Notes on A_∞-algebras (checked)

Our basic references are

[K] B. Keller "Introduction to A_∞-algebras and modules"

[P] M. Penkava "Infinity algebras, cohomology, ..."

[C] N. Carqueville PhD. thesis

[KS] M. Kontsevich, Y. Soibelman "Notes on A_∞-algebras, ..."

We mainly follow [K] with some deviations. Throughout k is a field, and we work with $\mathbb{Z}$-graded k-vector spaces. But everything has been checked also for $\mathbb{Z}_2$-graded k-spaces. We write $M(i)$ for the grading shift $M(i)_n = M_{i+n}$, and here to the Koszul sign rule $(f \otimes g)(x \otimes y) = (-1)^{|g||x|} fx \otimes gy$. Here an <u>algebra</u> over k need not be commutative or have a unit, and a $\mathbb{Z}$-graded algebra can have degrees ≤ 0 (so a graded algebra means $\mathbb{Z}$-graded)

<u>Defn</u> An <u>A_∞-algebra</u> over k is a $\mathbb{Z}$-graded vector space

$$A = \bigoplus_{p \in \mathbb{Z}} A^p$$

endowed with maps (k-linear)

$$m_n : A^{\otimes n} \longrightarrow A, \quad n \geq 1$$

of degree $2-n$ satisfying the relations

- $m_1^2 = 0$, i.e. (A, m_1) is a complex. Notice $|m_1| = 1$.

- $m_1 m_2 = m_2(m_1 \otimes \mathbb{1} + \mathbb{1} \otimes m_1)$ as maps $A^{\otimes 2} \longrightarrow A$, so m_1 is a (graded) derivation with respect to the multiplication m_2. Also that $m_2 : A^{\otimes 2} \longrightarrow A$ is a morphism of complexes.

- $m_2(\mathbb{1} \otimes m_2 - m_2 \otimes \mathbb{1}) = m_1 m_3 + m_3(m_1 \otimes \mathbb{1} \otimes \mathbb{1} + \mathbb{1} \otimes m_1 \otimes \mathbb{1} + \mathbb{1} \otimes \mathbb{1} \otimes m_1)$ as maps $A^{\otimes 3} \longrightarrow A$. Note the LHS is the associator for m_2 and the RHS is the boundary of m_3 in $\mathrm{Hom}_k(A^{\otimes 3}, A)$. Hence m_2 is associative up to homotopy.

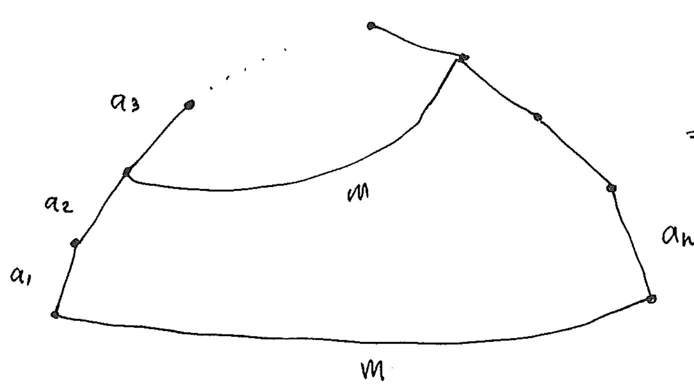
- More generally, for $n \geq 1$ we require

$$\sum (-1)^{r+st} m_u(\mathbb{1}^{\otimes r} \otimes m_s \otimes \mathbb{1}^{\otimes t}) = 0 \qquad (2.1)$$

where the sum runs over all decompositions $n = r+s+t$ and we put $u = r+1+t$, allowing $r=0$ or $t=0$, but $s \geq 1$ (hence both $r=t=0$) is OK. This is an equality of maps $A^{\otimes(r+s+t)} \longrightarrow A^{\otimes(r+1+t)} \longrightarrow A$. This recovers the above for $n = 1, 2, 3$.

The defining identities (2.1) are given pictorially by

$\sum \pm$



$= 0$

This is $m(\mathbb{1}^{\otimes 2} \otimes m \otimes \mathbb{1}^{n-3})$
i.e. sum over all ways to use m's twice on $A^{\otimes n}$

<u>Remark</u> Let $Z^*A = Z^*(A, m_1)$ and $H^*A = H^*(A, m_1)$.

