \mid a \leq x \leq b\}$ is a sublattice of $L$, called the interval between $a$ and $b$.

2. Ideals and Filters If $L$ is a lattice, then a subset $I$ of $L$ is an ideal if it satisfies

   (i) $a \in I$, $x \leq a$ implies $x \in I$
   (ii) $a, b \in I$ implies $a \vee b \in I$
   (iii) If $L$ has $0$, then $0 \in I$.

   In particular for any $a \in L$ one obtains the principal ideal $\{x \in L / x \leq a\}$. Dually one defines the notion of a filter: a subset $F$ of $L$ is a filter if

   (i) $a \in F$, $a \leq x$ implies $x \in F$
   (ii) $a, b \in F$ implies $a \wedge b \in F$
   (iii) If $L$ has $1$, then $1 \in F$.

3. The Lattice of Submodules If $M$ is a module, then the submodules of $M$ form a complete lattice, which we will denote by $L(M)$. When $K \subseteq L \subseteq M$, the interval $[K, L]$ in $L(M)$ is isomorphic to the lattice of submodules of $L/K$. The lattices of right resp. left ideals of the ring $A$ will be denoted by $L(A.)$ and $L(.A)$.

4. Completions If $L$ is a lattice, then an upper (resp. lower) completion of $L$ is a complete lattice $\hat{L}$, containing $L$ as a sublattice such that $x = \sup(L \cap [0, x])$ (resp. $x = \inf(L \cap [x, 1])$) for every $x \in \hat{L}$. If $\hat{L}$ is both an upper and a lower completion of $L$, then $\hat{L}$ is a completion of $L$. Eg. the chain of real numbers is a completion of the rational numbers. One may show that every lattice has a unique completion, the so-called Dedekind – MacNeille completion (for the existence, see §8; for unicity, see Schmidt [1]).

5. Closure systems Let $L$ be a complete lattice. If $C$ is a subset of $L$, and $C$ is closed under arbitrary meets (i.e. $S \subseteq C$ implies $\inf S \in C$), then it follows from Prop 1.2 that $C$ is a complete lattice. Such a subset $C$ of $L$ is called a closure system in $L$. Note that the meet in $C$ is the same as in $L$, but that the join in $C$ is given by

$$\sup_C S = \inf\{\text{upper bounds of } S \text{ in } C\}$$

## 2. MODULARITY

Let $L$ be a lattice. The operations $\wedge$ and $\vee$ are associative and commutative. Furthermore one clearly has for any $a \leq b$ in $L$ that

$$(x \wedge b) \vee a \leq (x \vee a) \wedge b \qquad \text{for all } x \in L$$

since $x \wedge b \leq x \leq x \vee a$, etc. The lattice is modular if the reverse inequality also holds, i.e.,

$$(c \wedge b) \vee a = (c \vee a) \wedge b \qquad \text{for all } a, b, c \text{ with } a \leq b$$

PROPOSITION 2.1 Let $a$ and $b$ be elements of a modular lattice. Then there is a lattice isomorphism $[a \wedge b, a] \to [b, a \vee b]$.

PROOF Define $\alpha : [a \wedge b, a] \longrightarrow [b, a \vee b]$ as $\alpha(x) = x \vee b$, and define $\beta : [b, a \vee b] \longrightarrow [a \wedge b, a]$ as $\beta(y) = y \wedge a$. Then $\beta\alpha(x) = (x \vee b) \wedge a = (a \wedge b) \vee x = x$ by modularity, since $x \leq a$. Dually, $\alpha\beta(y) = (y \wedge a) \vee b = (a \vee b) \wedge y = y$. The map $\beta$ is thus the inverse of $\alpha$. As an order iso., $\alpha, \beta$ are lattice isos. □

This result leads to a refinement of the notion of isomorphism between intervals of a modular lattice. Two intervals of $L$ are similar if there exist elements $a, b \in L$ s.t. one of the intervals can be written as $[a \wedge b, a]$ and the other one as $[b, a \vee b]$. Two intervals $I$ and $J$ of $L$ are projective if there exists a chain $I = I_0, I_1, \ldots, I_n = J$ of intervals s.t. $I_{i-1}$ and $I_i$ are similar. Projective intervals are of course isomorphic as lattices.

Let $L$ be a lattice with $0$ and $1$. If $a \in L$, then a complement of $a$ in $L$ is an element $c \in L$ s.t.

$$a \wedge c = 0, \qquad a \vee c = 1$$

the lattice $L$ is complemented if every element of $L$ has a complement in $L$.

PROPOSITION 2.2 If $L$ is a complemented modular lattice, then every interval of $L$ is complemented.

PROOF Let $a \leq b$ in $L$ and $d \in [a, b]$ and suppose $d$ has a complement $c$ in $L$. Then one verifies that

$$a \vee (c \wedge b) = b \wedge (a \vee c)$$

is a complement of $d$ in $[a, b]$. □

The modularity of $L$ depends on a unicity property of relative complements:

PROPOSITION 2.3 The lattice $L$ is modular if and only if every interval $I$ of $L$ has the following property: if $c \in I$ has two complements $a, b$ in $I$ with $a \leq b$, then $a = b$.

PROOF If $L$ is modular, then so is every interval, so for the necessity we may assume $I=L$. Then

$$b = b\wedge 1 = b\wedge(a\vee c) = a\vee(b\wedge c) = a\vee 0 = a$$

Conversely, if $a,b,c$ are elements of $L$ with $a\le b$, then we have the modular inequality

$$a_1 = (c\wedge b)\vee a \le (c\vee a)\wedge b = a_2$$

Then $a_1\wedge c = ((c\wedge b)\vee a)\wedge c \geq (c\wedge a)\vee(c\wedge b) = c\wedge b$, and $a_1\le b$ implies $a_1\wedge c = c\wedge b$. Also, $a_2\wedge c = (c\vee a)\wedge b\wedge c = b\wedge c$. Further we have $a_1\vee c = (c\wedge b)\vee a\vee c = a\vee c$ and finally $a_2\vee c = ((c\vee a)\wedge b)\vee c \le (b\vee c)\wedge(c\vee a) = a\vee c$, and $a\le a_2$ implies $a_2\vee c = a\vee c$. Thus $a_1$ and $a_2$ are complements of $c$ in $[b\wedge c, a\vee c]$, and by hypothesis they must be equal, which proves the modularity of $L$. $\square$

EXAMPLES

1. The lattice of submodules If $M$ is a module, then the lattice $L(M)$ of submodules is modular. For let $K, L, L'$ be submodules of $M$ such that $L'\subseteq L$. If $x\in(K+L')\cap L$, then $x = y+z$ with $y\in K$ and $z\in L'$. But $y = x-z\in K\cap L$, so $x\in(K\cap L)+L'$, which shows that $(K+L')\cap L = (K\cap L)+L'$.

If two intervals $[K',K]$ and $[L',L]$ in $L(M)$ are projective, then it follows from the Noether isomorphism theorem (Prop I.1.5) applied to each one of the similar pairs of intervals connecting $[K',K]$ and $[L',L]$, that the quotient modules $K/K'$ and $L/L'$ are isomorphic.

A member of $L(M)$ has a complement iff. it is a direct summand, and so $L(M)$ is complemented iff. it is semi-simple. (Prop I.7.2).

2. Coherent Rings

3. LATTICES WITH CHAIN CONDITION

Let $L$ be a modular lattice with $0$ and $1$. Two chains

$$a = a_0 \le a_1 \le \cdots \le a_m = b \qquad (1)$$

$$a = b_0 \le b_1 \le \cdots \le b_n = b \qquad (2)$$

between the same pair of elements of $L$ are called equivalent if $m=n$ and there is a permutation $\pi$ of $\{1,\dots,n\}$ s.t. the intervals $[a_{i-1}, a_i]$ and $[b_{\pi(i)-1}, b_{\pi(i)}]$ are projective. A refinement of a chain (1) is obtained by inserting further elements in the chain. The following result is known as the "Schreier refinement theorem":

PROPOSITION 3.1 *Any two finite chains between the same pair of elements in a modular lattice have equivalent refinements.*

PROOF Let (1) and (2) be the given chains. For each $j=0,\dots,n$ and $i=0,\dots,m$ we put

$$a_{ij} = (a_i\wedge b_j)\vee b_{j-1} = (a_i\vee b_{j-1})\wedge b_j$$

$$b_{ji} = (b_j\wedge a_i)\vee a_{i-1} = (b_j\vee a_{i-1})\wedge a_i$$

The interval

$$[(a_{i-1} \vee b_{j-1}) \wedge a_i \wedge b_j,\ a_i \wedge b_j] \qquad (3)$$

is then similar to $[a_{i-1,j}, a_{ij}]$ because we have

$$a_{i-1,j} \wedge (a_i \wedge b_j) = (a_{i-1} \vee b_{j-1}) \wedge b_j \wedge a_i \wedge b_j = (a_{i-1} \vee b_{j-1}) \wedge a_i \wedge b_j$$

while $a_{i-1} \wedge b_j \leq a_i$ gives

$$\begin{aligned} a_{i-1,j} \vee (a_i \wedge b_j) &= (a_{i-1} \wedge b_j) \vee b_{j-1} \vee (a_i \wedge b_j) \\ &= a_i \wedge ((a_{i-1} \wedge b_j) \vee b_j) \vee b_{j-1} = (a_i \wedge b_j) \vee b_{j-1} = a_{ij} \end{aligned}$$

By symmetry, the interval (3) is also similar to $[b_{j-1,i}, b_{ji}]$. So $[b_{j-1,i}, b_{ji}]$ is projective to $[a_{i-1,j}, a_{ij}]$, and it follows that the two chains

$$a = a_{01} \leq a_{11} \leq \cdots \leq a_{m1} \leq a_{12} \leq \cdots \leq a_{mn} = b$$
$$a = b_{01} \leq b_{11} \leq \cdots \leq b_{n1} \leq b_{12} \leq \cdots \leq b_{nm} = b$$

where $a_{mj} = (a_m \wedge b_j) \vee b_{j-1} = (b \wedge b_j) \vee b_{j-1} = b_j = a_{0,j+1}$ and $b_{ni} = a_i = a_{0,i+1}$ are equivalent. Visually,

$$a = b_0 \leq b_1 \leq \cdots \leq b_n = b$$

$$\begin{array}{ccccccccccc} a = a_{01} \leq a_{11} \leq & \cdots & & \leq a_{m1} & \leq a_{12} & \leq & \cdots & \leq a_{mn} & = b \\ = b_0 & & \leq & \| & & & & \| & \\ & & & b_1 & & \leq \cdots \leq & & b_n & = b \end{array}$$

and

$$a = a_0 \leq a_1 \leq \cdots \leq a_m = b$$

$$\begin{array}{ccccccccc} a = b_{01} \leq b_{11} \leq & \cdots & \leq b_{m1} & \leq b_{12} \leq & \cdots & & \leq b_{nm} & = b \\ \| & & \| & & & & \| & \\ a_0 & \leq & a_1 & \leq & \cdots & \leq & a_m & = b \end{array}$$

$i=1, j=1$ $\qquad$ $i=2, j=1$

so $[a_{01}, a_{11}]$ is projective to $[b_{01}, b_{11}]$, $[a_{11}, a_{21}]$ is projective to $[b_{02}, b_{12}]$, up to $[a_{m-1,1}, a_{m1}]$ proj. to $[b_{0m}, b_{1m}]$. Hence the two chains are equivalent, where $a_{01}, \ldots, a_{mn}$ is a refinement of (2), and $b_{01}, \ldots, b_{nm}$ is a refinement of (1). □

A composition chain between $a$ and $b$ is a chain

$$a = a_0 < a_1 < \cdots < a_m = b \qquad (4)$$

which has no refinement, except by introducing repetitions of the given elements $a_i$. The integer $m$ is the length of the chain. From Prop 3.1 we immediately get the "Jordan-Hölder" theorem.

COROLLARY 3.2 Any two composition chains between the same pair of elements in a modular lattice are equivalent.

A modular lattice $L$ is of finite length if there is a composition chain between $0$ and $1$, and by Cor 3.2 we can define the length of $L$ to be the length of such a composition chain. Prop 3.1 allows us to conclude:

PROPOSITION 3.3 In a modular lattice of finite length, every chain (4) can be refined to a composition chain.

A lattice $L$ is noethenan (or satisfies ACC) if there are no infinite strictly ascending chains $a_0 < a_1 < \cdots$ in $L$, and is artinian (or satisfies DCC) if there is no infinite strictly descending chain $a_0 > a_1 > \cdots$ in $L$. These chain conditions can also be formulated as maximum (minimum) conditions. If $S$ is a subset of $L$, then a maximal element of $S$ is an element $a \in S$ such that $a \leq x$ for $x \in S$ implies $x = a$. Dually one defines minimal element.

PROPOSITION 3.4 A lattice $L$ is noethenan (artinian) if and only if every nonempty subset of $L$ has a maximal (minimal) element.

PROPOSITION 3.5 A modular lattice is of finite length if and only if it is both noethenan and artinian.

PROOF If $L$ has finite length $m$, then every strictly ascending (descending) chain in $L$ consists of at most $m+1$ elements, so $L$ is noethenan and artinian. Suppose conversely that $L$ is noethenan. For every $a \neq 0$ in $L$ there exists by Prop 3.4 a maximal element $b$ s.t. $b < a$. By repeated use of this observation we get a descending chain $1 > a_1 > a_2 > \cdots$ If now $L$ is also artinian, this chain stops after a finite number of steps, and we have thus obtained a composition chain between $1$ and $0$. $L$ is thus of finite length. □

PROPOSITION 3.6 Let $a$ be an element of a modular lattice $L$. Then $L$ is noethenan (artinian) iff. both intervals $[0,a]$ and $[a,1]$ are noethenan (artinian).

PROOF If $L$ is noethenan or artinian, then clearly every interval of $L$ is likewise. Suppose conversely that the intervals $[0,a]$ and $[a,1]$ are noethenan, and let $b_1 < b_2 < \cdots$ be a strictly ascending chain in $L$. Then there exists an integer $n$ s.t.

$$b_n \wedge a = b_{n+1} \wedge a = c$$
$$b_n \vee a = b_{n+1} \vee a = d$$

Applying Prop 2.3. to the element $a$ in $[c,d]$, we obtain $b_n = b_{n+1}$. Hence $L$ must be noethenan. Similarly for the artinian case. □

EXAMPLES

1. Modules of Finite Length Let $M$ be a module. A composition chain between $\{0\}$ and $M$ in $L(M)$ is a chain of submodules

$$\{0\} = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

such that each quotient module $M_i/M_{i-1}$ is simple. In view of the meaning of projectivity of intervals in $L(M)$ (Ex. 2.1) the Jordan-Hölder theorem says:

PROPOSITION 3.7 Let

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$
$$0 = M'_0 \subset M'_1 \subset \cdots \subset M'_m = M$$

be two composition chains. Then $n = m$ and there exists a permutation of $\{1, \ldots, n\}$ s.t.

$$M_i/M_{i-1} \cong M_{\pi(i)}/M_{\pi(i)-1}$$

A module $M$ is called noethenan (artinian, of finite length) when the lattice $L(M)$ is noethenan (etc). From Prop 3.6 we obtain

PROPOSITION 3.8 Let $0 \longrightarrow M' \longrightarrow M \longrightarrow M \longrightarrow 0$ be an exact sequence of modules. Then $M$ is noethenan (artinian, resp. of finite length) if and only if both $M'$ and $M''$ are so.

2. Finite dimensional vector spaces A vector space $V$ is of finite length iff. it is finite dimensional, and its length is the dimension of $V$.

3. Irreducible decomposition An element $a$ of the lattice $L$ is (meet-) irreducible if $a = b \wedge c \Rightarrow a = b$ or $a = c$.
An irreducible decomposition of

# Chapter IV : Abelian Categories

PROPOSITION 5.1 Consider a pullback diagram

$$\begin{tikzcd} P \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & C_2 \arrow[d, "\alpha_2"] \\ C_1 \arrow[r, "\alpha_1"'] & C \end{tikzcd}$$

in an abelian category. Then:

(i) If $\alpha_1$ is a monomorphism, then so is $\pi_2$
(ii) If $\alpha_1$ is an epimorphism, then so is $\pi_2$
(iii) If $\alpha_1$ is the kernel of a morphism $\beta : C \longrightarrow D$, then $\pi_2$ is the kernel of $\beta\alpha_2$

PROOF (i) If $\xi : X \longrightarrow P$ is a morphism s.t. $\pi_2 \xi = 0$, then $\alpha_1 \pi_1 \xi = \alpha_2 \pi_2 \xi = 0$, and $\alpha_1$ monic gives $\pi_1 \xi = 0$. The unicity of the factorisation over $P$ gives $\xi = 0$.

(ii) Form the product $C_1 \times C_2$ and let $\kappa : P \longrightarrow C_1 \times C_2$ be $\operatorname{Ker}(\alpha_1 p_1 - \alpha_2 p_2)$

$$\begin{tikzcd} P \arrow[rr, "\pi_2"] \arrow[dr, "\kappa"] \arrow[dd, "\pi_1"'] & & C_2 \arrow[dd, "\alpha_2"] \arrow[dl, shift left, "i_2"] \\ & C_1 \times C_2 \arrow[ur, shift left, "p_2"] \arrow[dl, shift left, "p_1"] & \\ C_1 \arrow[rr, "\alpha_1"'] \arrow[ur, shift left, "i_1"] & & C \end{tikzcd}$$

then there is an exact sequence

$$0 \longrightarrow P \xrightarrow{\kappa} C_1 \times C_2 \xrightarrow{\mu} C \longrightarrow 0 \qquad (3)$$

where $\mu = \alpha_1 p_1 - \alpha_2 p_2$ is an epimorphism because $\mu i_1 = \alpha_1$ is an epimorphism. Suppose $\xi : C_2 \longrightarrow X$ is a morphism s.t. $\xi \pi_2 = 0$. Then $0 = \xi \pi_2 = \xi p_2 \kappa$ implies that there is $\eta : C \longrightarrow X$ with $\eta\mu = \xi p_2$. This gives $\eta\alpha_1 = \eta\mu i_1 = \xi p_2 i_1 = 0$, and $\alpha_1$ an epimorphism implies $\eta = 0$. But then $\xi p_2 = 0$, and so $\xi = 0$.

(iii) We certainly have $\beta\alpha_2\pi_2 = 0$, and if $\xi : X \longrightarrow C_2$ is a morphism s.t. $\beta\alpha_2\xi = 0$, then $\xi$ factors over $\alpha_1 = \ker\beta$ by a morphism $\eta : X \longrightarrow C_1$ with $\alpha_1\eta = \alpha_2\xi$. Hence $\xi$ factors over the pullback $P$, and does so uniquely since $\pi_2$ is a monomorphism.

When $\alpha_1$ is a monomorphism, one may think of the pullback $P$ as the "inverse image" of $C_1$ under $\alpha_2$, and one often writes $\alpha_2^{-1}(C)$ for $P$. In this language, (iii) of Prop 5.1 states that $\alpha_2^{-1}(\operatorname{Ker}\beta) = \operatorname{Ker}\beta\alpha_2$. If both $\alpha_1$ and $\alpha_2$ are monomorphisms, then it is quite clear that $P$ is the greatest lower bound of $C_1$ and $C_2$ in $C$, i.e. $P = C_1 \cap C_2$. The exact sequence in this case becomes

$$0 \longrightarrow C_1 \cap C_2 \longrightarrow C_1 \oplus C_2 \xrightarrow{\mu} C_1 + C_2 \longrightarrow 0 \qquad (4)$$

where $\mu$ is induced by the inclusions of $C_1$ and $C_2$ in $C$ (up to a minus sign). It follows that the sum $C_1 + C_2$ is direct iff. $C_1 \cap C_2 = 0$. After these remarks we can prove the Noether isomorphism theorem:

PROPOSITION 5.2 If $C_1$ and $C_2$ are subobjects of $C$, then

$$(C_1 + C_2)/C_1 \cong C_2/(C_1 \cap C_2)$$

PROOF With the use of Prop 5.1 (iii) we get a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & C_1 \cap C_2 \arrow[r] \arrow[d] & C_2 \arrow[r] \arrow[d] & (C_1 + C_2)/C_1 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & C_1 \arrow[r] & C_1 + C_2 \arrow[r] & (C_1 + C_2)/C_1 \arrow[r] & 0
\end{tikzcd}$$

since the left square is a pullback. The exactness of the upper row immediately leads to the desired conclusion. $\square$

Another consequence of Prop 5.1 that one can construct the pullback of a short exact sequence $0 \longrightarrow C_0 \longrightarrow C_1 \longrightarrow C \longrightarrow 0$ with respect to a morphism $C_2 \longrightarrow C$. For this we consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & C_0 \arrow[r, "\beta_0"] \arrow[d, equal] & P \arrow[r, "\beta"] \arrow[d, "\gamma"] & C_2 \arrow[r] \arrow[d, "\alpha_2"] & 0 \\
0 \arrow[r] & C_0 \arrow[r, "\alpha_0"'] & C_1 \arrow[r, "\alpha_1"'] & C \arrow[r] & 0
\end{tikzcd}$$

where the right square is a pullback diagram and $\beta_0$ is induced by $\alpha_0$ and the zero morphism $C_0 \longrightarrow C_2$. It remains to see that $\beta_0$ is the kernel of $\beta$. Suppose $\xi : X \longrightarrow P$ is s.t. $\beta\xi = 0$. Then $\alpha_1 \gamma \xi = 0$, so $\gamma\xi = \alpha_0 \lambda$ for some $\lambda : X \longrightarrow C_0$. Now $\beta_0\lambda$ and $\xi$ give the same results when composing them with $\gamma$ or $\beta$, so $\xi = \beta_0 \lambda$. Hence the upper row is exact.

When $\alpha_2 : C_2 \longrightarrow C$ is a monomorphism, this construction shows that each subobject $C_2$ of $C_1/C_0$ is of the form $P/C_0$ for some subobject $P$ of $C_1$. With the preceding results at hand, we return to the lattice $L(C)$ of subobjects of an object $C$, and prove:

PROPOSITION 5.3 The lattice $L(C)$ is modular.

PROOF We make use of Prop III 2.3 to show the modularity. Since an interval $[B_1, B_2]$ of $L(C)$ is isomorphic to the lattice $L(B_2/B_1)$, it suffices to consider subobjects $B_1 \subset B_2$ of $C$ with a common complement $C'$ in $C$. If $i_1 : B_1 \longrightarrow C$, $i_2 : B_2 \longrightarrow C$ and $i_3 : C' \longrightarrow C$ are the inclusions, this means that there are $p_1 : C \longrightarrow B_1$, $p_2 : C \longrightarrow B_2$ and $p_3 : C \longrightarrow C'$ s.t. $C$ is the coproduct of $i_1$ and $i_3$ with projections $p_1, p_3$ and resp. for $i_2, i_3$ and $p_2, p_3' : C \longrightarrow C'$

$$\begin{tikzcd}
B_1 \arrow[rr, shift left, "i_1"'] \arrow[dr, "\alpha"'] & & C \arrow[ll, shift left, "p_1"'] \arrow[dl, shift left, "p_2"] \arrow[d, shift left, "p_3"] \arrow[d, bend left, "p_3'"] \\
& B_2 \arrow[ur, shift left, "i_2"] & C' \arrow[u, shift left, "i_3"]
\end{tikzcd}$$

It follows that $i_1 p_1 + i_3 p_3 = 1_C = i_2 p_2 + i_3 p_3''$. On composing both sides with $p_2$ and replacing $i_1$ with $i_2 \alpha$, we have

$$p_2 i_2 \alpha p_1 + p_2 i_3 p_3 = p_2 i_2 p_2 + p_2 i_3 p_3''$$
$$\alpha p_1 = p_2$$

since $p_1$ and $p_2$ are both the cokernels of $i_3$, it follows that $\alpha$ is an isomorphism, and hence $B_1 = B_2$. $\square$

EXAMPLES 1. Module Categories In the category Mod $A$ for a ring $A$, the pushout is $C_1 \oplus C_2 / C'$ where $C' = \{(\beta_1(x), -\beta_2(x))\}$

2. Objects of Finite Length Let $C$ be an object of an abelian category. Since the lattice $C$ is modular, the Jordan-Hölder theorem holds, and it does so in the sharp form valid for modules (Prop III 3.7), because Prop 5.2 implies that projectivity of two intervals induces an isomorphism between corresponding quotient objects. The object $C$ is of finite length if the lattice $L(C)$ is of finite length, and this means there is a chain of subobjects

$$0 = C_0 \subset C_1 \subset \cdots \subset C_n = C$$

such that each $C_i/C_{i-1}$ is a simple object.

# 6. GENERATORS AND COGENERATORS

Consider two abelian categories $\mathcal{C}$ and $\mathcal{D}$ and an additive functor $T: \mathcal{C} \longrightarrow \mathcal{D}$. Recall that $T$ is said to be faithful if $T(\alpha) \neq 0$ for every non-zero morphism $\alpha$ in $\mathcal{C}$.

PROPOSITION 6.1 An exact functor $T$ is faithful if and only if $T(C) \neq 0$ for every non-zero object $C$.

PROOF If $T$ is faithful and $T(C) = 0$, then $T(1_C) = 0$ implies $1_C = 0$ and hence $C = 0$. Conversely, suppose $T(C) = 0$ implies $C = 0$. If $\alpha \neq 0$ in $\mathcal{C}$, then $\operatorname{Im}\alpha \neq 0$ and so $T(\operatorname{Im}\alpha) \neq 0$. But the exactness of $T$ implies $T(\operatorname{Im}\alpha) = \operatorname{Im}T(\alpha)$ so $T(\alpha) \neq 0$. $\square$

DEFINITION An object $C$ of $\mathcal{C}$ is a generator for $\mathcal{C}$ if $\operatorname{Hom}(C, -)$ is faithful, and is a cogenerator if $\operatorname{Hom}(-, C)$ is faithful.

PROPOSITION 6.2 Suppose $\mathcal{C}$ has coproducts. If $U$ is a generator, then for every object there is an epimorphism $U^{(I)} \longrightarrow C$ for some index set $I$.

PROPOSITION 6.3 A projective object $P$ is a generator if and only if there exists a nonzero morphism $P \longrightarrow C$ for any $C \neq 0$.

PROPOSITION 6.4 Suppose $\mathcal{C}$ has products. If $V$ is a cogenerator, then for every object $C$ there is a monomorphism $C \longrightarrow V^I$ for some index set $I$.

PROPOSITION 6.5 An injective object $E$ is a cogenerator if and only if there exists a non-zero morphism $C \longrightarrow E$ for each $C \neq 0$.

PROPOSITION 6.6 If $\mathcal{C}$ has a generator, then $\mathcal{C}$ is locally small.

EXAMPLES 1. Generators for Mod-$A$ A module $M$ is a generator for Mod-$A$ if and only if $A$ is a direct summand of of some direct sum of copies of $M$, as follows from Prop 6.2. This in a remarkable way is dual to the characterization of a projective module as a direct summand of a direct sum of copies of $A$. This duality has the following formal consequence:

PROPOSITION 6.7 Let $M$ be a right $A$-module with endomorphism ring $B = \operatorname{End}_A(M)$. Then:

(i) If $M_A$ is a generator, then ${}_BM$ is finitely generated projective.

(ii) If $M_A$ is finitely generated projective, then ${}_BM$ is a generator.

PROOF (i) If $M_A$ is a generator, then $M^m \cong A \oplus K$ for some integer $m$ and module $K$. Applying the functor $\operatorname{Hom}_A(-, M)$ to this, we get $B^m \cong M \oplus \operatorname{Hom}_A(K, M)$ and hence $M$ is finitely generated

## 10. MORITA EQUIVALENCE

Let $A$ and $B$ be two rings, In this section we will examine the implications of an equivalence between the module categories $\underline{Mod}-A$ and $\underline{Mod}-B$. Since $A$ is a small projective generator for $\underline{Mod}-A$, it follows immediately from Mitchell IV Lemma 5.1 that any ring homomorphism $A \longrightarrow Hom_B(M,M)$ for a $B$-module $M$ has a unique extension to an additive colimit preserving functor $\underline{Mod}-A \longrightarrow \underline{Mod}-B$. But this morphism of rings gives $M$ the structure of an $A-B$-bimodule, and we already know that $-\otimes_A M : \underline{Mod}-A \longrightarrow \underline{Mod}-B$ is additive, colimit preserving functor extending this morphism of rings.

But notice that any colimit preserving functor $S: \underline{Mod}-A \longrightarrow \underline{Mod}-B$ (which is automatically additive) induces a morphism of rings $A \longrightarrow Hom_B(S(A), S(A))$. Hence if we write ${}_AP_B = S(A)$, we have proven:

PROPOSITION 10.1 The following assertions are equivalent for a functor $S: \underline{Mod}-A \longrightarrow \underline{Mod}-B$

(a) $S$ has a right adjoint

(b) $S$ preserves colimits

(c) $S \cong -\otimes_A P$ for some $A-B$-bimodule $P$

Notice that $P$ is unique up to isomorphism since if $-\otimes_A P \cong -\otimes_A Q$, this equivalence restricts to an isomorphism of $P = A\otimes_A P$ with $Q = A\otimes_A Q$. By our earlier discussion, the right adjoint to $S$ must be $Hom_B(P,-): \underline{Mod}-B \longrightarrow \underline{Mod}-A$. We also have:

PROPOSITION The following assertions are equivalent for a functor $T: \underline{Mod}-B \longrightarrow \underline{Mod}-A$

(a) $T$ has a left adjoint

(b) $T$ preserves limits

(c) $T \cong Hom_B(Q,-)$ for some $A-B$-bimodule $Q$.

PROOF Follows immediately from Mitchell VI §2. □

Notice that $Q$ is again unique up to isomorphism, and the left adjoint must be $-\otimes_A Q$,

COROLLARY Any adjoint pair of functors $S \dashv T$, $S: \underline{Mod}-A \rightleftarrows \underline{Mod}-B$ has the form

$$S = -\otimes_A P$$
$$T = Hom_B(P,-)$$

for some uniquely determined $A-B$-bimodule $P = S(A)$. In particular any ring morphism $\varphi: A \longrightarrow B$ extends uniquely to $\varphi^*: \underline{Mod}-A \longrightarrow \underline{Mod}-B$ (Mitchell) which preserves colimits and is hence $-\otimes_A B$.

COROLLARY 10.2 The following assertions are equivalent for a functor $S: \underline{Mod}-B \longrightarrow \underline{Mod}-A$

(a) $S$ is an equivalence

(b) There exist bimodules ${}_AP_B$ and ${}_BQ_A$ with bimodule isomorphisms $\alpha: P\otimes_B Q \longrightarrow A$ and $\beta: Q\otimes_A P \longrightarrow B$, and $S = -\otimes_B Q$.

PROOF If $S$ is an equivalence, there is a functor $T: \underline{Mod}-A \longrightarrow \underline{Mod}-B$ s.t. $TS \simeq 1$, $ST \simeq 1$ and $S \dashv T$, $T \dashv S$. Hence we have bimodules $P, Q$ s.t. $S = -\otimes_B Q$, $T = -\otimes_A P$. Then $TS \cong 1$, $ST \cong 1$ implies

$$A \cong ST(A) = T(A)\otimes_B Q = P\otimes_B Q.$$
$$B \cong TS(B) \cong S(B)\otimes_A P = Q\otimes_A P$$

Now these isomorphisms are morphisms of right $A$-modules and right $B$-modules resp. But if we fix $S \dashv T$ and write $\eta: 1 \longrightarrow TS$, $\varepsilon: ST \longrightarrow 1$ for the unit and counit, naturality implies that for $a \in A$ (identified with $x \mapsto ax : A \longrightarrow A$)

$$\begin{aligned} a\cdot\varepsilon_A(p\otimes q) &= \varepsilon_A(ST(a)(p\otimes q)) \\ &= \varepsilon_A(T(a)(p)\otimes q) = \varepsilon_A(a\cdot p\otimes q) = \varepsilon_A(a\cdot(p\otimes q)) \end{aligned}$$

EXERCISES Ch IV Stenström

Q11 Let $\mathcal{C}$ be an abelian category and $F:\mathcal{C}\to\mathcal{C}$ an additive functor. Define a new category $\mathcal{C}\times F$ as follows. An object of $\mathcal{C}\times F$ is a couple $(C,\alpha)$ where $C\in\mathcal{C}$ and $\alpha: F(C)\to C$ is a morphism such that $\alpha F(\alpha)=0$. A morphism $\varphi:(C,\alpha)\to(C',\alpha')$ of $\mathcal{C}\times F$ is a morphism $\varphi: C\to C''$ in $\mathcal{C}$ such that the diagram

$$\begin{tikzcd} F(C) \arrow[r, "F(\varphi)"] \arrow[d, "\alpha"'] & F(C') \arrow[d, "\alpha'"] \\ C \arrow[r, "\varphi"'] & C' \end{tikzcd}$$

commutes. Then

(i) $\mathcal{C}\times F$ is additive (Stenström writes "preadditive") since given $\varphi,\phi:(C,\alpha)\to(C',\alpha')$ in $\mathcal{C}\times F$, the sum $\phi+\varphi$ in $\mathcal{C}$ qualifies for membership, since $(\phi+\varphi)\alpha = \phi\alpha+\varphi\alpha = \alpha'F(\phi)+\alpha'F(\varphi) = \alpha'F(\phi+\varphi)$ since $F$ is additive. That this makes $\mathcal{C}\times F$ additive follows from the fact that $\mathcal{C}$ is additive.

To show that $\mathcal{C}\times F$ has finite products it suffices to show it has binary products: if $(C,\alpha),(C',\alpha')\in\mathcal{C}\times F$ then let $C\times C'$ be the product in $\mathcal{C}$. Since $F$ is additive, it preserves finite products, so we get a diagram

$$\begin{tikzcd} C & & C' \\ & C\times C' \arrow[ul] \arrow[ur] & \\ F(C) \arrow[uu] & & F(C') \arrow[uu] \\ & F(C\times C') \arrow[ul] \arrow[ur] \arrow[uu, dotted, "\beta"] & \end{tikzcd}$$

and induce $\beta: F(C\times C')\to C\times C'$ so that both diagrams commute. Applying $F$ one more time to get

$$\begin{tikzcd} F(F(C)) \arrow[r, "F(\alpha)"] & F(C) \arrow[r, "\alpha"] & C \\ F(F(C\times C')) \arrow[u] \arrow[d] \arrow[r, "F(\beta)"] & F(C)\times F(C') \arrow[u] \arrow[d] \arrow[r, "\beta"] & C\times C' \arrow[u] \arrow[d] \\ F(F(C')) \arrow[r, "F(\alpha')"'] & F(C') \arrow[r, "\alpha'"'] & C' \end{tikzcd}$$

and using the fact that $\alpha F(\alpha)=0$ and $\alpha'F(\alpha')=0$, we find that $\beta F(\beta)=0$, so that $(C\times C',\beta)\in\mathcal{C}\times F$. By construction the morphisms $C\times C'\to C$, $C\times C'\to C'$ are morphisms in $\mathcal{C}\times F$, and if $(D,\delta)\in\mathcal{C}\times F$ with morphisms $\varphi_1:(D,\delta)\to(C,\alpha)$, $\varphi_2:(D,\delta)\to(C',\alpha')$ then we get a unique morphism $\varepsilon$ in $\mathcal{C}$, $\varepsilon: D\to C\times C'$, s.t. $D\to C\times C'\to C=\varphi_1$ and $D\to C\times C'\to C'=\varphi_2$. Using the fact that since $\varphi_1,\varphi_2$ are morphisms in $\mathcal{C}\times F$, $\varphi_1\delta=\alpha F(\varphi_1)$ and $\varphi_2\delta=\alpha'F(\varphi_2)$, consider

$$\begin{tikzcd} F(D) \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathrm{I}"] & F(C\times C') \arrow[d] \arrow[r] \arrow[dr] & F(C) \arrow[d] \\ D \arrow[r, "\varepsilon"'] & C\times C' \arrow[r] \arrow[dr] & C \\ & & F(C') \arrow[d] \\ & & C' \end{tikzcd}$$

the square I commutes upon composition with the two projections, and hence commutes. Uniqueness of $\varepsilon$ in $\mathcal{C}\times F$ follows from uniqueness in $\mathcal{C}$. Hence $\mathcal{C}\times F$ has finite products.

