Notes on A_∞-categories II (checked)

ainfcat2

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Our aim in this note is to check the proof of the A_∞ minimal model theorem in the papers (the former being an elaboration of the latter)

[L] C. Lazaroiu "Generating the superpotential on a D-brane category" 2006.

[KS] M. Kontsevich, Y. Soibelman "Homological mirror symmetry and torus fibrations" 2000.

Our notation is as in ainfcat and [L], with one exception: we allow k to be any commutative ring, $\otimes = \otimes_k$ (i.e. not necessarily a field, and no assumption on the characteristic). In ainf – ainf4 and ainfcat we made more restrictive assumptions, but everything said there holds in the present generality, as observed also at the beginning of ainf2 (with "vector space" replaced by "k-module").

<u>Cohomology</u> Let $\mathcal{A}$ be an A_∞-category. The <u>cohomology category</u> $H(\mathcal{A})$ is the (possibly non-unital) associative graded category with the same objects as $\mathcal{A}$, and morphism spaces

$$\mathrm{Hom}_{H(\mathcal{A})}(a,b) := H^*_{M_{ab}}(\mathrm{Hom}_{\mathcal{A}}(a,b))$$

and morphism compositions

$$\mathrm{Hom}_{H(\mathcal{A})}(b,c) \otimes \mathrm{Hom}_{H(\mathcal{A})}(a,b) \longrightarrow \mathrm{Hom}_{H(\mathcal{A})}(a,c)$$

given by $[x] * [y] = [\mu_{cba}(x \otimes y)]$. We denote by $H^0(\mathcal{A})$ the full subcategory of degree zero maps.

<u>Functors</u> Given A_∞-categories $\mathcal{A},\mathcal{B}$ an A_∞-functor $F:\mathcal{A}\longrightarrow\mathcal{B}$ is a map $F:\mathrm{Ob}\,\mathcal{A}\longrightarrow\mathrm{Ob}\,\mathcal{B}$ together with linear maps

$$F_{a_0\cdots a_n}:\mathrm{Hom}_{\mathcal{A}}(a_0,a_1)\otimes\cdots\otimes\mathrm{Hom}_{\mathcal{A}}(a_{n-1},a_n)\longrightarrow\mathrm{Hom}_{\mathcal{B}}(F(a_0),F(a_n))$$

of degree $1-n$ (here $n\geq 1$) such that the suspended maps

$$F^s_{a_0\cdots a_n}=s^{\mathcal{B}}_{F(a_0)F(a_n)}\circ F_{a_0\cdots a_n}\circ\left(s^{-1}_{a_0,a_1}\otimes\cdots\otimes s^{-1}_{a_{n-1},a_n}\right)$$

$$\mathrm{Hom}_{\mathcal{A}}(a_0,a_1)[1]\otimes\cdots\otimes\mathrm{Hom}_{\mathcal{A}}(a_{n-1},a_n)[1]\longrightarrow\mathrm{Hom}_{\mathcal{B}}(F(a_0),F(a_n))[1]$$

which are homogeneous of degree zero, satisfy (for $n\geq 1$)

$$\sum_{p=1}^{n}\sum_{0<i_1<\cdots<i_{p-1}<n} r^{\mathcal{B}}_{F(a_0)\cdots F(a_n)}\circ\left(F^s_{a_0\cdots a_{i_1}}\otimes F^s_{a_{i_1}\cdots a_{i_2}}\otimes\cdots\otimes F^s_{a_{i_{p-1}}\cdots a_n}\right) \tag{2.1}$$

$$=\sum_{0\leq i<j\leq n} F^s_{a_0\cdots a_i a_j\cdots a_n}\circ\left(\mathrm{id}^{\mathcal{A}}_{a_0a_1}\otimes\cdots\otimes\mathrm{id}^{\mathcal{A}}_{a_{i-1}a_i}\otimes r^{\mathcal{A}}_{a_i\cdots a_j}\otimes\mathrm{id}^{\mathcal{A}}_{a_ja_{j+1}}\cdots\right.$$
$$\left.\cdots\otimes\mathrm{id}^{\mathcal{A}}_{a_{n-1}a_n}\right)$$