(a) Then it is clear that

$$m_2(Z^{\otimes 2}) \subseteq Z$$

but there is no a priori reason for $m_n(Z^{\otimes n}) \subseteq Z$ for $n > 2$. Indeed evaluated on $Z^{\otimes 3}$ we have

$$m_2(\mathbb{1} \otimes m_2 - m_2 \otimes \mathbb{1}) = m_1 m_3$$

so $m_3(Z^{\otimes 3}) \subseteq Z$ would imply that m_2 is strictly associative on cocycles. In any case, m_2 also sends $B^{\otimes 2}$ to B (B for boundaries) so there is an induced product (again of degree zero)

$$m_2^H : (H^*A)^{\otimes 2} \longrightarrow H^*A$$

(actually what we need is that $m_2(Z \otimes B) \subseteq B$ and $m_2(B \otimes Z) \subseteq Z$, which is true). Moreover this product on H^*A is clearly <u>associative</u>, so an associative graded algebra.

(b) If $A^p = 0$ for $p \neq 0$ then $A = A^0$ is an ordinary associative algebra (all m's apart from m_2 vanish)

(c) If $m_n = 0$ for $n \geq 3$ then A is an associative dg-algebra, i.e. a $\mathbb{Z}$-graded k-vector space A with k-linear $A^{\otimes 2} \longrightarrow A$ and $d : A \longrightarrow A$ of degree one such that the product is associative and $d^2 = 0$, and

$$d(x \otimes y) = d(x) \otimes y + (-1)^{|x|} x \otimes dy).$$

Conversely each such dg-algebra yields an A_∞-algebra with $m_n = 0$, $n \geq 3$.

Morphisms

Def^N A *morphism* of A_∞-algebras $f : A \longrightarrow B$ is a family

$$f_n : A^{\otimes n} \longrightarrow B$$

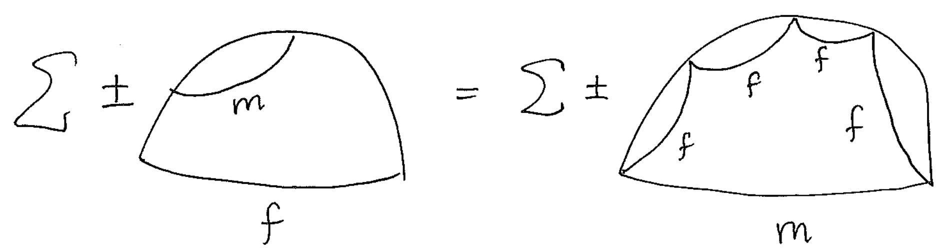
of maps k-linear of degree $1-n$ such that

- $f_1 m_1 = m_1 f_1$, i.e. f_1 is a morphism of complexes
- $f_1 m_2 = m_2(f_1 \otimes f_1) + m_1 f_2 + f_2(m_1 \otimes \mathbb{1} + \mathbb{1} \otimes m_1)$
  i.e. f_1 commutes with multiplication up to the homotopy f_2
- More generally for $n \geq 1$, we demand an equality of maps $A^{\otimes n} \longrightarrow B$

$$\sum (-1)^{r+st} f_u (\mathbb{1}^{\otimes r} \otimes m_s \otimes \mathbb{1}^{\otimes t}) = \sum (-1)^q m_r (f_{i_1} \otimes \cdots \otimes f_{i_r})$$

where the first sum is over all decompositions $n = r+s+t$, $u = r+1+t$ ($s \geq 1$ as before, $r,t \geq 0$), the second sum runs over all $1 \leq r \leq n$ and $n = i_1 + \cdots + i_r$ with $i_j \geq 1$, the sign q is

$$q = (r-1)(i_1 - 1) + (r-2)(i_2 - 1) + \cdots + 2(i_{r-2} - 1) + (i_{r-1} - 1)$$



NOTE The r on the RHS has nothing to do with that on the left.

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We say f is a quasi-isomorphism if f_1 is, and strict if $f_i = 0$ for all $i \neq 1$. The identity morphism is the strict morphism $A \longrightarrow A$ with $f_1 = 1_A$.

Rephrasing in terms of coalgebras

Recall from our notes (hopf) the notion of a coalgebra, although there coalgebras had units and that is not the case here. So to be clear, a graded coalgebra is a $\mathbb{Z}$-graded (or $\mathbb{Z}_2$-graded) k-vector space C together with a k-linear map $\Delta : C \to C \otimes C$ of degree zero which is coassociative, i.e.