To see that $\mathcal{C}\times F$ has kernels, first note that $0$, with the unique map $F(0)\to 0$, is certainly a $0$ for $\mathcal{C}\times F$. ?
If $\varphi:(C,\alpha)\to(C',\alpha')$ is a morphism of $\mathcal{C}\times F$ and $\phi:K\to C$ its kernel in $\mathcal{C}$, then we we

$$\begin{tikzcd} F(K) \arrow[r] \arrow[d, dotted, "k"'] & F(C) \arrow[r] \arrow[d] & F(C') \arrow[d] \\ K \arrow[r, "\phi"'] & C \arrow[r, "\varphi"'] & C' \end{tikzcd}$$

to induce $k: F(K) \longrightarrow K$, then $\phi k F(k) = \alpha F(\alpha) F(F(\phi)) = 0$, so since $\phi$ is monic $kF(k) = 0$ and $(K, k) \in \mathcal{C} \times F$. Clearly $\phi$ is then a morphism in $\mathcal{C} \times F$. If $\gamma : (T, t) \longrightarrow (C, \alpha)$ is s.t. $\varphi \gamma = 0$, then there is unique $T \longrightarrow K$ s.t. $T \to K \to C = T \to C$, and again using the fact that $\phi$ is monic it is easily shown that $T \longrightarrow K$ is a morphism of $\mathcal{C} \times F$. Hence $\mathcal{C}$ has kernels.

(ii) Suppose $F$ preserves cokernels. We show that $\mathcal{C} \times F$ is abelian by showing it has cokernels, is normal and conormal, and has epi-mono factorisations.

<u>Cokernels</u> Let $\varphi : (C, \alpha) \longrightarrow (C', \alpha')$ and $\phi : C' \longrightarrow D$ it's cokernel in $\mathcal{C}$. Since $F$ is right exact, we have a diagram with exact rows

$$\begin{tikzcd}
F(C) \arrow[r] \arrow[d] & F(C') \arrow[r] \arrow[d] & F(D) \arrow[r] \arrow[d, dashed, "\delta"'] & 0 \\
C \arrow[r, "\varphi"'] & C' \arrow[r, "\phi"'] & D \arrow[r] & 0
\end{tikzcd}$$

and we then induce $\delta$ in the obvious way. Extending this diagram to

$$\begin{tikzcd}
F(F(C)) \arrow[r] \arrow[d, "F(\alpha)"'] & F(F(C')) \arrow[r] \arrow[d, "F(\alpha')"'] & F(F(D)) \arrow[r] \arrow[d, "F(\delta)"'] & 0 \\
F(C) \arrow[r] \arrow[d, "\alpha"'] & F(C') \arrow[r] \arrow[d, "\alpha'"'] & F(D) \arrow[r] \arrow[d, "\delta"'] & 0 \\
C \arrow[r] & C' \arrow[r] & D \arrow[r] & 0
\end{tikzcd}$$

and using the fact that $F(F(C')) \longrightarrow F(F(D))$ is epi, we see that $\delta F(\delta) = 0$. Hence $(D, \delta) \in \mathcal{C} \times F$, and $\phi$ is a morphism of $\mathcal{C} \times F$. As with kernels, $\phi$ is then easily seen to be the cokernel in $\mathcal{C} \times F$.

<u>Normal and Conormal</u> If $\varphi : C \longrightarrow C'$ is monic in $\mathcal{C} \times F$, let $\phi : K \longrightarrow C$ be its kernel in $\mathcal{C}$ and hence in $\mathcal{C} \times F$. Since $\mathcal{C} \times F$ is additive, it follows that $\phi = 0$ in $\mathcal{C} \times F$ and hence in $\mathcal{C}$ — so $\varphi$ is also monic in $\mathcal{C}$. It is thus the kernel of its cokernel $C' \longrightarrow Q$, which is also the cokernel of $\varphi$ in $\mathcal{C} \times F$. Again, since $\varphi$ is the kernel of $C' \longrightarrow Q$ in $\mathcal{C}$, it is also the kernel in $\mathcal{C} \times F$. Hence $\mathcal{C}$ is normal (conormality follows similarly).

<u>Epi-Mono Factorisations</u> Let $\varphi : C \longrightarrow C'$ have the factorisation $C \twoheadrightarrow I \rightarrowtail C'$. Take kernels and cokernels of $\varphi$ in $\mathcal{C}$ and lift them to $\mathcal{C} \times F$ as above,

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] & C \arrow[rr] \arrow[dr, two heads] & & C' \arrow[r] & Q \arrow[r] & 0 \\
& & & I \arrow[ur] & & &
\end{tikzcd}$$

since $C \twoheadrightarrow I$ is the cokernel of $K \longrightarrow C$, it also lifts to $\mathcal{C} \times F$ (giving $i : F(I) \longrightarrow i$ s.t. $(I, i) \in \mathcal{C} \times F$). Similarly $I \longrightarrow C'$ becomes the kernel of $C' \longrightarrow Q$ in $\mathcal{C} \times F$, and it suffices to show that the two induced maps $F(I) \longrightarrow I$ are the same. It thus suffices to show that in the following diagram

$$\begin{tikzcd}
K \arrow[r] & C \arrow[r] & I \arrow[r] \arrow[dr] & 0 & \\
& & & & C' \\
F(K) \arrow[r] \arrow[uu] & F(C) \arrow[r] \arrow[uu] & F(I) \arrow[rr] \arrow[uu] \arrow[dr] \arrow[uur, phantom, "\mathrm{I}"] & & 0 \\
& & & F(C') \arrow[uur] &
\end{tikzcd}$$

the square (I) commutes. But using the fact that $C \longrightarrow I$ is epi and $C \longrightarrow C'$ is a morphism in $\mathcal{C} \times F$, this is easy. Hence $\mathcal{C} \times F$ has epi-mono factorisations.

Thus, $\mathcal{C} \times F$ is abelian, as required.

That the forgetful functor $\mathcal{C}\times F \longrightarrow \mathcal{C}$ is exact follows immediately from our construction of the kernels and cokernels in $\mathcal{C}\times F$.

(iii) Let $\mathcal{C} = \underline{\text{Mod}}\text{-}A$ and $F = - \otimes_A M$ for a bimodule $M$. If $A\times M$ is the trivial extension of $A$ by $M$, we claim that there is an isomorphism

$$\Phi : \mathcal{C}\times F \longrightarrow \underline{\text{Mod}}-(A\times M)$$

we have already defined the action of $\Phi$ on objects, and shown it sets up a bijection, in Ex 34 of Ch. I, so it only remains to check the behaviour of morphisms. Let $(C,\alpha)$ and $(C,\alpha') \in \mathcal{C}\times F$ and $\phi : (C,\alpha) \longrightarrow (C',\alpha')$. We make $C, C'$ into $A\times M$ modules via

$$\begin{aligned} x\cdot(a,m) &= xa + \alpha(x\otimes m) \qquad && C \\ x'\cdot(a,m) &= x'a + \alpha'(x'\otimes m) \qquad && C' \end{aligned}$$

and $\phi$ is then a morphism of $A\times M$-modules since

$$\begin{aligned} \phi(x\cdot(a,m)) &= \phi(xa+\alpha(x\otimes m)) \\ &= \phi(x)\cdot a + \phi(\alpha(x\otimes m)) \\ &= \phi(x)\cdot a + \alpha' F(\phi)(x\otimes m) \\ &= \phi(x)\cdot a + \alpha'(\phi(x)\otimes m) = \phi(x)\cdot(a,m) \end{aligned}$$

Similarly, if $\mathcal{S}: X \longrightarrow X'$ is a morphism of $A\times M$ modules, it becomes a morphism of the corresponding $A$-modules (I just realised we have also checked the morphisms in Q34, Ch. I). Hence we have the required isomorphism.

$\boxed{\text{Q28}}$ If $\mathcal{B}$ is a small additive category, $\mathcal{C}$ and $\mathcal{D}$ are additive and $S : \mathcal{C} \longrightarrow \mathcal{D}$ is left adjoint to $T: \mathcal{D} \longrightarrow \mathcal{C}$, we claim that the induced functors

$$\begin{tikzcd} \text{Hom}(\mathcal{B},\mathcal{C}) \arrow[r, bend left, "S^*"] & \text{Hom}(\mathcal{B},\mathcal{D}) \arrow[l, bend left, "T^*"] \end{tikzcd}$$

$$\begin{tikzcd} & \mathcal{B} \arrow[dl] \arrow[dr] & \\ \mathcal{C} \arrow[rr, bend left, "S"] & & \mathcal{D} \arrow[ll, bend left, "T"] \end{tikzcd}$$

are s.t. $S^* \dashv T^*$.

For each $B\in\mathcal{B}$, let $[S(F(B)), G(B)] \xrightarrow[\sim]{\Phi_{F(B),G(B)}} [F(B), T(G(B))]$ be the natural isomorphism arising from the fact that $S \dashv T$. For $F\in \text{Hom}(\mathcal{B},\mathcal{C})$ and $G\in\text{Hom}(\mathcal{B},\mathcal{D})$ we define

$$\begin{tikzcd} \Theta : [S^*(F), G] \arrow[r, "\sim"] \arrow[d, equal] & {[F, T^*(G)]} \arrow[d, equal] \\ {[SF, G]} & {[F, TG]} \end{tikzcd}$$

by letting $\Theta(\phi)_B = \Phi_{F(B),G(B)}(\phi_B)$. To see that $\Theta(\phi): F \longrightarrow TG$ is natural, let $\alpha : B \longrightarrow B'$ in $\mathcal{B}$, then

$$\begin{tikzcd} F(B) \arrow[r, "\Theta(\phi)_B"] \arrow[d, "F(\alpha)"'] & TG(B) \arrow[d, "TG(\alpha)"] \\ F(B') \arrow[r, "\Theta(\phi)_{B'}"'] & TG(B') \end{tikzcd}$$

Then

$$\begin{aligned} \Theta(\phi)_{B'} F(\alpha) &= \Phi_{F(B'),G(B')}(\phi_{B'})F(\alpha) = \Phi_{F(B),G(B')}(\phi_{B'}\, SF(\alpha)) \\ &= \Phi_{F(B),G(B')}(G(\alpha)\phi_B) = TG(\alpha)\,\Phi_B(\phi_B) = TG(\alpha)\Theta(\phi)_B \end{aligned}$$

using naturality of $\phi$ and $\Phi$. Hence $\Theta$ is well-defined: it remains to show that is bijective and natural.

It is injective since if $\Theta(\phi)=\Theta(\phi')$ then for $B\in\mathcal{B}$, $\Phi_{F(B),G(B)}(\phi_B) = \Theta(\phi)_B = \Theta(\phi')_B = \Phi_{F(B),G(B)}(\phi'_B)$, and hence since $\Phi$ is bijective, $\phi=\phi'$. We define for each $\psi: F\longrightarrow TG$ a transformation $\Theta'(\psi): SF\longrightarrow G$ using the same technique as before — this shows $\Theta$ is bijective.

To see that it is natural, suppose that $\phi: F\longrightarrow F'$ in $Hom(\mathcal{B},\mathcal{C})$. Then we have to show that the diagram

$$\begin{tikzcd}
{[S^*(F'), G]} \arrow[r, "\Theta_{F',G}"] \arrow[d, "{[S^*\phi, 1]}"'] & {[F', T^*(G)]} \arrow[d, "{[\phi, 1]}"] \\
{[S^*(F), G]} \arrow[r, "\Theta_{F,G}"'] & {[F, T^*G]}
\end{tikzcd}$$

commutes. This follows since for $q: SF'\longrightarrow G$

$$\begin{aligned}
\Theta_{F,G}[S^*\phi,1](q)_B &= \Theta_{F,G}(qS^*\phi)_B \\
&= \Phi_{F(B),G(B)}(\{qS^*\phi\}_B) \\
&= \Phi_{F(B),G(B)}(qS(\phi_B)) \\
&= \phi_B\,\Phi_{F'(B),G(B)}(q) \\
&= \{\phi\,\Theta_{F',G}(q)\}_B
\end{aligned}$$

as required. Similarly, to show naturality in $G$, let $\beta: G\longrightarrow G'$ in $Hom(\mathcal{B},\mathcal{D})$. We have to show that the diagram

$$\begin{tikzcd}
{[S^*F, G]} \arrow[r, "\Theta_{F,G}"] \arrow[d, "{[1,\beta]}"'] & {[F, T^*G]} \arrow[d, "{[1, T^*\beta]}"] \\
{[S^*F, G']} \arrow[r, "\Theta_{F,G'}"'] & {[F, T^*G']}
\end{tikzcd}$$

commutes. Again, for $g: SF\longrightarrow G$,

$$\begin{aligned}
\Theta_{F,G'}[1,\beta](g)_B &= \Theta_{F,G'}(\beta g)_B \\
&= \Phi_{F(B),G'(B)}(\{\beta g\}_B) \\
&= \Phi_{F(B),G'(B)}(\beta_B g_B) \\
&= T(\beta_B)\,\Phi_{F(B),G(B)}(g_B) \\
&= \{T^*\beta\,\Theta_{F,G}(g)\}_B
\end{aligned}$$

As required. □

# EXERCISES Ch IV Stenstrom

$\boxed{Q22}$ Let $\mathscr{C}$ be an abelian category. The <u>endomorphism category</u> $\underline{End}(\mathscr{C})$ of $\mathscr{C}$ is defined as follows. An object of $\underline{End}(\mathscr{C})$ is a couple $(C,\alpha)$, where $C$ is an object of $\mathscr{C}$ and $\alpha: C \longrightarrow C$. A morphism $\varphi:(C,\alpha)\longrightarrow(C',\alpha')$ in $\underline{End}(\mathscr{C})$ is a morphism $\varphi: C \longrightarrow C'$ such that the diagram

$$\begin{tikzcd} C \arrow[r, "\varphi"] \arrow[d, "\alpha"'] & C' \arrow[d, "\alpha'"] \\ C \arrow[r, "\varphi"'] & C' \end{tikzcd}$$

commutes. Similarly one defines the <u>automorphism category</u> $\underline{Aut}(\mathscr{C})$ as the full subcategory of $\underline{End}(\mathscr{C})$ consisting of $(C,\alpha)$ with $\alpha$ an automorphism. Then

(i) $\underline{End}(\mathscr{C}) \cong Fun(\mathbb{N},\mathscr{C})$ and $\underline{Aut}(\mathscr{C}) \cong Fun(\mathbb{Z},\mathscr{C})$.

<u>PROOF</u> We define $\Theta: [\mathbb{N},\mathscr{C}] \longrightarrow \underline{End}(\mathscr{C})$ by $\Theta(F) = (F(\mathbb{N}), F(1))$ and for $\phi: F \longrightarrow F'$, $\Theta(\phi):(F(\mathbb{N}),F(1)) \longrightarrow (F'(\mathbb{N}),F'(1))$, is $\Theta(\phi) = \phi_{\mathbb{N}} : F(\mathbb{N}) \longrightarrow F'(\mathbb{N})$, and

$$\begin{tikzcd} F(\mathbb{N}) \arrow[r, "\Theta(\phi)"] \arrow[d, "F(1)"'] & F'(\mathbb{N}) \arrow[d, "F(1)"] \\ F(\mathbb{N}) \arrow[r, "\Theta(\phi)"'] & F'(\mathbb{N}) \end{tikzcd}$$

obviously commutes. To see $\Theta$ is full, let $\alpha: F(\mathbb{N}) \longrightarrow F'(\mathbb{N})$ be any other morphism of $\mathscr{C}$ qualifying as a morphism $(F(\mathbb{N}),F(1)) \longrightarrow (F'(\mathbb{N}),F'(1))$. That is, $\alpha F(1) = F'(1)\alpha$. Then for any $n \geqslant 0$ in $\mathbb{N}$,

$$\begin{aligned}
\alpha F(n) &= \alpha F(\overbrace{1\cdot 1\cdot 1\cdot 1\cdots}^{n})\\
&= \alpha F(1)F(1)\cdots F(1)\\
&= F'(1)\alpha F(1)\cdots F(1)\\
&= \vdots\\
&\phantom{=}\ \vdots\\
&= F'(1)\cdots F'(1)\alpha\\
&= F'(1\cdots 1)\alpha\\
&= F'(n)\alpha
\end{aligned}$$

hence $\alpha$ defines a natural transformation $F \longrightarrow F'$ (since $F(0)=1$ and $F'(0)=1$, clearly $\alpha F(0) = F'(0)\alpha$). Hence $\Theta$ is full, and is trivially faithful. Now let $(C,\alpha)\in \underline{End}(\mathscr{C})$. We define $F:\mathbb{N} \longrightarrow \mathscr{C}$ by

$$\begin{aligned}
F(\mathbb{N}) &= C\\
F(0) &= 1_C\\
F(1) &= \alpha \qquad (\text{hence } F(n) = \alpha^n).
\end{aligned}$$

then $(C,\alpha) = (F(\mathbb{N}),F(1))$. Hence $\Theta$ is onto objects, and preserves distinct objects since if $F(\mathbb{N}) = F'(\mathbb{N})$ and $F(1) = F'(1)$, then trivially $F(0) = F'(0)$, and $F(n) = F(1)^n = F'(1)^n = F'(n)$, so $F = F'$. Hence $\Theta$ is an isomorphism of categories.

Similarly, we define $\Psi: [\mathbb{Z},\mathscr{C}] \longrightarrow \underline{Aut}(\mathscr{C})$ by $\Psi(F) = (F(\mathbb{Z}), F(1))$, and $\Psi(\phi) = \phi_{\mathbb{Z}}$. Note that, contrary to the usual setup of the category $\underline{\mathbb{Z}}$, where the additive structure is the additive structure of $\mathbb{Z}$, and multiplication is composition, here both $\underline{\mathbb{N}}$ and $\underline{\mathbb{Z}}$ denote categories with 1 object, resp. $|\mathbb{N}|$, $|\mathbb{Z}|$ endos, s.t. $\mathbb{N} = \{0,1,\dots\}$, and

$$a,b \in \mathbb{N} \qquad\qquad a,b\in\mathbb{Z}$$

$$a\cdot b = a+b \qquad\qquad a\cdot b = a+b.$$

(ie use the additive monoid structure)

The def$^N$ of $\Psi$ makes sense, since $F(-1): F(\mathbb{Z}) \longrightarrow F(\mathbb{Z})$ and $F(1)F(-1) = F(0) = F(-1)F(1)$, and $F(0) = 1_{F(\mathbb{Z})}$. (every morphism in $\mathbb{Z}$ is an automorphism). By the above proof for $\mathbb{N}$, $\Psi$ is also full and certainly faithful. Finally, if $(C,\alpha) \in \underline{\mathrm{Aut}}(\mathscr{C})$, define $F: \mathbb{Z} \longrightarrow \mathscr{C}$ by $F(\mathbb{Z}) = C$,

$$F(0) = 1_C$$
$$F(1) = \alpha$$
$$F(n) = \alpha^n$$

that is, for $n>0$, $F(n) = F(1)^n$, and for $n<0$, $F(n) = F(-1)^{-n}$. The only nontrivial case to check is $a<0, b>0$, with $|a|>b$. Then $F(ab) = F(-1)^{-(a+b)}$. But

$$\begin{aligned} F(a)F(b) &= F(-1)^{-a}F(1)^{b} \\ &= \overbrace{F(-1)F(-1)\cdots F(-1)}^{-a}\,\overbrace{F(1)\cdots F(1)}^{b} \\ &= \underbrace{F(-1)\cdots F(-1)}_{-a-b} \\ &= F(-1)^{-(a+b)} = F(ab) \end{aligned}$$

and $\Psi(F) = (F(\mathbb{Z}), F(1)) = (C,\alpha)$. Moreover, if $\Psi(F) = \Psi(F')$ then $F(\mathbb{Z}) = F'(\mathbb{Z})$ and $F(0) = F'(0)$, and $F(1) = F'(1)$ implies that for $n>0$, $F(n) = F(1)^n = F'(1)^n = F'(n)$ and for $n<0$, $n = (-1)^{|n|}$ in $\mathbb{Z}$, so $F(n) = F((-1)^{|n|}) = F(-1)^{|n|} = \{F(1)^{-1}\}^{|n|} = \{F'(1)^{-1}\}^{|n|} = F'(-1)^{|n|} = F'(n)$, so $F = F'$. Hence $\Psi$ is an isomorphism of categories.

(ii) If $\mathscr{C} = \underline{\mathrm{Mod}}\,A$, then $\underline{\mathrm{End}}(\mathscr{C}) \cong \underline{\mathrm{Mod}}\,A[x]$ and $\underline{\mathrm{Aut}}(\mathscr{C}) \cong \underline{\mathrm{Mod}}\,A[x,x^{-1}]$.

<u>PROOF</u> To $(C,\alpha) \in \underline{\mathrm{End}}(\underline{\mathrm{Mod}}\,A)$ we associate a ring morphism $\rho: A \longrightarrow \mathrm{End}_{\underline{Ab}}(C)$, where $r\cdot x = \rho(r)(x)$, for $r \in A$, $x \in C$. Then $\alpha \in \mathrm{End}_{Ab}(C)$ induces $\hat{\rho}: A[x] \longrightarrow \mathrm{End}_{\underline{Ab}}(C)$ defined by

$$\begin{aligned} &\hat{\rho}(a_0 + a_1x + \cdots + a_nx^n)(c) \\ &\quad = \{\rho(a_0) + \rho(a_1)\alpha + \cdots + \rho(a_n)\alpha^n\}(c) \\ &\quad = a_0\cdot c + a_1\cdot\alpha(c) + \cdots + a_n\cdot\alpha^n(c) \end{aligned}$$

this makes $C$ into an $A[x]$-module. If $\phi: (C,\alpha) \longrightarrow (C',\alpha')$ is a morphism in $\underline{\mathrm{End}}(\underline{\mathrm{Mod}}\,A)$, then it is a morphism $C \longrightarrow C'$ with the $A[x]$-module structure, since if $f \in A[x]$,

$$\begin{aligned} \phi(f\cdot c) &= \phi(a_0\cdot c + \cdots + a_n\alpha^n(c)) \\ &= \phi(a_0\cdot c) + \cdots + \phi(a_n\alpha^n(c)) \\ &= a_0\phi(c) + \cdots + a_n\phi(\alpha^n(c)) \\ &= a_0\phi(c) + \cdots + a_n\alpha'^n\phi(c) \qquad \phi\alpha = \alpha'\phi \\ &= \{a_0 + a_1\alpha' + \cdots + a_n\alpha'^n\}(\phi(c)) \\ &= f\cdot\phi(c). \end{aligned}$$

this defines a functor $X: \underline{\mathrm{End}}(\mathscr{C}) \longrightarrow \underline{\mathrm{Mod}}\,A[x]$. This functor is full since if $C, C' \in \underline{\mathrm{Mod}}\,A[x]$ are $X(C,\alpha)$ and $X(C',\alpha')$ resp. and $\phi: C \longrightarrow C'$ is a morphism of $A[x]$-modules, then $\phi$ is a morphism of $A$-modules, and by definition of the $A[x]$-module structure, $\phi\alpha = \alpha'\phi$, so $\phi: (C,\alpha) \longrightarrow (C',\alpha')$. In addition, $X$ is clearly faithful. $X$ is distinct on objects since if $X(C,\alpha) = X(C',\alpha')$ then $C$ and $C'$ are the same set with the same $A$-module structure, and for $x \in A[x]$ and $c \in C = C'$, $\alpha(c) = x\cdot c = \alpha'(c)$, so $\alpha = \alpha'$. To see that $X$ is onto objects, take an $A[x]$-module $D$, then $D$ is canonically an $A$-module, and $x$ determines an $A$-module endomorphism $d \longmapsto x\cdot d$ since by def$^N$ $xa = ax$ for $a \in A$ in $A[x]$, call this endomorphism $\alpha$. Then $X(D,\alpha)$ is the abelian group $D$ with $A[x]$-module structure

$$\begin{aligned} &\{a_0 + a_1x + \cdots + a_nx^n\}\cdot d \\ &\quad = a_0\cdot d + \cdots + a_n\cdot\alpha^n(d) \\ &\quad = a_0\cdot d + \cdots + a_n\cdot(x\cdot(x\cdot(\cdots x\cdot d)\cdots)) \\ &\quad = a_0\cdot d + \cdots + a_nx^n\cdot d \\ &\quad = (a_0 + \cdots + a_nx^n)\cdot d \end{aligned}$$

so $X(D,\alpha)$ is $D$. Hence $X$ is an isomorphism of categories.

(iii) Let $A$ be an $R$-algebra. We want to define a ring homomorphism $R \longrightarrow [1, 1]$, where $1 : \underline{\mathrm{Mod}}\, A \longrightarrow \underline{\mathrm{Mod}}\, A$.
But the structural morphism $\lambda : R \longrightarrow A$, $\lambda(r) = 1 \cdot r$ has its image in $\mathrm{cen}\, A$, which is isomorphic to $[1, 1]$.
Hence $\underline{\mathrm{Mod}}\, A$ has an $R$-category structure, where for $\phi : M \longrightarrow M'$, and $r \in R$,

$$(\phi \cdot r)(x) = (\phi \rho(r)_M)(x) = \phi(x \cdot (1 \cdot r))$$
$$= \phi(x) \cdot (1 \cdot r)$$

Similarly we define $\lambda : \underline{Aut}(\mathcal{B}) \longrightarrow \underline{Mod}\, A[x,x^{-1}]$ for $(C,\alpha) \in \underline{Aut}(\mathcal{B})$ by noting that $A[x,x^{-1}] = A[x]_x$, and the induced $\hat{\varphi} : A[x] \longrightarrow \mathrm{End}_{\underline{Ab}}(C)$ sends $x$ to $\alpha$, a unit in the ring of endomorphisms. Hence there is unique $\varphi' : A[x,x^{-1}] \longrightarrow \mathrm{End}_{\underline{Ab}}(C)$ s.t.

$$\begin{aligned}
&\varphi'(a_{-n}x^{-n} + \cdots + a_0 + \cdots + a_m x^m)(c) \\
&= \{\varphi'(a_{-n}x^{-n}) + \cdots + \varphi'(a_m x^m)\}(c) \\
&= a_{-n}\cdot \alpha^{-n}(c) + \cdots + a_0 \cdot c + \cdots + a_m \cdot \alpha^m(c)
\end{aligned}$$

where $\alpha^{-n}(c)$ denotes $(\alpha^{-1})^n(c)$. This defines the $A[x,x^{-1}]$-module $\lambda(C,\alpha)$. If $\phi : (C,\alpha) \longrightarrow (C',\alpha')$ in $\underline{Aut}(\mathcal{B})$, then $\phi\alpha = \alpha'\phi$ implies $\alpha'^{-1}\phi = \phi\alpha^{-1}$, so that in addition to $\phi(a\cdot c) = a\cdot\phi(c)$, $\phi(x^n \cdot c) = x^n\cdot\phi(c)$, $n \in \mathbb{Z}$. Hence $\phi : C \longrightarrow C'$ is a morphism of the induced $A[x,x^{-1}]$-modules. Thus $\lambda$ is faithful, and by the same reasoning as before it is full, and distinct on objects. To show it is onto, let $D$ be a $A[x,x^{-1}]$-module, and define $\alpha : D \longrightarrow D$ by $\alpha(d) = x\cdot d$. This is an $A$-module endomorphism of $D$, and $x^{-1}$ induces $\alpha'(d) = x^{-1}\cdot d$, with $\alpha\alpha' = \alpha'\alpha = 1$, so $\alpha$ is an automorphism. Again $\chi(D,\alpha)$ has the same abelian group structure as $D$, and since $\alpha^m(c) = x^m\cdot c$ and $\alpha^{-n}(c) = \alpha'^n(c) = (x^{-1})^n(c) = x^{-n}\cdot c$, $\chi(D,\alpha) = D$ as $A[x,x^{-1}]$-modules, so $\lambda$ is an isomorphism. $\square$

$\boxed{Q24}$ Let $R$ be a commutative ring. The abelian category $\mathcal{B}$ is an $\underline{R\text{-category}}$ if each group $\mathrm{Hom}_{\mathcal{B}}(C,C')$ has an $R$-module structure s.t. composition of morphisms is $R$-bilinear, i.e.

$$\alpha(\beta\cdot r + \gamma\cdot s) = \alpha\beta\cdot r + \alpha\gamma\cdot s$$
$$(\beta\cdot r + \gamma\cdot s)\alpha = \beta\alpha\cdot r + \gamma\alpha\cdot s$$

this is clearly equivalent to requiring that $\alpha(\beta\cdot r) = \alpha\beta\cdot r$ and $(\beta\cdot r)\alpha = \beta\alpha\cdot r$ for $r,s \in R$ $\alpha,\beta,\gamma \in \mathcal{B}$.

(i) Let $C \in \mathcal{B}$. Then $\mathrm{Hom}_{\mathcal{B}}(C,C)$ is a ring, an $R$-module and for $\alpha,\beta \in \mathrm{Hom}_{\mathcal{B}}(C,C)$,

$$(\alpha\cdot r)\beta = \alpha(\beta\cdot r) = \alpha\beta\cdot r$$

so by def$^N$ $\mathrm{Hom}_{\mathcal{B}}(C,C)$ forms an $R$-algebra. (most likely noncommutative)

(ii) Suppose $\mathcal{B}$ is an abelian $R$-category, and define $\varphi : R \longrightarrow [1,1]$ by

$$\varphi(a)_M = 1_M \cdot a$$

then $\varphi(ab)_M = 1\cdot(ab) = (1\cdot a)\cdot b = \{(1\cdot a)1\}\cdot b = (1\cdot a)(1\cdot b) = \varphi(a)_M\varphi(b)_M = (\varphi(a)\varphi(b))_M$, and $\varphi(a+b)_M = 1\cdot(a+b) = 1\cdot a + 1\cdot b = \varphi(a)_M + \varphi(b)_M = (\varphi(a)+\varphi(b))_M$. Also $\varphi(1)_M = 1\cdot 1 = 1 = 1_M$, so $\varphi$ is a morphism of rings, and for $a \in R$, $\varphi(a)$ is natural since if $\phi : C \longrightarrow C'$ in $\mathcal{B}$,

$$\begin{aligned}
\varphi(a)_{C'}\phi &= (1_{C'}\cdot a)\phi \\
&= (1_{C'}\phi)\cdot a = (\phi 1_C)\cdot a \\
&= \phi(1_C\cdot a) \\
&= \phi\,\varphi(a)_C
\end{aligned}$$

Conversely, given a morphism of rings $\varphi : R \longrightarrow [1,1]$, we get an $R$-algebra structure on every endomorphism ring $\mathrm{End}_{\mathcal{B}}(C)$, given by

$$\alpha\cdot a = \alpha\,\varphi(a)_C$$

this defines an $R$-module structure by bilinearity of composition and since $\varphi$ is a morphism of rings. Further,

$$\begin{aligned}
(\alpha\beta)\cdot a &= \alpha\beta\,\varphi(a)_C = \alpha\,\varphi(a)_C\beta \\
&= \alpha(\beta\cdot a) = (\alpha\cdot a)\beta
\end{aligned}$$

since $\varphi(a)$ is natural. For $C,C' \in \mathcal{B}$ and $\phi : C \longrightarrow C'$ we define $\phi\cdot a = \varphi(a)_{C'}\phi$ $(= \phi\varphi(a)_C)$, and this makes $\mathcal{B}$ into an $R$-category. Since for $M \in \mathcal{B}$ we have $1_M\cdot a = \varphi(a)_M$, these two processes are inverse, as required. $\square$

$\boxed{Q13}$ Let $C_1, \ldots, C_n$ be subobjects of $C$ in $\mathcal{C}$. We claim that the sum $C_1 + \cdots + C_n$ is direct iff.

$$C_i \cap \Big(\sum_{i \neq j} C_j\Big) = 0 \qquad \text{all } 1 \leq i \leq n.$$

PROOF Suppose the sum is direct, and hence $C_1 \oplus \cdots \oplus C_n$ is a subobject of $C$. In the subobject lattice of $\oplus_{i=1}^n C_i$, for $1 \leq i \leq n$ if $D \leq C_i$ and $D \leq \sum_{i \neq j} C_j$ then we can compose $D \longrightarrow \oplus_{i=1}^n C_i$ with the projection onto $C_i$ to find $D \longrightarrow C_i = 0$ (since $\oplus_{i \neq j} C_j$ is the sum). Hence since $D \longrightarrow \oplus_{i=1}^n C_i = D \longrightarrow C_i \longrightarrow \oplus_{i=1}^n C_i$, $D = 0$.

Conversely, recall that if $A, B$ are subobjects of an object $C$ and $A \cap B = 0$ then $A + B$ is direct, since considering

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & 0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathrm{I}"] & A \arrow[d] & & \\
0 \arrow[r] & B \arrow[r] & A+B \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathrm{II}"] & A+B/B \arrow[r] \arrow[d] & 0 \\
 & & \frac{A+B}{A} \arrow[r] \arrow[d] & 0 & \\
 & & 0 & &
\end{tikzcd}$$

where I is a pullback and recall that the pushout of $A+B/A$ and $A+B/B$ is $A+B/A+B = 0$. Hence by 13.2 and 13.2* $B \longrightarrow A+B \longrightarrow A+B/A$ is a monomorphism and an epimorphism, hence an isomorphism. Thus $A+B$ is the coproduct of $A$ and $B$.

Hence we proceed by induction on $n$, having considered already $n=2$. If $n > 2$ then

$$C_n \cap \Big(\sum_{i=1}^{n-1} C_i\Big) = 0$$

so $C_n + \sum_{i=1}^{n-1} C_i$ is direct. But for $j < n$, $C_j \cap \big(\sum_{i \neq j} C_i\big) = 0$ and contains $C_j \cap \big(\sum_{i \neq j}^{i<n} C_i\big)$, so that by the inductive hypothesis $\sum_{i=1}^{n-1} C_i$ is direct.

$$\begin{tikzcd}
 & C_n \arrow[d] \arrow[dr, bend left] & \\
 & C_n + \sum C_i \arrow[r] & C \\
C_i \arrow[r] & \sum_{i=1}^{n-1} C_i \arrow[u] \arrow[ur] & \\
C_j \arrow[ur] & &
\end{tikzcd}$$

but the coproduct of $C_n$ and $\oplus_{i=1}^{n-1} C_i$ is $\oplus_{i=1}^n C_i$, so $C_1 + \cdots + C_n$ is direct. $\square$

NOTE If $\{C_i\}_I$ is a family of objects in a Grothendieck category $\mathcal{C}$, then $\sum C_i$ is direct iff. for each finite $J \subseteq I$, $\sum_J C_i$ is direct. If $\sum_I C_i$ is direct, then clearly so is $\sum_J C_i$. For the converse, form a direct system out of the $\sum_J C_i$ and another on the same scheme from $C$. The $\sum_J C_i \longrightarrow C$ are monic, and since direct limits are exact,

$$\sum_I C_i \longrightarrow C$$

is also monic.

# CHAPTER V : GROTHENDIECK CATEGORIES

NOTE (Products of morphisms = Coproducts in Ab5)

Let $\mathcal{C}$ be an Ab5 abelian category with morphisms $\alpha : A \longrightarrow A'$, $\beta : B \longrightarrow B'$. Then we have

$$\begin{tikzcd}
A \arrow[r, shift left, "i_A"'] \arrow[d, "\alpha"'] & A \oplus B \arrow[l, shift left, "p_A"'] \arrow[r, shift left, "p_B"] \arrow[d, bend right, "\alpha\oplus\beta"'] \arrow[d, bend left, "\alpha\times\beta"] & B \arrow[l, shift left, "i_B"] \arrow[d, "\beta"] \\
A' \arrow[r, shift left, "i_{A'}"'] & A' \oplus B' \arrow[l, shift left, "p_{A'}"'] \arrow[r, shift left, "p_{B'}"] & B' \arrow[l, shift left, "i_{B'}"]
\end{tikzcd}$$

$$p_{A'}(\alpha \times \beta) = \alpha p_A$$
$$p_{B'}(\alpha \times \beta) = \beta p_B$$
$$i_{B'}\beta = (\alpha \oplus \beta) i_B$$
$$i_{A'}\alpha = (\alpha \oplus \beta) i_A$$

Hence, $p_{A'}(\alpha\times\beta) i_A = \alpha = p_{A'}(\alpha \oplus \beta) i_A$. Since $p_{B'}(\alpha\times\beta) i_A = \beta p_B i_A = 0$ and $p_{B'}(\alpha\oplus\beta) i_A = p_{B'} i_{A'} \alpha = 0$, and since in an Ab5 category the morphism $A' \oplus B' \longrightarrow A' \times B'$ is monic, we conclude that $(\alpha\times\beta) i_A = (\alpha\oplus\beta) i_A$ and hence $\alpha \times \beta = \alpha \oplus \beta$ since $i_A$ is monic.