Together with F_{ab}, the map on objects induces a (possibly non-unital) functor $H(F):H(\mathcal{A})\longrightarrow H(\mathcal{B})$ of graded associative categories. F is called a <u>quasi-isomorphism</u> if $H(F)$ is an isomorphism. It is called <u>strict</u> if $F_{a_0\cdots a_n}=0$ unless $n=1$. In this case (2.1) reduces to

$$r^{\mathcal{B}}_{F(a_0)\cdots F(a_n)}\circ\left(F^s_{a_0a_1}\otimes F^s_{a_1a_2}\otimes\cdots\otimes F^s_{a_{n-1}a_n}\right) \tag{2.2}$$

$$=F^s_{a_0a_n}\circ r^{\mathcal{A}}_{a_0\cdots a_n}$$

Sector decomposition Consider the commutative associative k-algebra $R := R_{\mathcal{A}}$ for an A_∞-algebra $\mathcal{A}$, generated by $\{\epsilon_a\}_{a \in \mathrm{Ob}\mathcal{A}}$ subject to $\epsilon_a \epsilon_b = \delta_{ab}\epsilon_a$. Note that R is unital iff. $\mathrm{Ob}\mathcal{A}$ is finite. Since the ϵ_a are commuting idempotents, we have $R \cong \bigoplus_{a\in \mathrm{Ob}\mathcal{A}} k$ as associative algebras. Consider the k-module

$$\mathcal{H} = \mathcal{H}_{\mathcal{A}} = \bigoplus_{a,b\in \mathrm{Ob}\mathcal{A}} \mathrm{Hom}_{\mathcal{A}}(a,b) \tag{3.1}$$

with the grading

$$\mathcal{H}^n = \bigoplus_{a,b\in \mathrm{Ob}\mathcal{A}} \mathrm{Hom}^n_{\mathcal{A}}(a,b). \tag{3.2}$$

We let $\pi_{ab} : \mathcal{H} \longrightarrow \mathrm{Hom}_{\mathcal{A}}(a,b)$ be the projectors onto the subspace $\mathrm{Hom}_{\mathcal{A}}(a,b)$. The binary decomposition (3.1) defines an R-bimodule structure on $\mathcal{H}$. Namely, ϵ_a acts on the left by the projector ${}_a\pi$ of $\mathcal{H}$ onto

$${}_a\mathcal{H} = \bigoplus_{b\in \mathrm{Ob}\mathcal{A}} \mathrm{Hom}_{\mathcal{A}}(a,b) \tag{3.3}$$

and ϵ_b acts on the right by the projector π_b

$$\mathcal{H}_b = \bigoplus_{a\in \mathrm{Ob}\mathcal{A}} \mathrm{Hom}_{\mathcal{A}}(a,b) \tag{3.4}$$

Lemma The k-module $\mathcal{H}^{\otimes_R n} = \mathcal{H} \otimes_R \cdots \otimes_R \mathcal{H}$ is given by

$$\mathcal{H}^{\otimes_R n} = \bigoplus_{a_0,\dots,a_n} \mathrm{Hom}_{\mathcal{A}}(a_0,a_1) \otimes \mathrm{Hom}_{\mathcal{A}}(a_1,a_2) \otimes \cdots \otimes \mathrm{Hom}_{\mathcal{A}}(a_{n-1},a_n) \tag{3.5}$$

with the obvious R-bimodule structure.

<u>Proof</u> The relation imposed by the tensor product is

$$\mathcal{H} \otimes_R \mathcal{H} = (\mathcal{H} \otimes \mathcal{H}) \big/ \big(x\epsilon_b \otimes y - x \otimes \epsilon_b y \big)_{x,y \text{ homogeneous}}$$

from which the claim is clear. □

We define the <u>total products</u> $r_n : \mathcal{H}[1]^{\otimes_R n} \longrightarrow \mathcal{H}[1]$ via

$$r_n(x^{(1)} \otimes \cdots \otimes x^{(n)}) := \bigoplus_{a_0, a_n} \sum_{a_1, \ldots, a_{n-1}} r_{a_0, \ldots, a_n}(x^{(1)}_{a_0 a_1} \otimes \cdots \otimes x^{(n)}_{a_{n-1} a_n}) \tag{3.5}$$

where $x^{(j)} = \bigoplus_{a,b \in \mathrm{ob}\mathcal{A}} x^{(j)}_{ab} \in \mathcal{H}[1]$ with $x^{(j)}_{ab} \in \mathrm{Hom}_{\mathcal{A}}(a,b)[1]$. Since r_n is clearly R-bilinear, we can view r_n as an element of

$$r_n \in \mathrm{Hom}^1_{{}_R\mathrm{Mod}_R}(\mathcal{H}[1]^{\otimes_R n}, \mathcal{H}[1]).$$

These maps obey the A_∞-relations

$$\sum_{\substack{i \geq 0, j \geq 1 \\ 1 \leq i+j \leq n}} (-1)^{\tilde{x}_1 + \cdots + \tilde{x}_i} r_{n-j+1}(x_1 \otimes \cdots \otimes x_i \otimes r_j(x_{i+1} \otimes \cdots \otimes x_{i+j}) \otimes x_{i+j+1} \otimes \cdots \otimes x_n) = 0 \tag{3.6}$$

Composing with the quotient $\mathcal{H}[1]^{\otimes n} \longrightarrow \mathcal{H}[1]^{\otimes_R n}$ defines r_n on $\mathcal{H}[1]^{\otimes n}$, and $(\mathcal{H}, \{r_n\}_{n \geq 1})$ is thus an A_∞-algebra over k (not over R because $\mathcal{H}$ is an R-bimodule with left and right actions do not necessarily agree).