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes C \arrow[d, "id \otimes \Delta"] \\ C \otimes C \arrow[r, "\Delta \otimes id"'] & C \otimes C \otimes C \end{tikzcd}$$

commutes. One calls Δ the comultiplication.

Def^N Let V be a $\mathbb{Z}$-graded vector space. Then the reduced tensor algebra

$$\overline{T}V = V \oplus V^{\otimes 2} \oplus \cdots$$

is obviously $\mathbb{Z}$-graded and becomes a coalgebra with

$$\Delta : \overline{T}V \longrightarrow \overline{T}V \otimes \overline{T}V$$

$$\Delta(v_1 \otimes \cdots \otimes v_n) = \sum_{1 \le i \le n-1} (v_1 \otimes \cdots \otimes v_i) \otimes (v_{i+1} \otimes \cdots \otimes v_n)$$

$$= v_1 \otimes (\cdots) + \cdots + (v_1 \otimes \cdots \otimes v_{n-1}) \otimes v_n$$

Note that $\Delta(v) = 0$ and $\Delta(v_1 \otimes v_2) = v_1 \otimes v_2$. It is easy to check that $(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$.

At this point [P] is a good reference. A <u>coderivation</u> on a graded coalgebra C is a map $d : C \to C$ s.t.

$$\Delta \circ d = (d \otimes 1 + 1 \otimes d) \circ \Delta \qquad (7.1)$$

We assume that d is homogeneous of some degree $|d|$, so $1 \otimes d$ has an appropriate Koszul sign. We call d a <u>codifferential</u> if $|d| = 1$ and $d^2 = 0$, and it is a coderivation. (Note <u>degree</u> always means internal to V, not weight of tensors)

<u>Lemma</u> For $q \in \mathbb{Z}$ There is a bijection between coderivations $d : \bar{T}V \to \bar{T}V$ of degree $\cancel{1}q$ and families of k-linear maps $d_n : V^{\otimes n} \to V$, $n \geq 1$ of degree $\cancel{1}q$. In one direction the bijection sends d to

$$d_n : V^{\otimes n} \hookrightarrow \bar{T}V \xrightarrow{d} \bar{T}V \twoheadrightarrow V.$$

In the other direction, given $\{d_n\}_{n \geq 1}$ we define d with components $V^{\otimes n} \to V^{\otimes u}$ by

$$\sum 1^{\otimes r} \otimes d_s \otimes 1^{\otimes t} \qquad (n \geq u) \qquad (7.2)$$

where the sum is over $r, t \geq 0$ and $s \geq 1$ with $r + 1 + t = u$, $(r + s + t = n)$ forces $s = n - u + 1$ and there are Koszul signs. We claim this is a coderivation of degree $\cancel{1}q$ and that these two maps give a bijection.

<u>Proof</u> It obviously suffices to show that (7.2) really defines a coderivation. It is enough to check for $v_1 \otimes \cdots \otimes v_n \in V^{\otimes n}$ that

$$\Delta d(\underline{v}) = (d \otimes 1 + 1 \otimes d)\Delta(\underline{v}).$$

and for this we can check component by component in $\bar{T}V$, so let us fix $u \leq n$ and ignore tensors of weight $\neq u$.

The u-component of Δd is

$$\sum_{r+1+t=u} \Delta\left(V_{\text{first } r \text{ terms}} \otimes d_s(\text{next } s \text{ terms}) \otimes V_{\text{rest}} \right)\cdot(-1)^r$$

$$= \sum_{r+1+t=u} (-1)^r \sum_{\substack{\text{groupings} \\ (\text{either before } d, \text{ or after } d)}} V_{\text{first } r \text{ terms}} \otimes d_s(\text{next } s \text{ terms}) \otimes V_{\text{rest}}$$

whereas the u-component of $(d\otimes 1 + 1\otimes d)\Delta(\underline{V})$ is

$$(d\otimes 1)\Big(\sum_{\text{groupings}} V_{\text{initial}} \otimes V_{\text{rest}}\Big) + (1\otimes d)\Big(\sum_{\text{groupings}} V_{\text{init}} \otimes V_{\text{rest}}\Big)$$