**NOTE** Let $\mathcal{C}$ be an abelian category, satisfying Ab5. If $M \rightarrowtail M'$ and $N \rightarrowtail N'$ are subobjects, then since finite direct sums of monics are monic, $M \oplus N$ is canonically a subobject of $M' \oplus N'$.

**PROPOSITION** Let $M \rightarrowtail M'$, $N \rightarrowtail N'$ be subobjects in an Ab5 abelian category $\mathcal{C}$. If $p_1 : M' \oplus N' \longrightarrow M'$ and $p_2 : N' \oplus M' \longrightarrow N'$ are the projections, then

$$p_1^{-1}(M) \cap p_2^{-1}(N) = M \oplus N$$

**PROOF** Consider the diagram

$$\begin{tikzcd}
M \arrow[rr, shift left] \arrow[dd] \arrow[ddrr, phantom, "\mathrm{I}"] & & M \oplus N \arrow[ll, shift left] \arrow[rr, shift left] \arrow[dl, dashed] \arrow[dr, dashed] \arrow[dd] & & N \arrow[ll, shift left] \arrow[dd] \\
& M'' \arrow[ul] \arrow[dr, bend right] & & N'' \arrow[ur] \arrow[dl, bend left] & \\
M' \arrow[rr, shift left] & & M' \oplus N' \arrow[ll, shift left, "p_1"'] \arrow[rr, shift left, "p_2"] \arrow[uurr, phantom, "\mathrm{II}"] & & N' \arrow[ll, shift left]
\end{tikzcd}$$

where the squares (I) and (II) are pullbacks. Then the dashed arrows are induced by $M \oplus N \rightarrowtail M' \oplus N'$ and the projections from $M \oplus N$. Notice that since $\mathcal{C}$ is Ab5, we need not worry about the difference between $(M \rightarrowtail M') \times (N \rightarrowtail N')$ and $(M \rightarrowtail M') \oplus (N \rightarrowtail N')$. To prove the square

$$\begin{tikzcd}
Q \arrow[r, bend left] \arrow[dr, bend right] & M \oplus N \arrow[r] \arrow[d] & N'' \arrow[d] \\
& M'' \arrow[r] & M' \oplus N'
\end{tikzcd}$$

is a pullback, let $Q \longrightarrow M''$, $Q \longrightarrow N''$ make the square commute. We induce $Q \longrightarrow M \oplus N$ into the product by $Q \longrightarrow N'' \longrightarrow N$ and $Q \longrightarrow M'' \longrightarrow M$. Using once again the fact that $M' \oplus N' \longrightarrow M' \times N'$ is monic, and the pullbacks (I) and (II) we show this morphism $Q \longrightarrow M \oplus N$ is indeed a factorisation through the pullback. Uniqueness follows from the uniqueness of the factorisation through the product. $\square$.

**LEMMA** Let $f : A \longrightarrow B$ in an abelian category $\mathcal{C}$, with subobjects $A' \rightarrowtail A$ and $B' \rightarrowtail B$. Then

$$f(A') \cap B' = 0 \iff f^{-1}(B') \cap A' = 0$$

**PROOF** Let

$$\begin{tikzcd}
A' \arrow[r, two heads, "f'"] \arrow[d, tail] & A''' = f(A') \arrow[d, tail] \\
A \arrow[r, "f"] & B
\end{tikzcd}$$

denote the image factorisation of $f(A')$. Then $f^{-1}(B') \cap A'$ is the pullback of $B' \rightarrowtail B$ along $A' \rightarrowtail A \xrightarrow{f} B$, or $A' \twoheadrightarrow f(A') \rightarrowtail B$. Hence $f^{-1}(B') \cap A' = f'^{-1}(B' \cap f(A'))$, where $f'$ is an epimorphism. This makes the result clear. $\square$

In view of Prop 2.3. it is legitimate to talk about the injective envelope of an object $C$ (when it exists!), and we denote it by $E(C)$. However, one should bear in mind that $E(C)$ is determined only up to isomorphism. $E(C)$ can be embedded in any injective object containing $C$, but the imbedding is usually not unique. It is known that every finite direct sum of injective objects is injective. For injective envelopes one has the following result:

PROPOSITION 2.6 The monomorphism $C_1 \oplus \cdots \oplus C_n \longrightarrow E(C_1) \oplus \cdots \oplus E(C_n)$ induces an isomorphism

$$E(C_1 \oplus \cdots \oplus C_n) \cong E(C_1) \oplus \cdots \oplus E(C_n)$$

Note that an Ab5 category is s.t. canonical $\oplus \longrightarrow \Pi$ is monic whenever the product exists – in particular, for finite products, and it follows (see start Ch III, M) that finite direct sums of monics are monic. The proposition is then an immediate consequence of

LEMMA 2.7 If $C_i \longrightarrow C_i'$ $(i=1,\ldots,n)$ are essential monomorphisms, then $C_1 \oplus \cdots \oplus C_n \longrightarrow C_1' \oplus \cdots \oplus C_n'$ is essential.

PROOF By induction it suffices to do the proof for the case $n=2$, so we show that each of the two monomorphisms $C_1 \oplus C_2 \longrightarrow C_1' \oplus C_2 \longrightarrow C_1' \oplus C_2'$ is essential. Consider the first one. Let $\pi : C_1' \oplus C_2 \longrightarrow C_1'$ denote the canonical projection, and suppose $\beta : B \longrightarrow C_1' \oplus C_2$ is an arbitrary non-zero monomorphism. Then either $\pi\beta = 0$, in which case $\operatorname{Im}\beta \subseteq C_2$ and hence $\operatorname{Im}\beta \cap (C_1 \oplus C_2) \neq 0$, or $\pi\beta \neq 0$ in which case $\operatorname{Im}\pi\beta \cap C_1 \neq 0$ and hence by the Lemma on the previous page, $\pi^{-1}C_1 \cap \operatorname{Im}\beta \neq 0$. Since if $\pi' : C_1' \oplus C_2 \longrightarrow C_2$ is the other projection, $\pi'^{-1}C_2 = C_1' \oplus C_2$, we have

$$\operatorname{Im}\beta \cap (\pi^{-1}C_1 \cap \pi'^{-1}C_2) \neq 0$$

but by an earlier Proposition, this means $\operatorname{Im}\beta \cap (C_1 \oplus C_2) \neq 0$, as required. □

When direct limits in $\mathcal{C}$ are exact, one can extend Lemma 2.7 to arbitrary families of essential monomorphisms (see Exercise 6).

DEFINITION An object of $\mathcal{C}$ is called indecomposable if it cannot be written as the direct sum of two non-zero subobjects. A subobject $B$ of $C$ is irreducible in $\mathcal{C}$ if it cannot be written as the intersection of two strictly bigger subobjects of $C$ (cf. Example III 3.3)

It is clear that $B$ is irreducible in $C$ iff. $C/B$ is a coirreducible object, i.e. any two non-zero subobjects of $C/B$ have nonzero intersection. With the use of Prop 2.6 we have

PROPOSITION 2.8 The following properties of an injective object $E$ are equivalent:

(a) $E$ is indecomposable
(b) Each subobject of $E$ is coirreducible
(c) $E$ is the injective envelope of a coirreducible object
(d) $E$ is an injective envelope of each one of its non-zero subobjects.

PROOF If $E=0$ all of these hold trivially, so suppose $E \neq 0$, and clearly every subobject of $E$ is coirreducible iff. $E$ is coirreducible. Now recall that an injective object is a direct summand of any object containing it.

(a) $\Rightarrow$ (b) Let $C$ be a subobject of $E$, and let $E(C)$ be the injective envelope of $C$ (considered as a subobject of $E$ via Lemma 2.2). Then, as a direct summand of $E$, $E(C)$ is either $0$ or $E$. Hence if $C \neq 0$, $E$ is an essential extension of $C$, so $C \cap C' \neq 0$ for any $C' \neq 0$ in $E$, hence $E$ is coirreducible.

(b) $\Rightarrow$ (c) Then $E$ is an essential extension of itself, hence the injective envelope of a coirreducible object.

(c) $\Rightarrow$ (d) Let $E$ be an essential extension of a coirreducible subobject $C$ (which must be $\neq 0$ since $E$ is). Let $0 \neq D \leq E$ and $0 \neq D' \leq E$. We must show $D \cap D' \neq 0$. But $D \cap C \neq 0$ and $D' \cap C \neq 0$, and since $C$ is coirreducible, $(D \cap C) \cap (D' \cap C) \neq 0$. But then $0 \neq (D \cap D') \cap C \leq D \cap D'$, as required.

(d) $\Rightarrow$ (a) Thus $E$ is an essential extension of every nonzero subobject, so no two non-zero subobjects can have zero intersection. Hence $E$ is indecomposable. □

We know that an $A$-module $E$ is injective iff. homomorphisms $\mathfrak{a} \longrightarrow E$, where $\mathfrak{a}$ is a right ideal of $A$, can be extended to $A$ (Prop. I.6.5). This can be generalised to Grothendieck categories.

<u>PROPOSITION 2.9</u> Let $\mathscr{C}$ be a Grothendieck category with a family of generators $(U_i)_I$. An object $E$ is injective if and only if for every monomorphism $\alpha : C \longrightarrow U_i$ and morphism $\varphi : C \longrightarrow E$, there exists $\varphi' : U_i \longrightarrow E$ s.t. $\varphi'\alpha = \varphi$.

<u>PROOF</u> Consider an arbitrary monomorphism $\alpha : C \longrightarrow C'$ and a morphism $\varphi : C \longrightarrow E$. If $C \leq D \leq C'$ and $\varphi' : D \longrightarrow E$ is s.t. the diagram

$$\begin{tikzcd} & E \\ C \arrow[ur, "\varphi"] \arrow[r] & D \arrow[u, "\varphi'"'] \end{tikzcd}$$

commutes then we call $\varphi'$ an <u>extension</u> of $\varphi$. The set $\mathcal{M}$ of all such extensions is partially ordered (to get a set, we pick representatives for each class of subobjects and only consider extensions defined on such representatives.) where $\varphi' \leq \varphi''$ iff. $\varphi''$ extends $\varphi'$. If $\{\varphi_j\}_j$ is a chain of these extensions, $\varphi_j : C_j \longrightarrow E$, then the $C_j$ form a direct family of subobjects of $D$, and hence the direct limit $\varinjlim C_i$ is a subobject of $D$ ($\mathscr{C}$ is Ab5) and there is an induced morphism $\varphi' : \varinjlim C_i \longrightarrow E$ extending all the $\varphi_j$. Hence every chain in $\mathcal{M}$ has an upper bound, and hence by Zorn's Lemma there is a maximal extension $\varphi' : C' \longrightarrow E$. This reduces the problem to the case where $\varphi$ cannot be extended within $C'$, and we show that $\alpha$ is an isomorphism.

Suppose that $\alpha$ is not an isomorphism. Then there exists some morphism $\gamma : U_i \longrightarrow C'$ such that $\operatorname{Im}\gamma$ is not contained in $C$. Construct the pullback diagram

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] \arrow[d, equal] & \gamma^{-1}(C) \arrow[r] \arrow[d] & \operatorname{Im}\gamma \cap C \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & K \arrow[r] & U_i \arrow[r] & \operatorname{Im}\gamma \arrow[r] & 0 \end{tikzcd}$$

Since $\gamma^{-1}C$ is a subobject of $U_i$, the composed morphism

$$\gamma^{-1}C \longrightarrow \operatorname{Im}\gamma \cap C \longrightarrow C \longrightarrow E$$

may be extended to $\beta : U_i \longrightarrow E$. Since the restriction of $\beta$ to $K$ is zero, $\beta$ can be factored over $\operatorname{Im}\gamma$ to give $\beta' : \operatorname{Im}\gamma \longrightarrow E$. The diagram

$$\begin{tikzcd} \operatorname{Im}\gamma \cap C \arrow[r] \arrow[d] & C \arrow[d, "\varphi"] \\ \operatorname{Im}\gamma \arrow[r, "\beta'"'] & E \end{tikzcd} \qquad (1)$$

then commutes because $\gamma^{-1}C \longrightarrow \operatorname{Im}\gamma \cap C$ is an epimorphism. By Chap IV (§5, (4)) there is an exact sequence

$$0 \longrightarrow \operatorname{Im}\gamma \cap C \longrightarrow \operatorname{Im}\gamma \oplus C \longrightarrow \operatorname{Im}\gamma + C \longrightarrow 0$$

Using the commutativity of (1), it follows that $(-\beta', \varphi) : \operatorname{Im}\gamma \oplus C \longrightarrow E$ induces a morphism $\operatorname{Im}\gamma + C \longrightarrow E$ which strictly extends $\varphi$. This is the desired contradiction. □

EXAMPLES 3. Modules of Fractions Let $S$ be a right denominator set in a ring $A$. Let $M$ be an $S$-torsion-free module. Also the injective envelope $E(M)$ is then $S$-torsion-free, for otherwise the torsion submodule of $E(M)$ would intersect $M$ trivially. The canonical embedding $M \longrightarrow M[S^{-1}]$ is essential, for if $xs^{-1}$ is a nonzero element of $M[S^{-1}]$, then $0 \neq xs^{-1} \cdot s = x \in M$. Therefore $M[S^{-1}]$ may be considered as a submodule of an injective envelope $E(M)$ of $M$. More precisely we have:

PROPOSITION 2.11 $M[S^{-1}] \cong \{u \in E(M) \mid us \in M \text{ for some } s \in S\}$

PROOF If $us = x \in M$, then $us = xs^{-1} \cdot s$ implies $u = xs^{-1}$ since $E(M)$ is $S$-torsion-free. $\square$ ?

## 3. FINITELY GENERATED OBJECTS

Let $\mathscr{C}$ be a Grothendieck category. An object $C$ of $\mathscr{C}$ is finitely generated if the lattice $L(C)$ is compact, i.e., whenever $C = \sum C_i$ for a direct family of subobjects $C_i$ of $C$, there exists an index $i_0$ s.t. $C = C_{i_0}$.

LEMMA 3.1 Let $0 \to C' \to C \to C'' \to 0$ be an exact sequence in $\mathscr{C}$. Then:

(i) If $C$ is finitely generated, so is $C''$.
(ii) If $C'$ and and $C''$ are finitely generated, so is $C$.

PROOF (i) Suppose $C''$ is expressed as a direct union $C'' = \sum C''_i$. Each $C''_i$ can be written as $C_i'' = C_i/C'$ for a unique subobject $C_i$ of $C$ containing $C'$. The family $(C_i)$ is also direct, and since $\mathscr{C}$ is Ab 5 we have $C = \sum C_i$. Hence $C = C_{i_0}$ for some $i_0$, and then $C'' = C_{i_0}/C' = C_{i_0}''$.

(ii) Suppose $C$ is written as a direct union $C = \sum C_i$. Then $C'$ finitely generated implies that $C' \subseteq C_{i_0}$ for some $i_0$. This gives $C'' = \sum_{i \geq i_0} (C_i/C')$, and $C''$ finitely generated implies that

LEMMA 3.3 Let $\mathscr{C}$ be locally finitely generated. If $\alpha : B \to C$ is an epimorphism and $C$ is finitely generated, there there exists a finitely generated subobject $B'$ of $B$ such that $\alpha(B') = C$.

First, a Grothendieck category $\mathscr{C}$ is locally finitely generated if it has a family of finitely generated generators.

LEMMA A Grothendieck category $\mathscr{C}$ is locally finitely generated iff. every object is the union of finitely generated subobjects.

PROOF If $\{G_i\}_{i\in I}$ is the family of f.g. generators, then for $C \in \mathscr{C}$, $C = \bigcup \operatorname{Im}\alpha$, where $\alpha$ ranges over all morphisms $\alpha : G_i \to C$ for all $i$. If not, then let $\mu : C \to C/\bigcup \operatorname{Im}\alpha$ be the cokernel, which is nonzero by assumption. Then there is $\beta : G_i \to C$ some $i$ s.t. $\mu\beta \neq 0$. Hence we obtain a contradiction. Since all the $\operatorname{Im}\alpha$ are finitely generated, this proves ($\Rightarrow$). Conversely, let $U$ be a generator for $\mathscr{C}$, so $U = \bigcup V_i$, the union of f.g. subobjects.

# 4. Locally Noetherian Categories

Let $\mathcal{C}$ be a Grothendieck category. An object $C$ is called <u>noetherian</u> (<u>artinian</u>) if the lattice $L(C)$ of subobjects is noetherian (artinian).

<u>PROPOSITION 4.1</u> The object $C$ is noetherian iff. every subobject of $C$ is finitely generated.

<u>PROOF</u> If $C$ is noetherian, every subobject is noetherian and hence finitely generated (if $D \leq C$ and $D = \sum_i D_i$, let $D_k = \max\{D_i\}$. Then $D_k = \sum_i D_i = D$). Conversely, if $D_0 < D_1 < \cdots$ is an ascending chain of subobjects then $\{D_i\}_{i \geq 0}$ is a direct family of submodules with union $\sum_i D_i$. This finitely generated, so $D_k = \sum_i D_i$ some $k$, whence the chain stabilises at $k$. $\square$

From Prop III 3.6 we have

<u>PROPOSITION 4.2</u> Let $0 \longrightarrow C' \longrightarrow C \longrightarrow C'' \longrightarrow 0$ be an exact sequence in $\mathcal{C}$. Then $C$ is noetherian iff. both $C'$ and $C''$ are noetherian.

The category $\mathcal{C}$ is <u>locally noetherian</u> if it has a family of noetherian generators. In that case, every object is a direct union of noetherian subobjects, and every finitely generated object is noetherian. (by the fact that if $D$ is f.g and $D = \sum N_i$, $1_D : D \longrightarrow \sum N_i$ factors through some $N_i$). For locally noetherian categories one has a very satisfactory decomposition theory for injective objects.

<u>PROPOSITION 4.3</u> Suppose $\mathcal{C}$ is locally finitely generated and has enough injective objects. Then $\mathcal{C}$ is locally noetherian iff. every coproduct of injective objects is injective.

<u>PROOF</u> Suppose $\mathcal{C}$ is locally noetherian, and let $\{E_i\}_I$ be a family of injective objects. To show that $\oplus_I E_i$ is injective it suffices by Prop 2.9 to consider a monomorphism $\alpha : B \longrightarrow C$ of noetherian objects and extend every morphism $\varphi : B \longrightarrow \oplus_I E_i$ to $C$. But since $B$ is finitely generated, the image of $\varphi$ is contained in a coproduct $\oplus_J E_i$ for a finite subset $J$ of $I$ (recall Example IV 8.4). Since $\oplus_J E_i$ is certainly injective, there is no problem extending $\varphi$ to $C \longrightarrow \oplus_J E_i \longrightarrow \oplus_I E_i$.

Assume conversely that every coproduct of injective objects is injective. We will show that every finitely generated object $C$ is noetherian. Let $C_1 < C_2 < \cdots$ be an ascending chain of subobjects of $C$, and put $E_i = E(C/C_i)$. For each $C_n$ and each $j \leq n$ we let $\varphi_{nj} : C_n \longrightarrow E_j$ denote the composition of the canonical morphisms $C_n \longrightarrow C \longrightarrow C/C_j \longrightarrow E_j$. For each $n$, these morphisms $\varphi_{nj}$ induce $\varphi_n : C_n \longrightarrow \overset{\infty}{\oplus} E_j$. The family $(\varphi_n)$ is compatible and therefore induces $\varphi : \sum_n C_n \longrightarrow \oplus_1^\infty E_j$. Since $\oplus_1^\infty E_j$ is injective, $\varphi$ extends to a morphism $\bar{\varphi} : C \longrightarrow \oplus E_j$. But since $C$ is finitely generated, $\bar{\varphi}$ factors through a finite coproduct $E_{i_1} \oplus \cdots \oplus E_{i_m}$. If $e = \max\{i_1, \ldots, i_m\}$ then for $m \geq e+1$ we have $C_m = C_{e+1}$. Hence the chain stabilises. $\square$

The argument used in the first half of the proof shows more generally

<u>COROLLARY 4.4</u> If $\mathcal{C}$ is locally noetherian, then every direct union of injective objects is injective.

Still more generally it can be shown that every direct limit of injective objects is injective. in a locally noetherian category.

<u>PROPOSITION 4.5</u> (Matlis) If $\mathcal{C}$ is locally noetherian, then every injective object is a coproduct of indecomposable injective objects.

<u>PROOF</u> Let $E$ be an injective object. Consider all independent families $\{E_i\}_I$ of indecomposable injective subobjects of $E$ (independent means the sum is direct). We show this collection is non-empty by showing that any non-zero injective object $E'$ in $\mathcal{C}$ contains a non-zero indecomposable direct summand (of course, it is already nonempty since $0$ is indecomposable injective). Let $C$ be a non-zero noetherian subobject of $E'$. Consider all injective subobjects $E''$ of $E'$ s.t. $C \nleq E''$ (nonempty since $0$ works). By Zorn's Lemma, together with Cor 4.4 and the proof of Prop 3.2, there is a maximal such subobject $E''$.

Then $E' = E'' \oplus D$, and we assert that $D$ is indecomposable. For if $D = D' \oplus D''$ with $D', D''$ nonzero, then $E' = E'' \oplus D' \oplus D''$, so $D'' \cap (E'' + D') = 0$ and hence by modularity of Sub$E'$,

$$\begin{aligned}(E''+D') \cap (E''+D'') &= E'' \cup (D'' \cap (E'' \cup D')) \\ &= E''\end{aligned}$$

and hence one of the objects $E''+D'$ or $E''+D''$ does not contain $C$, which contradicts the maximality of $E''$. Hence $E'$ contains a nonzero, injective indecomposable subobject. The set of all families $\{E_i\}_I$ of indecomposable injective subobjects whose sum $\sum E_i$ is direct is nonempty, partially ordered under inclusion, and if $\mathcal{E}$ is a chain of such families, let $\mathcal{F}$ be the union (of all the sets involved). This is still independent by IV, Ex 13 and the attached comment. Hence by Zorn's Lemma there is a maximal such family $\{E_i\}_I$. The sum $\sum_I E_i$ is an injective object by Prop 4.3, and so we can write

$$E = \left(\sum_I E_i\right) \oplus E'$$

But if $E' \neq 0$ then we have already shown that $E'$ will contain a nonzero indecomposable injective subobject, which contradicts the maximality of the family $\{E_i\}_I$. $\square$

EXAMPLES 1. Module Categories Let $A$ be a ring. The category Mod-$A$ is locally finitely generated since if $A \longrightarrow \sum D_i$ selects the element $d_1 + \cdots + d_n$, $d_i \in D_i$ then let $i \leq k$ $1 \leq i \leq n$, then $A \longrightarrow \sum D_i$ clearly factors through $D_k$. Mod-$A$ is locally noetherian iff. $A$ is a right noetherian ring, and is locally artinian iff. $A$ is a right artinian ring. It is known (see Chap VIII) that every right artinian ring is right noetherian. However, it is not true in general that every locally artinian Grothendieck category is locally noetherian.

2. Commutative Noetherian rings For a commutative noetherian ring it is possible to classify all indecomposable injective modules, and thereby all injective modules.

PROPOSITION 4.6 (Matlis) Let $A$ be a commutative noetherian ring. There is a bijective correspondence between prime ideals of $A$ and isomorphism classes of indecomposable injective modules, given by
nonzero

$$\mathfrak{p} \longmapsto E(A/\mathfrak{p})$$

PROOF Every prime ideal is irreducible, so $E(A/\mathfrak{p})$ is indecomposable (Prop 2.8). If $\mathfrak{p}$ and $\mathfrak{q}$ are prime ideals s.t. $E(A/\mathfrak{p}) \cong E(A/\mathfrak{q})$, then we may consider both $A/\mathfrak{p}$ and $A/\mathfrak{q}$ as submodules of $E(A/\mathfrak{p})$. Then $A/\mathfrak{p} \cap A/\mathfrak{q} \neq 0$, but $\operatorname{Ann}(x) = \mathfrak{p}$ for every nonzero $x \in A/\mathfrak{p}$ and $\operatorname{Ann}(y) = \mathfrak{q}$ for every nonzero $y \in A/\mathfrak{q}$, so we must have $\mathfrak{p} = \mathfrak{q}$. Finally we must show that if $E$ is an indecomposible injective module, then there exists $x \in E$ s.t. $\operatorname{Ann}(x)$ is a prime ideal. The family of ideals $\operatorname{Ann}(y)$, for $0 \neq y \in E$, has a maximal member $\operatorname{Ann}(x)$ since $A$ is noetherian. We assert that $\operatorname{Ann}(x)$ is a prime ideal. Suppose $a, b$ with $ab \in \operatorname{Ann}(x)$ but $b \notin \operatorname{Ann}(x)$. Then $bx \neq 0$ while $abx = 0$. So $a \in \operatorname{Ann}(bx)$. But $\operatorname{Ann}(x) \subseteq \operatorname{Ann}(bx)$, so by maximality we have $\operatorname{Ann}(x) = \operatorname{Ann}(bx)$, hence $a \in \operatorname{Ann}(x)$. $\square$

By Ex 7 (from Mitchell p90, but the result is standard), in $\mathbf{Ab} = \mathbf{Mod}\,\mathbb{Z}$ the injective envelope of $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ is $\mathbb{Z}_{n^\infty}$. For $n > 0$. ($E(\mathbb{Z}) = \mathbb{Q}$). Hence any f.g abelian group has an injective envelope of the form (using Prop 2.6)

$$\bigoplus_{i=1}^{s} \mathbb{Z}_{n_i^\infty} \oplus \mathbb{Q}^t$$

and by Prop 4.6 for $p > 0$ a prime, $\mathbb{Z}_{p^\infty}$ is an indecomposable injective abelian group and if $D$ is a divisible abelian group, then by Prop 4.5 there is a (possibly infinite) collection of primes $\{p_i\}_I$ s.t.

$$D \cong \bigoplus_i \mathbb{Z}_{p_i^\infty}$$

# 5. THE KRULL–REMAK–SCHMIDT–AZUMAYA THEOREM

Let $\mathcal{C}$ be a Grothendieck (with enough injectives). In this section we prove the uniqueness of direct sum decompositions $C_1 \oplus \cdots \oplus C_n$, when each $C_i$ has a local endomorphism ring. First some words on local rings:

DEFINITION A ring is local if all the non-invertible elements form a proper ideal. (non-invertible $\equiv$ not retraction or coretraction)

A local ring thus has precisely one maximal ideal, which also is the unique maximal right ideal.

PROPOSITION 5.1 If $E$ is an indecomposable injective object of $\mathcal{C}$, then $\mathrm{Hom}_{\mathcal{C}}(E, E)$ is a local ring.

PROOF It suffices to show that if $\alpha$ and $\beta$ are non-invertible endomorphisms of $E$, then also $\alpha + \beta$ is non-invertible.

# EXERCISES Ch V Stenstrom

$\boxed{Q7}$ (p 91 Mitchell)

The abelian group $\mathbb{Q}$ is clearly divisible, hence injective. If $x = a/b \in \mathbb{Q}$ then $b \cdot x = a \in \mathbb{Z}$, so $\mathbb{Q}$ is an injective envelope for $\mathbb{Z}$. Let $p$ be any positive integer, then note that $\mathbb{Z}_p$ is isomorphic to the subgroup

$$\mathbb{Z}_p \cong \{ n/p + \mathbb{Z} \in \mathbb{Q}/\mathbb{Z} \mid n \in \mathbb{Z} \}$$

via $x \longmapsto x/p$. Let $\mathbb{Z}_{p^\infty}$ be the set of cosets of $\mathbb{Q}/\mathbb{Z}$ of the form $n/p^s$ for some $n \in \mathbb{Z}$, $s \geq 1$. We claim that $\mathbb{Z}_{p^\infty}$ is divisible. This will follow if we can show that, given $x = 1/p^m$ and $n > 0$ there is $y = s/p^{k+m}$ s.t. $ny - x \in \mathbb{Z}$. Equivalently, we need positive integers $s, k$ and $t$ s.t.

$$ns = p^k(1 + tp^m)$$

Let $n_1$ be the product of all the prime powers $q_i^{n_i}$ occurring in the factorisation of $n$ with $q_i | p$. Then we can make the integer $k$ large enough so that $n_1 | p^k$. Then since $n/n_1$ and $p^m$ share no prime factors, their gcd is $1$ and hence we can find $s, t \in \mathbb{Z}$ s.t.

$$\frac{n}{n_1} s + tp^m = 1$$
$$\Rightarrow \quad \frac{sp^k}{n_1} n + tp^m p^k = p^k$$
$$\Rightarrow \quad \left\{ \frac{sp^k}{n_1} \right\} n = p^k(1 + (-t)p^m)$$

as required. Hence $\mathbb{Z}_{p^\infty}$ is injective, clearly contains $\mathbb{Z}_p$ and is an essential extension of $\mathbb{Z}_p$

If $n/p^s \in \mathbb{Z}_{p^\infty}$ then we may assume $p \nmid n$, otherwise say $n = n'p^{s'}$, $p \nmid n'$, and then either $s \leq s'$ in which case $p^s | n$ and $n/p^s = 0$ in $\mathbb{Q}/\mathbb{Z}$, or $s > s'$ and $n/p^s = n'/p^{s-s'}$ in $\mathbb{Z}_{p^\infty}$. Then $p^{s-s'-1} \cdot n'/p^{s-s'} = n'/p$ which is nonzero since $p \nmid n'$ and is an element of $\mathbb{Z}_p$. Hence $\mathbb{Z}_p \hookrightarrow \mathbb{Z}_{p^\infty}$ is essential, so

$$E(\mathbb{Z}_p) = \mathbb{Z}_{p^\infty}$$

$\boxed{Q9}$ Let $\mathfrak{a}$ be an irreducible right ideal of $A$, and let $b \notin \mathfrak{a}$. We claim that $E(A/\mathfrak{a}) = E(A/(\mathfrak{a}:b))$ and that $(\mathfrak{a}:b)$ is irreducible. Since $\mathfrak{a}$ is irreducible, $A/\mathfrak{a}$ is coirreducible, so by Prop 2.8 $E(A/\mathfrak{a})$ is indecomposable. If we can show that $E(A/\mathfrak{a}) = E(A/(\mathfrak{a}:b))$ it will then follow from Prop 2.8 (b) that $A/(\mathfrak{a}:b)$ is coirreducible, hence $(\mathfrak{a}:b)$ is irreducible.

We need only show that for $b \notin \mathfrak{a}$ the cyclic submodule $(b+\mathfrak{a})$ of $A/\mathfrak{a}$, which is isomorphic canonically to $A/(\mathfrak{a}:b)$, is an essential subobject of $E(A/\mathfrak{a})$ via $(b+\mathfrak{a}) \longrightarrow A/\mathfrak{a} \longrightarrow E(A/\mathfrak{a})$. Let $0 \neq M$ be a submodule of $E(A/\mathfrak{a})$. Then $M \cap A/\mathfrak{a}$ is a nonzero submodule of $A/\mathfrak{a}$. Since $A/\mathfrak{a}$ is coirreducible and $(b+\mathfrak{a}) \neq 0$ $(b \notin \mathfrak{a})$, we must have $M \cap A/\mathfrak{a} \cap (b+\mathfrak{a}) \neq 0$ — hence $M \cap (b+\mathfrak{a}) \neq 0$ and $(b+\mathfrak{a})$ is essential in $E(A/\mathfrak{a})$. Hence $E(A/(\mathfrak{a}:b)) = E(A/\mathfrak{a})$, as required.

$\boxed{Q16}$ Let $\alpha$ be an endomorphism of an object $C$ of $\mathcal{C}$. (a Groth. cat. with enough injectives).

(i) If $\mathcal{C}$ were Mod$A$ for some ring $A$, the proof would go like this: let $K_n = \operatorname{Ker}\alpha^n$. Then $K_n \leq K_{n+1}$ for each $n$ and so eventually $K_m = K_{m+1}$ some $m$. This means that $\alpha^{m+1}(x) = 0 \Rightarrow \alpha^m(x) = 0$, or $\alpha(\alpha^m(x)) = 0 \Rightarrow \alpha^m(x) = 0$. But each $\alpha^i$ is still surjective since if $z \in C$, since $\alpha$ is epi, $z = \alpha(y)$. But then $y = \alpha(x)$, so $z = \alpha^2(x)$, etc.

For the general case, suppose $\alpha$ is an epimorphism and put $K_n = \operatorname{Ker}\alpha^n$. First, each $\alpha^n$, $n \geq 1$ is epi since by induction (for $i=2$ if $z\alpha^2 = 0$, $z\alpha\alpha = 0$ so $z\alpha = 0$ hence $z = 0$) for case $i > 2$ if $z\alpha^{i+1} = 0$ then $z\alpha^i\alpha = 0$ so $z\alpha^i = 0$ and hence by assumption $z = 0$. Also $K_n \leq K_{n+1}$ since if $k_n : K_n \longrightarrow C$ and $k_{n+1} : K_{n+1} \longrightarrow C$ are the resp. kernels, then $\alpha^{n+1}k_n = \alpha\alpha^n k_n = 0$, so $K_n \leq K_{n+1}$.

Since $C$ is noetherian, there is some $m$ with $\mathrm{Ker}\,\alpha^m = \mathrm{Ker}\,\alpha^{m+1}$. To show that $\alpha$ is monic it suffices (since $\mathcal{C}$ has a generator $U$) to show that for any $x : U \longrightarrow C$ if $\alpha x = 0$ then $x = 0$. Let $x$ be such a morphism and form the pullback

$$\begin{tikzcd} V \arrow[r, twoheadrightarrow, "t"] \arrow[d, "y"'] & U \arrow[d, "x"] \\ C \arrow[r, twoheadrightarrow, "\alpha^m"'] & C \end{tikzcd}$$

Then if $\alpha x = 0$, $\alpha x t = 0$ so $\alpha \alpha^m y = 0$ and so $\alpha^{m+1} y = 0$. Hence $y$ factors through $K_{m+1} = K_m$, so $\alpha^m y = 0$. But then $xt = 0$ and since as the pullback of $\alpha^m$ (an epimorphism) is epi, this implies that $x = 0$. Hence $\alpha$ is epi and monic, and is thus an isomorphism.

(ii) Now suppose that $C$ is artinian and that $\alpha$ is a monomorphism. We actually didn't need to use the generator in (i), so this follows by duality. Explicitly, by the same argument as before $\alpha^n$ is monic each $n \geqslant 1$ and if $C_n = \mathrm{Coker}\,\alpha^n$, then the cokernels obey $C_n \geqslant C_{n+1}$, hence so their kernels and thus by artinian-ness, $C_m = C_{m+1}$ some $m$. Suppose that $z\alpha = 0$, and form the pushout

$$\begin{tikzcd} C \arrow[r, "\alpha^m"] \arrow[d, "z"'] & C \arrow[d, "t'"] \\ D \arrow[r, "t"'] & E \end{tikzcd}$$

Since $\mathcal{C}$ is abelian and $\alpha^m$ is monic, $t$ is monic. Then $z\alpha = 0$ implies $tz\alpha = 0$, hence $t'\alpha^m\alpha = 0$. Since $C_m = C_{m+1}$, this implies $t'\alpha^m = 0$ and hence $tz = 0$. Since $t$ is monic, $z = 0$ as required.