Minimal models (§3.3 of [L])

Let $\mathcal{A}$ be an A_∞-category, and R, $\mathcal{H} = \mathcal{H}_{\mathcal{A}}$ as above. We view r_n as defined on $\mathcal{H}$ with the tilde grading.

Def^N A *strict homotopy retraction* of $\mathcal{A}$ is a homotopy retract of the R-complex $(\mathcal{H}, r_1)$ (notice $r_1 = m_1$), i.e a pair (P, G) with $P \in \mathrm{Hom}_{R\,\mathrm{GrMod}_R}(\mathcal{H}, \mathcal{H})$ and $G \in \mathrm{Hom}_{R\,\mathrm{GrMod}_R}(\mathcal{H}, \mathcal{H}[-1])$ such that

(1) $P^2 = P$

(2) $\mathrm{id}_{\mathcal{H}} - P = r_1 G + G r_1$

Note that by $\mathrm{Hom}_{R\,\mathrm{GrMod}_R}(-,-)$ we mean degree zero maps, and (2) implies $Pr_1 = r_1 P$. The R-bilinearity means P, G amount to the data of

$$P_{ab} : \mathrm{Hom}_{\mathcal{A}}(a,b) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(a,b)$$
$$G_{ab} : \mathrm{Hom}_{\mathcal{A}}(a,b) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(a,b)[-1]$$

such that $P_{ab}^2 = P_{ab}$ and $\mathrm{id} - P_{ab} = (r_1)_{ab} G_{ab} + G_{ab} (r_1)_{ab}$. The submodule (graded, R-bimodule)

$$B := \mathrm{Im}\, P \subseteq \mathcal{H}$$

is given by $\bigoplus_{a,b \in \mathrm{ob}\mathcal{A}} B_{ab}$ where $B_{ab} = \mathrm{Im}\, P_{ab}$. We let $i : B \longrightarrow \mathcal{H}$ be the inclusion and $p : \mathcal{H} \longrightarrow B$ the map induced by P, so that $i \circ p = P$. Clearly $r_1(B) \subseteq B$, so B is a subcomplex of $\mathcal{H}$.

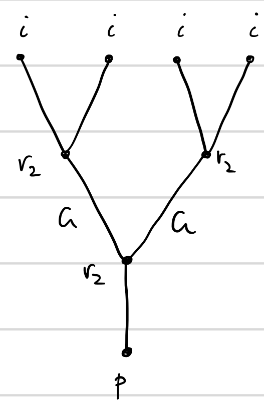
Important For our conventions on trees see ainfcat3. $\mathcal{J}_n$ is defined on p. 2 there.

Note on Koszul signs Recall that for k-linear maps ϕ, ψ of degree a, b the map $\phi \otimes \psi$ is defined by $(\phi \otimes \psi)(x \otimes y) = (-1)^{b|x|} \phi(x) \otimes \psi(y)$. This is the Koszul sign convention.

Given a valid plane tree T let $E(T)$ denote the set of all edges, $E_i(T)$ all internal edges, and $E_e(T)$ all external edges. Set $e_i(T) := \operatorname{Card} E_i(T)$ the number of internal edges. For each $T \in \mathcal{J}_n$ we define a morphism of graded R-bimodules $\rho_T \in \operatorname{Hom}_{{}_R\mathrm{Mod}_R}(B^{\otimes_R n}, B)$ as follows:

(a) associate the inclusion i with every leaf of T.
(b) associate the surjection p with the root of T. (6.1)
(c) associate r_k with each internal vertex of valency $k+1$ (note $k \geq 2$)
(d) associate G with each internal edge of T.