$$= \sum_{a=1}^{n-1}\sum_{r_1,t_1} \big(\underbrace{V_{r_1 \text{ terms}} \otimes d_{s_1}(s_1 \text{ terms}) \otimes V_{t_1 \text{ terms}}}_{\substack{\text{was } a \text{ terms, now } u-(n-a) \\ r_1+1+t_1 = u-(n-a)}}\big) \otimes \underbrace{(V_{\text{rest}})}_{n-a \text{ terms}} \cdot (-1)^n$$

$$+ \sum_{a=1}^{n-1}\sum_{r_2,t_2} (-1)^{a+r_2} \underbrace{(V_{a \text{ terms}})}_{a \text{ terms}} \otimes \big(\underbrace{V_{r_2 \text{ terms}} \otimes d_{s_2}(s_2 \text{ terms}) \otimes V_{t_2 \text{ terms}}}_{\substack{\text{was } n-a \text{ terms, now } u-a \\ r_2+1+t_2 = u-a}}\big)$$

Now using the grading on $TV\otimes TV$ (i.e. $V_{a \text{ things}} \otimes V_{b \text{ things}}$ has degree $a+b$) and projecting onto e.g. ~~$V^{\otimes a}\otimes V^{\otimes}$~~ $V^{\otimes a}\otimes V^{\otimes b}$ one checks the above are equal.

Also we have to check that any coderivation $d: TV \to TV$ is determined by the maps d_n. But notice that $\mathrm{Ker}(\Delta) = V \subseteq TV$, so we can proceed by induction using (7.1). At the same we prove by induction that d never increases TV-degree (apply (7.1) to V for base case)

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Now for the version with the full tensor coalgebra

Defⁿ Let V be a $\mathbb{Z}$-graded vector space. The tensor algebra

$$TV = k \oplus V \oplus V^{\otimes 2} \oplus \cdots$$

is obviously $\mathbb{Z}$-graded and is a coalgebra with Δ defined by ~~on $\overline{T}V \subseteq TV$ as on p 6 and~~

Now let A be a graded vectorspace and $A[1]$ its suspension. Define a bijection between families of maps $b_n : (A(1))^{\otimes n} \to A(1)$ of degree $+1$ and maps $m_n : A^{\otimes n} \longrightarrow A$, $n \geq 1$ by the square

$$\begin{tikzcd} (A(1))^{\otimes n} \arrow[r, "b_n"] & A(1) \\ A^{\otimes n} \arrow[u, "s^{\otimes n}"] \arrow[r, "m_n"'] & A \arrow[u, "s"'] \end{tikzcd} \qquad (9.1)$$

where $s : A \longrightarrow A(1)$ is $a \mapsto a$ of degree -1. Note that m_n is degree $2-n$.
Hence $s^{\otimes n}$ has signs by Koszul $s^{\otimes 2}(a_1 \otimes \cdots \otimes a_n) = (-1)^{(|a_1|+\cdots+|a_{n-1}|)+(|a_1|+\cdots+|a_{n-2}|)+\cdots+|a_1|}$
$= (-1)^{(n-1)|a_1|+(n-2)|a_2|+\cdots+|a_{n-1}|} a_1 \otimes \cdots \otimes a_n$

<u>Lemma</u> The following are equivalent:

(i) The maps $m_n : A^{\otimes n} \longrightarrow A$ define an A_∞-structure on A.

(ii) The coderivation $TA(1) \xrightarrow{b} TA(1)$ satisfies $b^2 = 0$.

(iii) For each $n \geq 1$

$$\sum b_u (\mathbb{1}^{\otimes r} \otimes b_s \otimes \mathbb{1}^{\otimes t}) = 0 \quad \text{as maps } A(1)^{\otimes n} \to A(1) \qquad (9.2)$$

where the sum runs over all decompositions $n = r+s+t$ and we put $u = r+1+t$. We allow $r, t \geq 0$ and $s \geq 1$.

<u>Proof</u> (i) is $\sum (-1)^{r+st} m_u(\mathbb{1}^{\otimes r} \otimes m_s \otimes \mathbb{1}^{\otimes t}) = 0$

(ii) b^2 is a coderivation, so the same argument given on p. 7 says that $b^2 = 0$ (equality of two coderivations) iff they agree on $A^{\otimes n}$ that is, iff. (take $V = A(1)$)

$$A(1)^{\otimes n} \hookrightarrow TA(1) \xrightarrow{b^2} TA(1) \twoheadrightarrow A(1)$$

vanishes for all $n \geq 1$. But this is exactly (9.2), proving (ii) $\Leftrightarrow$ (iii).
Since m_n is b_n, ignoring gradings, up to signs. (i) $\Leftrightarrow$ (iii) is clear, just account for the sign.