(iii) If $C$ is both artinian and noetherian and $\alpha : C \longrightarrow C$, then for $n \geqslant 1$ $\mathrm{Ker}\,\alpha^{n+1} \supseteq \mathrm{Ker}\,\alpha^n$ and $\mathrm{Im}\,\alpha^{n+1} \subseteq \mathrm{Im}\,\alpha^n$. Using the properties of $C$, let $k$ be an integer so that

$$\mathrm{Im}\,\alpha^k = \mathrm{Im}\,\alpha^n \qquad n \geqslant k$$
$$\mathrm{Ker}\,\alpha^k = \mathrm{Ker}\,\alpha^n \qquad n \geqslant k$$

(notice that $\mathrm{Im}\,\alpha^{n+1} \subseteq \mathrm{Im}\,\alpha^n$ since as in (ii), $\mathrm{Coker}\,\alpha^n \geqslant \mathrm{Coker}\,\alpha^{n+1}$, hence since $\mathrm{Im}\,\alpha^n = \mathrm{Ker}\,\mathrm{Coker}\,\alpha^n$, it follows).
We claim that

$$\begin{tikzcd} & & & & \mathrm{Im}(\alpha_k|_{\mathrm{Im}\alpha_k}) \arrow[dd] \\ & \mathrm{Im}\,\alpha^k \arrow[urrr] \arrow[dr] & & \mathrm{Im}\,\alpha^k \arrow[dr] & \\ C \arrow[ur] \arrow[rr, "\alpha^k"'] \arrow[rrrr, bend right, "\alpha^{2k}"'] & & C \arrow[ur] \arrow[rr, "\alpha^k"'] & & C \end{tikzcd}$$

this makes it clear that $\mathrm{Im}(\alpha|_{\mathrm{Im}\alpha}) = \mathrm{Im}(\alpha^{2k}) = \mathrm{Im}\,\alpha^k$. Hence $\mathrm{Im}\,\alpha^k \longrightarrow \mathrm{Im}\,\alpha^k$ induced by $\alpha_k$ is epi.
By (i) if $\mathrm{Im}\,\alpha^k$ is noetherian (which it is since $C$ is noetherian) and this morphism is epi it is iso.
(For modules this is trivial since if $y \in \mathrm{Im}\,\alpha^k = \mathrm{Im}\,\alpha^{2k}$, say $y = \alpha^{2k}(x) = \alpha^k(\alpha^k(x))$, so $\alpha^k|_{\mathrm{Im}\alpha^k}$ is onto $\mathrm{Im}\,\alpha^k$).
The upshot of all this is that $\alpha^k|_{\mathrm{Im}\alpha^k}$ is monic. This implies that $\mathrm{Im}\,\alpha^k \cap \mathrm{Ker}\,\alpha^k = 0$, since if we form the pullback:

$$\begin{tikzcd} \mathrm{Im}\,\alpha^k \cap \mathrm{Ker}\,\alpha^k \arrow[r] \arrow[d, tail] & \mathrm{Ker}\,\alpha^k \arrow[d, tail, "k"] & \\ \mathrm{Im}\,\alpha^k \arrow[r] & C \arrow[r, "\alpha^k"] & C \end{tikzcd}$$

and use the fact that the bottom row is monic, we see that $\mathrm{Im}\,\alpha^k \cap \mathrm{Ker}\,\alpha^k \longrightarrow \mathrm{Im}\,\alpha^k$ is zero. Since it is also monic (as the pullback of $k$), this shows that $\mathrm{Im}\,\alpha^k \cap \mathrm{Ker}\,\alpha^k = 0$.

It only remains to show that $\operatorname{Im}\alpha^k + \operatorname{Ker}\alpha^k = C$, since we have just shown that the sum is direct (see IV. Q13).
(For modules, let $C/\operatorname{Ker}\alpha^k \longrightarrow C/\operatorname{Ker}\alpha^k$ be $a + \operatorname{Ker}\alpha^k \longmapsto \alpha^k(a) + \operatorname{Ker}\alpha^k$ which is monic since $\operatorname{Ker}\alpha^{2k} = \operatorname{Ker}\alpha^k$.
Hence since $C/\operatorname{Ker}\alpha^k$ is noetherian, this map is also epi and so for any $x \in C$, $x = z + \alpha^k(a)$ for some $a \in C$ and $z \in \operatorname{Ker}\alpha^k$.
This shows that $\operatorname{Im}\alpha^k + \operatorname{Ker}\alpha^k = C$).

Well, I've just realised once we know that $\operatorname{Im}\alpha^k \rightarrowtail C \twoheadrightarrow \operatorname{Im}\alpha^k$ is iso, we have proven that

$$0 \longrightarrow \operatorname{Ker}\alpha^k \longrightarrow C \longrightarrow \operatorname{Im}\alpha^k \longrightarrow 0$$

is split exact, and hence that $C = \operatorname{Ker}\alpha^k \oplus \operatorname{Im}\alpha^k$. (To get it split, canonical $\operatorname{Im}\alpha^k \rightarrowtail C$ may get an iso tached on, but since everything is independent of changes of equiv. subobjects, doesn't matter). $\square$

$\boxed{Q20}$ (i) Let $A$ be an abelian group, that is indecomposable and injective. We also assume that $A \neq 0$.
Let $0 \neq x \in A$. Either $(x)$ is infinite, or it is finite. In the former case $A$ has $\mathbb{Z}$ as a subgroup. In the latter case, suppose $x$ has order $m$ and let $p$ be a prime divisor of $m$. Then $A$ has a cyclic subgroup of order $p$, hence $\mathbb{Z}_p$ as a subgroup. By Prop 2.8 (d) and the included Ex. from Mitchell, it follows that $A \cong \mathbb{Q}$ or $A \cong E(\mathbb{Z}_p) = \mathbb{Z}_{p^\infty}$

(ii)

# CHAPTER VI : TORSION THEORY

We have seen in Chap II to each ring of fractions of a ring $A$ there is associated a notion of torsion for $A$-modules. The same will be true when we get to consider general rings of quotients of $A$, but here we will follow a converse course. We start by axiomatising the concept of torsion, and then to each torsion theory we associate a ring of quotients. This chapter is devoted to a comprehensive study of the general aspects of torsion. The basic result will be that the particular notion of torsion, used in the theory of rings of quotients, can be described in three equivalent ways

1) By the class of torsion modules
2) By the right ideals which serve as annihilators of torsion elements
3) By the functor assigning to each module its torsion submodule.

## 1. PRERADICALS

One way of introducing a torsion concept for $A$-modules is to prescribe a functor on Mod-$A$ which to each module associates a torsion submodule. In order to make formal duality arguments available, we will work in an abelian category, although all our applications deal with module categories.

Let $\mathcal{C}$ be an abelian category, which we assume to be complete, cocomplete and locally small. A preradical $r$ of $\mathcal{C}$ assigns to each $C$ a subobject $r(C)$ in such a way that every morphism $C \longrightarrow D$ induces $r(C) \longrightarrow r(D)$ by restriction. In other words, a preradical is a subfunctor of the identity functor on $\mathcal{C}$. The class of all preradicals of $\mathcal{C}$ is a complete lattice, because there is a partial ordering in which $r_1 \leq r_2$ means $r_1(C) \leq r_2(C)$ for all objects $C$, and any family $(r_i)$ of preradicals has a least upper bound $\sum r_i$ and a greatest lower bound $\bigcap r_i$, defined in the obvious ways. (NB $r_1 \leq r_2$ iff. $r_1 \longrightarrow 1$ factors through $r_2 \longrightarrow 1$). That is,

$$\left(\sum r_i\right)(C) = \sum r_i(C)$$

[since for $f, g : \sum C_i \longrightarrow B$, $\mathrm{Ker}(f-g) \leq C$ so if $f|_{C_i} = g|_{C_i}$ each $i$, $C_i \leq \mathrm{Ker}(f-g)$, implies $\sum C_i \leq \mathrm{Ker}(f-g)$, so $f|_{C_i} = g|_{C_i}$ each $i$ iff. $f = g$]

and

$$\left(\bigcap r_i\right)(C) = \bigcap r_i(C)$$

[for each $i$, $\bigcap r_i(C) \longrightarrow r_i(C) \longrightarrow r_i(D) \longrightarrow D$ $= \bigcap r_i(C) \longrightarrow C \longrightarrow D$, and so $\bigcap r_i(C) \longrightarrow D$ must factor through $\bigcap r_i(D)$]

we have the zero radical $r(C) = 0$ and the improper radical $1$. If $r_1$ and $r_2$ are preradicals, one defines preradicals $r_1 r_2$ and $r_1 : r_2$ as

$$r_1 r_2(C) = r_1(r_2(C))$$
$$(r_1 : r_2)(C)/r_1(C) = r_2(C/r_1(C))$$

That is, form the subobject $r_1(C) \longrightarrow C$. Then the torsion subobject $r_2(C/r_1(C))$ of $C/r_1(C)$ corresponds to a subobject $(r_1 : r_2)(C)$ of $C$ containing $r_1(C)$. Thus $r_1 : r_2$ is defined up to a natural equivalence.

A preradical $r$ is idempotent if $rr \cong r$ and is called a radical if $r : r \cong r$, i.e. if $r(C/r(C)) = 0$ for every object $C$.
(note that this means $r(r(C)) \cong r(C)$ as subobjects)

LEMMA 1.1 If $r$ is a radical and $D \leq r(C)$ then $r(C/D) = r(C)/D$

PROOF The canonical morphism $C \longrightarrow C/D$ induces $r(C) \longrightarrow r(C/D)$ making

$$\begin{tikzcd}
D \arrow[r] \arrow[dr] & C \arrow[r] & C/D \\
r(D) \arrow[u] \arrow[r] & r(C) \arrow[u] \arrow[r] & r(C/D) \arrow[u]
\end{tikzcd}$$

commute. Hence $r(C)/D = \mathrm{Im}(r(C) \longrightarrow C \longrightarrow C/D)$, we find $r(C)/D \leq r(C/D)$. On the other hand, by the First Isomorphism Theorem — that is, the 9 Lemma applied to

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & D \arrow[r, equal] \arrow[d] & D \arrow[r] \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & r(C) \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & C/r(C) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \frac{r(C)}{D} \arrow[r] \arrow[d] & \frac{C}{D} \arrow[r] \arrow[d] & C/r(C) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

the bottom row is exact. But since $r(C/r(C)) = 0$ and the square

$$\begin{tikzcd}
C/D \arrow[r] & C/r(C) \\
r(C/D) \arrow[u] \arrow[r] & r(C/r(C)) \arrow[u]
\end{tikzcd}$$

commutes, $r(C/D) \leq r(C)/D$. Hence we have the desired equality. □

If $r$ is a preradical of $\mathscr{C}$, then one can define a preradical $r^{-1}$ of $\mathscr{C}^{op}$ by setting $r^{-1}(M) = M/r(M)$. It is clear that $r$ is idempotent (resp. a radical) of $\mathscr{C}$ iff. $r^{-1}$ is a radical (resp. an idempotent preradical) of $\mathscr{C}^{op}$. This gives rise to a useful duality principle. Notice that $r^{-1}$ is only determined up to a natural equivalence.

[$r$ idempotent iff. $r^{-1}$ radical] Suppose $r$ is idempotent and let $C \in \mathscr{C}$. Then $r^{-1}C$ is the dual of some cokernel $C \longrightarrow C/rC$ of $rC \longrightarrow C$ hence we can take $r(C) \longrightarrow C$'s dual as the cokernel

$\mathscr{C}^{op}$

$$\begin{tikzcd}
r^{-1}C \arrow[r] & C \arrow[r] & C/r^{-1}C \\
 & & r^{-1}\left(C/r^{-1}(C)\right) \arrow[u]
\end{tikzcd}$$

$\mathscr{C}$

$$\begin{tikzcd}
 & & r(C)/rr(C) = 0 \\
C/r(C) & C \arrow[l] & r(C) \arrow[l] \arrow[u] \\
 & & rr(C) = r(C) \arrow[u]
\end{tikzcd}$$

hence $r^{-1}\, C/r^{-1}(C) = 0$. Conversely $r^{-1}$ radical implies that $rr(C) \cong C$, so $rr \cong r$ is idempotent. One proves that $r$ is radical iff. $r^{-1}$ is idempotent in a similar way. Notice that $(r^{-1})^{-1} \cong r$.

To the preradical $r$ one can associate two classes of objects of $\mathcal{C}$, namely

$$\mathcal{T}_r = \{C \mid r(C) = C\}$$

$$\mathcal{F}_r = \{C \mid r(C) = 0\} \qquad\qquad (\mathcal{F} = \mathcal{F})$$

$(r^{-1})^{-1} = r.$

where equality here is understood as equality of subobjects. Note that $\mathcal{F}_r$ in $\mathcal{C}$ is equal to $\mathcal{T}_{r^{-1}}$ in $\mathcal{C}^{op}$. Also $\mathcal{T}_r = \mathcal{F}_{r^{-1}}$

PROPOSITION 1.2 $\mathcal{T}_r$ is closed under quotients and coproducts, while $\mathcal{F}_r$ is closed under subobjects and products.

PROOF It is easy to see that $\mathcal{T}_r$ is closed under quotient objects by Lemma 1.1. Let $(C_i)_I$ be an arbitrary family of objects in $\mathcal{T}_r$. Since $r(C_i) = C_i$, the image of each canonical morphism $C_i \longrightarrow \oplus_I C_i$ is contained in $r(\oplus_I C_i)$, and it follows from the definition of coproducts that $r(\oplus_I C_i) = \oplus_I C_i$. The corresponding results for $\mathcal{F}_r$ follow by duality. □

COROLLARY 1.3 If $C \in \mathcal{T}_r$ and $D \in \mathcal{F}_r$ then $Hom(C, D) = 0$.

A class of objects $\hat{\mathcal{C}}$ is called a pretorsion class if it closed under quotient objects and coproducts, and a pretorsion-free class if it is closed under subobjects and products. Let $\hat{\mathcal{C}}$ be a pretorsion class. If $C$ is an arbitrary object of $\mathcal{C}$ and $t(C)$ denotes the sum of all the subobjects of $C$ belonging to $\hat{\mathcal{C}}$, then clearly also $t(C) \in \hat{\mathcal{C}}$. (presuming $\hat{\mathcal{C}}$ is nonempty). Hence every object $C$ contains a largest subobject $t(C)$ belonging to $\hat{\mathcal{C}}$. In this way $\hat{\mathcal{C}}$ gives rise to a preradical $t$ of $\mathcal{C}$, and $t$ is clearly idempotent. (we assume $0 \in \hat{\mathcal{C}}$) ($t$ is a functor since for $C \longrightarrow D$, $Im(t(C) \to C \to D)$ is in $\hat{\mathcal{C}}$, hence factors through $t(D)$). Combining this procedure with the previous assignment $r \longmapsto \mathcal{T}_r$, restricted to idempotent $r$, we obtain:

PROPOSITION 1.4 There is a bijective correspondence between idempotent preradicals of $\mathcal{C}$ and pretorsion classes of objects of $\mathcal{C}$. Dually, there is a bijective correspondence between radicals of $\mathcal{C}$ and pretorsion-free classes of objects of $\mathcal{C}$.

PROOF More precisely, the bijection is between isomorphism classes of idempotent preradicals of $\mathcal{C}$ and pretorsion classes. Take $r$, produce $\mathcal{T}_r$. For $C \in \mathcal{C}$, $t(C) \in \mathcal{T}_r$ means $r(t(C)) \cong t(C)$. Since

$$\begin{tikzcd}
t(C) \arrow[r] \arrow[d, equal] & C \\
r(t(C)) \arrow[r] & r(C) \arrow[u]
\end{tikzcd}$$

commutes, and $r(C) \leq t(C)$ by def$^N$ of $t$, so $t(C) = r(C)$ as subobjects and hence $t \cong r$. Given a pretorsion class $\mathcal{T}$ and the induced $t$ (defined only up to n.e), $C \in \mathcal{T}_t$ iff. $t(C) = C$ iff. $C \in \mathcal{T}$. □

PROPOSITION 1.7 The following assertions are equivalent for a preradical $r$

(a) $r$ is a left exact functor

(b) If $D \leq C$ then $r(D) = r(C) \cap D$ as subobjects of $D$

(c) $r$ is idempotent and $\mathcal{T}_r$ is closed under subobjects.

PROOF (a) $\Leftrightarrow$ (b). Since the kernel of the morphism $r(C) \longrightarrow r(C/D)$ induced by $C \longrightarrow C/D$ is equal to $r(C) \cap D$, it is clear that (b) is equivalent to the left exactness of (a), considering

$$\begin{tikzcd}
 & C & \\
r(C) \arrow[ur] & & D \arrow[ul] \\
r(C)\cap D \arrow[u] \arrow[urr] \arrow[r, phantom, "="] & & r(D) \arrow[u] \arrow[ull]
\end{tikzcd}$$

(b) $\Rightarrow$ (c). By applying (b) to $r(C) \longrightarrow C$ one sees that $r$ is idempotent. It is obvious that $\mathcal{T}_r$ is closed under subobjects.

(c) $\Rightarrow$ (b) The inclusions $r(D) \leq r(C) \cap D \leq D$ are trivial. On the other hand, $r(C) \cap D$ belongs to $\mathcal{T}_r$ as a subobject of $r(C)$, and $r$ idempotent implies $r(C) \cap D = r(D)$, since $r(r(C) \cap D) \leq rr(C) \cap r(D)$ $= C \cap r(D) = r(D)$. $\square$

A pretorsion class is called __hereditary__ if it is closed under subobjects.

COROLLARY 1.8 There is a bijective correspondence between isomorphism classes of left-exact preradicals and hereditary pretorsion classes.

EXAMPLES 2. __S-torsion and S-divisibility__ Let $S$ be a right denominator set in a ring $A$. For every module $M$ we let $t(M)$ be the $S$-torsion submodule of $M$, while $d(M)$ denotes the largest $S$-divisible submodule of $M$ (which is easily seen to exist). Both $t$ and $d$ are idempotent radicals. Furthermore $t$ is left exact, which is not the case with $d$. It thus appears as if the preradicals connected to the customary torsion concepts should be left exact radicals (which are therefore often called "torsion radicals").

3. __The Pretorsion Ideal__ If $r$ is a preradical of $\underline{\text{Mod}}$-$A$, then $r(A)$ is a two sided ideal (it is by defn a right ideal, and each $a \in A$ determines the endomorphism $\alpha_a : A \longrightarrow A$, $x \longmapsto ax$, and since

$$\begin{tikzcd}
A \arrow[r, "\alpha_a"] & A \\
r(A) \arrow[u] \arrow[r] & r(A) \arrow[u]
\end{tikzcd}$$

commutes, $r(A)$ is closed under $\alpha_a$ and is hence also a left ideal.

## 2. Torsion Theories

DEFINITION A torsion theory for $\mathcal{C}$ is a pair $(\mathcal{T}, \mathcal{F})$ of classes of objects of $\mathcal{C}$ such that

(i) $\mathrm{Hom}(T,F)=0$ for all $T\in\mathcal{T}$, $F\in\mathcal{F}$

(ii) If $\mathrm{Hom}(C,F)=0$ for all $F\in\mathcal{F}$, then $C\in\mathcal{T}$

(iii) If $\mathrm{Hom}(T,C)=0$ for all $T\in\mathcal{T}$, then $C\in\mathcal{F}$.

$\mathcal{T}$ is called a torsion class and its objects are torsion objects, while $\mathcal{F}$ is a torsion-free class consisting of torsion-free objects. Notice $\mathcal{T}\cap\mathcal{F}=\{0\}$.

Any given class $\mathcal{C}$ of objects generates a torsion theory in the following way:

$$\mathcal{F}=\{F \mid \mathrm{Hom}(C,F)=0 \text{ for all } C\in\mathcal{C}\}$$
$$\mathcal{T}=\{T \mid \mathrm{Hom}(T,F)=0 \text{ for all } F\in\mathcal{F}\}$$

Clearly this pair $(\mathcal{T},\mathcal{F})$ is a torsion theory, and $\mathcal{T}$ is the smallest torsion class containing $\mathcal{C}$. To verify this, let $(\mathcal{T}',\mathcal{F}')$ be another torsion theory on $\mathcal{C}$ with $\mathcal{C}\subseteq\mathcal{T}'$. Then clearly $\mathcal{F}'\subseteq\mathcal{F}$ and hence $\mathcal{T}\subseteq\mathcal{T}'$. Notice that in general for two torsion theories $(\mathcal{T},\mathcal{F})$ and $(\mathcal{T}',\mathcal{F}')$, $\mathcal{T}\subseteq\mathcal{T}'$ iff. $\mathcal{F}'\subseteq\mathcal{F}$, so enlarging the torsion objects shrinks the torsion-free objects and vice-versa. Dually, the class $\mathcal{C}$ cogenerates a torsion theory $(\mathcal{T},\mathcal{F})$ such that $\mathcal{F}$ is the smallest torsion-free class containing $\mathcal{C}$. Explicitly

$$\mathcal{T}=\{T \mid \mathrm{Hom}(T,C)=0 \text{ for all } C\in\mathcal{C}\}$$
$$\mathcal{F}=\{F \mid \mathrm{Hom}(T,F)=0 \text{ for all } T\in\mathcal{T}\}$$

PROPOSITION 2.1 The following properties of a class $\mathcal{T}$ of objects are equivalent:

(a) $\mathcal{T}$ is a torsion class for some torsion theory

(b) $\mathcal{T}$ is closed under quotient objects, coproducts and extensions

PROOF A class $\mathcal{C}$ is said to be "closed under extensions" if for every exact sequence $0\longrightarrow C'\longrightarrow C\longrightarrow C''\longrightarrow 0$ with $C'$ and $C''$ in $\mathcal{C}$, also $C\in\mathcal{C}$.

Suppose $(\mathcal{T},\mathcal{F})$ is a torsion theory. $\mathcal{T}$ is obviously closed under quotient objects, and it is closed under coproducts because $\mathrm{Hom}(\bigoplus T_i, F)\cong\prod \mathrm{Hom}(T_i,F)$. Let $0\longrightarrow C'\longrightarrow C\longrightarrow C''\longrightarrow 0$ be exact with $C'$ and $C''$ in $\mathcal{T}$. If $F$ is torsion-free and there is a morphism $\alpha: C\longrightarrow F$, then $\alpha'$ is zero on $C'$, so $\alpha$ factors through $C''$. But also $\mathrm{Hom}(C'',F)=0$, so $\alpha=0$. Hence $C\in\mathcal{T}$.

Conversely, assume that $\mathcal{T}$ is closed under quotient objects, coproducts and extensions. Let $(\mathcal{T}',\mathcal{F})$ be the torsion theory generated by $\mathcal{T}$. We want to show that $\mathcal{T}=\mathcal{T}'$, so suppose $\mathrm{Hom}(C,F)=0$ for all $F\in\mathcal{F}$. Since $\mathcal{T}$ is a pretorsion class, there is a largest subobject $T$ of $C$ belonging to $\mathcal{T}$. To show that $C\cong T$, it suffices to show that $C/T\in\mathcal{F}$ (then since $\mathcal{T}$ is closed under isos, $C\in\mathcal{T}$). Suppose we have $\alpha: T''\longrightarrow C/T$ for $T''\in\mathcal{T}$. The image of $\alpha$ also belongs to $\mathcal{T}$ and if $\alpha\neq 0$ then we would get a subobject of $C$ which strictly contains $T$ and belongs to $\mathcal{T}$, since $\mathcal{T}$ is closed under extensions. This would contradict the maximality of $T$, and so we must have $\alpha=0$, and $C/T\in\mathcal{F}$. $\square$

By duality one also has

PROPOSITION 2.2 The following properties of a class $\mathcal{F}$ of objects are equivalent:

(a) $\mathcal{F}$ is a torsion-free class for some torsion theory

(b) $\mathcal{F}$ is closed under subobjects, products and extensions.

If $(\mathcal{T}, \mathcal{F})$ is a torsion theory, then $\mathcal{T}$ is in particular a pretorsion class, so every object $C$ contains a largest subobject $t(C)$ belonginging to $\mathcal{T}$, called the <u>torsion subobject</u> of $C$. An object $C$ is torsion-free iff. $t(C)=0$, because $C \in \mathcal{F}$ iff. $Hom(T, C) = 0$ for all $T \in \mathcal{T}$. The idempotent preradical $t$ is actually a radical, as is easily seen from the fact that $\mathcal{T}$ is closed under extensions.

$$\begin{tikzcd}
t(C) \arrow[r] \arrow[dr] & C \arrow[r] & C/t(C) \\
& (t:t)(C) \arrow[u] \arrow[r] & t(C/t(C)) \arrow[u]
\end{tikzcd}$$

hence as by def$^N$ $t(C) \leq (t:t)(C)$ and since $t(C)$ is the largest subobject in $\mathcal{T}$, $(t:t)(C) \leq t(C)$, so $t(C) = (t:t)(C)$ (we have to use the fact that $\mathcal{T}$ is closed under extensions to show $(t:t)(C) \in \mathcal{T}$). Conversely, if $t$ is an idempotent radical of $\mathcal{C}$, then one obtains a torsion theory $(\mathcal{T}_t, \mathcal{F}_t)$ with

$$\mathcal{T}_t = \{ C / t(C) = C \}, \qquad \mathcal{F}_t = \{ C / t(C) = 0 \}$$

we know that $Hom(T, F) = 0$ for $T \in \mathcal{T}_t$, $F \in \mathcal{F}_t$ by Cor 1.3. If $C \in \mathcal{C}$ and $Hom(C, F) = 0$ for all $F \in \mathcal{F}_t$, then in particular since $t$ is radical, $Hom(C, C/t(C)) = 0$, so $t(C) = C$ and $C \in \mathcal{T}_t$. Conversely if $Hom(T, C) = 0$ for all $T \in \mathcal{T}_t$, then since $t$ is idempotent, $Hom(t(C), C) = 0$, so $t(C) = 0$ and $C \in \mathcal{F}_t$. Hence $(\mathcal{T}_t, \mathcal{F}_t)$ is a torsion theory.

<u>PROPOSITION 2.3</u> There is a bijective correspondence between torsion theories and isomorphism classes of idempotent radicals.

<u>PROOF</u> Prop 2.1 shows that any torsion theory can be recovered from the class of torsion objects, (since $\mathcal{T} = \mathcal{T}' \Leftrightarrow \mathcal{F} = \mathcal{F}'$) and by Prop 1.4 we are done. □

<u>PROPOSITION 2.5</u> Let $\mathcal{C}$ be a class of objects closed under quotient objects. The torsion class generated by $\mathcal{C}$ consists of all objects $C$ such that each non-zero quotient object of $C$ has a non-zero subobject in $\mathcal{C}$.

<u>PROOF</u> Let $(\mathcal{T}, \mathcal{F})$ be the torsion class generated by $\mathcal{C}$. Since $\mathcal{C}$ is closed under quotients, an object belongs to $\mathcal{F}$ iff. it has no non-zero subobjects in $\mathcal{C}$. Therefore the assertion is that an object $C$ belongs to $\mathcal{T}$ iff. $C$ has no non-zero quotient object in $\mathcal{F}$, and this is an obvious property of a torsion theory $(\mathcal{T}, \mathcal{F})$. □

<u>EXAMPLES</u> 3. <u>Divisible Objects</u> Let $K$ be an object of $\mathcal{C}$. The torsion theory $(\mathcal{D}, \mathcal{R})$ generated by $K$ may be described by means of Prop 2.5 as

$$\mathcal{D} = \{ C \mid Hom(K, C') \neq 0 \text{ for every non-zero quotient } C' \text{ of } C \}$$
$$\mathcal{R} = \{ C \mid Hom(K, C) = 0 \}$$

The objects in $\mathcal{D}$ are called <u>K-divisible</u>, and those in $\mathcal{R}$ are <u>K-reduced</u>. Each object $C$ contains a largest K-divisible object $d(C)$. We may also define a preradical $q$ by

$$q(C) = \sum Im\,\alpha \qquad \alpha \text{ ranging over } Hom(K, C)$$

Clearly $q$ is idempotent.

Of particular interest is the case where $K = E(A)$ in <u>Mod-A</u>. Every injective module is $E(A)$-divisible, and in fact the $E(A)$-divisible modules constitute the torsion class generated by the injective modules. If $(\mathcal{T}, \mathcal{F})$ denotes this torsion theory, clearly $\mathcal{T} \subseteq \mathcal{D}$ by the previous comment, so to show $\mathcal{T} = \mathcal{D}$ we need only show $\mathcal{F} \subseteq \mathcal{R}$. But this is trivial since $E(A)$ is injective.

## 3. HEREDITARY TORSION THEORIES

At this stage we have to impose on $\mathcal{C}$ conditions which are not of a self-dual nature, e.g. that $\mathcal{C}$ is a Grothendieck category with enough injectives. For the sake of simplicity we henceforth assume $\mathcal{C}$ to be a module category <u>Mod</u>$-A$ for a ring $A$. A torsion theory is called <u>hereditary</u> if $\mathcal{T}$ is hereditary, i.e. $\mathcal{T}$ is closed under submodules. From Prop 1.7 we recall that this occurs iff. the associated radical $t$ is left exact. Combining Cor 1.8 and Prop 2.3 we get

<u>PROPOSITION 3.1</u> *There is a bijective correspondence between hereditary torsion theories and left exact radicals.*

<u>PROPOSITION 3.2</u> *A torsion theory $(\mathcal{T}, \mathcal{F})$ is hereditary if and only if $\mathcal{F}$ is closed under injective envelopes.*

<u>PROOF</u> If $t$ is left exact and $F \in \mathcal{F}$, then $t(E(F)) \cap F = t(F) = 0$, which implies that $E(F) \in \mathcal{F}$ since $F$ is essential in $E(F)$. Suppose conversely that $\mathcal{F}$ is closed under injective envelopes. If $T \in \mathcal{T}$ and $C \leq T$ then there is a morphism $\beta : T \longrightarrow E(C/t(C))$ such that

$$\begin{tikzcd} C \arrow[r] \arrow[d, "\alpha"'] & T \arrow[d, "\beta"] \\ C/t(C) \arrow[r] & E(C/t(C)) \end{tikzcd}$$

commutes. But $E(C/t(C))$ is torsion-free, so $\beta = 0$. This implies $\alpha = 0$ and hence $C = t(C) \in \mathcal{T}$. $\square$

<u>PROPOSITION 3.3</u> *Let $\mathcal{C}$ be a class of objects closed under subobjects and quotient objects. The torsion theory generated by $\mathcal{C}$ is hereditary.*

<u>PROOF</u> We show that the class of torsion-free objects is closed under injective envelopes. Suppose $F$ is torsion free and there exists a non-zero $\alpha : C \longrightarrow E(F)$ with $C \in \mathcal{C}$. Then $\operatorname{Im}\alpha \in \mathcal{C}$ and $F \cap \operatorname{Im}\alpha$ is a nonzero subobject of $F$ belonging to $\mathcal{C}$, a contradiction. $\square$

<u>PROPOSITION 3.6</u> *A hereditary torsion theory for <u>Mod</u>$-A$ is generated by the family of those cyclic modules $A/\mathfrak{a}$ which are torsion modules.*

<u>PROOF</u> A module $M$ is torsion iff. every cyclic submodule is torsion. Clearly if $\mathcal{T}'$ denotes the torsion class generated by the cyclic torsion modules, then $\mathcal{T}' \subseteq \mathcal{T}$ where $\mathcal{T}$ is the original torsion class. But if $M \in \mathcal{T}$ is torsion, then as the sum of all its cyclic torsion submodules, $M \in \mathcal{T}'$, as required. $\square$

A hereditary torsion theory is thus uniquely determined by the family of right ideals $\mathfrak{a}$ for which $A/\mathfrak{a}$ is a torsion module. (notice that the list of ideals is not uniquely determined, since we may have $\mathfrak{a} \neq \mathfrak{b}$ but $A/\mathfrak{a} \cong A/\mathfrak{b}$). These families of right ideals will be characterised in §5. A kind of dual statement to Prop 3.6 is:

<u>PROPOSITION 3.7</u> *A torsion theory is hereditary if and only if it can be cogenerated by an injective module.*

<u>PROOF</u> Let $E$ be an injective module and put $\mathcal{T} = \{M \mid \operatorname{Hom}(M, E) = 0\}$. If $M \in \mathcal{T}$ and $L$ is a submodule of $M$ with a nonzero morphism $L \longrightarrow E$, call it $\alpha$, then $\alpha$ extends to a morphism $M \longrightarrow E$, which is impossible. Hence $L \in \mathcal{T}$, and the torsion theory cogenerated by $E$ is hereditary.

Conversely, assume that $(\mathcal{T}, \mathcal{F})$ is a hereditary torsion theory. Put $E = \prod E(A/\mathfrak{a})$ with the product taken over all right ideals $\mathfrak{a}$ such that $A/\mathfrak{a} \in \mathcal{F}$. Then $E$ is a torsion-free module, so $\operatorname{Hom}(M, E) = 0$ for all $M \in \mathcal{T}$. On the other hand, if $M \notin \mathcal{T}$, then there exists a cyclic submodule $C$ of $M$ with a non-zero morphism $\alpha : C \longrightarrow F$ for some $F \in \mathcal{F}$. The image of $\alpha$ is cyclic torsion-free, so $\alpha$ induces a morphism $C \longrightarrow E$ which can be extended to a nonzero map $M \longrightarrow E$. We have thus shown that $M \in \mathcal{T}$ iff. $\operatorname{Hom}(M, E) = 0$, and this means that $E$ cogenerates the torsion theory.

When applying this result to an injective module of the form $E(M)$, it is useful to have available the following easily verified fact:

<u>LEMMA 3.8</u> If $L$ and $M$ are modules, then $\mathrm{Hom}(L, E(M)) = 0$ iff. $\mathrm{Hom}(C, M) = 0$ for every cyclic submodule $C$ of $L$.

<u>PROPOSITION 3.9</u> Consider a hereditary torsion theory cogenerated by the injective module $E$. A module $M$ is torsion free iff. it is a submodule of a direct product of copies of $E$.

<u>PROOF</u> Every submodule of a product $E^I$ is of course torsion-free. Conversely, if $M$ is torsion-free and $0 \neq x \in M$, then $xA$ is not a torsion module, so $\mathrm{Hom}(xA, E) \neq 0$. Since $E$ is injective, this means that for every non-zero $x \in M$ there is $\mu : M \longrightarrow E$ such that $\mu(x) \neq 0$. If one defines $\eta : M \longrightarrow E^I$, where $I = \mathrm{Hom}(M, E)$, as $\eta(x) = (\mu(x))_{\mu \in I}$, then $\eta$ is a monomorphism. □

<u>EXAMPLES</u>

1. <u>Injective Cogenerator</u> An injective cogenerator for <u>Mod</u>-$A$ is the same as an injective module cogenerating the torsion theory $(0, \mathcal{F})$ with $\mathcal{F}$ consisting of all $A$-modules.

2. <u>S-torsion</u> If $S$ is a right denominator set in $A$, then the $S$-torsion and $S$-torsion-free modules, as defined in Chap I, form a hereditary torsion theory.

3. <u>Commutative Localisation</u> Let $A$ be a commutative ring and $\mathfrak{p}$ a prime ideal of $A$. Then if $S = \{a \in A \mid a \notin \mathfrak{p}\}$, then the $S$-torsion theory is cogenerated by $E(A/\mathfrak{p})$. In fact, $0 \neq x \in A/\mathfrak{p}$ implies that $\mathrm{Ann}(x) = \mathfrak{p}$, so $A/\mathfrak{p}$ is $S$-torsion-free. On the other hand, if $\mathrm{Hom}(M, E(A/\mathfrak{p})) = 0$, then $\mathrm{Ann}(x) \nsubseteq \mathfrak{p}$ holds for every nonzero $x \in M$, so $M$ is an $S$-torsion-module.