Example



However notice that due to Koszul signs, with r_n of degree 1, and G of degree -1, we have

$$p \circ r_2 \circ (G \otimes G) \circ (r_2 \otimes r_2) \circ (i \otimes i \otimes i \otimes i) \qquad (6.2)$$ ← height

$$\neq p \circ r_2 \circ (G \circ r_2 \circ i^{\otimes 2} \otimes G \circ r_2 \circ i^{\otimes 2}).$$ ← branch

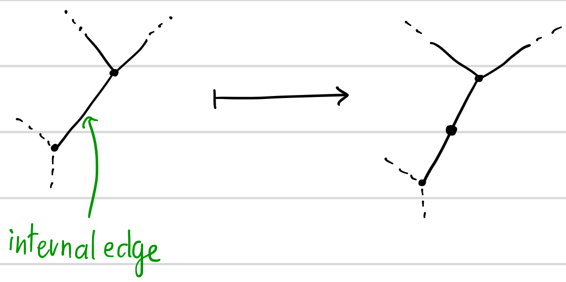
It is therefore important that we fix a convention for which of the two alternatives in (6.2) that we will use. The two choices are called in ainfcat3 the height and branch denotations. After some effort we are convinced the former is very difficult to make work in the context of the minimal model theorem, so we use branch.

To be more precise about what (6.1) means we introduce the notion of <u>augmented</u> plane trees.

<u>Def</u>

Def^n Given a valid plane tree T the <u>augmentation</u> $A(T)$ of T is the plane tree obtained from T by inserting a new vertex (of valency 2) on each internal edge of T.

i.e.



Given $T \in \mathcal{T}_n$ a valid plane tree, we decorate the augmented tree $A(T)$ according to (6.1), that is we define D to be:

- we assign $L_v := B[1]$, a graded R-bimodule, for every leaf v (inc. the root).
- to all edges e of $A(T)$ we assign $M_e := \mathcal{H}[1]$. (8.1)
- The maps $B[1] = L_v \longrightarrow M_e = \mathcal{H}[1]$ for each edge e incident at a non-root leaf v are all i, and the map $\mathcal{H}[1] = M_e \longrightarrow L_r = B[1]$ at the root is p.
- to vertices of $A(T)$ coming from an internal edge of T we assign G.
- to internal vertices of $A(T)$ of valency $k+1$ for $k \geqslant 2$ we assign r_k, of degree $+1$.

<u>Def^n</u> Given $T \in \mathcal{T}_n$ and the homotopy retract data for $\mathcal{A}$ on p. 5 we define the homogeneous R-bilinear map $\rho_T : B[1]^{\otimes_R n} \longrightarrow B[1]$ to be the denotation $\langle D \rangle$ of the decoration (8.1) of the augmented plane tree $A(T)$, multiplied by a sign factor $(-1)^{e_i(T)}$. Here $\langle D \rangle$ is the <u>branch</u> denotation $\langle D \rangle_B$ of ainfcat3.

<u>Lemma</u> ρ_T is homogeneous of degree $2-n$, and hence degree $+1$ as $B[1]^{\otimes_R n} \longrightarrow B[1]$.

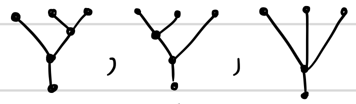
<u>Proof</u> The degree of ρ_T is $e_i(T) \cdot (-1) + \sum_{\text{int. vertex } v \text{ of } T} \big(3 - \text{valency}(v)\big)$, but this is $3\#\text{int. vertices} - 3\#\text{int. edges} - \#\text{ext. edges}$. There is an injection, $\{\text{int. edges}\} \longrightarrow \{\text{int. vertices}\}$ sending an edge to its source, which only misses one vertex (the one adjacent to the root). Hence $|\rho_T| = 3 - \#\text{ext. edges} = 2-n$. $\square$

<u>Def^n</u> For $n \geqslant 2$ we define the degree $+1$ map

$$\rho_n := \sum_{T \in \mathcal{T}_n} \rho_T \in \mathrm{Hom}_{R\,\mathrm{Mod}\,R}\left(B[1]^{\otimes_R n}, B[1] \right). \tag{9.1}$$

and we set $\rho_1 := p \circ r_1 \circ i$ (i.e. $r_1|_B$). Note that $\rho_2 = p \circ r_2 \circ i^{\otimes 2} = p \circ r_2|_{B \otimes B}$.

<u>Example</u> We have $\mathcal{T}_3 = \{$



T_1, T_2, T_3

$\}$ and thus

$$\begin{aligned} \rho_{T_1} &= (-1)^1 p \circ r_2 \circ (i \otimes a r_2 i^{\otimes 2}), \\ \rho_{T_2} &= (-1)^1 p \circ r_2 \circ (a r_2 i^{\otimes 2} \otimes i), \\ \rho_{T_3} &= (-1)^0 p \circ r_3 \circ i^{\otimes 3}. \end{aligned} \tag{9.2}$$

and finally $\rho_3 = \rho_{T_1} + \rho_{T_2} + \rho_{T_3}$.

Recall eval from p. 6 ainfcat3.

