Consider the category $\text{Top}$, which is complete and cocomplete and has a terminal object $1$ consisting of the singleton set $\{\ast\}$ with the discrete topology. Hence we can apply the ideas of this Chapter to topological spaces. This example is particularly important since the axioms for a Gabriel topology (and hence for Grothendieck topologies) arise naturally in the study of topological rings.

## 1 Topological Groups

To begin with, we define a topological group to be a group object in $\text{Top}$. Since any morphism $1 \to A$ is continuous, this reduces to the following definition:

**Definition 1.** A topological group is an abelian group $A$ together with a topology on $A$ such that the maps

$$a : A \times A \to A, \quad (a, b) \mapsto a + b$$

$$v : A \to A, \quad a \mapsto -a$$

are continuous. For any subsets $U, V \subseteq A$ we define $U + V = \{u + v \mid u \in U, v \in V\}$, and $-U = \{-u \mid u \in U\}$. The map $v$ is clearly a homeomorphism, so if $U$ is open then $-U$ is also open.

**Lemma 1.** Let $A$ be a topological group. If $c \in A$ then the map $A \to A$ defined by $x \mapsto c + x$ is a homeomorphism.

**Proof.** The subspace $\{c\} \times A$ of $A \times A$ is clearly homeomorphic to $A$ via $(c, b) \mapsto b$, and the restriction of $a$ to $\{c\} \times A$ is continuous, and obviously bijective. Thus there is a continuous bijection $A \to A$ defined by $x \mapsto c + x$. Clearly the morphism $x \mapsto -c + x$ is an inverse, and so we have a homeomorphism $x \mapsto c + x$ for each $c \in A$. \qed

**Remark 1.** Let $A$ be a topological group. If $U \subseteq A$ is open and $c \in A$ then the set $U + c = \{u + c \mid u \in U\}$ is also open. Taking unions, we see that the sum $U + V$ of any two open sets $U, V$ is open. If $c \in A$ then $U$ is an open neighborhood of $c$ if and only if $U - c$ is an open neighborhood of $0$, so the topology of $A$ is completely determined by the open neighborhoods of $0$.

**Definition 2.** Let $X$ be a topological space. If $x \in X$ then a fundamental system of neighborhoods of $x$ is a nonempty set $\mathcal{M}$ of open neighborhoods of $x$ with the property that if $U$ is open and $x \in U$, then there is $V \in \mathcal{M}$ with $V \subseteq U$.

**Proposition 2.** Let $A$ be a topological group. Then the set $\mathcal{N}$ of open neighborhoods of $0$ satisfies

1. For $U \in \mathcal{N}$ and $c \in U$ there exists $V \in \mathcal{N}$ such that $c + V \subseteq U$.

2. For each $U \in \mathcal{N}$, there exists $V \in \mathcal{N}$ such that $V + V \subseteq U$.

3. If $U \in \mathcal{N}$ then $-U \in \mathcal{N}$.\[N0\]
If $A$ is any abelian group and $N$ a nonempty set of subsets of $A$ which satisfies $N_0, N_1, N_2$ and has the property that (a) every element of $N$ contains $0$ and (b) if $U, V \in N$ then there is $W \in N$ with $W \subseteq U \cap V$ then there is a unique topology on $A$ making $A$ into a topological group in such a way that $N$ is a fundamental system of neighborhoods of $0$.

Proof. Let $A$ be a topological group with $N$ as described. Condition $N_0$ follows from the fact that the map $x \mapsto x - c$ is a homeomorphism. For $N_1$, let $U \in N$ be given. Since $a$ is continuous and $(0, 0) \in a^{-1}U$, there are open sets $V_1, V_2$ with $(0, 0) \in V_1 \times V_2 \subseteq a^{-1}U$. Set $V = V_1 \cap V_2$. The condition $N_2$ follows from the fact that $v$ is continuous.

For the converse, let $A$ be an abelian group and $N$ a nonempty set of subsets of $A$ with the given properties. We define a subset $U \subseteq A$ to be open if for every $x \in U$ there is $W \in N$ with $x + W \subseteq U$. It is easy to check that this is a topology. Condition $N_0$ implies that the elements of $N$ are open sets.

Next, we claim that if $U$ is open then $c + U$ is open for any $c \in A$. This is clear since if $b \in c + U$ then $b - c \in U$ and so there is $V \in N$ with $b - c + V \subseteq U$. Thus $b + V \subseteq c + U$, and $c + U$ is open.