By Corollary 1.8 there corresponds to $\mathcal{F}$ a left exact preradical $t$, defined as $t(M)$ = union of all the $\mathcal{F}$-discrete submodules of $M$. Clearly

$$t(M) = \{x \in M \mid Ann(x) \in \mathcal{F}\}$$

which is a submodule since $Ann(x+y) \supseteq Ann(x) \cap Ann(y)$ and for $a \in A$, $Ann(x \cdot a) = (Ann(x) : a)$. It will be convenient to call $t(M)$ the $\mathcal{F}$-pretorsion submodule of $M$. We will also use alternatingly the terms "$\mathcal{F}$-discrete module" and "$\mathcal{F}$-pretorsion module". Corollary 1.8 can now be completed to:

PROPOSITION 7.2 There is a bijective correspondence between

(1) Right linear topologies on $A$

(2) Hereditary pretorsion classes of $A$-modules

(3) Isomorphism classes of left exact preradicals of Mod–$A$

PROOF (We tend to identify a right linear topology with its collection $\mathcal{F}$ of open right ideals). To each linear topology $\mathcal{F}$ we have associated a pretorsion class $\mathcal{C} = \{M \mid Ann(x) \in \mathcal{F} \ \forall x \in M\}$. Conversely, if $\mathcal{C}$ is a hereditary pretorsion class, then we let $\mathcal{F}$ be the family of right ideals $\mathfrak{a}$ for which $A/\mathfrak{a} \in \mathcal{C}$. This family $\mathcal{F}$ satisfies T1 because $\mathcal{C}$ is closed under quotients, T2 because $A/\mathfrak{a} \cap \mathfrak{b}$ is a submodule of $A/\mathfrak{a} \oplus A/\mathfrak{b}$, and satisfies T3 because if $\mathfrak{a} \in \mathcal{F}$ and $a \in A$, then left multiplication by $a$ induces an exact sequence

$$0 \longrightarrow (\mathfrak{a} : a) \longrightarrow A \longrightarrow A/\mathfrak{a}$$

more precisely, it induces $A \longrightarrow A/\mathfrak{a}$ by $x \mapsto ax + \mathfrak{a}$, which has kernel $(\mathfrak{a} : a)$, and this induces $A/(\mathfrak{a}:a) \longrightarrow A/\mathfrak{a}$, $x + (\mathfrak{a} : a) \mapsto ax + \mathfrak{a}$, which is clearly monic. This shows that $A/(\mathfrak{a}:a) \subseteq A/\mathfrak{a}$, Therefore $\mathcal{F}$ defines a right linear topology on $A$.

It remains to verify that we have obtained a bijective correspondence (1) $\longleftrightarrow$ (2). Starting with a linear topology with the set $\mathcal{F}$ of open right ideals, we get $\mathcal{C} = \{M \mid Ann(x) \in \mathcal{F}$ for all $x \in M\}$, and then $\{\mathfrak{a} \mid A/\mathfrak{a} \in \mathcal{C}\} = \{\mathfrak{a} \mid (\mathfrak{a} : a) \in \mathcal{F}$ for all $a \in A\} = \mathcal{F}$ by T3. On the other hand, if we start with the class $\mathcal{C}$, we first get $\mathcal{F} = \{\mathfrak{a} \mid A/\mathfrak{a} \in \mathcal{C}\}$ and then obtain $\{M \mid Ann(x) \in \mathcal{F}$ for all $x \in M\} = \{M \mid$ each cyclic submodule $\in \mathcal{C}\}$ $= \mathcal{C}$ because of the closure properties of $\mathcal{C}$. □

If $A$ is an arbitrary ring and $\mathcal{F}$ is a set of right ideals of $A$ satisfying T1–3, we will by abuse of language call $\mathcal{F}$ a (right) topology. The corresponding linear topology on $A$ is called the $\mathcal{F}$-topology on $A$, in accordance with terminology already introduced. By a basis for the topology $\mathcal{F}$ we mean a subset $\mathcal{B}$ of $\mathcal{F}$ such that every right ideal in $\mathcal{F}$ contains some $\mathfrak{b} \in \mathcal{B}$.

EXAMPLES 1. Separated topologies A topological group is called separated if every point is closed. A topology $\mathcal{F}$ of right ideals is separated iff. $\bigcap_{\mathfrak{a} \in \mathcal{F}} \mathfrak{a} = (0)$. For, if $\mathcal{F}$ is separated and $0 \neq a \in \bigcap_{\mathfrak{a} \in \mathcal{F}} \mathfrak{a}$, then $A \setminus \{a\}$ cannot be open, because there is no $\mathfrak{a} \in \mathcal{F}$ s.t. $0 + \mathfrak{a} \subseteq A \setminus \{a\}$. Hence $\bigcap \mathfrak{a} = 0$. Conversely, if $\bigcap_{\mathfrak{a} \in \mathcal{F}} \mathfrak{a} = 0$ and $x \neq y$ then there is some $\mathfrak{a} \in \mathcal{F}$ with $x - y \notin \mathfrak{a}$. Hence $x \notin y + \mathfrak{a}$ and so $y + \mathfrak{a} \subseteq A \setminus \{x\}$. This shows that for all $x \in A$, $A \setminus \{x\}$ is open. Hence $A$ is separated.

2. The $\mathfrak{a}$-adic topology Let $\mathfrak{a}$ be a two-sided ideal in a ring $A$. The powers $\mathfrak{a}^n$ ($n \geq 1$) form a basis for a linear topology on $A$ – that is, $\mathcal{F} = \{\mathfrak{b} \subseteq A \mid \mathfrak{a}^n \subseteq \mathfrak{b}$ for some $n \geq 1\}$. T1 is trivial, and since $\mathfrak{a}^{n+m} \subseteq \mathfrak{a}^n \cap \mathfrak{a}^m$, so is T2. If $\mathfrak{a}^n \subseteq \mathfrak{b}$ and $a \in A$, then $(\mathfrak{b} : a) = \{x \in A \mid ax \in \mathfrak{b}\} \supseteq \{x \in A \mid ax \in \mathfrak{a}^n\} \supseteq \mathfrak{a}^n$. This topology is usually called the $\mathfrak{a}$-adic topology. The $\mathfrak{a}$-adic topology is separated iff.

$$\bigcap_{n=1}^{\infty} \mathfrak{a}^n = 0$$

# 5. GABRIEL TOPOLOGIES

A hereditary torsion theory corresponds to a linear topology for which the class of discrete modules is closed under extensions. In order to characterise such topologies, we introduce a further axiom:

T4. If $\mathfrak{a}$ is a right ideal and there is $\mathfrak{b} \in \mathcal{F}$ s.t. $(\mathfrak{a} : b) \in \mathcal{F}$ for every $b \in \mathfrak{b}$, then $\mathfrak{a} \in \mathcal{F}$.

A family $\mathcal{F}$ of right ideals of $A$ satisfying axioms T1 – 4 is a (right) Gabriel topology on $A$. We can now state the main result of this chapter:

THEOREM 5.1 There is a bijective correspondence between

(1) Right Gabriel topologies on $A$
(2) Hereditary torsion theories for $A$
(3) Left exact radicals of $A$ (that is, iso. classes)

PROOF We have already established the correspondence (2) $\longleftrightarrow$ (3) in Prop 3.1. Suppose $\mathcal{F}$ is a Gabriel top, and let $0 \to L \to M \to N \to 0$ be an exact sequence of modules s.t. $L$ and $N$ are $\mathcal{F}$-discrete. For each $x \in M$, put $\mathfrak{b} = Ann(\bar{x})$, where $\bar{x}$ is the image of $x$ in $N$. Then $\mathfrak{b} \in \mathcal{F}$, and for each $b \in \mathfrak{b}$ we have $xb \in L$ so $Ann(xb) \in \mathcal{F}$. Since $Ann(xb) = (Ann(x) : b)$, the axiom T4 then implies that $Ann(x) \in \mathcal{F}$. Hence $M$ is $\mathcal{F}$-discrete. The class of $\mathcal{F}$-discrete modules is thus closed under extensions, and consequently it is a hereditary torsion class.

On the other hand, if $\mathcal{T}$ is a hereditary torsion class, then the corresponding topology $\mathcal{F} = \{\mathfrak{a} \mid A/\mathfrak{a} \in \mathcal{T}\}$ satisfies T4. For if $\mathfrak{a}$ is a right ideal and $(\mathfrak{a} : b) \in \mathcal{F}$ for all $b \in \mathfrak{b}$ for some $\mathfrak{b} \in \mathcal{F}$, we consider the exact sequence

$$0 \longrightarrow \mathfrak{b}/\mathfrak{a} \cap \mathfrak{b} \longrightarrow A/\mathfrak{a} \longrightarrow A/\mathfrak{a}+\mathfrak{b} \longrightarrow 0$$

where $A/\mathfrak{a}+\mathfrak{b} \in \mathcal{T}$ since it is a quotient of $A/\mathfrak{b} \in \mathcal{T}$ and also $\mathfrak{b}/\mathfrak{a}\cap\mathfrak{b} \in \mathcal{T}$ since $b \in \mathfrak{b}$ implies

$$((\mathfrak{a} \cap \mathfrak{b}) : b) = (\mathfrak{a} : b) \in \mathcal{F}$$

$\leftarrow$ Hence $\mathfrak{b}/\mathfrak{a}\cap\mathfrak{b}$ is $\mathcal{F}$-torsion, and use Prop 4.2 to see $\mathfrak{b}/\mathfrak{a}\cap\mathfrak{b} \in \mathcal{T}$

Since $\mathcal{T}$ is closed under extensions, it follows that $A/\mathfrak{a} \in \mathcal{T}$ and hence $\mathfrak{a} \in \mathcal{F}$. □

Thus if $\mathcal{F}$ is a Gabriel topology on $A$, the corresponding hereditary torsion class consists of all modules which are discrete in their $\mathcal{F}$-topology, or equivalently, for which all elements have annihilators in $\mathcal{F}$. These modules will be called $\mathcal{F}$-torsion modules. The task of checking axioms T1 – 4 is simplified by the fact that T1 and 2 actually follow from T3, T4.

LEMMA 5.2 If $\mathcal{F}$ is a nonempty set of right ideals of $A$ satisfying T3 and T4, then $\mathcal{F}$ also satisfies T1 and T2.

PROOF T1: We first note that T3, together with the fact that $\mathcal{F}$ is non-empty, implies that $A \in \mathcal{F}$. Then suppose $\mathfrak{a} \in \mathcal{F}$ and $\mathfrak{b} \supseteq \mathfrak{a}$. For each $a \in \mathfrak{a}$ we have $(\mathfrak{b} : a) = A \in \mathcal{F}$, so $\mathfrak{b} \in \mathcal{F}$ by T4.
T2: Suppose $\mathfrak{a}$ and $\mathfrak{b}$ belong to $\mathcal{F}$. If $b \in \mathfrak{b}$, then $((\mathfrak{a} \cap \mathfrak{b}) : b) = (\mathfrak{a} : b) \cap (\mathfrak{b} : b) = (\mathfrak{a} : b) \in \mathcal{F}$ by T3, so $\mathfrak{a} \cap \mathfrak{b} \in \mathcal{F}$ by T4. □

We also remark that a Gabriel topology is closed under products:

LEMMA 5.3 Let $\mathcal{F}$ be a Gabriel topology. If $\mathfrak{a}$ and $\mathfrak{b}$ belong to $\mathcal{F}$, then $\mathfrak{a}\mathfrak{b} \in \mathcal{F}$.

PROOF For each $a \in \mathfrak{a}$ we have $(\mathfrak{a}\mathfrak{b} : a) \supseteq \mathfrak{b}$, so $\mathfrak{a}\mathfrak{b} \in \mathcal{F}$ by T1 and T4. □

If $\mathcal{F}_1$ and $\mathcal{F}_2$ are topologies on $A$, we say that $\mathcal{F}_1$ is weaker than $\mathcal{F}_2$ (and $\mathcal{F}_2$ is stronger than $\mathcal{F}_1$) if $\mathcal{F}_1 \subseteq \mathcal{F}_2$. Since it is clear that any intersection of topologies is a topology, the topologies on $A$ form a complete lattice Top $A$. ?
Since also every intersection of Gabriel topologies is a Gabriel topology,

We recall that every hereditary torsion theory may be cogenerated by an injective module (Prop 3.7). If this injective module is given as the injective envelope of some module $M$, the corresponding Gabriel topology can be described as follows:

PROPOSITION 5.5 Let $\mathcal{F}$ be the Gabriel topology corresponding to the torsion theory cogenerated by $E(M)$. Then $\mathfrak{a} \in \mathcal{F}$ if and only if $x(\mathfrak{a}:a) \neq 0$ for every $a \in A$ and nonzero $x \in M$.

PROOF From Lemma 3.8 we obtain that $\mathfrak{a} \in \mathcal{F}$ if and only if $Hom(C,M)=0$ for every cyclic submodule $C$ of $A/\mathfrak{a}$. But the cyclic submodules of $A/\mathfrak{a}$ have the form $A/(\mathfrak{a}:a)$ for $a \in A$, since if $c+\mathfrak{a} \in A/\mathfrak{a}$, consider

$$A \longrightarrow A/\mathfrak{a} \qquad 1 \longmapsto c+\mathfrak{a}$$

whose kernel is $(\mathfrak{a}:c)$, Hence $A/(\mathfrak{a}:c) \cong (c+\mathfrak{a})$ as A-modules. A-module morphisms $Hom_A(A/(\mathfrak{a}:c), M)$ are in bijection with morphisms $\phi: A \longrightarrow M$ taking $(\mathfrak{a}:c)$ to $0$, which are precisely elements $x \in M$ with $x(\mathfrak{a}:c)=0$. Hence $\mathfrak{a} \in \mathcal{F}$ iff. $Hom(C,M)=0$ $C$ cyclic submodule $A/\mathfrak{a}$ iff. $Hom(A/(\mathfrak{a}:c), M)=0$ every $c \in A$ iff. $x(\mathfrak{a}:a) \neq 0$ for $x \neq 0$ in $M$, $a \in A$. □

PROPOSITION 5.6 The Gabriel topology corresponding to the torsion theory cogenerated by $E(M)$ is the strongest Gabriel topology for which $M$ is torsion-free.

PROOF If $\mathcal{F}_1, \mathcal{F}_2$ are Gabriel topologies and $\mathcal{T}_1, \mathcal{T}_2$ the corresponding hereditary torsion classes, then $\mathcal{F}_1 \subseteq \mathcal{F}_2 \Rightarrow \mathcal{T}_1 \subseteq \mathcal{T}_2$ since under $\mathcal{F}_2$ it is easier for an element to be torsion. Since enlarging the torsion class shrinks the torsion-free class, it makes sense to say a Gabriel topology is strongest s.t. $M$ is torsion-free. Indeed: let $\mathcal{F}$ be the Gabriel topology cogenerated by $E(M)$. Clearly since $E(M)$ is torsion-free, $M$ is also. Suppose $\mathcal{F} \subseteq \mathcal{F}'$ and $M$ is still-torsion-free in $\mathcal{F}'$. Since the torsion-free class of $\mathcal{F}'$ is closed under injective envelopes, $E(M)$ is $\mathcal{F}'$-torsion free – implying $\mathcal{F}=\mathcal{F}'$. Alternatively, if $\mathfrak{a} \in \mathcal{F}'$ but $\mathfrak{a} \notin \mathcal{F}$, then $\exists\, 0 \neq x \in M$ and $a \in A$ with $(\mathfrak{a}:a) \subseteq Ann(x)$. Hence $Ann(x) \in \mathcal{F}'$, contradicting the fact that $M$ is $\mathcal{F}'$-torsion-free. □

# 6. Examples of Gabriel Topologies

1. 1-topologies A 1-topology on $A$ is a nonempty Gabriel topology with a basis consisting of principal right ideals. That is, for $\mathfrak{a} \in \mathcal{F}$ there is $x \in A$ with $(x) \in \mathcal{F}$ and $x \in \mathfrak{a}$. A 1-topology is determined by the set

$$\Sigma(\mathcal{F}) = \{ s \in A \mid sA \in \mathcal{F} \}$$

PROPOSITION 6.1 The map $\mathcal{F} \longmapsto \Sigma(\mathcal{F})$ defines a bijective correspondence between 1-topologies $\mathcal{F}$ on $A$ and multiplicatively closed subsets of $A$ satisfying:

S0. If $ab \in S$, then $a \in S$

S1. If $s \in S$ and $a \in A$, then $\exists t \in S, b \in A$ s.t. $sb = at$

PROOF Suppose $\mathcal{F}$ is a 1-topology. Then $\Sigma(\mathcal{F})$ is multiplicatively closed, for if $s, t \in \Sigma(\mathcal{F})$, then $(stA : sa) \supseteq (tA : a)$ for every $a \in A$ (and $A \in \mathcal{F}$ since $\mathcal{F}$ is nonempty), and $(tA : a) \in \mathcal{F}$ by T3, so $stA \in \mathcal{F}$ by T4. The property S0 for $\Sigma(\mathcal{F})$ is immediate from T1, and also S1 is clear, because T3 implies $(sA : a) \supseteq tA$ for some $tA \in \mathcal{F}$.

Conversely, let $S$ be a multiplicatively closed set satisfying S1, and put $\mathcal{F} = \{ \mathfrak{a} \mid \mathfrak{a} \cap S \neq \emptyset \}$. To check T3, suppose $\mathfrak{a} \in \mathcal{F}$, with $s \in \mathfrak{a} \cap S$. Let $A \ni a$ and find $t \in S, b \in A$ s.t. $sb = at$. Then $at \in sA \subseteq \mathfrak{a}$, and hence $t \in (\mathfrak{a} : a)$ – so $(\mathfrak{a} : a) \cap S \neq \emptyset$ and $(\mathfrak{a} : a) \in \mathcal{F}$. For T4, suppose $\mathfrak{a} \in \mathcal{F}$ and $\mathfrak{b}$ is a right ideal s.t. $(\mathfrak{b} : a) \in \mathcal{F}$ for $a \in \mathfrak{a}$. Let $s \in \mathfrak{a} \cap S$. – thus in particular $(\mathfrak{b} : s) \in \mathcal{F}$ – so there is $t \in S$ with $st \in \mathfrak{b}$. Since $st \in S$, $\mathfrak{b} \in \mathcal{F}$. Hence $\mathcal{F}$ is a Gabriel topology.

Clearly $\mathcal{F} = \{ \mathfrak{a} \mid \mathfrak{a} \cap \Sigma(\mathcal{F}) \neq \emptyset \}$, so this assignment $\mathcal{F} \longmapsto \Sigma(\mathcal{F})$ is injective. It will be bijective provided we restrict the codomain to multiplicatively closed sets satisfying S0, S1, since if $S$ is such a set, $\mathcal{F}$ the associated topology

$$\begin{aligned} \Sigma(\mathcal{F}) &= \{ t \in A \mid tA \cap S \neq \emptyset \} \\ &= \{ t \in A \mid ta \in S \text{ for some } a \in A \} \\ &= S \qquad (\text{by S0}). \square \end{aligned}$$

For a 1-topology $\mathcal{F}$, the $\mathcal{F}$-torsion modules $M$ are those s.t. $\forall x \in M$, $\mathrm{Ann}(x) \in \mathcal{F}$, so iff. $xs = 0$ for some $s \in \Sigma(\mathcal{F})$.

Combining 6.13(c) and 6.14(ii) we obtain:

<u>COROLLARY 6.15</u> If $\mathcal{F}$ is a Gabriel topology and $\mathcal{F}$ has a basis of finitely generated ideals, then $\mathcal{F} = \mathcal{F}_{\mathcal{P}}$ for certain $\mathcal{P} \subseteq \operatorname{Spec} A$.

So for a commutative noetherian ring $A$, all Gabriel topologies are of the form $\mathcal{F}_{\mathcal{P}}$ with $\mathcal{P} \subseteq \operatorname{Spec} A$.

<u>NOTE</u> For a prime ideal $\mathfrak{p}$, $\mathcal{F}_{\mathfrak{p}}$ is the 1-topology with $\Sigma(\mathcal{F}_{\mathfrak{p}}) = A \setminus \mathfrak{p}$. For $\mathcal{P} \subseteq \operatorname{Spec} A$, $\mathcal{F}_{\mathcal{P}}$ has

$$\begin{aligned}\Sigma(\mathcal{F}_{\mathcal{P}}) &= \{ s \in A \mid V(sA) \cap \mathcal{P} = \phi \} \\ &= \{ s \in A \mid s \notin \mathfrak{p} \text{ for all } \mathfrak{p} \in \mathcal{P} \} \\ &= A \setminus \textstyle\bigcup_{\mathcal{P}} \mathfrak{p}\end{aligned}$$

But nonetheless, $\mathcal{F}_{\mathcal{P}}$ may not be a 1-topology, because that requires $\mathfrak{a} \nsubseteq \mathfrak{p}$, all $\mathfrak{p} \in \mathcal{P}$, to imply $\exists s \in A$, $s \notin \bigcup \mathfrak{p}$, which is not necessarily the case. (see Ex. 23). (works for finite $\mathcal{P}$, since $\mathfrak{a} \subseteq \bigcup_{i=1}^{n} \mathfrak{p}_i \Rightarrow \mathfrak{a} \subseteq \mathfrak{p}_i$, some $i$)

# 6. Commutative Rings

Let $A$ be a commutative ring. Denote by $\operatorname{Spec} A$ the set of all prime ideals of $A$. For each ideal $\mathfrak{a}$, put

$$V(\mathfrak{a}) = \{\mathfrak{p} \mid \mathfrak{a} \subseteq \mathfrak{p}\}$$

If $\mathfrak{p}$ is a prime ideal, its complement in $A$ is a multiplicatively closed set. For a module $M$ one denotes the module of fractions $M[S^{-1}]$ by $M_{\mathfrak{p}}$. There is a corresponding Gabriel Topology

$$\mathcal{F}_{\mathfrak{p}} = \{\mathfrak{a} \mid \mathfrak{p} \notin V(\mathfrak{a})\}$$

This follows because for a right ideal $\mathfrak{a}$, $A/\mathfrak{a}$ is $S$-torsion iff. $1+\mathfrak{a} \in A/\mathfrak{a}$ is $S$-torsion, which is iff. $\mathfrak{a} \cap S \neq \phi$, which is iff. $\mathfrak{a} \not\subseteq \mathfrak{p}$. The $\mathcal{F}_{\mathfrak{p}}$-torsion modules $M$ are characterised by the property that $M_{\mathfrak{p}} = 0$.

More generally, let $\mathcal{P}$ be a subset of $\operatorname{Spec} A$. To $\mathcal{P}$ we associate a Gabriel topology

$$\begin{aligned} \mathcal{F}_{\mathcal{P}} &= \{\mathfrak{a} \mid V(\mathfrak{a}) \cap \mathcal{P} = \phi\} \\ &= \bigcap_{\mathfrak{p} \in \mathcal{P}} \mathcal{F}_{\mathfrak{p}} \end{aligned}$$

The corresponding torsion class consists of all modules $M$ with $M_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \in \mathcal{P}$.

Conversely, to each Gabriel topology $\mathcal{F}$ on $A$ we associate

$$D(\mathcal{F}) = \{\mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{p} \notin \mathcal{F}\} \subseteq \operatorname{Spec} A$$

Then $\mathcal{F}_{D(\mathcal{F})} = \{\mathfrak{a} \mid V(\mathfrak{a}) \cap D(\mathcal{F}) = \phi\} = \{\mathfrak{a} \mid V(\mathfrak{a}) \subseteq \mathcal{F}\}$. Also, $D(\mathcal{F}_{\mathcal{P}}) \supseteq \mathcal{P}$ for any $\mathcal{P} \subseteq \operatorname{Spec} A$.
(First claim implies $\mathcal{F}_{D(\mathcal{F})} \supseteq \mathcal{F}$ since if $\mathfrak{a} \in \mathcal{F}$, certainly $V(\mathfrak{a}) \subseteq \mathcal{F}$).

<u>PROPOSITION 6.13</u> The following properties of a Gabriel topology $\mathcal{F}$ on a commutative ring $A$ are equivalent

(a) $\mathcal{F}$ is $\mathcal{F}_{\mathcal{P}}$ for some $\mathcal{P} \subseteq \operatorname{Spec} A$
(b) $\mathcal{F} = \mathcal{F}_{D(\mathcal{F})}$
(c) For every right ideal $\mathfrak{a}$, with $\mathfrak{a} \notin \mathcal{F}$, there is $\mathfrak{p} \in V(\mathfrak{a})$ s.t. $\mathfrak{p} \notin \mathcal{F}$.

<u>PROOF</u> (a) $\Rightarrow$ (b). Suppose $\mathcal{F} = \mathcal{F}_{\mathcal{P}}$. If $\mathfrak{a} \in \mathcal{F}_{D(\mathcal{F})}$ then $V(\mathfrak{a}) \subseteq \mathcal{F} = \mathcal{F}_{\mathcal{P}}$. Suppose $\mathfrak{q} \in V(\mathfrak{a}) \cap \mathcal{P}$. Then $\mathfrak{q} \in \mathcal{F}_{\mathcal{P}}$, so $\mathfrak{q} \notin \mathcal{P}$ a contradiction. Hence $V(\mathfrak{a}) \cap \mathcal{P} = \phi$ and $\mathfrak{a} \in \mathcal{F}_{\mathcal{P}}$.

(b) $\Rightarrow$ (c) Suppose $\mathcal{F} = \mathcal{F}_{D(\mathcal{F})}$. If $\mathfrak{a} \notin \mathcal{F}$ then $\mathfrak{a} \notin \mathcal{F}_{D(\mathcal{F})}$ and hence $V(\mathfrak{a}) \not\subseteq \mathcal{F}$, so there is $\mathfrak{p} \in V(\mathfrak{a})$ with $\mathfrak{p} \notin \mathcal{F}$.

(c) $\Rightarrow$ (a) We claim $\mathcal{F} = \mathcal{F}_{\mathcal{P}}$ with $\mathcal{P} = D(\mathcal{F})$. If $\mathfrak{a} \in \mathcal{F}_{D(\mathcal{F})}$ then $V(\mathfrak{a}) \subseteq \mathcal{F}$. By (c), this implies $\mathfrak{a} \in \mathcal{F}$.

<u>LEMMA 6.14</u> Let $\mathcal{F}$ be a Gabriel topology on $A$. Then:

(i) If $\mathfrak{a}$ is an ideal which is maximal w.r.t. $\mathfrak{a} \notin \mathcal{F}$, then $\mathfrak{a}$ is prime
(ii) If $\mathcal{F}$ has a basis consisting of finitely generated ideals and $\mathfrak{a} \notin \mathcal{F}$, then there exists $\mathfrak{p} \in V(\mathfrak{a})$ s.t. $\mathfrak{p} \notin \mathcal{F}$.

<u>PROOF</u> (i) Suppose $a, b \in A \setminus \mathfrak{a}$. Then $\mathfrak{a} + Aa$ and $\mathfrak{a} + Ab$ must be members of $\mathcal{F}$, and also $(\mathfrak{a}+Aa)(\mathfrak{a}+Ab) \in \mathcal{F}$ by Lemma 5.3. But $(\mathfrak{a}+Aa)(\mathfrak{a}+Ab) \subseteq \mathfrak{a} + Aab$, and therefore $ab \notin \mathfrak{a}$.

(ii) Since $\mathfrak{a} \notin \mathcal{F}$, one can use Zorn's Lemma to find an ideal $\mathfrak{b} \supseteq \mathfrak{a}$ which is maximal w.r.t. $\mathfrak{b} \notin \mathcal{F}$. (This follows from the f.g basis assumption, since if $\mathfrak{b}_i$ is a chain and $\bigcup \mathfrak{b}_i \in \mathcal{F}$, then there is f.g. $\mathfrak{z} \in \mathcal{F}$ $\mathfrak{z} \subseteq \bigcup \mathfrak{b}_i$. Hence $\mathfrak{z} \subseteq \mathfrak{b}_j$ some $j$. Thus $\mathfrak{b}_j \in \mathcal{F}$, a contradiction). $\mathfrak{b}$ is a prime ideal by (i).

EXERCISES Ch VI Stenstrom

Q8 Let $(\mathcal{T},\mathcal{F})$ be a torsion theory. We already know (Props 2.1, 2.2) that (1), (2), (3) hold ((1) holds since clearly $0\in\mathcal{T}\cap\mathcal{F}$ and conversely if $C\in\mathcal{T}\cap\mathcal{F}$ then $\mathrm{End}\,C=0$, so $C=0$). For (4), let $C$ be an object, and $t$ the idempotent radical corresponding to $(\mathcal{T},\mathcal{F})$. Then $t(C)\in\mathcal{T}$, since $t$ is radical $t(C/t(C))=0$, so $C/t(C)\in\mathcal{F}$ and there is the exact sequence

$$0\longrightarrow t(C)\longrightarrow C\longrightarrow C/t(C)\longrightarrow 0$$

Conversely, suppose $\mathcal{T},\mathcal{F}$ are classes of objects of $\mathcal{C}$ which satisfy (1) — (4). Then we check the axioms of a torsion theory:

(i) Let $T\in\mathcal{T}$ and $F\in\mathcal{F}$ and let $\alpha: T\longrightarrow F$. Then $\mathrm{Im}\,\alpha$ is a quotient of $T$, hence by (2) belongs to $\mathcal{T}$, and it is also a subobject of $F$ and hence belongs to $\mathcal{F}$. By (1), $\mathrm{Im}\,\alpha=0$ so $\alpha=0$. Hence $\mathrm{Hom}(\mathcal{T},\mathcal{F})=0$.

(ii), (iii) If $X\in\mathcal{C}$ and $\mathrm{Hom}(X,F)=0\ \forall F\in\mathcal{F}$ (resp. $\mathrm{Hom}(T,X)=0$ all $T\in\mathcal{T}$) find $T\in\mathcal{T}$ and $F\in\mathcal{F}$ forming an exact sequence

$$0\longrightarrow T\longrightarrow X\longrightarrow F\longrightarrow 0$$

by (4). Then $X\longrightarrow F=0$ (resp $T\longrightarrow X=0$) so $T\longrightarrow X$ is epi (resp. $X\longrightarrow F$ is monic) so by (2), (3) it follows that $X\in\mathcal{T}$ (resp $X\in\mathcal{F}$).

Q10 Let $\mathcal{D}$ be a class of objects in $\mathcal{C}$. Let $T(\mathcal{D})$ be class of objects $T$ with the property that each non-zero quotient of $T$ has a nonzero subobject in $\mathcal{D}$. Then

(i) $T(\mathcal{D})$ is a torsion class.

<u>PROOF</u> By Prop 2.1 it suffices to show that $T(\mathcal{D})$ is closed under quotients, coproducts and extensions. Let $X\in T(\mathcal{D})$ and let $X\twoheadrightarrow Y$ be a quotient of $X$. If $Y$ has a nonzero quotient $Y\twoheadrightarrow Z$ then it is also a nonzero quotient of $X$, hence has a nonzero subobject in $\mathcal{D}$.

Suppose $X_i\in T(\mathcal{D})$ for $i\in I$ and there is a nonzero quotient $\oplus_i X_i \overset{y}{\twoheadrightarrow} Y$. Then some $yu_i\neq 0$ $u_i: X_i\longrightarrow\oplus X_i$. Form the epi-mono factorisation

$$\begin{tikzcd}
\oplus X_i \arrow[r, two heads] & Y \\
X_i \arrow[u, tail] \arrow[r, two heads] & Y' \arrow[u, tail]
\end{tikzcd}$$

Then $Y'\neq 0$ since $yu_i\neq 0$ hence $Y'$ has a nonzero subobject in $\mathcal{D}$, which is then a subobject of $Y$.

If finally $X, X'\in T(\mathcal{D})$ and $0\longrightarrow X\longrightarrow C\longrightarrow X'\longrightarrow 0$ is exact, form the pullback of any nonzero quotient $C\twoheadrightarrow Y$ along $X\longrightarrow C$:

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, tail] \arrow[d, two heads] & C \arrow[r] \arrow[d, two heads] & X' \arrow[r] & 0 \\
& Y' \arrow[r, tail] & Y & &
\end{tikzcd}$$

if $Y'=0$ then $C\twoheadrightarrow Y$ factors through $C\twoheadrightarrow X'$. Hence $Y$ is a nonzero quotient of $X'$, and thus has a nonzero subobject in $\mathcal{D}$. If $Y'\neq 0$ then it has such a subobject, which is then a subobject of $Y$.

(ii) Suppose that $\mathcal{T}$ is a torsion class containing $\mathcal{S}$ and let $X \in T(\mathcal{S})$. Suppose that $X \notin \mathcal{T}$ so that we have an exact sequence

$$0 \longrightarrow t(X) \longrightarrow X \longrightarrow X/t(X) \longrightarrow 0$$

where $X/t(X) \neq 0$. Since $X \in T(\mathcal{S})$, $X/t(X)$ has a nonzero subobject in $\mathcal{S} \subseteq \mathcal{T}$. But this contradicts the fact that since $t$ is a radical, $t(X/t(X)) = 0$. Hence $X \in \mathcal{T}$, as required.

# CHAPTER IX : RINGS AND MODULES OF QUOTIENTS

## 1. CONSTRUCTION OF MODULES OF QUOTIENTS (All Gabriel topologies in this section are nonempty)

Let $\mathcal{F}$ be a Gabriel topology of right ideals on the ring $A$. For each right module $M$ we will define its module of quotients with respect to $\mathcal{F}$. This we do in two steps. The first step we take is to define

$$M_{(\mathcal{F})} = \varinjlim \mathrm{Hom}_A(\mathfrak{a}, M) \qquad \mathfrak{a} \in \mathcal{F},$$

(in comparison to the nonadditive case, $M : R^{op} \longrightarrow \underline{Ab}$ is a presheaf, for a sieve $\mathfrak{a} \in \mathcal{F}$ $\mathrm{Hom}_A(\mathfrak{a}, M)$ is the collection of matching families. The above defines $M^+$). where the direct limit is taken over the downwards directed family $\mathcal{F}$ of right ideals. Every element in $M_{(\mathcal{F})}$ is thus represented by a homomorphism $\xi : \mathfrak{a} \longrightarrow M$ for some $\mathfrak{a} \in \mathcal{F}$, with the understanding that if $\eta : \mathfrak{b} \longrightarrow M$, $\xi = \eta$ iff. $\xi$ and $\eta$ coincide on some $\mathfrak{c} \in \mathcal{F}$ with $\mathfrak{c} \subseteq \mathfrak{a} \cap \mathfrak{b}$. We want to give $A_{(\mathcal{F})}$ the structure of a ring and $M_{(\mathcal{F})}$ that of a right $A_{(\mathcal{F})}$-module. For this we need:

<u>LEMMA 1.1</u> If $\mathfrak{a}, \mathfrak{b} \in \mathcal{F}$ and $\alpha : \mathfrak{a} \longrightarrow A$ is a homomorphism, then $\alpha^{-1}(\mathfrak{b}) \in \mathcal{F}$

<u>PROOF</u> For each $a \in \mathfrak{a}$ we have $(\alpha^{-1}(\mathfrak{b}) : a) = \{ b \mid \alpha(ab) \in \mathfrak{b} \} = (\mathfrak{b} : \alpha(a)) \in \mathcal{F}$ by T3, so $\alpha^{-1}(\mathfrak{b}) \in \mathcal{F}$ by T4. □

We define a pairing $M_{(\mathcal{F})} \times A_{(\mathcal{F})} \longrightarrow M_{(\mathcal{F})}$ as follows: suppose $x \in M_{(\mathcal{F})}$, $a \in A_{(\mathcal{F})}$ are represented by $\xi : \mathfrak{b} \longrightarrow M$ and $\alpha : \mathfrak{a} \longrightarrow A$; we then define $xa \in M_{(\mathcal{F})}$ to be represented by the composed map

$$\alpha^{-1}\mathfrak{b} \longrightarrow \mathfrak{b} \longrightarrow M$$

using Lemma 1.1. This is well-defined, since if $\xi' : \mathfrak{b}' \longrightarrow M$ and $\alpha' : \mathfrak{a}' \longrightarrow A$ are elements of $M_{(\mathcal{F})}, A_{(\mathcal{F})}$ resp. with $\xi' = \xi$ (say $\mathfrak{c} \in \mathcal{F}$, $\mathfrak{c} \subseteq \mathfrak{b} \cap \mathfrak{b}'$ s.t. $\xi|_{\mathfrak{c}} = \xi'|_{\mathfrak{c}}$) and $\alpha = \alpha'$ (say $\mathfrak{d} \in \mathcal{F}$, $\mathfrak{d} \subseteq \mathfrak{a} \cap \mathfrak{a}'$ s.t. $\alpha|_{\mathfrak{d}} = \alpha'|_{\mathfrak{d}}$), then $\mathfrak{d} \cap \alpha^{-1}\mathfrak{c} \cap \alpha'^{-1}\mathfrak{c} \in \mathcal{F}$ and the diagram

$$\begin{tikzcd}
 & M & \\
\mathfrak{b} \arrow[ur] & & \mathfrak{b}' \arrow[ul] \\
\alpha^{-1}\mathfrak{b} \arrow[u] & & \alpha'^{-1}\mathfrak{b}' \arrow[u] \\
 & \mathfrak{d} \cap \alpha^{-1}\mathfrak{c} \cap \alpha'^{-1}\mathfrak{c} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

commutes, so the same element of $M_{(\mathcal{F})}$ is produced. Now let $\xi : \mathfrak{b} \longrightarrow M \in M_{(\mathcal{F})}$ and $\psi : \mathfrak{b}' \longrightarrow M \in M_{(\mathcal{F})}$, $\alpha : \mathfrak{a} \longrightarrow A \in A_{(\mathcal{F})}$. $\xi + \psi$ is defined to be the homomorphism $\mathfrak{b} \cap \mathfrak{b}' \longrightarrow M$, $x \longmapsto \xi(x) + \psi(x)$. Then $(\xi + \psi) \cdot \alpha$ is the composite

$$\alpha^{-1}(\mathfrak{b} \cap \mathfrak{b}') \longrightarrow \mathfrak{b} \cap \mathfrak{b}' \xrightarrow{\xi + \psi} M$$

whereas $\xi\alpha + \psi\alpha$ is the composite $\alpha^{-1}(\mathfrak{b}) \cap \alpha^{-1}(\mathfrak{b}') \longrightarrow M$. Since $\alpha^{-1}(\mathfrak{b} \cap \mathfrak{b}') = \alpha^{-1}(\mathfrak{b}) \cap \alpha^{-1}(\mathfrak{b}')$, and

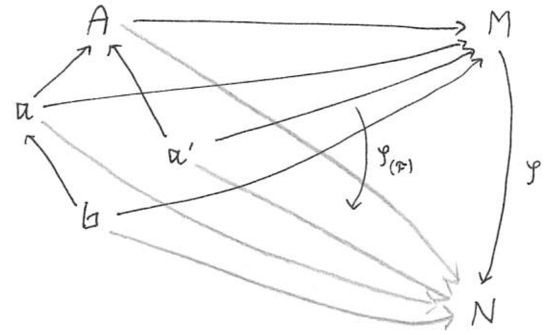
$$\begin{aligned}(\xi + \psi)\alpha(x) &= \xi(\alpha(x)) + \psi(\alpha(x)) \\ &= (\xi\alpha)(x) + (\psi\alpha)(x)\end{aligned}$$

we have $(\xi + \psi)\alpha = \xi\alpha + \psi\alpha$. Similarly, $\xi(\alpha + \beta) = \xi\alpha + \xi\beta$. In the case $M_{(\mathcal{F})} = A_{(\mathcal{F})}$ this defines multiplication (one only need check associativity and that $1 : A \longrightarrow A$ is identity). Hence $A_{(\mathcal{F})}$ is a ring. In the general case, $M_{(\mathcal{F})}$ is thus an $A_{(\mathcal{F})}$-module. (one need only check $\xi(\alpha\beta) = (\xi\alpha)\beta$).