<u>Lemma</u> For any $T \in \mathcal{T}_n$ and $a_1, \dots, a_n \in B[1]$ we have

$$\rho_T(a_1 \otimes \cdots \otimes a_n) = (-1)^{e_i(T)} \mathrm{eval}_D(a_1 \otimes \cdots \otimes a_n). \tag{9.3}$$

<u>Proof</u> By p. 6.5 ainfcat3 the difference is a sign $\sum_{i=1}^n \sum_{v > \ell_i} \tilde{a}_i |\phi_v|$ where $\tilde{a}_i = |a_i| + 1$, v runs over all vertices in $A(T)$, ℓ_i means the ith non-root leaf, and $|\phi_v|$ is the degree of the insertion in D at v. But vertices $v > \ell_i$ in $A(T)$ can be paired up (vertices from internal edges in T with their source vertex) in a way that makes it clear this sign is zero. $\square$

<u>Upshot</u> We can evaluate ρ_T ignoring <u>all Koszul signs</u>!

Theorem The maps $\{\rho_n\}_{n\geq 1}$ satisfy the forward suspended A_∞-relations ((3.6) above or ainfcat (4.3)), i.e. for $n \geq 1$

$$\sum_{\substack{i\geq 0, j\geq 1\\ 1\leq i+j\leq n}} \rho_{n-j+1} \circ \left(id_{B[1]}^{\otimes i} \otimes \rho_j \otimes id_{B[1]}^{\otimes n-i-j} \right) = 0. \qquad (10.1)$$

Notes For $n=1$ the relation we need is $\rho_1^2 = 0$, which is immediate since $r_1^2 = 0$. The relation for $n=2$ is $\rho_1\rho_2 + \rho_2(\rho_1\otimes 1) + \rho_2(1\otimes\rho_1) = 0$, which is $r_1 p r_2 + p r_2(r_1\otimes 1) + p r_2(1\otimes r_1) = 0$, which follows by multiplying $r_1 r_2 + r_2(r_1\otimes 1) + r_2(1\otimes r_1) = 0$ on the left by p.

Proof For $n>1$ consider the following R-bilinear map $\mathcal{H}[1]^{\otimes n} \longrightarrow \mathcal{H}[1]$,

$$(r)_1^n := r_1 \circ r_n + \sum_{i=0}^{n-1} r_n \circ \left(id_{\mathcal{H}[1]}^{\otimes i} \otimes r_1 \otimes id_{\mathcal{H}[1]}^{\otimes (n-i-1)} \right) \qquad (10.2)$$

and similarly define $(\rho)_1^n$ for the products ρ on $B[1]$. The A_∞-relations are equivalent to $r_1^2 = 0$ together with

$$(r)_1^n = - \sum_{\substack{i\geq 0, j\geq 2\\ i+j\leq n,\, j\leq n-1}} r_{n-j+1} \circ \left(id_{\mathcal{H}[1]}^{\otimes i} \otimes r_j \otimes id_{\mathcal{H}[1]}^{\otimes (n-j-i)} \right) \qquad n\geq 2. \qquad (10.3)$$

Since $\rho_1^2 = 0$ to complete the proof it suffices to show for $n\geq 2$ that

$$(\rho)_1^n = - \sum_{\substack{i\geq 0, j\geq 2\\ i+j\leq n,\, j\leq n-1}} \rho_{n-j+1} \circ \left(id_{B[1]}^{\otimes i} \otimes \rho_j \otimes id_{\mathcal{H}[1]}^{\otimes (n-j-i)} \right). \qquad (10.4).$$

Now by definition $\rho_n = \sum_{T\in\mathcal{T}_n} (-1)^{e_i(T)} \langle D_T \rangle$ where D_T is the canonical decoration of $A(T)$. The strategy is to expand the RHS of (10.4) using this definition, "merge" the trees from ρ_{n-j+1} and ρ_j then use $r_1 G + G r_1 = id_{\mathcal{H}} - P$ to recover the LHS.

Given $f \in \mathrm{End}_{R\mathrm{Mod}R}(\mathcal{H})$ of degree zero, $T \in \mathcal{T}_n$ and an internal edge e, let $D_{f,e}$ be the decoration of $A(T)$ which puts f rather than G at the vertex corresponding to e. We define

$$\rho^f_{T,e} := (-1)^{e_i(T)} \langle D_{f,e} \rangle \qquad (11.1)$$

We also set

$$\rho^f_n := \sum_{\substack{T \in \mathcal{T}_n \\ e_i(T) \geq 1}} \sum_{e \in E_i(T)} \rho^f_{T,e} \in \mathrm{Hom}_{R\mathrm{Mod}R}\big(B[1]^{\otimes_R n}, B[1] \big).$$