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We have, for $a_1, \dots, a_n \in A$ homogeneous

$$b_n(a_1 \otimes \cdots \otimes a_n) = m_n(s^{-1})^{\otimes n}(a_1 \otimes \cdots \otimes a_n)$$

$$= (-1)^{(n-1)|a_1| + (n-2)|a_2| + \cdots + |a_{n-1}|} m_n(a_1 \otimes \cdots \otimes a_n)$$

Hence for $r, t \geq 0$, $s \geq 1$, $u = r+1+t$

$$m_u(\mathbb{1}^{\otimes r} \otimes m_s \otimes \mathbb{1}^{\otimes t})(a_{[1,r]} \otimes a_{[r+1,r+s]} \otimes a_{rest})$$

$$\|$$

$$(-1)^{(2-s)(\sum_{i=1}^{r} |a_i|)} m_u(a_{[1,r]} \otimes m_s(\cdots) \otimes a_{rest})$$

$$\|$$

$$(-1)^{s\sum_{i=1}^{r} |a_i| + (s-1)|a_{r+1}| + (s-2)|a_{r+2}| + \cdots + |a_{r+s-1}|}$$

$$(-1)^{(u-1)|a_1| + \cdots + (u-r)|a_r| + (u-r-1)|\overset{m}{\cancel{b}}_s(\cdots)| + (u-r-2)|a_{r+s+1}| + \cdots + |a_{n-1}|} \leftarrow \text{A-degree}$$

$$b_u(a_{[1,r]} \otimes b_s(\cdots) \otimes a_{rest})$$

$$\|$$

$$(-1)^{s\sum_{i=1}^{r} |a_i| + (s-1)|a_{r+1}| + (s-2)|a_{r+2}| + \cdots + |a_{r+s-1}| + (u-1)|a_1| + \cdots + (u-r)|a_r| + (u-r-1)(2-s+|a_{r+1}| + \cdots + |a_{r+s}|) + (u-r-2)|a_{r+s+1}| + \cdots + |a_{n-1}| + \sum_{i=1}^{r} |a_i| + r}$$

$$b_u(\mathbb{1}^{\otimes r} \otimes b_s \otimes \mathbb{1}^{\otimes t})(a_1 \otimes \cdots \otimes a_n)$$

Let us collect the sign terms in the above

For $1 \leq i \leq r$ $|a_i|$ has coefficient $s+u-i+1 \equiv n+i$

$r+1 \leq i \leq r+s-1$ $|a_i|$ has coeff. $s+r-i+u-r-1 \equiv s+i+u+1 \equiv n+i$

$i = r+s$ $|a_i|$ has coeff. $u-r-1 \equiv t \equiv n+i$

$r+s+1 \leq i \leq n-1$ $|a_i|$ has coeff $u+s-1-i \equiv n+i$

additional

$$(u-r-1)(2-s)+r \equiv s(u+r+1)+r \equiv st+r$$

So the sign is $st+r+\sum_{i=1}^{n-1}(n+i)|a_i|$.

Here then is the proof of (i) $\Leftrightarrow$ (iii). With $\sum$ as usual taken over $r \geq 0, t \geq 0, s \geq 1$

$$\sum m_u(\mathbb{1}^{\otimes r} \otimes m_s \otimes \mathbb{1}^{\otimes t})(a_1 \otimes \cdots \otimes a_n)$$
$$= (-1)^{st+r+\sum_{i=1}^{n-1}(n+i)|a_i|} b_u(\mathbb{1}^{\otimes r} \otimes b_s \otimes \mathbb{1}^{\otimes t})(a_1 \otimes \cdots \otimes a_n)$$

Hence

$$\sum (-1)^{st+r} m_u(\mathbb{1}^{\otimes r} \otimes m_s \otimes \mathbb{1}^{\otimes t})(a_1 \otimes \cdots \otimes a_n)$$
$$\shortparallel$$
$$\sum b_u(\mathbb{1}^{\otimes r} \otimes b_s \otimes \mathbb{1}^{\otimes t})\left((-1)^{\sum_{i=1}^{n-1}(n+i)|a_i|} a_1 \otimes \cdots \otimes a_n\right)$$

and it follows that one vanishes iff the other does, so (i) $\Leftrightarrow$ (iii). $\square$