There is a canonical homomorphism

$$\varphi_M : M \cong \mathrm{Hom}_A(A,M) \longrightarrow \varinjlim \mathrm{Hom}_A(\mathfrak{a},M) = M_{(\mathcal{F})}$$

In particular, $\varphi_A$ is a ring homomorphism. By pullback along $\varphi_A$ we may consider each $M_{(\mathcal{F})}$ as an $A$-module, $\varphi_M$ is then $A$-linear. The assignment $M \longmapsto M_{(\mathcal{F})}$ is a functor $\underline{\mathrm{Mod}}-A \longrightarrow \underline{\mathrm{Mod}}-A_{(\mathcal{F})}$. A morphism $\varphi : M \longrightarrow N$ of $A$-modules induces a morphism of diagrams



$\mathrm{Hom}_A(\mathfrak{a},M) \longrightarrow \mathrm{Hom}_A(\mathfrak{a},N)$, each $\mathfrak{a} \in \mathcal{F}$, and taking the direct limit we get $\varphi_{(\mathcal{F})} : M_{(\mathcal{F})} \longrightarrow N_{(\mathcal{F})}$. This is unique s.t. for $\xi : \mathfrak{a} \longrightarrow M$, $\varphi_{(\mathcal{F})}(\xi) = \varphi\xi$. Clearly $1_{(\mathcal{F})} = 1$ and $(\varphi\psi)_{(\mathcal{F})} = \varphi_{(\mathcal{F})}\psi_{(\mathcal{F})}$. The functor is additive and preserves kernels since Hom is left-exact and direct limits are exact in $\underline{\mathrm{Ab}}$. We denote by $L : \underline{\mathrm{Mod}}-A \longrightarrow \underline{\mathrm{Mod}}-A$ this functor followed by the forgetful functor $\underline{\mathrm{Mod}}-A_{(\mathcal{F})} \longrightarrow \underline{\mathrm{Mod}}-A$. (N.B. above defines a morphism $\varphi_{(\mathcal{F})}$ of groups. Need to check this is also a morphism of $A_{(\mathcal{F})}$-modules). Let $t$ denote the torsion radical associated with the Gabriel topology $\mathcal{F}$.

<u>LEMMA 1.2</u> $\operatorname{Ker}\varphi_M = t(M)$

<u>PROOF</u> $x \in \operatorname{Ker}\varphi_M$ iff. there is $\mathfrak{a} \in \mathcal{F}$ with $x \cdot \mathfrak{a} = 0$ so iff. there is $\mathfrak{a} \in \mathcal{F}$ s.t. $\mathfrak{a} \subseteq \mathrm{Ann}(x)$, which is iff. $\mathrm{Ann}(x) \in \mathcal{F}$. □

<u>LEMMA 1.3</u> $M$ is an $\mathcal{F}$-torsion module iff. $M_{(\mathcal{F})} = 0$.

<u>PROOF</u> If $M_{(\mathcal{F})} = 0$, then $t(M) = \operatorname{Ker}\varphi_M = M$ by Lemma 1.2. Suppose on the other hand that $M$ is a torsion module. Let $x \in M_{(\mathcal{F})}$ be represented by $\xi : \mathfrak{b} \longrightarrow M$ with $\mathfrak{b} \in \mathcal{F}$. If we can show that $\operatorname{Ker}\xi \in \mathcal{F}$, it will follow that $x = 0$. For each $b \in \mathfrak{b}$ there exists $\mathfrak{a}_b \in \mathcal{F}$ s.t. $\xi(b)\mathfrak{a}_b = 0$. Put $\mathfrak{c} = \sum_{b \in \mathfrak{b}} b\mathfrak{a}_b$. Then $\mathfrak{c} \subseteq \operatorname{Ker}\xi$, and for each $b \in \mathfrak{b}$ we have $(\mathfrak{c} : b) \supseteq \mathfrak{a}_b$, so $(\mathfrak{c} : b) \in \mathcal{F}$ and it follows that $\mathfrak{c} \in \mathcal{F}$ by T4. Hence $\operatorname{Ker}\xi \in \mathcal{F}$. □

<u>LEMMA 1.4</u> If $x \in M_{(\mathcal{F})}$ is represented by $\xi : \mathfrak{a} \longrightarrow M$ with $\mathfrak{a} \in \mathcal{F}$, then the diagram

$$\begin{tikzcd}
\mathfrak{a} \arrow[r] \arrow[d, "\xi"'] & A \arrow[d, "\beta"] \\
M \arrow[r, "\varphi_M"'] & M_{(\mathcal{F})}
\end{tikzcd}$$

commutes, where $\beta(a) = xa$.

<u>PROOF</u> If $a \in \mathfrak{a}$, then $xa$ is represented by the composed morphism $A = (\mathfrak{a} : a) \longrightarrow \mathfrak{a} \longrightarrow M$ given by $b \longmapsto \xi(ab) = \xi(a)b$. Hence $xa = \varphi_M\xi(a)$. □

An immediate consequence of this is:

<u>LEMMA 1.5</u> $\operatorname{Coker}\varphi_M$ is an $\mathcal{F}$-torsion module.

<u>PROOF</u> $\operatorname{Coker}\varphi_M$ is $M_{(\mathcal{F})}$ modulo the equivalence classes of $m : A \longrightarrow M$ for $m \in M$. Let $\xi : \mathfrak{a} \longrightarrow M$ be a rep. of some coset. Then in $\operatorname{Coker}\varphi_M$,

$$\mathrm{Ann}(\xi) = \{a \mid \xi a \in M\} = \{a \mid (\mathfrak{a} : a) = A\} = \mathfrak{a} \in \mathcal{F}$$

Hence $\operatorname{Coker}\varphi_M$ is $\mathcal{F}$-torsion. □

The second step in our construction of modules of quotients is to apply the construction once more to $A_{(\mathcal{F})}$ and $M_{(\mathcal{F})}$. That is, we apply the functor $L$ to the $A$-modules $A_{(\mathcal{F})}$ and $M_{(\mathcal{F})}$ to obtain $A$-modules $A_{\mathcal{F}}$ and $M_{\mathcal{F}}$.

LEMMA 1.6 $L$ carries the monomorphism $M/t(M) \longrightarrow M_{(\mathcal{F})}$ into an isomorphism $(M/t(M))_{(\mathcal{F})} \cong M_{\mathcal{F}}$.

PROOF Apply the left exact functor $L$ to the diagram

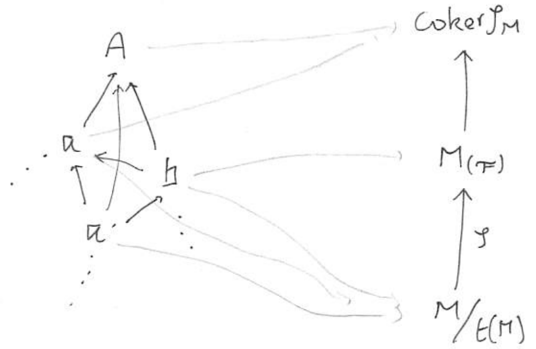
$$0 \longrightarrow M/t(M) \longrightarrow M_{(\mathcal{F})} \longrightarrow \operatorname{Coker} \rho_M \longrightarrow 0$$

and use Lemmas 1.5, 1.3. The isomorphism identifies a morphism $\mathfrak{a} \longrightarrow M/t(M)$ with the composite $\mathfrak{a} \longrightarrow M/t(M) \longrightarrow M_{(\mathcal{F})}$. $\square$

We have thus obtained the formula

$$M_{\mathcal{F}} = \varinjlim \operatorname{Hom}_A(\mathfrak{a}, M/t(M)) \qquad \mathfrak{a} \in \mathcal{F}$$

since, considering the diagram




any $\mathfrak{a} \longrightarrow M/t(M)$ is mapped to $\mathfrak{a} \longrightarrow M/t(M) \xrightarrow{\rho} M_{(\mathcal{F})}$ by the isomorphism of Lemma 1.6. The reverse direction takes $\mathfrak{b} \longrightarrow M_{(\mathcal{F})}$ and says that since $\operatorname{Coker} \rho_M$ is $\mathcal{F}$-torsion, there is $\mathfrak{c} \in \mathcal{F}$, $\mathfrak{c} \leq \mathfrak{b}$, s.t. $\mathfrak{c} \longrightarrow \mathfrak{b} \longrightarrow M_{(\mathcal{F})} \longrightarrow \operatorname{Coker} \rho_M$ is $0$. Then $\mathfrak{c} \longrightarrow M_{(\mathcal{F})} = \mathfrak{b} \longrightarrow M_{(\mathcal{F})}$ in $M_{\mathcal{F}}$, and since the sequence is exact, $\mathfrak{c} \longrightarrow M_{(\mathcal{F})}$ factors through $\rho$ — this factorisation gives the inverse direction of $\rho$. So the formula

$$M_{\mathcal{F}} = \varinjlim \operatorname{Hom}_A(\mathfrak{a}, M/t(M)) \qquad \mathfrak{a} \in \mathcal{F}$$

gives the $A$-module $M_{\mathcal{F}}$ with injections $\operatorname{Hom}_A(\mathfrak{a}, M_{(\mathcal{F})}) \longrightarrow M_{\mathcal{F}}$ given by $\mathfrak{a} \longrightarrow M_{(\mathcal{F})} \longmapsto$ unique $\mathfrak{c} \longrightarrow M/t(M)$ s.t. $\mathfrak{c} \longrightarrow M/t(M) \xrightarrow{\rho} M_{(\mathcal{F})} = \mathfrak{a} \longrightarrow M_{(\mathcal{F})}$. We define $M_{\mathcal{F}} \times A_{\mathcal{F}} \longrightarrow M_{\mathcal{F}}$ by

$$\text{let } x \in M_{\mathcal{F}} \text{ be } \xi : \mathfrak{b} \longrightarrow M/t(M)$$
$$a \in A_{\mathcal{F}} \text{ be } \alpha : \mathfrak{a} \longrightarrow A/t(A)$$

then $\xi$ induces $\mathfrak{b}/t(\mathfrak{b}) \longrightarrow M/t(M)$ and $\mathfrak{b}/t(\mathfrak{b})$ is a submodule of $A/t(A)$ by the left exactness of $t$; $xa \in M_{\mathcal{F}}$ is represented by

$$\alpha^{-1}(\mathfrak{b}/t(A)) \longrightarrow \mathfrak{b}/t(\mathfrak{b}) \xrightarrow{\xi} M/t(M)$$

this means that $\mathfrak{b}/t(A) = \{ b + t(A) \mid b \in \mathfrak{b} \}$ is isomorphic to $\mathfrak{b}/t(\mathfrak{b})$, and we define $\xi'(b + t(A)) = \xi(b)$.

We need to verify that

(i) This operation is well-defined
(ii) ~~It~~ is associative, bidistributive, etc.

(i) Suppose $x$ is also represented by $\xi' : \mathfrak{b}' \longrightarrow M/t(M)$ and $a$ by $\alpha' : \mathfrak{a}' \longrightarrow A/t(A)$, where $\mathfrak{c} \leq \mathfrak{b} \cap \mathfrak{b}'$ and $\mathfrak{d} \leq \mathfrak{a} \cap \mathfrak{a}'$ are s.t. $\xi, \xi'$ agree on $\mathfrak{c}$, and $\alpha, \alpha'$ agree on $\mathfrak{d}$. Then form the ideal

$$\alpha^{-1}(\mathfrak{c}/t(A)) \cap \mathfrak{d}$$

LEMMA If $\mathfrak{a},\mathfrak{b}\in\mathcal{F}$ and $\alpha:\mathfrak{a}\longrightarrow A/t(A)$ is a homomorphism, then $\alpha^{-1}(\mathfrak{b}/t(A))\in\mathcal{F}$.

PROOF For each $a\in\mathfrak{a}$ we have $(\alpha^{-1}(\mathfrak{b}/t(A)):a)=\{b\mid\alpha(ab)\in\mathfrak{b}/t(A)\}=\{b\mid\alpha(a)b\in\mathfrak{b}/t(A)\}$ $=\{b\mid cb\in\mathfrak{b}+t(A)\}$ where $\alpha(a)=c+t(A)$. But this is $(\mathfrak{b}+t(A):c)\in\mathcal{F}$, since $\mathfrak{b}\in\mathcal{F}$. Hence $\alpha^{-1}(\mathfrak{b}/t(A))\in\mathcal{F}$. $\square$

Hence $\alpha^{-1}(\mathfrak{c}/t(A))\cap\mathfrak{d}\in\mathcal{F}$, and the two compositions are equal when restricted to this ideal. Hence the operation is well-defined.

Next we check that for $\xi:\mathfrak{b}\longrightarrow M/t(M)$ and $\xi':\mathfrak{b}'\longrightarrow M/t(M)$, $\alpha:\mathfrak{a}\longrightarrow A/t(A)$, that $(\xi+\xi')\alpha=\xi\alpha+\xi'\alpha$.
Now $\xi+\xi':\mathfrak{b}\cap\mathfrak{b}'\longrightarrow M/t(M)$ and so

$$(\xi+\xi')\alpha:\alpha^{-1}\left(\frac{\mathfrak{b}\cap\mathfrak{b}'}{t(A)}\right)\longrightarrow\frac{\mathfrak{b}\cap\mathfrak{b}'}{t(A)}\longrightarrow M/t(M)\qquad a\mapsto\alpha(a)\mapsto(\xi+\xi')(\alpha(a))$$

$$\xi\alpha+\xi'\alpha:\alpha^{-1}\left(\frac{\mathfrak{b}}{t(A)}\right)\cap\alpha^{-1}\left(\frac{\mathfrak{b}'}{t(A)}\right)\longrightarrow M/t(M)\qquad a\mapsto(\xi\alpha)(a)+(\xi'\alpha)(a)$$

Since $\alpha^{-1}\left(\frac{\mathfrak{b}\cap\mathfrak{b}'}{t(A)}\right)\subseteq\alpha^{-1}(\mathfrak{b}/t(A))\cap\alpha^{-1}(\mathfrak{b}'/t(A))$, let $a\in\mathfrak{a}$ be s.t. $\alpha(a)=b+t(A)\in\frac{\mathfrak{b}\cap\mathfrak{b}'}{t(A)}$ (that is, let $b\in\mathfrak{b}\cap\mathfrak{b}'$). The first map takes $a\longmapsto b+t(A)\longmapsto\xi(b)+\xi'(b)$. Second map takes $a$ to $\xi(b)+\xi'(b)$, so this property holds.

Next let $\xi:\mathfrak{b}\longrightarrow M/t(M)$ and $\alpha:\mathfrak{a}\longrightarrow A/t(A)$, $\alpha':\mathfrak{a}'\longrightarrow A/t(A)$. Then $\alpha+\alpha':\mathfrak{a}\cap\mathfrak{a}'\longrightarrow A/t(A)$. Then

$$\xi(\alpha+\alpha'):(\alpha+\alpha')^{-1}(\mathfrak{b}/t(A))\longrightarrow\mathfrak{b}/t(A)\longrightarrow A/t(A)\qquad \begin{array}{l}a\mapsto\alpha(a)+\alpha'(a)\\ \mapsto\xi(\alpha(a)+\alpha'(a))\end{array}$$

and

$$\xi\alpha:\alpha^{-1}(\mathfrak{b}/t(A))\longrightarrow\mathfrak{b}/t(A)\longrightarrow A/t(A)$$
$$\xi\alpha':\alpha'^{-1}(\mathfrak{b}/t(A))\longrightarrow\mathfrak{b}/t(A)\longrightarrow A/t(A)$$

hence if $\alpha(a)=b+t(A)$, $\alpha'(a)=b'+t(A)$, since $\alpha^{-1}(\mathfrak{b}/t(A))\cap\alpha'^{-1}(\mathfrak{b}/t(A))\subseteq(\alpha+\alpha')^{-1}(\mathfrak{b}/t(A))$, for an element $a$ of this intersection $\{\xi(\alpha+\alpha')\}(a)=\xi(b)+\xi(b')=(\xi\alpha+\xi\alpha')(a)$, as required.

We claim that this operation $A_{\mathcal{F}}\times A_{\mathcal{F}}\longrightarrow A_{\mathcal{F}}$ makes $A_{\mathcal{F}}$ into a ring. We have shown the operation is left, right distributive. We show associativity (the proof also shows that $\xi\cdot(\alpha\beta)=(\xi\alpha)\cdot\beta$ for $M_{\mathcal{F}}$). Let $x,y,z\in A_{\mathcal{F}}$ be represented by $\xi:\mathfrak{a}\longrightarrow A/t(A)$, $\eta:\mathfrak{b}\longrightarrow A/t(A)$ and $\theta:\mathfrak{c}\longrightarrow A/t(A)$. Then $\eta\theta:\theta^{-1}(\mathfrak{b}/t(A))\longrightarrow\mathfrak{b}/t(A)\longrightarrow A/t(A)$ and $\xi(\eta\theta)$:

$$\xi(\eta\theta):(\eta\theta)^{-1}(\mathfrak{a}/t(A))\longrightarrow\mathfrak{a}/t(A)\longrightarrow A/t(A)$$
$$\{q\in\theta^{-1}(\mathfrak{b}/t(A))\mid(\eta\theta)(q)\in\mathfrak{a}/t(A)\}$$
$$=\{q\in\mathfrak{c}\mid\theta(q)\in\mathfrak{b}/t(A)\text{ and }\eta\theta(q)\in\mathfrak{a}/t(A)\}$$

also $\xi\eta:\eta^{-1}(\mathfrak{a}/t(A))\longrightarrow\mathfrak{a}/t(A)\longrightarrow A/t(A)$ and

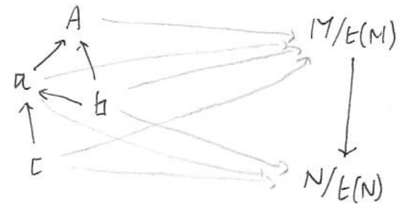
$$(\xi\eta)\theta:\theta^{-1}\left(\frac{\eta^{-1}(\mathfrak{a}/t(A))}{t(A)}\right)\longrightarrow\frac{\eta^{-1}(\mathfrak{a}/t(A))}{t(A)}\longrightarrow A/t(A)$$
$$\{q\in\mathfrak{c}\mid\theta(q)\in\eta^{-1}(\mathfrak{a}/t(A))/t(A)\}$$

Note that $\eta^{-1}(\mathfrak{a}/t(A))\subseteq\mathfrak{b}$, so $\eta^{-1}(\mathfrak{a}/t(A))/t(A)\subseteq\mathfrak{b}/t(A)$. Say $q\in\mathfrak{c}$ and $\theta(q)=b+t(A)$. Then $\theta(q)\in\eta^{-1}(\mathfrak{a}/t(A))/t(A)$ iff. there is $s\in t(A)$ s.t. $b-s\in\eta^{-1}(\mathfrak{a}/t(A))$, i.e. $\eta(b-s)\in\mathfrak{a}/t(A)$. Thus $b-s\in\mathfrak{b}$, so $s\in t(\mathfrak{b})$ and hence $\eta(s)\in t(A)$. Thus $\eta(b)\in\mathfrak{a}/t(A)$. Hence the two ideals are the same. If $q\in\mathfrak{c}$ the first maps $q$ to $b+t(A)$ and then to $\eta(b)\in\mathfrak{a}/t(A)$, and then to $\xi(\eta(b))$ — so it is $\xi(\eta(\theta(q)))$. So is the second map.

LEMMA If $\mathfrak{a} \in \mathcal{F}$ then $\mathfrak{a} + t(A) \in \mathcal{F}$. [Duh! $\mathfrak{a} + t(A) \supseteq \mathfrak{a}$]

PROOF This is the previous lemma with $\alpha : A \longrightarrow A/t(A)$ canonical. $\square$

The canonical morphism $A \longrightarrow A/t(A)$ is the identity. Hence $A_{\mathcal{F}}$ is a ring. The above also shows that $M_{\mathcal{F}}$ is an $A_{\mathcal{F}}$-module. For each homomorphism $f : M \longrightarrow N$ in Mod-$A$ one gets the induced morphism



$$\begin{array}{c} M \\ \big\downarrow \\ N \end{array}$$

⌈It is easy to see that $f_{\mathcal{F}}$ is a morphism of $A_{\mathcal{F}}$-modules⌋

which we denote by $f_{\mathcal{F}} : M_{\mathcal{F}} \longrightarrow N_{\mathcal{F}}$. There are canonical homomorphisms $\psi_M : M \longrightarrow M_{\mathcal{F}}$ of $A$-modules, where $\psi_M(x)$ is defined to be $A \longrightarrow M/t(M)$ defined by $b \longmapsto xb + t(M)$:

$$\psi_M(x)(b) = xb + t(M)$$

this is obtained from the isomorphisms, and morphisms, of

$$M \cong \mathrm{Hom}_A(A, M) \longrightarrow \varinjlim \mathrm{Hom}_A(\mathfrak{a}, M) = M_{(\mathcal{F})} \cong \mathrm{Hom}_A(A, M_{(\mathcal{F})}) \longrightarrow \varinjlim \mathrm{Hom}_A(\mathfrak{a}, M_{(\mathcal{F})})$$

$$\varinjlim \mathrm{Hom}_A(\mathfrak{a}, M_{(\mathcal{F})}) \cong M_{\mathcal{F}} = \varinjlim \mathrm{Hom}_A(\mathfrak{a}, M/t(M))$$

$$x \dashrightarrow \begin{pmatrix} A \longrightarrow M \\ a \longmapsto xa \end{pmatrix} \dashrightarrow \begin{pmatrix} A \longrightarrow M_{(\mathcal{F})} \\ b \longmapsto \varphi_M(xb), \text{ that is} \\ d \longmapsto x(bd) \end{pmatrix}$$

the bubbles go up to $\varinjlim \mathrm{Hom}_A(\mathfrak{a}, M_{(\mathcal{F})})$. Then notice that $b \mapsto \varphi_M(xb)$ (clearly) has its image in the image of $\varphi_M$, hence $A \longrightarrow M_{(\mathcal{F})} \longrightarrow \mathrm{Coker}\, \varphi_M = 0$, and we factor through $M/t(M)$ as $b \longmapsto xb + t(M)$. Hence, finally, $x$ is taken to $A \longrightarrow M/t(M)$ defined by $b \longmapsto xb + t(M)$, or, $\psi_M(x)(b) = xb + t(M)$, as claimed. $\psi_M$ is clearly a morphism of groups, and if $x \in M$, $a \in A$, then $\psi_M(x) : A \longrightarrow M/t(M)$, and if $\alpha : A \longrightarrow A$ denotes $b \mapsto ab$, then $\psi_M(x) \cdot a$ is the composite $A \longrightarrow A \longrightarrow M/t(M)$, $b \longmapsto ab \longmapsto \psi_M(x)(ab) = x \cdot (ab) + t(M) = (x \cdot a) \cdot b + t(M)$ $= \psi_M(ax)(b)$. Hence $\psi_M$ is also a morphism of $A$-modules.

In particular, $\psi_A : A \longrightarrow A_{\mathcal{F}}$ is a morphism of groups, and if $a, b \in A$ then

$$\psi_A(ab) : A \longrightarrow A/t(A) \qquad c \longmapsto (ab)c + t(A)$$

and $\psi_A(a) : A \longrightarrow A/t(A)$, $\psi_A(b) : A \longrightarrow A/t(A)$, then $\psi_A(a)\psi_A(b)$ is

$$\begin{aligned} A \longrightarrow A/t(A) \longrightarrow A/t(A) \qquad c \longmapsto \psi_A(b)(c) = bc + t(A) &\longmapsto \psi_A(a)(bc) \\ &= a(bc) + t(A) \\ &= (ab)c + t(A) \end{aligned}$$

and so $\psi_A$ is a morphism of rings (clearly $\psi_A(1)(b) = b + t(A)$ is the identity).

Hence we have a functor $q : \text{Mod-}A \longrightarrow \text{Mod-}A_{\mathcal{F}}$ defined by $q(M) = M_{\mathcal{F}}$ and $q(f) = f_{\mathcal{F}}$. It is clear that $q(1) = 1$, and that $q(fg) = q(f)q(g)$, so $q$ is indeed a functor. Note that for $f \in \mathrm{Hom}_A(M, N)$, and $x : \mathfrak{a} \longrightarrow M/t(M) \in M_{\mathcal{F}}$,

$$f_{\mathcal{F}}(x) : \mathfrak{a} \longrightarrow N/t(N) \qquad f_{\mathcal{F}}(x)(a) = \tilde{f}(x(a)) \qquad \tilde{f} : M/t(M) \longrightarrow N/t(N)$$

Note also that $f_{\mathcal{F}}$ is a morphism of the $A$-module structures, since

$$\begin{aligned} f_{\mathcal{F}}(xa)(b) = \tilde{f}((xa)(b)) &= \tilde{f}(x(ab)) \\ &= (f_{\mathcal{F}}(x) \cdot a)(b) \end{aligned}$$

For each $f \in \mathrm{Hom}_A(M,N)$ there is a commutative diagram

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\psi_M"'] & N \arrow[d, "\psi_N"] \\ M_{\mathcal{F}} \arrow[r, "f_{\mathcal{F}}"'] & N_{\mathcal{F}} \end{tikzcd}$$

$$\begin{aligned}(f_{\mathcal{F}}\psi_M)(x)(b) &= f_{\mathcal{F}}(\psi_M(x))(b) \\ &= \tilde{f}(\psi_M(x)(b)) \\ &= \tilde{f}(xb + t(M)) \\ &= f(xb) + t(N) \\ &= f(x)b + t(N) \\ &= \psi_N(f(x))(b) \\ &= (\psi_N f)(x)(b)\end{aligned}$$

which means that $\psi : 1 \longrightarrow \psi_* q$ is natural, where $\psi_* : \underline{\mathrm{Mod}}\text{-}A_{\mathcal{F}} \longrightarrow \underline{\mathrm{Mod}}\text{-}A$ is forgetful.

<u>LEMMA</u> For an $A$-module $M$, $\mathrm{Ker}\,\psi_M = t(M)$.

<u>PROOF</u> Suppose $\psi_M(x) = 0$ — then for all $b \in A$, $xb \in t(M)$. In particular $b = 1$ gives $x \in t(M)$. Conversely, if $x \in t(M)$ then $\mathrm{Ann}(x) \in \mathcal{F}$. Since for $b \in A$

$$\begin{aligned}(\mathrm{Ann}(x) : b) &= \{a \in A \mid ba \in \mathrm{Ann}(x)\} \\ &= \{a \in A \mid xba = 0\} = \mathrm{Ann}(xb)\end{aligned}$$

for each $b \in A$, $\mathrm{Ann}(xb) \in \mathcal{F}$ and hence $xb \in t(M)$ — thus $\psi_M(x) = 0$. □

<u>LEMMA 1.4</u> If $x \in M_{\mathcal{F}}$ is represented by $\xi : \mathfrak{a} \longrightarrow M/t(M)$ with $\mathfrak{a} \in \mathcal{F}$ then the diagram

$$\begin{tikzcd} \mathfrak{a} \arrow[r] \arrow[d, "\xi"'] & A \arrow[d, "\beta"] \\ \frac{M}{t(M)} \arrow[r, "\tilde{\psi}_M"'] & M_{\mathcal{F}} \end{tikzcd}$$

commutes, where $\beta(a) = xa$ and $\psi_M : M \longrightarrow M_{\mathcal{F}}$ induces $\tilde{\psi}_M$ by the previous Lemma.

<u>PROOF</u> Let $a \in \mathfrak{a}$, and suppose $\xi(a) = m + t(M)$. Then $\tilde{\psi}_M(\xi(a)) : A \longrightarrow M/t(M)$, $b \longmapsto mb + t(M) = \xi(a)b$. Also, $xa$ is the composite $(\mathfrak{a} : a) \longrightarrow \mathfrak{a} \longrightarrow M/t(M)$, $b \longmapsto \xi(ab) = \xi(a)b$. Hence the two are equal. □

<u>LEMMA 1.5</u> $\mathrm{Coker}\,\psi_M$ is an $\mathcal{F}$-torsion module (as an $A$-module).

<u>PROOF</u> Obvious, because $\mathrm{Coker}\,\psi_M = M_{\mathcal{F}}/\mathrm{Im}\,\psi_M$, and if $x \in M_{\mathcal{F}}$ is represented by $\xi : \mathfrak{a} \longrightarrow M/t(M)$, then for $a \in A$, $\xi a : (\mathfrak{a} : a) \longrightarrow M/t(M)$, $b \longmapsto \xi(a)b$ is $\psi_M(x)|_{(\mathfrak{a}:a)}$ where $\xi(a) = x + t(M)$. Hence in $\mathrm{Coker}\,\psi_M$, $\xi a = 0$ — so $\mathrm{Ann}(\xi) = A \in \mathcal{F}$ and $\mathrm{Coker}\,\psi_M$ is $\mathcal{F}$-torsion. □

<u>LEMMA</u> $M$ is $\mathcal{F}$-torsion iff. $M_{\mathcal{F}} = 0$.

<u>PROOF</u> Recall that as an $A$-module, $M_{\mathcal{F}}$ is isomorphic to $(M/t(M))_{(\mathcal{F})}$, so $M_{\mathcal{F}} = 0$ iff. $(M/t(M))_{(\mathcal{F})} = 0$, so iff. $M/t(M)$ is $\mathcal{F}$-torsion. But $t$ is a radical, so $t(M/t(M)) = 0$. Hence $M = t(M)$ iff. $(M/t(M))$ is $\mathcal{F}$-torsion. Hence $M$ is $\mathcal{F}$-torsion iff. $M_{\mathcal{F}} = 0$. □

We wish to give a characterisation of those $A$-modules which may be considered as modules of quotients with respect to $\mathcal{F}$ (similar to the characterisation of modules of fractions in Prop II. 3.7). For this purpose we introduce:

**DEFINITION** An $A$-module $M$ is $\mathcal{F}$-closed (sheaf) (resp $\mathcal{F}$-injective) if the canonical homomorphisms

$$M \cong Hom_A(A, M) \longrightarrow Hom_A(\mathfrak{a}, M)$$

N.B. $0$ is trivially $\mathcal{F}$-closed,

are isomorphisms (resp. epimorphisms) for all $\mathfrak{a} \in \mathcal{F}$.

Since $Hom_A(A,M) \longrightarrow Hom_A(\mathfrak{a}, M)$ is a monomorphism iff. $\forall 0 \neq x \in M$, $a \in \mathfrak{a} \longmapsto xa$ is not the zero morphism, which is iff. $\mathfrak{a} \nsubseteq Ann(x)$ $\forall x \in M \setminus 0$, a module $M$ is $\mathcal{F}$-closed iff. it is $\mathcal{F}$-torsion-free and $\mathcal{F}$-injective. In this case, $\psi_M : M \longrightarrow M_{\mathcal{F}}$ is an isomorphism of $A$-modules: we know that $\psi_M$ is the composition of the canonical morphisms

$$M \longrightarrow M_{(\mathcal{F})} \longrightarrow \varinjlim Hom_A(\mathfrak{a}, M_{(\mathcal{F})}) \cong M_{\mathcal{F}}.$$

if $M$ is $\mathcal{F}$-closed, then $M \longrightarrow M_{(\mathcal{F})}$ is an isomorphism (it is injective since $\mathcal{F}$-torsion-free $\Rightarrow t(M) = Ker\, M \longrightarrow M_{(\mathcal{F})} = 0$ and is obviously surjective).

No. Just do it manually: if $\psi_M(x) = 0$, then $\forall a \in A$, $\psi_M(x)(a) = xa + t(M) = 0$. In particular $a = 1$ gives $x \in t(M)$. But by assumption $t(M) = 0$, so $x = 0$. To see $\psi_M$ is an epimorphism, let $\alpha = \mathfrak{a} \longrightarrow M/t(M)$ be in $M_{\mathcal{F}}$. Since by assumption $t(M) = 0$, we can find $A \longrightarrow M$ extending $\alpha$, as required. Conversely, every module of quotients $M_{\mathcal{F}}$ is $\mathcal{F}$-closed (and is therefore its own module of quotients)

**PROPOSITION 1.8** $M_{\mathcal{F}}$ is an $\mathcal{F}$-closed $A$-module for every $A$-module $M$.

**PROOF** To show $M$ is torsion-free, it suffices to show that if $M$ is torsion-free, then so is also $M_{(\mathcal{F})}$. Suppose $x \in M_{(\mathcal{F})}$ and $x\mathfrak{b} = 0$ for $\mathfrak{b} \in \mathcal{F}$. Let $x$ be represented by $\xi : \mathfrak{a} \longrightarrow M$. By Lemma 1.4 there is a commutative diagram

$$\begin{tikzcd} \mathfrak{a} \arrow[r] \arrow[d, "\xi"'] & A \arrow[d, "\beta"] \\ M \arrow[r, "\varphi_M"'] & M_{(\mathcal{F})} \end{tikzcd} \qquad \beta(a) = xa$$

so $\varphi_M \xi$ is zero when restricted to $\mathfrak{a} \cap \mathfrak{b} \in \mathcal{F}$. But $\varphi_M$ is a monomorphism, since by assumption $M$ is torsion-free, and hence $\xi|\mathfrak{a} \cap \mathfrak{b} = 0$. Hence $x = 0$.