Given $e \in E_i(T)$ we write $\hat{\rho}_{T,e}$ for $\rho^{r_1 G + G r_1}_{T,e}$. Given $e \in E_e(T)$ let v be the leaf to which e is adjacent (possibly $v = r$ the root). Define D_e to be the decoration replacing $\phi_{\tilde{v}}$ (which is either i or p) by $r_1 \circ \phi_{\tilde{v}} = \phi_{\tilde{v}} \circ \rho_1$ if v is a non-root leaf, and by $\rho_1 \circ \phi_{\tilde{v}} = \phi_{\tilde{v}} \circ r_1$ if $v = r$, and set

$$\hat{\rho}_{T,e} := (-1)^{e_i(T)} \langle D_e \rangle,$$

$$\hat{\rho}_n = \sum_{T \in \mathcal{T}_n} \sum_{e \in E(T)} \hat{\rho}_{T,e}.$$

We begin by proving

<u>Claim A</u> For $0 \leq i \leq n-1$

$$\rho_n \circ \big(id^{\otimes i}_{B[1]} \otimes \rho_1 \otimes id^{\otimes (n-i-1)}_{B[1]} \big) = \sum_{T \in \mathcal{T}_n} \hat{\rho}_{T,e}$$

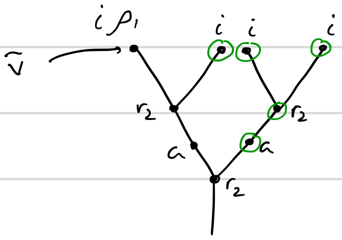
where e is the edge in T adjacent to the ith leaf.

<u>Proof of claim</u> Fix $0 \le i \le n-1$, $T \in \mathcal{T}_n$ and let D_e as defined above. Then

$$\langle D_e \rangle = (-1)^s \langle D_T \rangle \circ (1^{\otimes i-1} \otimes \rho_1 \otimes 1^{\otimes (n-i)})$$

(D_T means the standard denotation)

where s is the sum of the degrees $|\phi_q|$ of insertions on $A(T)$ at vertices q which satisfy $q > \tilde{v}$ according to the relation "$>$" of p. 2 ainfcat3, where $\tilde{v}$ is the ith leaf vertex in T, e.g. for $i=1$ the indicated vertices are $> \tilde{v}$



total sign $|a| + |r_2| \equiv 0$

$$r_2(a r_2(i\rho_1 \otimes i) \otimes a r_2(i \otimes i))$$

$$\|$$

$$r_2(a r_2(i \otimes i) \otimes a r_2(i \otimes i)) \circ (\rho_1 \otimes 1^{\otimes 3})$$

But now observe that the vertices q contributing to this sum come in pairs (like the r_2, a in the above example) of a vertex in $A(T)$ originating from an internal edge in T, and its source vertex. Since these pairs cancel, we conclude $s = 0$. $\square$

It is obvious that

$$\rho_1 \circ \rho_n = \sum_T (-1)^{e_i(T)} \rho_1 \circ \langle D_T \rangle = \sum_T \hat{\rho}_{T,e} \quad (e \text{ adjacent to } r).$$

Hence

$$(\rho)_1^n = \sum_{T \in \mathcal{T}_n} \sum_{e \in E_e(T)} \hat{\rho}_{T,e} = \hat{\rho}_n - \sum_{T \in \mathcal{T}_n} \sum_{e \in E_i(T)} \hat{\rho}_{T,e}$$

$$= \hat{\rho}_n - \sum_{T \in \mathcal{T}_n} \sum_{e \in E_i(T)} \rho_{T,e}^{r_1 a + a r_1}$$

Now using $r_1 G + G r_1 = \mathrm{id}_{\mathcal{H}} - P$ we have

$$= \hat{\rho}_n - \sum_{T \in \mathcal{T}_n} \sum_{e \in E_i(T)} \left[\rho_{T,e}^{\mathrm{id}_{\mathcal{H}}} - \rho_{T,e}^{P} \right]$$

$$= \hat{\rho}_n - \rho_n^{\mathrm{id}_{\mathcal{H}}} + \rho_n^{P}$$

That is,

$$\hat{\rho}_n = (\rho)_1^n + \rho_n^{\mathrm{id}_{\mathcal{H}}} - \rho_n^{P}. \qquad (12.1)$$