To show that $A$ is a topological group we have to show that the maps $a : A \times A \to A$ and $v : A \to A$ are continuous. Let $U$ be an open set, and suppose $(c, d) \in a^{-1}U$. Then $c + d \in U$, so there is $W \in N$ such that $c + d + W \subseteq U$. Using $N_1$, let $Q \in N$ be such that $Q + Q \subseteq W$. Then $(c, d) \in (c + Q) \times (d + Q) \subseteq a^{-1}U$ which shows that $a^{-1}U$ is open. Therefore $a$ is continuous. To see that $v$ is continuous, we have to show that if $U$ is open then so is $U$. But if $c \in -U$ there is $V \in N$ such that $c - V \subseteq U$. Hence $c - V \subseteq -U$, and since $-V \in N$ by $N_2$, we are done. This shows that $A$ is a topological group. Suppose that $\mathcal{J}$ is another topology on $A$ with respect to which $A$ is a topological group, and suppose further that the set $N$ is a fundamental system of neighborhoods of $0$ in this topology. It is not hard to see that this topology must agree with the one we have just defined, which is therefore unique with these properties.

2 Topological Rings

Definition 3. A topological ring is a ring $A$ with a topology making $A$ into an additive topological group, such that the multiplication $m : A \times A \to A, (b, c) \mapsto bc$ is a continuous map. For any subsets $V, W \subseteq A$ we define $V \cdot W = \{vw \mid v \in V, w \in W\}$.

Lemma 3. Let $A$ be a topological ring. If $c \in A$ then the maps $A \to A$ defined by $x \mapsto cx$ and $x \mapsto xc$ are continuous.

Proof. The subspace $\{c\} \times A$ of $A \times A$ is clearly homeomorphic to $A$ via $(c, b) \mapsto b$, and the restriction of $m$ to $\{c\} \times A$ is continuous. The same argument works on the right.

Proposition 4. Let $A$ be a topological ring. Then the set $N$ of open neighborhoods of $0$ satisfies $N_0, N_1, N_2$ and also

$N_3$. For $c \in A$ and $U \in N$ there is $V \in N$ such that $cV \subseteq U$ and $Vc \subseteq U$.

$N_4$. For each $U \in N$ there is $V \in N$ such that $V \cdot V \subseteq U$.

Conversely, if $A$ is any ring and $N$ a nonempty set of subsets of $A$ which satisfies $N_0, N_1, N_2, N_3$ and $N_4$ and has the property that (a) every element of $N$ contains $0$ and (b) if $U, V \in N$ then there is $W \in N$ with $W \subseteq U \cap V$ then there is a unique topology on $A$ making $A$ into a topological ring in such a way that $N$ is a fundamental system of neighborhoods of $0$.

Proof. Suppose that $A$ is a topological ring. Then by Proposition 2 the set $N$ satisfies $N_0, N_1, N_2$. Condition $N_3$ follows easily from continuity of the maps $x \mapsto cx$ and $x \mapsto xc$. For $N_4$, let $U \in N$ be given and use the fact that $m$ is continuous to find $V_1, V_2 \in N$ such that $(0, 0) \in V_1 \times V_2 \subseteq m^{-1}U$. Then set $V = V_1 \cap V_2$. 

2
Conversely, suppose we are given a ring \( A \) and a nonempty set of subsets of \( A \) with the given properties. With the topology defined in Proposition 2, \( A \) becomes a topological group. We have to show that with this topology, \( A \) is a topological ring.

First we show that for \( c \in A \) the map \( \theta : A \rightarrow A \) defined by \( x \mapsto cx \) is continuous. Let \( U \subseteq A \) be open and suppose that \( x \in \theta^{-1}U \), that is, \( cx \in U \). By definition there is \( W \in \mathcal{N} \) with \( cx + W \subseteq U \). Let \( V \in \mathcal{N} \) be such that \( cV \subseteq W \). Then \( x + V \subseteq \theta^{-1}U \). Therefore \( \theta^{-1}U \) is open and \( \theta \) is continuous. Similarly we show that the map \( x \mapsto xc \) is continuous.

We are now ready to show that the product \( m : A \times A \rightarrow A \) is continuous. Let \( U \subseteq A \) be open and suppose \((c, d) \in m^{-1}U \). Let \( \theta : A \rightarrow A \) be left multiplication by \( c \) and \( \psi : A \rightarrow A \) right multiplication by \( d \). Then \( cd \in U \), so there is \( W \in \mathcal{N} \) such that \( cd + W \subseteq U \). Let \( Q \in \mathcal{N} \) satisfy \( Q + Q \subseteq W \), and using \( \mathcal{N}4 \), let \( V \in \mathcal{N} \) be such that \( V \cdot V \subseteq Q \). Let \( P \in \mathcal{N} \) be such that \( P + P \subseteq Q \), and set \( P_{d} = \psi^{-1}(P) \cap Q \cap V \) and \( P_{c} = \theta^{-1}(P) \cap Q \cap V \). Then

\[
(c, d) \in (c + P_{d}) \times (d + P_{c}) \subseteq m^{-1}U
\]

since for \( v \in P_{d}, v' \in P_{c}, (c + v)(d + v') = cd + cv' + vd + vv' \in cd + W \subseteq U \). It follows that \( m \) is continuous, as required. It follows from Proposition 2 that this topology is the unique topology making \( A \) into a topological group with \( \mathcal{N} \) a fundamental system of neighborhoods of 0. \( \square \)

**Definition 4.** Let \( A \) be a ring. A nonempty set \( \mathcal{N} \) of subsets of \( A \) is **fundamental** if it satisfies the following conditions

(a) Every element of \( \mathcal{N} \) contains 0.