Next, we show that $M_{\mathcal{F}}$ is $\mathcal{F}$-injective. Suppose we are given $f : \mathfrak{a} \longrightarrow M_{\mathcal{F}}$ with $\mathfrak{a} \in \mathcal{F}$. Then since $Coker\,\psi_M$ is a torsion module by Lemma 1.5, we induce $g : \mathfrak{b} \longrightarrow M/t(M)$, $\mathfrak{b} \in \mathcal{F}$, making the diagram

$$\begin{tikzcd} & \mathfrak{b} \arrow[r] \arrow[d, "g"'] & \mathfrak{a} \arrow[d, "f"] & & \\ 0 \arrow[r] & M/t(M) \arrow[r] & M_{\mathcal{F}} \arrow[r] & Coker\,\psi_M \arrow[r] & 0 \end{tikzcd}$$

commute. Since by Lemma 1.4 the diagram

$$\begin{tikzcd} \mathfrak{b} \arrow[r] \arrow[d, "g"'] & A \arrow[d, "h"] \\ M/t(M) \arrow[r, "\tilde{\psi}_M"'] & M_{\mathcal{F}} \end{tikzcd} \qquad h(a) = ga$$

commutes, we see that $h$ restricts to give $f$ using the following Lemma:

LEMMA 1.9 Let $\mathfrak{b} \subseteq \mathfrak{a}$ be right ideals in $\mathcal{F}$ and $M$ a torsion-free module. Suppose $f$ and $g$ are homomorphisms $\mathfrak{a} \longrightarrow M$ such that $f|\mathfrak{b} = g|\mathfrak{b}$. Then $f = g$.

PROOF Since $f$ and $g$ are equal on $\mathfrak{b}$, their difference $f-g$ is zero on $\mathfrak{b}$. It suffices to prove that if $h : \mathfrak{a} \longrightarrow M$, $h|\mathfrak{b} = 0$, then $h = 0$. To prove this, let $a \in \mathfrak{a}$. Then $(\mathfrak{b}:a) \in \mathcal{F}$, and $h(a)(\mathfrak{b}:a) = 0$, since for $x \in (\mathfrak{b}:a)$, $h(a)x = h(ax) = 0$, since $ax \in \mathfrak{b}$. Hence $\mathrm{Ann}(h(a)) \supseteq (\mathfrak{b}:a) \in \mathcal{F}$, so $\mathrm{Ann}(h(a)) \in \mathcal{F}$, and since $M$ is torsion-free, $h(a) = 0$. $\square$

So we have shown:

- If $M$ is $\mathcal{F}$-closed, then $\psi_M : M \longrightarrow M_{\mathcal{F}}$ is an isomorphism of $A$-modules
- For any $A$-module $M$, $M_{\mathcal{F}}$ is an $\mathcal{F}$-closed $A$-module.

Note also that the $A$-module structure on $M_{\mathcal{F}}$ (which is by def$^n$ the $A$-module structure on $(M/t(M))_{(\mathcal{F})}$) is the same as the $A$-module structure given by $\psi_A : A \longrightarrow A_{\mathcal{F}}$, since if $a \in A$, the first method given for $\xi : \mathfrak{a} \longrightarrow M/t(M)$ the result

$$(\mathfrak{a}:a) \longrightarrow \mathfrak{a} \longrightarrow M/t(M) \qquad b \longmapsto \xi(a)b$$

and the second, which is $\xi \cdot \psi_A(a)$, is

$$\psi_A(a)^{-1}\left(\mathfrak{a}/t(A)\right) \longrightarrow \mathfrak{a}/t(A) \xrightarrow{\hat{\xi}} M/t(M) \qquad b \longmapsto ab + t(A) \longmapsto \hat{\xi}(ab + t(A))$$

now, clearly $(\mathfrak{a}:a) \subseteq \psi_A(a)^{-1}(\mathfrak{a}/t(A))$, and for $b \in (\mathfrak{a}:a)$, $\hat{\xi}(ab + t(A)) = \xi(ab) = \xi(a)b$. Hence the $A$-module structures are the same.

Let $M, N$ be $\mathcal{F}$-closed, and suppose $f \in \mathrm{Hom}_A(M,N)$. Then $f_{\mathcal{F}} : M_{\mathcal{F}} \longrightarrow N_{\mathcal{F}}$ is unique in $\mathrm{Hom}_{A_{\mathcal{F}}}(M_{\mathcal{F}}, N_{\mathcal{F}})$ with the property that

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\psi_M \cong"'] & N \arrow[d, "\cong \psi_N"] \\ M_{\mathcal{F}} \arrow[r, "f_{\mathcal{F}}"'] & N_{\mathcal{F}} \end{tikzcd}$$

commutes (as a diagram of $A$-modules and $A$-linear morphisms). This is obvious since as a function, $f_{\mathcal{F}} = \psi_N f \psi_M^{-1}$. Now, let $g : M_{\mathcal{F}} \longrightarrow N_{\mathcal{F}}$ be a morphism of $A_{\mathcal{F}}$-modules. Let $g' = \psi_N^{-1} g \psi_M : M \longrightarrow N$ – then $g' \in \mathrm{Hom}_A(M,N)$. But since $g'_{\mathcal{F}} = \psi_N g' \psi_M^{-1} = \psi_N \psi_N^{-1} g \psi_M \psi_M^{-1} = g$, it follows that $g = g'_{\mathcal{F}}$. Hence the functor

$$j : \underline{\mathrm{Mod}}\text{-}(A,\mathcal{F}) \longrightarrow \underline{\mathrm{Mod}}\text{-}A_{\mathcal{F}} \qquad j(M) = M_{\mathcal{F}} \quad j(f) = f_{\mathcal{F}}.$$

from the full subcategory $\underline{\mathrm{Mod}}\text{-}(A,\mathcal{F})$ of $\underline{\mathrm{Mod}}\text{-}A$ consisting of the $\mathcal{F}$-closed modules, to $\underline{\mathrm{Mod}}\text{-}A_{\mathcal{F}}$, is full. It is also faithful, since if $f_{\mathcal{F}} = g_{\mathcal{F}}$, $\psi_N f \psi_M^{-1} = \psi_N g \psi_M^{-1}$, so $f = g$. (One could attempt to claim that $j$ is also distinct on objects. Since this will depend on the choice of direct limit, details of forming equiv. classes, etc. it is not really interesting).

Suppose $M \in \underline{\mathrm{Mod}}\text{-}A$ and hence $M_{\mathcal{F}} \in \underline{\mathrm{Mod}}\text{-}A_{\mathcal{F}}$. By Prop 1.8 $M_{\mathcal{F}}$ is an $\mathcal{F}$-closed $A$-module. Hence $M_{\mathcal{F}} \in \underline{\mathrm{Mod}}\text{-}(A,\mathcal{F})$. We claim $j(M_{\mathcal{F}}) \cong M_{\mathcal{F}}$. We know $\psi_{M_{\mathcal{F}}} : M_{\mathcal{F}} \longrightarrow (M_{\mathcal{F}})_{\mathcal{F}} = j(M_{\mathcal{F}})$ is an isomorphism of $A$-modules. It is an isomorphism of $A_{\mathcal{F}}$-modules (see next page).

Let $\Psi_{M_{\mathcal{F}}} : M_{\mathcal{F}} \longrightarrow (M_{\mathcal{F}})_{\mathcal{F}}$ and $x \in M_{\mathcal{F}}$, $a \in A_{\mathcal{F}}$ be represented by $\xi : \mathfrak{b} \longrightarrow M/t(M)$ and $\alpha : \mathfrak{a} \longrightarrow A/t(A)$ respectively. Then

$$xa : \quad \alpha^{-1}(\mathfrak{b}/t(A)) \longrightarrow \mathfrak{b}/t(A) \xrightarrow{\tilde{\xi}} M/t(M) \qquad \in M_{\mathcal{F}}$$
$$b \longmapsto \tilde{\xi}(\alpha(b))$$

$$\Psi_{M_{\mathcal{F}}}(xa) : A \longrightarrow M_{\mathcal{F}}/t(M_{\mathcal{F}}) = M_{\mathcal{F}}$$
$$c \longmapsto (xa)c$$
$$\|$$
$$(\alpha^{-1}(\mathfrak{b}/t(A)) : c) \longrightarrow \alpha^{-1}(\mathfrak{b}/t(A)) \longrightarrow M/t(M)$$
$$d \longmapsto cd \longmapsto \tilde{\xi}(\alpha(cd))$$

and also

$$\Psi_{M_{\mathcal{F}}}(x) : A \longrightarrow M_{\mathcal{F}}$$
$$c \longmapsto xc$$
$$\|$$
$$(\mathfrak{b} : c) \longrightarrow \mathfrak{b} \longrightarrow M/t(M)$$
$$d \longmapsto cd \longmapsto \xi(cd)$$

$$\Psi_{M_{\mathcal{F}}}(x)\cdot a : \quad \mathfrak{a} \longrightarrow A/t(A) \xrightarrow{\widetilde{\Psi_{M_{\mathcal{F}}}(x)}} M_{\mathcal{F}}$$
$$b \longmapsto \alpha(b) \longmapsto \widetilde{\Psi_{M_{\mathcal{F}}}(x)}(\alpha(b))$$

We show $\Psi_{M_{\mathcal{F}}}(xa) = \Psi_{M_{\mathcal{F}}}(x)\cdot a$ by showing they agree on $\mathfrak{a}$. For, let $c \in \mathfrak{a}$. Then if $\alpha(c) = g + t(A)$, we see that $(\mathfrak{b} : g) \subseteq (\alpha^{-1}(\mathfrak{b}/t(A)) : c)$, since if $gy \in \mathfrak{b}$, $\alpha(cy) = \alpha(c)y = gy + t(A) \in \mathfrak{b}/t(A)$. We must show $\Psi_{M_{\mathcal{F}}}(xa)(c) = (\Psi_{M_{\mathcal{F}}}(x)\cdot a)(c)$ :

$$(xa)c : (\alpha^{-1}(\mathfrak{b}/t(A)) : c) \longrightarrow M/t(M)$$
$$\widetilde{\Psi_{M_{\mathcal{F}}}(x)}(\alpha(c)) = \Psi_{M_{\mathcal{F}}}(x)(g) = xg : (\mathfrak{b} : g) \longrightarrow M/t(M).$$

For $z \in (\mathfrak{b} : g)$, $\{(xa)c\}(z) = \tilde{\xi}(\alpha(cz)) = \tilde{\xi}(gz + t(A)) = \xi(gz)$.
$$\{xg\}(z) = \xi(gz)$$

hence we have shown that $\Psi_{M_{\mathcal{F}}}$ is a morphism of $A_{\mathcal{F}}$-modules. Hence

<u>PROPOSITION</u> The functor $j : \underline{\mathrm{Mod}}\text{-}(A,\mathcal{F}) \longrightarrow \underline{\mathrm{Mod}}\text{-}A_{\mathcal{F}}$ defined by

$$j(M) = M_{\mathcal{F}} \quad j(f) = f_{\mathcal{F}}$$

is full, faithful, and hence defines an equivalence with the full subcategory of $\underline{\mathrm{Mod}}\text{-}A_{\mathcal{F}}$ consisting of $M_{\mathcal{F}}$ for $M \in \underline{\mathrm{Mod}}\text{-}A$.

We let $\underline{\text{Mod}}$-$(A,\mathcal{F})$ denote the full subcategory of $\underline{\text{Mod}}$-$A$ consisting of $\mathcal{F}$-closed modules, and call it the <u>quotient category</u> of $\underline{\text{Mod}}$-$A$ with respect to $\mathcal{F}$. In practice we will usually not make any distinction between this category and the category of $A_\mathcal{F}$-modules of the form $M_\mathcal{F}$. It is important to notice that with this convention, every $A$-linear map between $A_\mathcal{F}$-closed modules is automatically $A_\mathcal{F}$-linear. There are a number of interesting functors:

$$\begin{tikzcd}
\underline{\text{Mod}}\text{-}A \arrow[rr, bend left, "q"] \arrow[rr, bend right, "\psi^*"'] \arrow[dr, bend left, "a"] & & \underline{\text{Mod}}\text{-}A_\mathcal{F} \arrow[ll, "\psi_*"'] \\
& \underline{\text{Mod}}\text{-}(A,\mathcal{F}) \arrow[ul, bend left, "i"] \arrow[ur, "j"'] &
\end{tikzcd}$$

$\psi_*$ is the forgetful functor

$\psi^*$ is $M \longmapsto M \otimes_A A_\mathcal{F}$

$q$ is $M \longmapsto M_\mathcal{F}$

$i$ is the inclusion functor

$a$ is the functor $M \longmapsto M_\mathcal{F}$

$j$ is as defined earlier.

We already know that $\psi^* \dashv \psi_*$. We have $ja \cong q$ (this the content of the previous page) and $i \cong \psi_* j$.? (On the other hand, $ia \cong LL = \psi_* q$ ... no) We prove directly that $q$ is left exact. Let

$$0 \longrightarrow M \xrightarrow{\lambda} N \xrightarrow{\alpha} M'$$

be exact in $\underline{\text{Mod}}$-$A$. It is not hard to see that $q(0)=0$ and $q$ preserve monics. We omitted mentioning it earlier, but

$$\begin{tikzcd}
N \arrow[r, "\rho_N"] \arrow[d, "\alpha"'] & N_{(\mathcal{F})} \arrow[d, "L(\alpha)"] \\
M' \arrow[r, "\rho_{M'}"'] & M'_{(\mathcal{F})}
\end{tikzcd}$$

commutes. Hence consider the following diagram, where $x \in N_\mathcal{F}$ is represented by $\xi : \mathfrak{b} \longrightarrow N/t(N)$.

$$\begin{tikzcd}
 & & A & & & 0 \arrow[d] \\
 & & & & M/t(M) \arrow[r] \arrow[d, "\tilde{\lambda}"'] & M_{(\mathcal{F})} \arrow[d] \\
\mathfrak{a} \arrow[urr] & & & \mathfrak{b} \arrow[ul] \arrow[r, "\xi"] & N/t(N) \arrow[r] \arrow[d, "\tilde{\alpha}"'] & N_{(\mathcal{F})} \arrow[d] \\
 & \mathfrak{c} \arrow[ul] \arrow[uuur, bend left] \arrow[urr] & & & M'/t(M') \arrow[r] & M'_{(\mathcal{F})} \\
\mathfrak{d} \arrow[ur] & & & & &
\end{tikzcd}$$

Since $q(\alpha\lambda) = q(0) = 0$, and $q(\lambda)$ is monic, we need only show that $q(\alpha)(x) = 0$ implies $x \in \operatorname{Im} q(\lambda)$. If $q(\alpha)(x) = 0$, then there is $\mathfrak{c} \in \mathcal{F}$, $\mathfrak{c} \subseteq \mathfrak{b}$, s.t. $\forall c \in \mathfrak{c}$, $\tilde{\alpha}\xi(c) = 0$. Using the fact that $L$ is left exact, so $0 \to M_{(\mathcal{F})} \to N_{(\mathcal{F})} \to M'_{(\mathcal{F})}$ is exact, $\mathfrak{c} \to \mathfrak{b} \to N/t(N) \to N_{(\mathcal{F})}$ factors through $M_{(\mathcal{F})}$. As usual, this means there is $\mathfrak{d} \subseteq \mathfrak{c}$ s.t. $\mathfrak{d} \to \mathfrak{c} \to M_{(\mathcal{F})}$ factors through $M/t(M) \to M_{(\mathcal{F})}$. Using the fact that $N/t(N) \to N_{(\mathcal{F})}$ is monic, this implies that $\mathfrak{d} \to M/t(M) \xrightarrow{\tilde{\lambda}} N/t(N) = \mathfrak{d} \to \mathfrak{c} \to \mathfrak{b} \to N/t(N)$. Hence $x \in \operatorname{Im} q(\lambda)$, and $q$ is left exact.

Define a mapping $\mu : M \times A_{\mathcal{F}} \longrightarrow M_{\mathcal{F}}$ for each $A$-module $M$ by

$$\mu(x, q) = \psi_M(x) q$$

this is bilinear in $A$ since $\mu(x+x', q) = \psi_M(x+x')q = \psi_M(x)\cdot q + \psi_M(x')q$, similarly $\mu(x, q+q') = \mu(x,q) + \mu(x,q')$, and

$$\mu(x\cdot a, q) = \psi_M(xa)q = (\psi_M(x)\cdot a)\cdot q = \psi_M(x)\cdot(\psi_A(a)q) = \psi_M(x)\cdot(a\cdot q) = \mu(x, a\cdot q)$$

which thus induces $\Theta_M : M \otimes_A A_{\mathcal{F}} \longrightarrow M_{\mathcal{F}}$, $\Theta_M(x \otimes q) = \psi_M(x) q$. This is a morphism of $A_{\mathcal{F}}$-modules, since

$$\begin{aligned} \Theta_M((x\otimes q)a) &= \Theta_M(x \otimes qa) \\ &= \psi_M(x)\cdot(qa) \\ &= (\psi_M(x)\cdot q)\cdot a \end{aligned}$$

It is also natural in $M$ — suppose $\alpha : M \longrightarrow N$ is a morphism of $A$-modules:

$$\begin{tikzcd} M \otimes_A A_{\mathcal{F}} \arrow[r, "\Theta_M"] \arrow[d, "\alpha\otimes 1"'] & M_{\mathcal{F}} \arrow[d, "q(\alpha)"] \\ N \otimes_A A_{\mathcal{F}} \arrow[r, "\Theta_N"'] & N_{\mathcal{F}} \end{tikzcd}$$

For $x\otimes q \in M \otimes_A A_{\mathcal{F}}$, $\Theta_M(x\otimes q) = \psi_M(x) q$. If $q : \mathfrak{a} \longrightarrow A/t(A)$, $\psi_M(x) : A \longrightarrow M/t(M)$ then $\psi_M(x)q$ is

$$\mathfrak{a} \longrightarrow A/t(A) \xrightarrow{\widetilde{\psi_M(x)}} M/t(M)$$

and $q(\alpha)\Theta_M(x\otimes q)$ is $\mathfrak{a} \longrightarrow A/t(A) \longrightarrow M/t(M) \longrightarrow N/t(N)$. On the other hand

$$\begin{aligned} \Theta_N(\alpha(x)\otimes q) &= \psi_N(\alpha(x)) q \\ &= q(\alpha)(\psi_M(x)) q \\ &= q(\alpha)(\psi_M(x) q) \end{aligned}$$

as required. In many important cases it turns out that $\Theta$ is a natural equivalence $\Theta : \psi^{*} \longrightarrow q$, and then the right exactness of $q$ follows from the right exactness of $\psi^{*}$. (this will be discussed in Chap. XI). In any case, there is always a commutative diagram

$$\begin{tikzcd} & M \otimes_A A_{\mathcal{F}} \arrow[d, "\Theta_M"] \\ M \arrow[ur] \arrow[r] & M_{\mathcal{F}} \end{tikzcd}$$

<u>PROPOSITION 1.11</u> $a$ is a left adjoint of $i$.

<u>PROOF</u> Let $M \in \underline{\mathrm{Mod}}\text{-}A$ and $N$ be $\mathcal{F}$-closed. For any $\alpha : M \longrightarrow N \in \mathrm{Hom}_A(M,N)$ the diagram

$$\begin{tikzcd} M \arrow[r, "\psi_M"] \arrow[d, "\alpha"'] & M_{\mathcal{F}} \arrow[d, "q(\alpha)"] \\ N \arrow[r, "\psi_N"'] & N_{\mathcal{F}} \end{tikzcd}$$

and hence there is $M_{\mathcal{F}} \longrightarrow N$ s.t. $M \longrightarrow M_{\mathcal{F}} \longrightarrow N = M \longrightarrow N$. To show uniqueness, consider the exact sequence

$$0 \longrightarrow M/t(M) \longrightarrow M_{\mathcal{F}} \longrightarrow \operatorname{Coker}\Psi_M \longrightarrow 0$$

which induces

$$0 \longrightarrow \operatorname{Hom}(\operatorname{Coker}\Psi_M, N) \longrightarrow \operatorname{Hom}(M_{\mathcal{F}}, N) \longrightarrow \operatorname{Hom}(M/t(M), N)$$

where the first term is zero since $\operatorname{Coker}\Psi_M$ is a torsion-module. Since $\operatorname{Hom}(M/t(M), N) = \operatorname{Hom}(M, N)$, this shows that $\operatorname{Hom}(M_{\mathcal{F}}, N) \longrightarrow \operatorname{Hom}(M, N)$ is monic. Hence $\Psi$ is the unit of an adjunction $? \dashv i$ to show it is a we show that for $\alpha : M \longrightarrow M'$ the diagram

$$\begin{tikzcd} M \arrow[r, "\alpha"] \arrow[d] & M' \arrow[d] \\ M_{\mathcal{F}} \arrow[r, "a(\alpha)"'] & M'_{\mathcal{F}} \end{tikzcd}$$

commutes. But this is known. □

It will be shown in the following chapter that the quotient category $\underline{\text{Mod}} - (A, \mathcal{F})$ is very well-behaved, in fact it is a Grothendieck category, although the inclusion functor $i$ is not exact in general.

EXAMPLES

2. Rings of Fractions Let $S$ be a multiplicatively closed subset of $A$ consisting of non-zero-divisors. Assume $S$ satisfies the generalised Ore condition $S1$:

$$S1: \text{ given } s \in S \text{ and } a \in A, \text{ there is } t \in S, b \in A \text{ s.t. } sb = at$$

The ring of fractions $A[S^{-1}]$ then exists (Chap II, Prop 1.4) and the family $\mathcal{F}$ of right ideals intersecting $S$ is a Gabriel topology. (Example VI.6.1). Let $M$ be a module – we will show that $M_{\mathcal{F}} \cong M[S^{-1}]$. Define

$$\phi : M[S^{-1}] \longrightarrow M_{\mathcal{F}}$$

by setting $\phi(x,s) : sA \longrightarrow M/t(M)$, $sa \longmapsto xa + t(M)$. This map is well-defined because $s$ is a non-zero divisor, and if $(x,s) = (x', s')$, say $c, d \in A$ s.t. $xc = x'd$ and $sc = s'd \in S$, then on $scA = s'dA \subseteq sA \cap s'A$,

$$\begin{aligned} x'(da) + t(M) &= (x'd)\cdot a + t(M) \\ &= (xc)\cdot a + t(M) \\ &= x(ca) + t(M) \end{aligned}$$

once again using the fact that $S$ consists of non-zero divisors to see that if $b \in scA$, $\exists a \in A$ s.t. $b = sca$ and $b = s'da$. Hence $\phi$ is well-defined. It is a morphism of $A$-modules, since for $(x,s) \in M[S^{-1}]$ and $z \in A$,

$$(x,s)\cdot(z,1) = (xc, u) \quad \text{where} \quad sc = zu, \ u \in S$$

hence $\phi((x,s)\cdot z) = \phi(xc, u) : uA \longrightarrow M/t(M)$, $ua \longmapsto (xc)a + t(M)$. Whereas $\phi(x,s)\cdot z$ is the composite

$$(sA : z) \longrightarrow sA \longrightarrow M/t(M)$$

but $zu = sc \in sA$, so $uA \subseteq (sA : z)$, and $\phi(x,s)\cdot z$ takes $ua \longmapsto zua = sca \longmapsto x(ca) + t(M) = (xc)a + t(M)$. Hence $\phi(x,s)\cdot z = \phi((x,s)\cdot z)$, and $\phi$ is a morphism of $A$-modules.

To see that $\phi$ is injective, suppose $\phi(x,s) = \phi(y,t)$. Then there is $zA \subseteq sA \cap tA$, say $z = sa = ta'$, s.t. for all $b \in A$, $x(ab) + t(M) = x(a'b) + t(M)$. In particular, $xa - xa' \in t(M)$, so there is $u \in S$ with $x(au) = x(a'u)$. But then also $sau = ta'u$, and $sau \in S$ – so by def$^{n}$ $(x,s) = (y,t)$.

If $x \in M_{\mathcal{F}}$ is represented by $\xi : B \longrightarrow M/t(M)$, let $u \in S \cap B$, so $x = \xi|_{uA}$ ($uA \in \mathcal{F}$ trivially) and $\xi|_{uA} : uA \longrightarrow M/t(M)$ is equal to $\phi(m,u)$ where $\xi(u) = m + t(M)$. Hence $\phi$ is an isomorphism of $A$-modules.

When $M = A$, $\phi : A[S^{-1}] \longrightarrow A_{\mathcal{F}}$ becomes a morphism of rings (which is an isomorphism since it is bijective). The identity $1 \in A[S^{-1}]$ is $(1,1)$, mapped to $\phi(1,1) : A \longrightarrow A/t(A)$, $b \longmapsto b + t(A)$, which is $1 \in A_{\mathcal{F}}$. For $(x,s), (y,t) \in A[S^{-1}]$, let $sc = yu$, $u \in S$. Then $(x,s)(y,t) = (xc, tu)$ and $\phi(xc, tu) : tuA \longrightarrow A/t(A)$ is given by $tua \longmapsto (xc)a + t(A)$. Conversely, $\phi(x,s)\phi(y,t)$ is the composition

$$\phi(y,t)^{-1}\left(sA/t(A)\right) \longrightarrow sA/t(A) \longrightarrow A/t(A)$$

but $tuA \subseteq \phi(y,t)^{-1}(sA/t(A))$, and for $a \in A$ the above composite maps $tua \longmapsto yua + t(A) = sca + t(A) \longmapsto x(ca) + t(A)$. Hence $\phi$ is a ring morphism.

## 2. $\mathcal{F}$-Injective Envelopes

In Example V.2.3 we saw that the module of fractions of a torsion-free module $M$ can be canonically embedded as a submodule of the injective envelope $E(M)$ of $M$, namely $M[S^{-1}] = \{x \in E(M) \mid (M:x) \cap S \neq \phi\}$. We will now see that the same thing holds for modules of quotients with respect to a Gabriel topology $\mathcal{F}$, and that $M_{\mathcal{F}}$ for an $\mathcal{F}$-torsion free module $M$ can be interpreted as an "$\mathcal{F}$-injective envelope" of $M$. Recall that $\mathcal{F}(M)$ denotes the filter of open submodules of $M$ in the $\mathcal{F}$-topology, i.e. the submodules $L$ of $M$ for which $M/L$ is an $\mathcal{F}$-torsion module (Chap VI. §4).

PROPOSITION 2.1 The following properties of a module $E$ are equivalent:

(a) $E$ is $\mathcal{F}$-injective

(b) If $M$ is a module and $L \in \mathcal{F}(M)$, then every morphism $L \longrightarrow E$ may be extended to a morphism $M \longrightarrow E$.

PROOF (a) $\Rightarrow$ (b) Let $L \in \mathcal{F}(M)$ and $\beta : L \longrightarrow E$ be given. In the usual way (cf. Prop I. 6.5) we may assume that there is a maximal extension $\beta' : L' \longrightarrow E$ of $\beta$, where $L \subseteq L' \subseteq M$. Then also $L' \in \mathcal{F}(M)$ by TM1. Suppose there exists $x \in M$ such that $x \notin L'$. Put $\mathfrak{a} = (L':x) \in \mathcal{F}$ and let $\alpha : \mathfrak{a} \longrightarrow E$ be the homomorphism $\alpha(a) = \beta'(xa)$. By (a) we can extend $\alpha$ to $A$, i.e. there exists $y \in E$ such that $\alpha(a) = ya$. We may then define $\gamma : L' + xA \longrightarrow E$ as $\gamma(z + xa) = \beta'(z) + ya$. This is well-defined since if $z + xa = z' + xa'$ then $z - z' = x(a' - a)$, so $\beta'(z) - \beta'(z') = \beta'(x(a'-a)) = \alpha(a') - \alpha(a), = ya' - ya$. Then $\gamma$ extends $\beta'$, which is a contradiction.

(b) $\Rightarrow$ (a) is obvious since $\mathcal{F}(A) = \mathcal{F}$. □

DEFINITION An $\mathcal{F}$-injective envelope of $M$ is an essential monomorphism $M \longrightarrow E$ s.t. $E$ is $\mathcal{F}$-injective and $M \in \mathcal{F}(E)$.

An $\mathcal{F}$-injective envelope of $M$ is unique up to isomorphism ..

PROPOSITION 2.3 If $\alpha : C \longrightarrow E$ and $\alpha' : C \longrightarrow E'$ are $\mathcal{F}$-injective envelopes of $C$, then there is an isomorphism $\gamma : E \longrightarrow E'$ s.t. $\gamma\alpha = \alpha'$.

PROOF Since $C \in \mathcal{F}(E)$, by Prop 2.1 the morphism $\alpha'$ can be extended to $\gamma : E \longrightarrow E'$ s.t. $\gamma\alpha = \alpha'$, and $\gamma$ is essential by Lemma 2.1 of Chap V. ($\gamma$ is monic since $\operatorname{Ker}\gamma \cap \operatorname{Im}\alpha = \operatorname{Ker}\gamma\alpha = \operatorname{Ker}\alpha' = 0$, hence since $\alpha$ is essential, $\operatorname{Ker}\gamma = 0$). Since $\operatorname{Im}\alpha' \in \mathcal{F}(E')$ and $\operatorname{Im}\gamma \supseteq \operatorname{Im}\alpha'$, $\operatorname{Im}\gamma \in \mathcal{F}(E')$. Hence by the usual argument, but using Prop 2.1, $\gamma$ splits and hence since $\gamma$ is essential, $\operatorname{Im}\gamma' = E'$ and $\gamma$ is an isomorphism. □

We will denote the $\mathcal{F}$-injective envelope of $M$ by $E_{\mathcal{F}}(M)$. It can of course be imbedded in $E(M)$, and as a submodule of $E(M)$ it can be described in the following manner (which also establishes the existence of $E_{\mathcal{F}}(M)$)

PROPOSITION 2.2 $E_{\mathcal{F}}(M) = \{x \in E(M) \mid (M:x) \in \mathcal{F}\}$

PROOF Since $(M : x+y) \supseteq (M:x) \cap (M:y)$ and $(M : x \cdot a) = ((M:x) : a)$, $E' = \{x \in E(M) \mid (M:x) \in \mathcal{F}\}$ is a submodule of $E(M)$, and $M \in \mathcal{F}(E')$. The inclusion $M \longrightarrow E'$ is also trivially essential, since $M$ is essential in $E(M)$. (Notice if $y \in E(M)$ is $\mathcal{F}$-torsion, so $\operatorname{Ann}(y) \subseteq (M:y)$ and $\operatorname{Ann}(y) \in \mathcal{F}$, then $y \in E'$). It only remains to show that $E'$ is $\mathcal{F}$-injective. Suppose we are given $\alpha : \mathfrak{a} \longrightarrow E'$ with $\mathfrak{a} \in \mathcal{F}$. $\alpha$ may in any case be extended to a morphism $\beta : A \longrightarrow E(M)$, so we get a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{a} \arrow[r] \arrow[d, "\alpha"'] & A \arrow[r] \arrow[d, "\beta"'] & A/\mathfrak{a} \arrow[r] \arrow[d, "\gamma"'] & 0 \\
0 \arrow[r] & E' \arrow[r] \arrow[d] & E(M) \arrow[r] \arrow[d] & E(M)/E' \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & E'/M \arrow[r] & E(M)/M \arrow[r] & E(M)/E' \arrow[r] & 0
\end{tikzcd}$$

If $Y \neq 0$ then $E(M)/E'$ contains a nonzero-torsion submodule, and from the lowest row we see that this contradicts the fact that $E'$ is the maximal submodule of $E(M)$ containing $M$ as an $\mathcal{F}$-open submodule. More precisely, if $\bar{x} \in E(M)/E'$ is nonzero-torsion, say $\mathfrak{a} = (E' : x) \in \mathcal{F}$ then for all $a \in \mathfrak{a}$, $xa \in E'$ means $(M : xa) \in \mathcal{F}$ – but $(M : xa) = ((M : x) : a)$, and hence by T4, $(M : x) \in \mathcal{F}$ and $x \in E'$ – a contradiction. Hence $Y = 0$, and $\beta$ actually maps $A$ into $E'$. $\square$

## 3. Rational completion

For a given module $M$ we let $\mathcal{F}_M^{\circ}$ denote the strongest Gabriel topology on $A$ for which $M$ is torsion-free. (Example VI. 6.4). It is the Gabriel topology corresponding to the hereditary torsion theory cogenerated by $E(M)$.

<u>4. The canonical topology</u>

For a given module $M$ we let $\mathcal{F}'_M$ denote the Gabriel topology corresponding to the hereditary torsion theory cogenerated by $E(M) \oplus E(E(M)/M)$ (this is $n=1$ in Example VI.6.4).

<u>PROPOSITION 2.15</u> $\mathcal{F}'_M$ is the strongest Gabriel topology $\mathcal{F}$ for which $M$ is $\mathcal{F}$-closed.

<u>PROOF</u> If $M$ is $\mathcal{F}$-injective, then certainly $M \in \mathcal{F}(M)$ and $1_M$ is essential, so since by Prop 2.2 $E_{\mathcal{F}}(M)$ is another $\mathcal{F}$-injective envelope, we get an isomorphism $\gamma : M \longrightarrow E_{\mathcal{F}}(M)$ s.t.

$$\begin{tikzcd} M \arrow[r] & E' \arrow[r] & E(M) \\ M \arrow[u, equal] \arrow[ur, "\gamma"'] & & \end{tikzcd}$$

commutes. Hence $M = E_{\mathcal{F}}(M)$ in $E(M)$, and hence $E(M)/M$ is $\mathcal{F}$-torsion free. Conversely, if $E(M)/M$ is $\mathcal{F}$-torsion free, if $x \in E_{\mathcal{F}}(M)$ then $(M:x) \in \mathcal{F}$, whence $x+M$ is torsion in $E(M)/M$, so $x \in M$. Hence $M$ is $\mathcal{F}$-injective.
If $\mathcal{F}'$ is a Gabriel topology for which $M$ is $\mathcal{F}$-closed, equiv. $\mathcal{F}$-torsion-free and $\mathcal{F}$-injective, then $E(M)$ and $E(M)/M$ are both torsion-free, hence so is $E(M) \oplus E(E(M)/M)$. Hence $\mathcal{F}' \leq \mathcal{F}'_M$. Hence it only remains to show that $M$ is $\mathcal{F}'_M$-closed — but since the torsion-free class is closed under subobjects, both $E(M)$, $E(M)/M$ are torsion-free, so $M$ is $\mathcal{F}'_M$-closed. $\square$

In particular, the topology $\mathcal{F}'_A$ is called <u>the canonical topology</u> on $A$, and it is thus the strongest Gabriel topology on $A$ for which $A$ is its own ring of quotients.

<u>NOTE</u> Let $\mathcal{F}$ be a Gabriel topology on $A$. If $\mathfrak{a} \in \mathcal{F}$, we think of $a \in \mathfrak{a}$ as an arrow $a: A \longrightarrow A$. Here our presheaves are functors $F: A^{op} \longrightarrow \underline{Ab}$ (i.e. right $A$-modules).

Recall that for a general small category $\mathcal{C}$ and $C \in \mathcal{C}$, a <u>sieve</u> on $C$ is a subobject of $H_C$ in $\underline{Sets}^{\mathcal{C}^{op}}$ additive. In $\underline{Ab}^{A^{op}}$, $H_A$ is the ring $A$ considered as a right $A$-module, and subobjects of $H_A$ are right ideals. A Grothendieck topology on $\mathcal{C} = A$ is thus precisely a Gabriel topology. Let $\mathfrak{a} \in \mathcal{F}$ (that is, $\mathfrak{a}$ is a collection of arrows which "cover" $A$). A matching family on $\mathfrak{a}$ for a module $F$ consists of $x_a \in F(A) = M$ for each $a \in \mathfrak{a}$, such that for any $b \in A$, $x_a \cdot b = x_{ab}$. As usual, these are precisely module morphisms $Hom_A(\mathfrak{a}, M)$. An amalgamation is $q \in M$ s.t. $q \cdot a = x_a$, all $a \in \mathfrak{a}$. In particular, let $\mathcal{F}$ be a 1-topology. If $s \in \Sigma(\mathcal{F})$, then $(s)$ is a cover – pick the module $A$ – we claim $\phi: (s) \longrightarrow A$, $\phi(sa) = a$ is a matching family (assume $s$ regular). Hence if $A$ is an $\mathcal{F}$-sheaf (i.e. $\mathcal{F}$-closed) $\phi$ has an amalgamation $s' \in A$ s.t. $s'(sa) = a$, $a \in A$. In particular, $s's = 1$! So there is a connection with normal localisation. The plus-construction can be thought of as "adjoining inverses".