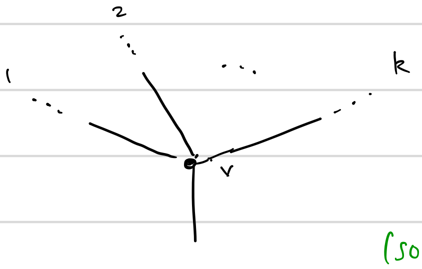
Next we calculate $\hat{\rho}_n$ in a different way. Set $\hat{\rho}_T := \sum_{e \in E(T)} \hat{\rho}_{T,e}$ so $\hat{\rho}_n = \sum_T \hat{\rho}_T$. For $n \geqslant 2$ and $T \in \mathcal{T}_n$ we can organise the sum $\hat{\rho}_T = \sum_{e \in E(T)} \hat{\rho}_{T,e}$ as a sum over internal vertices of T,

$$\hat{\rho}_T = \sum_{\substack{v \text{ internal} \\ \text{vertex}}} \hat{\rho}_{T,v}$$

where $\hat{\rho}_{T,v}$ is the sum of $\rho_{T,e}^{r_1 G}$ for every internal edge e which is incoming to v, $\rho_{T,e}^{G r_1}$ for every internal edge outgoing from v, and $\hat{\rho}_{T,e}$ for every external edge incident at v. That is,

$$\hat{\rho}_{T,v} = \sum_{\substack{e \in E_i(T) \\ e \text{ ends at } v}} \rho_{T,e}^{r_1 G} + \sum_{\substack{e \in E_i(T) \\ e \text{ begins at } v}} \rho_{T,e}^{G r_1} + \sum_{\substack{e \in E_e(T) \\ e \text{ incident at } v}} \hat{\rho}_{T,e} \qquad (12.2)$$

Recall $\rho^{r_1 G}_{T,e}$, $\rho^{G r_1}_{T,e}$, $\hat{\rho}_{T,e}$ are signed versions of $\langle D_{r_1 G, e}\rangle_H$, $\langle D_{G r_1, e}\rangle_H$, $\langle D_e \rangle_H$ resp.
Suppose that v has valency $k+1$ with incident edges (in T)



(13.1)

(some or all of which may be external edges)

The idea is to recognise $(r)^k_1$ from earlier on the RHS of (13.1), apply (10.3), and then the result on p. 12 of ainfcat3.

The contribution of (13.1) to $\langle D_T \rangle$ is via the operator

$$r_k \circ (T_1 \otimes \cdots \otimes T_k) \tag{13.2}$$

where T_j is the denotation of the ith subtree (of $A(T)$). Let us write $\langle\!\langle - \rangle\!\rangle$ somewhat ambiguously to reflect denotations of decorations of $A(T)$ where we only describe the deviation from D_T near v (and since we are in the branch denotation, not height, it is genuinely safe to do so). Then,

$$\langle D_T \rangle = \langle\!\langle\, r_k \circ (T_1 \otimes \cdots \otimes T_k) \,\rangle\!\rangle \tag{13.3}$$

$$\langle D_{r, G, e_i} \rangle = \langle\!\langle\, r_k \circ (T_1 \otimes \cdots \otimes r_1 T_i \otimes \cdots \otimes T_k) \,\rangle\!\rangle$$

$$\langle D_{G r_1, e} \rangle = \langle\!\langle\, G\, r_1 \circ r_k \circ (T_1 \otimes \cdots \otimes T_k) \,\rangle\!\rangle$$

For the same reason elaborated above, $|T_j| = 0$ for all j, since insertions on edges and internal vertices precisely cancel. Thus

$$\langle D_{r_1,a,e_i}\rangle = \langle\!\langle r_k \circ (T_1 \otimes \cdots \otimes r_1 T_i \otimes \cdots \otimes T_k) \rangle\!\rangle$$
$$= \langle\!\langle r_k \circ (\mathbb{1}^{\otimes(i-1)} \otimes r_1 \otimes \mathbb{1}^{\otimes(k-i)}) \circ (T_1 \otimes \cdots \otimes T_k) \rangle\!\rangle$$

Hence from (12.2), we conclude by (10.3),

$$\hat{\rho}_{T,v} = (-1)^{e_i(T)} \langle\!\langle (r)_1^k \rangle\!\rangle_{T,v} \leftarrow \text{meaning } (r)_1^k \text{ is inserted at } v \text{ in } A(T)$$
$$= (-1)^{e_i(T)} \left\langle\!\!\left\langle - \sum_{\substack{i\geq 0, j\geq 2 \\ i+j\leq k,\ j\leq k-1}} r_{k-j+1} \circ \left(\mathrm{id}_{\mathfrak{X}[1]}^{\otimes i} \otimes r_j \otimes \mathrm{id}_{\mathfrak{X}[1]}^{\otimes(k-j-i)} \right) \right\rangle\!\!\right\rangle_{T,v} \qquad (14.1)$$
$$= (-1)^{e_i(T)+1} \sum_{\substack{i\geq 0, j\geq 2 \\ i+j\leq k,\ j\leq k-1}} \langle\!\langle r_{k-j+1} \circ \left(\mathrm{id}_{\mathfrak{X}[1]}^{\otimes i} \otimes r_j \otimes \mathrm{id}_{\mathfrak{X}[1]}^{\otimes(k-j-i)} \right) \rangle\!\rangle_{T,v}$$