(b) If \( U, V \in \mathcal{N} \) then there is \( W \in \mathcal{N} \) with \( W \subseteq U \cap V \).

N0. For \( U \in \mathcal{N} \) and \( c \in U \) there exists \( V \in \mathcal{N} \) such that \( c + V \subseteq U \).

N1. For each \( U \in \mathcal{N} \), there exists \( V \in \mathcal{N} \) such that \( V + V \subseteq U \).

N2. If \( U \in \mathcal{N} \) then \( -U \in \mathcal{N} \).

N3. For \( c \in A \) and \( U \in \mathcal{N} \) there is \( V \in \mathcal{N} \) such that \( cV \subseteq U \) and \( Vc \subseteq U \).

N4. For each \( U \in \mathcal{N} \) there is \( V \in \mathcal{N} \) such that \( V \cdot V \subseteq U \).

By Proposition 4 if \( A \) is a topological ring, then the set \( \mathcal{N} \) of open neighborhoods of 0 is fundamental. Conversely, if \( A \) is a ring and \( \mathcal{N} \) a fundamental set of subsets of \( A \), then there is a unique topology on \( A \) making \( A \) into a topological ring in such a way that \( \mathcal{N} \) is a fundamental system of neighborhoods of 0. We call this the topology generated by \( \mathcal{N} \).

**Proposition 5.** Let \( A \) be a ring and \( \mathcal{G} \) a nonempty set of right ideals. Suppose that the following conditions are satisfied

\( T1. \) If \( a \in \mathcal{G} \) and \( a \subseteq b \) for a right ideal \( b \), then \( b \in \mathcal{G} \).

\( T2. \) If \( a \) and \( b \) belong to \( \mathcal{G} \), then \( a \cap b \in \mathcal{G} \).

\( T3. \) If \( a \in \mathcal{G} \) and \( a \in A \), then \( (a : a) \in \mathcal{G} \).

Then the set \( \mathcal{G} \) is fundamental, and \( \mathcal{G} \) is precisely the set of open right ideals in the generated topology on \( A \).

**Proof.** The axioms (a), N0, N1, N2, N4 are trivially verified and (b) follows from T2. To check N3, let \( a \in A \) and \( a \in \mathcal{N} \) be given. By assumption the right ideal \( (a : a) = \{ x \in A | ax \in a \} \) belongs to \( \mathcal{G} \). Set \( a \mathcal{G} = (a : a) \cap a \), which is in \( \mathcal{G} \) by T2. Clearly \( a \mathcal{G} \subseteq a \) and \( a \mathcal{G} \subseteq a \), which shows that \( \mathcal{G} \) is fundamental. Give \( A \) the topology generated by \( \mathcal{G} \). Every ideal in \( \mathcal{G} \) is open in this topology. If \( b \) is an open right ideal of \( A \) then \( b \supseteq a \) for some \( a \in \mathcal{G} \), and therefore \( b \in \mathcal{G} \) by T1. \( \square \)

**Lemma 6.** Let \( A \) be a topological ring. Then the set \( \mathcal{G} \) of all open right ideals of \( A \) satisfies the conditions \( T1, T2, T3 \).
Proof. If $a$ is open and $a \subseteq b$, then for $x \in b$ we have $x + a \subseteq b$. As the union of such open sets, $b$ is open. The condition $T2$ is trivial. If $a$ is open, $a \in A$ and $x \in (a : a)$, then $ax \in a$, and since the whole collection of open neighborhoods of 0 satisfies the condition $N3$, we may find a neighborhood $V$ of 0 such that $aV \subseteq a$. Hence $x + V \subseteq (a : a)$, which is thus open.

Definition 5. A right linear topological ring is a topological ring $A$ which admits a fundamental system of neighborhoods of 0 consisting of right ideals (necessarily open).

Lemma 7. Let $A$ be a topological ring. Then $A$ is right linear topological if and only if the topology on $A$ is the one generated by the set of all open right ideals.

Proof. Let $A$ be a topological ring, and let $\mathcal{G}$ be the set of all open right ideals. If the topology on $A$ is the one generated by $\mathcal{G}$, then of course $A$ is a right linear topological ring. For the converse, suppose that $A$ is right linear topological. Then $\mathcal{G}$ must be a fundamental system of open neighborhoods of 0, and the uniqueness part of Proposition 4 implies that the topology on $A$ must be the one generated by $\mathcal{G}$. 

\[ \Box \]