# CHAPTER X : THE CATEGORY OF MODULES OF QUOTIENTS

<u>PROPOSITION 1.4</u> Let $\mathcal{C}$ be a Giraud subcategory of $\mathcal{A}$. An object $C$ of $\mathcal{C}$ is injective iff. $i(C)$ is injective in $\mathcal{A}$.

<u>PROOF</u> Since $i$ preserves monics, it is clear that if $C$ is injective in $\mathcal{A}$, then $C$ is injective also in $\mathcal{C}$. Suppose conversely that $C$ is injective in $\mathcal{C}$. Let there be given a monomorphism $\beta: B \longrightarrow B'$ in $\mathcal{A}$ and $\gamma: B \longrightarrow C$. Then $\gamma$ induces $a(B) \longrightarrow C$ which may be extended in $\mathcal{C}$ to a morphism $a(B') \longrightarrow C$. The composed morphism $B' \longrightarrow a(B') \longrightarrow C$ then extends $\gamma$. □

If $\mathcal{C}$ is a Giraud subcategory of $\mathcal{A}$, and $a: \mathcal{C} \longrightarrow \mathcal{A}$ is the left adjoint to the inclusion, then we consider the class $\mathcal{T}$ of objects $B$ of $\mathcal{A}$ for which $a(B) = 0$, and also the class $\mathcal{F}$ of objects $B$ of $\mathcal{A}$ for which the reflection $B \longrightarrow iaB$ is a monomorphism. (Notice if $\mathcal{A} = \underline{Mod}\text{-}A$, $\mathcal{C} = \underline{Mod}\text{-}(A,\mathcal{F})$ then $a(M) = M_{\mathcal{F}}$ is $0$ iff. $M$ is $\mathcal{F}$-torsion, and $M \longrightarrow iaM = \psi_M$ is monic iff. $M$ is $\mathcal{F}$-torsion free).

<u>PROPOSITION 1.5</u> $(\mathcal{T}, \mathcal{F})$ is a hereditary torsion theory for $\mathcal{A}$.

<u>PROOF</u> From the exactness of $a$ it follows immediately that $\mathcal{T}$ is closed under subobjects, quotient objects and extensions. Since $a$ has a right adjoint, $a$ preserves coproducts, and hence $\mathcal{T}$ is closed under coproducts. $\mathcal{T}$ is thus a hereditary torsion class.

Clearly $\mathrm{Hom}(T, C) = 0$ for $T \in \mathcal{T}$ and $C \in \mathcal{C}$, and it follows that $\mathrm{Hom}(T, F) = 0$ for $T \in \mathcal{T}$ and $F \in \mathcal{F}$. Conversely, if $B$ is an object such that $\mathrm{Hom}(T, B) = 0$ for all $T \in \mathcal{T}$, then $B \in \mathcal{F}$ because the kernel of $B \longrightarrow iaB$ belongs to $\mathcal{T}$. (since we may assume for $C \in \mathcal{C}$, $aC = C$, and $C \longrightarrow iaC$ is $1_C$.). □

Let us now specialise to the case when $\mathcal{A}$ is the category of right modules over a ring $A$. Let $\mathcal{F}$ be a Gabriel topology on $A$.

<u>THEOREM 1.6</u> $\underline{Mod}\text{-}(A,\mathcal{F})$ is a Giraud subcategory of $\underline{Mod}\text{-}A$

<u>PROOF</u> It was proved in Prop IX.1.11 that $\underline{Mod}\text{-}(A,\mathcal{F})$ is a reflective subcategory of $\underline{Mod}\text{-}A$. (It is also replete). It only remains to show that $a$ preserves kernels. But $q: \underline{Mod}\text{-}A \longrightarrow \underline{Mod}\text{-}A_{\mathcal{F}}$ is left exact, as is $\psi_*: \underline{Mod}\text{-}A_{\mathcal{F}} \longrightarrow \underline{Mod}\text{-}A$. Since $ia = \psi_* q$, it follows that if

$$0 \longrightarrow M \longrightarrow N \longrightarrow M'$$

is exact in $A$, then

$$0 \longrightarrow aM \longrightarrow aN \longrightarrow aM'$$

is s.t. its image under $i$ is exact. Hence $a$ preserves kernels. □

$\underline{Mod}\text{-}(A,\mathcal{F})$ is thus a Grothendieck category, and the functor $a: \underline{Mod}\text{-}A \longrightarrow \underline{Mod}\text{-}(A,\mathcal{F})$ is exact. By Prop 1.4:

<u>PROPOSITION 1.7</u> An $\mathcal{F}$-closed module is an injective object in $\underline{Mod}\text{-}(A,\mathcal{F})$ if and only if $M$ is injective in $\underline{Mod}\text{-}A$.

<u>PROPOSITION 1.9</u> Let the injective module $E$ cogenerate the torsion theory associated to $\mathcal{F}$. $E$ is then an injective cogenerator for $\underline{Mod}\text{-}(A,\mathcal{F})$.

<u>PROOF</u> Since injective $\Rightarrow$ $\mathcal{F}$-injective and $E$ is $\mathcal{F}$-torsion-free, $E \in \underline{Mod}\text{-}(A,\mathcal{F})$ and is injective there. It is a cogenerator since $\mathrm{Hom}_{\underline{Mod}\text{-}(A,\mathcal{F})}(M, E) = \mathrm{Hom}_A(M, E) \neq 0$ for every $\mathcal{F}$-closed module $M \neq 0$.

## 2. Gabriel Topologies and Giraud Subcategories

To each Gabriel topology $\mathcal{F}$ on a ring $A$ we have associated the Giraud subcategory $\underline{\mathrm{Mod}}-(A,\mathcal{F})$ of $\underline{\mathrm{Mod}}-A$. We will now show that every Giraud subcategory of $\underline{\mathrm{Mod}}-A$ arises in this way.

<u>THEOREM 2.1</u> There is a bijective correspondence between Gabriel topologies on $A$ and replete Giraud subcategories of $\underline{\mathrm{Mod}}-A$, given by

$$\begin{tikzcd} \text{Gabriel topologies} \arrow[r, bend left, "T"] & \text{Giraud subcategories } \underline{\mathrm{Mod}}-A \arrow[l, bend left, "\Phi"] \end{tikzcd}$$

$$T(\mathcal{F}) = \underline{\mathrm{Mod}}-(A,\mathcal{F})$$
$$\Phi(\mathcal{D}, a : \underline{\mathrm{Mod}}-A \longrightarrow \mathcal{D}) = \{ \mathfrak{a} \mid a(A/\mathfrak{a}) = 0 \}$$

<u>PROOF</u> We have seen that $\underline{\mathrm{Mod}}-(A,\mathcal{F})$ is a Giraud subcategory of $\underline{\mathrm{Mod}}-A$ and is replete. Conversely, given $\mathcal{A}$ and $\mathcal{C}$ replete Giraud subcategory of $\mathcal{A}$, we just saw how $\mathcal{T} = \{ B \mid a(B) = 0 \}$ is a hereditary torsion class, which produces the specified Gabriel topology by Prop VI, 4.2. We have to verify that these two maps are inverse.

If $\mathcal{F}$ is a Gabriel topology then $\Phi T(\mathcal{F}) = \mathcal{F}$ since

$$\begin{aligned} \Phi T(\mathcal{F}) &= \{ \mathfrak{a} \mid \underline{a}(A/\mathfrak{a}) = 0 \} \\ &= \{ \mathfrak{a} \mid A/\mathfrak{a} \text{ is } \mathcal{F}\text{-torsion} \} \\ &= \{ \mathfrak{a} \mid \forall b \in A,\ (\mathfrak{a} : b) \in \mathcal{F} \} \\ &= \mathcal{F}. \end{aligned}$$

Conversely, let $\mathcal{D}$ be a replete, Giraud subcategory of $\underline{\mathrm{Mod}}-A$ with inclusion $i'$ and $a' \dashv i'$. Let

$$\mathcal{F} = \Phi(\mathcal{D}) = \{ \mathfrak{a} \mid a'(A/\mathfrak{a}) = 0 \}$$

and let $i : \underline{\mathrm{Mod}}-(A,\mathcal{F}) \longrightarrow \underline{\mathrm{Mod}}-A$ and $a \dashv i$. We have to show that $\mathcal{D} = \underline{\mathrm{Mod}}-(A,\mathcal{F})$. Firstly we show that $\mathcal{D} \subseteq \underline{\mathrm{Mod}}-(A,\mathcal{F})$ by showing that $D \in \mathcal{D}$ is $\mathcal{F}$-closed. Indeed, if $\mathfrak{a} \in \mathcal{F}$, then $a'(A/\mathfrak{a}) = 0$ and since $a'$ is exact,

$$0 \longrightarrow \mathfrak{a} \longrightarrow A \longrightarrow A/\mathfrak{a} \longrightarrow 0$$
$$0 \longrightarrow a'\mathfrak{a} = a'A \longrightarrow 0 \longrightarrow 0$$

we find $a'\mathfrak{a} = a'A$. But then, considering

$$\begin{tikzcd} \mathfrak{a} \arrow[r] \arrow[d] \arrow[rrd, bend left] & A \arrow[d] \arrow[rd, bend left] & \\ i'a'\mathfrak{a} \arrow[r, equal] \arrow[rr, bend right] & i'a'A & D \end{tikzcd}$$

we easily see that $D$ is s.t. $\mathrm{Hom}_A(A,D) \cong \mathrm{Hom}_A(\mathfrak{a},D)$, using the uniqueness properties of the unit of $a' \dashv i'$. Hence $D \in \underline{\mathrm{Mod}}-(A,\mathcal{F})$. To see that $\underline{\mathrm{Mod}}-(A,\mathcal{F}) \subseteq \mathcal{D}$, we first need to show that if $M$ is an $A$-module, $t(M) = \mathrm{Ker}(M \longrightarrow i'a'M)$. Let $\phi : M \longrightarrow i'a'M$ be canonical — for $x \in M$ let $A \longrightarrow M$ be $a \longmapsto xa$ and let $\mathfrak{a} = \mathrm{Ann}(x)$ be the kernel of this map. Then $i'a'\mathfrak{a} \longrightarrow i'a'A$ is the kernel of $i'a'A \longrightarrow i'a'M$. Consider the diagram

$$\begin{tikzcd} & 0 \arrow[r] & i'a'\mathfrak{a} \arrow[r] & i'a'A \arrow[r] \arrow[d] & i'a'(A/\mathfrak{a}) \\ & & & i'a'M & \\ 0 \arrow[r] & \mathfrak{a} \arrow[r] & A \arrow[r] \arrow[ruu] \arrow[d] & A/\mathfrak{a} \arrow[r] \arrow[u] & 0 \\ & & M \arrow[ru, dash] & & \end{tikzcd}$$

If $\phi(x)=0$ then $A \longrightarrow i'a'A$ factors uniquely through $i'a'\mathfrak{a} \longrightarrow i'a'A$. This induces $a'A \longrightarrow a'\mathfrak{a}$ s.t.

$$\begin{tikzcd}
 & i'a'\mathfrak{a} \arrow[d, shift left] \\
A \arrow[ur] \arrow[dr] & \\
 & i'a'A \arrow[uu, shift left]
\end{tikzcd}$$

and it follows that $i'a'\mathfrak{a} \longrightarrow i'a'A$ is also a retraction – hence an isomorphism. This implies $a'(A/\mathfrak{a})=0$, so $\mathfrak{a}\in\mathcal{F}$ and $x$ is torsion. Conversely if $\mathfrak{a}\in\mathcal{F}$ then $a'(A/\mathfrak{a})=0$, hence $a'\mathfrak{a} \longrightarrow a'A$ is an isomorphism. Hence $i'a'\mathfrak{a} \longrightarrow i'a'A$ is iso, and since it is the kernel of $i'a'A \longrightarrow i'a'M$, this morphism is $0$. Hence $\phi(x)=0$, as required. Thus $\phi(x)=0$ iff. $x\in t(M)$.

Consider the exact sequence

$$0 \longrightarrow \mathrm{Ker}\,\phi_M \longrightarrow M \xrightarrow{\phi_M} i'a'(M) \longrightarrow \mathrm{Coker}\,\phi_M \longrightarrow 0$$

for an $A$-module $M$. Since $a'(\phi_M)$ is the identity, $a'\mathrm{Ker}\,\phi_M$ and $a'\mathrm{Coker}\,\phi_M$ are $0$. Hence, they are $\mathcal{F}$-torsion, and it follows that $a\mathrm{Ker}\,\phi_M$ and $a\,\mathrm{Coker}\,\phi_M$ are $0$. Hence $a(\phi_M): aM \longrightarrow ai'a'(M)$ is an isomorphism. If $M$ is $\mathcal{F}$-closed, then $aM=M$ and since $\mathcal{D}\subseteq \underline{\mathrm{Mod}}\text{-}(A,\mathcal{F})$, also $ai'a'(M)=i'a'(M)$. Hence $a(\phi_M)=\phi_M$ is then an isomorphism of $M$ with $i'a'(M)\in\mathcal{D}$ – hence since $\mathcal{D}$ is replete, $M\in\mathcal{D}$. Hence $\underline{\mathrm{Mod}}\text{-}(A,\mathcal{F})\subseteq\mathcal{D}$, as required. □

(ii) We know that the hereditary torsion theory cogenerated by $E(T(C)) \oplus E\left(E(T(C))/_{T(C)}\right)$ is the strongest one for which $T(C)$ is closed. It similarly follows that if we let $\mathcal{F}$ be the Gabriel topology

$$\mathcal{F} = \bigcap_{C \in \mathcal{C}} \mathcal{F}'_{T(C)} = \{ \mathfrak{a} \subseteq A \mid \mathfrak{a} \in \mathcal{F}'_{T(C)} \text{ all } C \in \mathcal{C} \}$$

then $\mathcal{F} \subseteq \mathcal{F}'_{T(C)}$ each $C \in \mathcal{C}$, so $\mathcal{F}'_{T(C)}$-torsion-free $\Rightarrow$ $\mathcal{F}$-torsion-free. Hence $T(C)$ is $\mathcal{F}$-closed for each $C \in \mathcal{C}$. ($T(C)$ is $\hat{\mathcal{F}}$-closed iff. $E(T(C)) \oplus E(T(C))/T(C)$ is $\hat{\mathcal{F}}$-torsion-free). $\mathcal{F}$ is the strongest Gabriel topology for which all the $T(C)$ are closed, since if $\mathcal{G}$ is another such Gabriel topology, then $T(C)$ being $\mathcal{G}$-closed implies $\mathcal{G} \subseteq \mathcal{F}'_{T(C)}$. Hence each $\mathfrak{a} \in \mathcal{G}$ belongs to each $\mathcal{F}'_{T(C)}$, $C \in \mathcal{C}$, hence to $\mathcal{F}$, so $\mathcal{G} \subseteq \mathcal{F}$. Since every module $T(C)$ is $\mathcal{F}$-closed, there results a diagram of functors:

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, bend left, "T"] \arrow[d, "T'"'] & & \underline{\text{Mod}}\text{-}A \\
\underline{\text{Mod}}\text{-}(A,\mathcal{F}) \arrow[rr, bend left, "a"] & & \arrow[ll, bend left, "i"]
\end{tikzcd}$$

with $T = iT'$. We want to show that $T'$ is an equivalence. Since $iT' = T$ is full and faithful by (i), also $T'$ is full and faithful, and we only have to show that every $\mathcal{F}$-closed module is isomorphic to a module of the form $T(C)$. For each $A$-module $M$ we choose an exact sequence (assume $M$ f.g. so $J$ is finite)

$$A^{(I)} \xrightarrow{\alpha = (\alpha_{ij})} A^{(J)} \longrightarrow M \longrightarrow 0 \qquad \text{(coproducts taken in } \underline{\text{Mod}}\text{-}(A,\mathcal{F}))$$

(we may as well assume $M$ is $\mathcal{F}$-closed). Since $A$ is the endomorphism ring of $U$, $\alpha$ induces in a natural way $\beta_{ij} : U \longrightarrow U$ s.t. $T(\beta_{ij}) = \alpha_{ij}$. We define $\tilde{M} \in \mathcal{C}$ by the exact sequence

$$U^{(I)} \xrightarrow{\beta} U^{(J)} \longrightarrow \tilde{M} \longrightarrow 0 \qquad (2)$$

(we can form $\beta$ since $J$ is finite, hence $U^{(J)}$ a biproduct). We now apply the functor $T'$ to (2). If we knew that $T'$ preserved colimits, we would then get a commutative diagram in $\underline{\text{Mod}}\text{-}(A,\mathcal{F})$.

$$\begin{tikzcd}
A^{(I)} \arrow[r, "\alpha"] \arrow[d, equal] & A^{(J)} \arrow[r] \arrow[d, equal] & M \arrow[r] & 0 \\
A^{(I)} \arrow[r, "T'(\beta) = \alpha"'] & A^{(J)} \arrow[r] & T'(\tilde{M}) \arrow[r] & 0
\end{tikzcd}$$

with exact rows (exact in $\underline{\text{Mod}}\text{-}(A,\mathcal{F})$, not in $\underline{\text{Mod}}\text{-}A$!), and hence $M \cong T'(\tilde{M})$. For general $M$ (not f.g) in $\underline{\text{Mod}}\text{-}(A,\mathcal{F})$, we know that $M$ is the direct limit of its finitely generated submodules in $\underline{\text{Mod}}\text{-}A$. Suppose that $M$ is also $\mathcal{F}$-closed, then reflecting all these f.g. subobjects into $\underline{\text{Mod}}\text{-}(A,\mathcal{F})$ we have $M$ as the colimit of objects all isomorphic to some $T'(\tilde{A})$, and since $T'$ is full, and (by assumption) preserves colimits, we have $M \cong T'(Z)$ for some $Z \in \mathcal{C}$ as well. To conclude the proof, it thus remains to show:

<u>LEMMA 4.2</u> *The functor $T' : \mathcal{C} \longrightarrow \underline{\text{Mod}}\text{-}(A,\mathcal{F})$ is exact and preserves direct sums.*

<u>PROOF</u> $T' = aT$ is left exact ($T = \text{Hom}_{\mathcal{C}}(U,-)$), so to prove exactness it suffices to show that $T'$ preserves epimorphisms. This means that if $\varphi : C' \longrightarrow C''$ is an epimorphism in $\mathcal{C}$, then we should show that $\text{Coker}\, T(\varphi)$ is a $\mathcal{F}$-torsion module. Since $\mathcal{F}$ is cogenerated by the class

$$E(T(C)) \oplus E\left(E(T(C))/_{T(C)}\right) \qquad C \in \mathcal{C}$$

a module $Z$ in $\underline{\text{Mod}}\text{-}A$ is $\mathcal{F}$-torsion iff. $\text{Hom}_A\left(Z, E(T(C)) \oplus E\left(E(T(C))/_{T(C)}\right)\right) = 0$, all $C \in \mathcal{C}$. But since $\oplus$ is a biproduct,

# 4. Representation of Grothendieck Categories

The Theorem of Gabriel and Popescu reduces (at least in principal) the theory of Grothendieck categories to a study of module categories and their quotients.

THEOREM 4.1 (Popescu and Gabriel) Let $\mathcal{C}$ be a Grothendieck category with a generator $U$. Put $A = \mathrm{Hom}_{\mathcal{C}}(U,U)$, and let $T : \mathcal{C} \longrightarrow \underline{\mathrm{Mod}}\text{-}A$ be the functor $T(C) = \mathrm{Hom}_{\mathcal{C}}(U, C)$. Then

(i) $T$ is full and faithful

(ii) $T$ induces an equivalence between $\mathcal{C}$ and the category $\underline{\mathrm{Mod}}\text{-}(A,\mathcal{F})$ where $\mathcal{F}$ is the strongest Gabriel topology on $A$ for which all modules $T(C)$ are $\mathcal{F}$-closed.

PROOF (i) Since $U$ is a generator, the functor $\mathrm{Hom}(U,-)$ is faithful. To see that it is full, we must show that if $C$ and $D$ are objects in $\mathcal{C}$ and $\Phi : \mathrm{Hom}_{\mathcal{C}}(U,C) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(U,D)$ is $A$-linear, then $\Phi$ is of the form

$$\Phi(f) = \varphi f$$

for some $\varphi : C \longrightarrow D$. Let $I$ be the set of all morphisms $U \longrightarrow C$. For each $\alpha \in I$ we let $i_\alpha : U \longrightarrow U^{(I)}$ denote the corresponding injection. There is a unique morphism $\lambda : U^{(I)} \longrightarrow C$ s.t. $\lambda i_\alpha = \alpha$ for $\alpha \in I$, and $\lambda$ is an epimorphism since $U$ is a generator. Similarly there is a unique morphism $\mu : U^{(I)} \longrightarrow D$ such that $\mu i_\alpha = \Phi(\alpha)$ for each $\alpha \in I$. Let $\kappa : K \longrightarrow U^{(I)}$ be the kernel of $\lambda$. If we can show that $\mu\kappa = 0$, then $\mu$ factors as $\mu = \varphi\lambda$ for some $\varphi : C \longrightarrow D$, and for each $\alpha : U \longrightarrow C$ we get

$$\Phi(\alpha) = \mu i_\alpha = \varphi \lambda i_\alpha = \varphi\alpha$$

and our assertion would be proved.

So we set about proving that $\mu\kappa = 0$. For each finite subset $J$ of $I$ and each $\alpha \in J$ there are canonical morphisms $\pi'_\alpha : U^{(J)} \longrightarrow U$, $i'_\alpha : U \longrightarrow U^{(J)}$ and $i_J : U^{(J)} \longrightarrow U^{(I)}$. Let $\kappa_J : K_J \longrightarrow U^{(J)}$ be the kernel of the composed morphism $\lambda i_J : U^{(J)} \longrightarrow C$:

$$\begin{tikzcd}
 & K_J \arrow[r, "\kappa_J"] \arrow[d] & U^{(J)} \arrow[r, shift right, "\pi'_\alpha"'] \arrow[d] & U \arrow[l, shift right, "i'_\alpha"'] \arrow[dl, "i_\alpha"] \arrow[d, "\alpha"] & \\
0 \arrow[r] & K \arrow[r, "\kappa"'] & U^{(I)} \arrow[r, "\lambda"'] & C \arrow[r] & 0
\end{tikzcd}$$

Since $K$ is the direct limit of the kernels $K_J$ for all finite subsets $J$ of $I$, (this is a simple consequence of direct limits being exact), it suffices to show that $\mu i_J \kappa_J = 0$ (since $\mathrm{Ker}(\mu\kappa) = \sum \mathrm{Ker}(\mu\kappa) \cap K_J = 0$, since $\mathrm{Ker}(\mu\kappa) \cap K_J = \mathrm{Ker}(\mu i_J \kappa_J) = 0$). Now for each $\beta : U \longrightarrow K_J$ we have, using the $A$-linearity of $\Phi$, that

$$\begin{aligned}
\mu i_J \kappa_J \beta &= \mu i_J \Big(\sum_{\alpha \in J} i'_\alpha \pi'_\alpha\Big) \kappa_J \beta \\
&= \sum_{\alpha \in J} \mu i_\alpha \pi'_\alpha \kappa_J \beta = \sum_{\alpha \in J} \Phi(\alpha)\, \pi'_\alpha \kappa_J \beta \\
&= \sum_{\alpha \in J} \Phi(\alpha \pi'_\alpha \kappa_J \beta) = \Phi\Big(\sum_{\alpha \in J} \lambda i_J i'_\alpha \pi'_\alpha \kappa_J \beta\Big) \\
&= \Phi(\lambda i_J \kappa_J \beta) = 0
\end{aligned}$$

since $\lambda i_J \kappa_J = 0$. Since this holds for all $\beta : U \longrightarrow K_J$, it follows that $\mu i_J \kappa_J = 0$.

PROPOSITION Let $M$ be an $A$-module and $\mathcal{F}'_M$ the strongest topology for which $M$ is $\mathcal{F}$-closed — that is, $\mathcal{F}'_M$ is cogenerated by $E(M)$. Then $\mathfrak{a}\in\mathcal{F}'_M$ iff. $\mathfrak{a}\longrightarrow A$ induces

$$\mathrm{Hom}_A(A,M)\cong\mathrm{Hom}_A(\mathfrak{a},M) \qquad (1)$$

PROOF Let $\mathcal{F}=\mathcal{F}'_M$. Since $M$ is $\mathcal{F}$-closed, the condition is clearly necessary. To prove sufficiency, note that $\mathcal{F}'_M$ is $\mathcal{F}^0_{M\oplus E(M)/M}$, so by Prop 5.5, $\mathfrak{a}\in\mathcal{F}$ iff. $x(\mathfrak{a}:a)\neq 0$ for every $a\in A$ and $0\neq x\in M\oplus E(M)/M$. Or, by cases, we show that for $0\neq x\in M$ and $a\in A$, $x(\mathfrak{a}:a)\neq 0$. Let $a\in A$ and consider the exact sequence of $A$-modules

$$0\longrightarrow(\mathfrak{a}:a)\longrightarrow A\xrightarrow{1\mapsto a+\mathfrak{a}} A/\mathfrak{a}$$

applying $\mathrm{Hom}_A(-,M)$ we have the exact sequence

$$\mathrm{Hom}_A((\mathfrak{a}:a),M)\longleftarrow\mathrm{Hom}_A(A,M)\longleftarrow\mathrm{Hom}_A(A/\mathfrak{a},M)$$

since $\mathrm{Hom}_A(A/\mathfrak{a},M)$ consists of $x\in M$ s.t. $x\mathfrak{a}=0$, and by (1) for $x\in M$ $x=0$ iff. $x\mathfrak{a}=0$, we find $\mathrm{Hom}_A(A/\mathfrak{a},M)=0$. Hence $\mathrm{Hom}_A(A,M)\longrightarrow\mathrm{Hom}_A((\mathfrak{a}:a),M)$ is monic. That is, for $x\in M$, $x=0$ iff. $x(\mathfrak{a}:a)=0$. Hence if $x\neq 0$ in $M$, $x(\mathfrak{a}:a)\neq 0$, as required.

Secondly, we have to show that if $y\in E(M)$ and $y\notin M$ then for any $a\in A$, $y(\mathfrak{a}:a)\notin M$.

$Z$ is $\mathcal{F}$-torsion iff. $\mathrm{Hom}_A(Z, E(T(C))) = 0$ (equiv. $\mathrm{Hom}_A(C, T(C)) = 0$, each cyclic submodule $C$ of $Z$, Lemma 3.8, ChVI), and $\mathrm{Hom}_A(Z, E(E(T(C))/T(C))) = 0$, equiv. $\mathrm{Hom}_A(C, E(T(C))/T(C)) = 0$ each cyclic submodule $C \subseteq Z$.

THEOREM (Gabriel and Popescu) Let $\mathcal{C}$ be a Grothendieck category with generator $U$. Put $A = \mathrm{Hom}_{\mathcal{C}}(U,U)$ and let $T : \mathcal{C} \longrightarrow \text{Mod-}A$ be $T(C) = \mathrm{Hom}_{\mathcal{C}}(U,C)$. Then $T$ is fully faithful and has an exact left adjoint.

PROOF Since $U$ is a generator $T$ is faithful, and it is clearly distinct on objects. To prove fullness we prove the following Lemma:

LEMMA Consider a diagram of $A$-modules

$$\begin{tikzcd} M \arrow[r, hook] \arrow[d, "f"'] & \mathrm{Hom}_{\mathcal{C}}(U,B) \\ \mathrm{Hom}_{\mathcal{C}}(U,C) & \end{tikzcd}$$

where the top arrow is the inclusion of a submodule. Consider also the induced diagram in $\mathcal{C}$

$$\begin{tikzcd} {}^{M}U \arrow[r, "\psi"] \arrow[d, "\phi"'] & B \\ C & \end{tikzcd}$$

where, denoting the coproduct injections by $u_m$, we have

$$\psi u_m = m, \qquad \phi u_m = f(m)$$

Then we claim $\phi$ factors through $\mathrm{Im}\,\psi$.

PROOF Let $\mu : K \longrightarrow {}^{M}U$ be the kernel of $\psi$. The assertion is equivalent to $\phi\mu = 0$. If $F \subseteq M$ is a finite subset, form the pullback

$$\begin{tikzcd} K' \arrow[r, "\mu'"] \arrow[d, "\lambda"'] & {}^{F}U \arrow[d, "\Sigma_F u_m p_m"] \\ K \arrow[r, "\mu"'] & {}^{M}U \end{tikzcd}$$

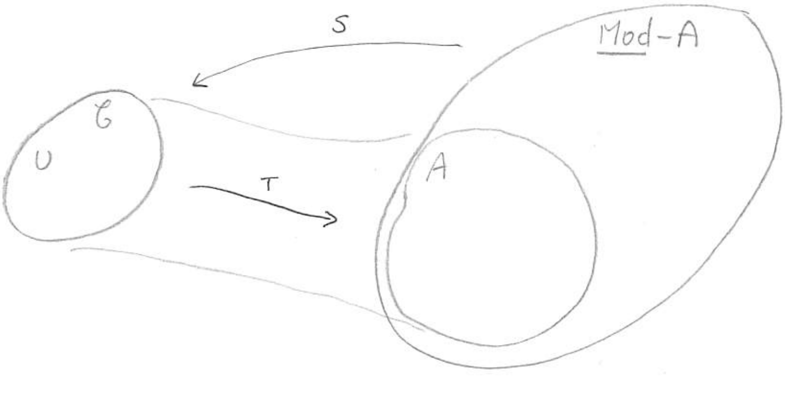
where $p_m$ denotes $m$th projection for the finite coproduct. By Ab5 it suffices to show $\phi\mu\lambda = 0$, and for this, since $U$ is a generator, it suffices to show $\phi\mu\lambda\alpha = 0$ for all $\alpha \in \mathrm{Hom}_{\mathcal{C}}(U, K')$. But we have

$$\phi\mu\lambda\alpha = \phi \textstyle\sum u_m p_m \mu'\alpha = \sum f(m)(p_m\mu'\alpha) = \sum f(m \cdot p_m\mu'\alpha)$$
$$= f\left(\textstyle\sum m \cdot p_m\mu'\alpha\right) = f\left(\textstyle\sum \psi u_m p_m \mu'\alpha\right) = f(\psi\mu\lambda\alpha) = f(0) = 0. \square$$

To see $T$ is full, put $M = \mathrm{Hom}_{\mathcal{C}}(U,B)$, then $\psi$ is epi so $\phi$ factors through $\psi$, as required.

This also shows that $T$ preserves injectives, since if $C$ is injective then using the Lemma and injectivity of $C$, again $\phi$ factors through $\psi$. Then taking $B = U$ we see that every homomorphism from a right ideal $M$ to $T(U) = A$ extends to the whole ring. Thus $T(C)$ is injective. Since $T$ has a left adjoint (see Mitchell notes), and $\mathcal{C}$ has enough injectives, the fact that $T$ preserves injectives implies that the left adjoint is exact, as required. $\square$

We have the following situation:




using $T$, identify $\mathscr{C}$ with the full subcategory of Mod-$A$ consisting of the right $A$-modules $\mathrm{Hom}_{\mathscr{C}}(U,C)$, $C \in \mathscr{C}$. Then this subcategory's inclusion has an exact left adjoint: the reflection of an $A$-module $M$ is the $A$-module

$$\mathrm{Hom}_{\mathscr{C}}(U, M \otimes_A U)$$

together with $\eta_M : M \longrightarrow \mathrm{Hom}_{\mathscr{C}}(U, M \otimes_A U)$ given by

$$\eta_M(x) = x\text{th coordinate of the canonical epi } {}^{M}U \longrightarrow M \otimes_A U.$$

Now let $\mathscr{C}'$ be the replete closure of this subcategory. We know it is reflective, where the reflection $\underline{a} : \text{Mod-}A \longrightarrow \mathscr{C}'$ is just the reflection to $\mathscr{C} \subseteq$ Mod-$A$ followed by the inclusion $\mathscr{C} \hookrightarrow \mathscr{C}'$. To show it is Giraud, notice that if $\phi : M \longrightarrow M'$ is monic in Mod-$A$, then its reflection to $\mathscr{C}$ (which is $TS$) is monic. Hence it is monic in $\mathscr{C}$, so $\mathscr{C}'$ is Giraud. One can also draw:

$$\begin{tikzcd}
\mathscr{C} \arrow[rr, bend right, "T"'] \arrow[dr, bend left, "i"] & & \text{Mod-}A \arrow[ll, bend right, "S"'] & \\
& \mathscr{A} \arrow[ur, "o"] \arrow[ul, bend left, "i'"] \arrow[rr, bend left, "j \cong"] & & \mathscr{A}' \arrow[ul, "q"'] \arrow[ll, bend left, "j'"]
\end{tikzcd}$$

where $S \dashv T$, $\mathscr{C} \longrightarrow \mathscr{A}$ is an isomorphism, hence has an adjoint $i'$ and $\mathscr{A} \longrightarrow \mathscr{A}'$ is an equivalence, hence has an adjoint $j'$, so $i \dashv i'$, $j \dashv j'$, and hence $jiS \dashv Ti'j'$, but $Ti' = \mathscr{A} \hookrightarrow$ Mod-$A$, so $jiS \dashv oj' \cong q$. Since $\mathscr{A}'$ is a Giraud subcategory, it is Mod-$(A,\mathcal{F})$ where $\mathcal{F}$ is the Gabriel topology

$$\mathcal{F} = \{\mathfrak{a} \mid \underline{a}(A/\mathfrak{a}) = 0\}$$

where $\underline{a} : \text{Mod-}A \longrightarrow \mathscr{A}'$ is the reflection. But $\underline{a}(A/\mathfrak{a})$ is the right $A$-module $\mathrm{Hom}_{\mathscr{C}}(U, A/\mathfrak{a} \otimes_A U)$, and since the zero object in Mod-$(A,\mathcal{F})$ is the same as the zero in Mod-$A$, $\underline{a}(A/\mathfrak{a}) = 0$ iff. the only morphism in $\mathrm{Hom}_{\mathscr{C}}(U, A/\mathfrak{a} \otimes_A U)$ is the zero morphism. But $U$ is a generator, so this is iff. $A/\mathfrak{a} \otimes_A U = 0$ in $\mathscr{C}$. So

$$\mathcal{F} = \{\mathfrak{a} \mid A/\mathfrak{a} \otimes_A U = 0\}$$

Also notice that for any right ideal $\mathfrak{a}$ of $A$, since $- \otimes_A U$ is exact, we have an exact sequence in $\mathscr{C}$

$$0 \longrightarrow \mathfrak{a} \otimes_A U \longrightarrow A \otimes_A U \longrightarrow A/\mathfrak{a} \otimes_A U \longrightarrow 0$$

so $A/\mathfrak{a} \otimes_A U = 0$ iff. $\mathfrak{a} \otimes_A U \longrightarrow A \otimes_A U$ is an isomorphism. Since $T$ certainly gives an equivalence of $\mathscr{C}$ with Mod-$(A,\mathcal{F})$, we have

THEOREM Let $\mathscr{C}$ be a Grothendieck category with a generator $U$. Let $A = \mathrm{Hom}_{\mathscr{C}}(U,U)$ and $H^U : \mathscr{C} \longrightarrow$ Mod-$A$, $- \otimes_A U :$ Mod-$A \longrightarrow \mathscr{C}$, and

$$\mathcal{F} = \{\mathfrak{a} \mid A/\mathfrak{a} \otimes_A U = 0\}$$

then $\mathscr{C}$ is equivalent to Mod-$(A,\mathcal{F})$.