<u>Claim B</u> Given $T \in \mathcal{T}_n$, an internal vertex v, and integers $i\geq 0, j\geq 2$ with $i+j\leq k$, $j\leq k-1$ where v has valency $k+1$, let $T' = \mathrm{ins}(T,v,i,j)$ as defined on p.10 of aincat3 and let e' be the created edge, $D'_{\mathrm{id}_{\mathfrak{X}},e'}$ the decoration of $A(T')$ obtained from the standard one (i.e. p.8) by inserting $\mathrm{id}_{\mathfrak{X}}$ rather than a at e'.
We claim that

$$\langle\!\langle r_{k-j+1} \circ \left(\mathrm{id}_{\mathfrak{X}[1]}^{\otimes i} \otimes r_j \otimes \mathrm{id}_{\mathfrak{X}[1]}^{\otimes(k-j-i)} \right) \rangle\!\rangle_{T,v} = \langle D'_{\mathrm{id}_{\mathfrak{X}},e'} \rangle. \qquad (14.1)$$

<u>Proof of claim</u> with the vicinity of v as in (13.1) and the notation T_j as above,

$$\text{LHS of (14.1)} = \langle\!\langle r_{k-j+1} \circ (\mathrm{id} \otimes r_j \otimes \mathrm{id}) \circ (T_1 \otimes \cdots \otimes T_k) \rangle\!\rangle$$
$$= \langle\!\langle r_{k-j+1} \circ (T_1 \otimes \cdots \otimes r_j T_j \otimes \cdots \otimes T_k) \rangle\!\rangle$$
$$= \text{RHS of (14.1)}. \ \square$$

From Claim B and (14.1) we obtain

$$\hat{\rho}_{T,v} = (-1)^{e_i(T)+1} \sum_{\substack{i \geq 0,\, j \geq 2 \\ i+j \leq k,\, j \leq k-1}} \langle D'_{id_{\partial e}, e'} \rangle_{ins(T,v,i,j)}$$

We can partition $\mathcal{T}_n$ by the number of internal edges, writing

$$\mathcal{T}_n = \coprod_{c \geq 0} \mathcal{T}_n^{(c)} \quad \text{where } T \in \mathcal{T}_n^{(c)} \text{ iff. } e_i(T) = c.$$

By the Lemma on p. 12 of ainfcat3 there is a bijection for $n > 2$ and $c \geq 0$

$$(\mathcal{T}_n^{(c+1)})^+ \longrightarrow \{ (Q, v, i, j) \mid Q \in \mathcal{T}_n^{(c)},\ v \text{ an internal vertex},$$
$$i \geq 0,\ i+j \leq |v|-1,\ 2 \leq j \leq |v|-2 \}.$$

where $(\mathcal{T}_n^{(c+1)})^+$ denotes the set of pairs (T, e) where $T \in \mathcal{T}_n^{(c+1)}$ and $e \in E_i(T)$.
Hence, assuming $n > 2$

$$\mathcal{T}_n^+ = \coprod_{c \geq 0} (\mathcal{T}_n^{(c+1)})^+ \cong \coprod_{c \geq 0} \{ (Q, v, i, j) \mid Q \in \mathcal{T}_n^{(c)}, \ldots \}$$

which shows

$$\hat{\rho}_n = \sum_{T \in \mathcal{T}_n} \sum_{v} \hat{\rho}_{T,v} = \sum_{T \in \mathcal{T}_n} \sum_{v} \sum_{\substack{i \geq 0,\, j \geq 2 \\ i+j \leq k,\, j \leq k-1}} (-1)^{e_i(T)+1} \langle D'_{id_{\partial e}, e'} \rangle_{ins(T,v,i,j)}$$

$$= \sum_{T' \in \mathcal{T}_n} \sum_{e' \in E_i(T')} (-1)^{e_i(T')} \langle D'_{id_{\partial e}, e'} \rangle$$

in T'

$$= \rho_n^{id_{\partial e}}$$

Comparing with (12.1) we conclude for $n>2$ that

$$(\rho)_1^n = \rho_n^P. \qquad (16.1)$$

Recall $P = i \circ p$. To show (10.4) and complete the proof, it is therefore enough to check

<u>Claim C</u> $$\rho_n^P = -\sum_{\substack{i\geq 0,\, j\geq 2\\ i+j\leq n,\, j\leq n-1}} \rho_{n-j+1} \circ \left(id_{B[1]}^{\otimes i} \otimes \rho_j \otimes id_{\mathcal{H}[1]}^{\otimes(n-j-i)} \right).$$

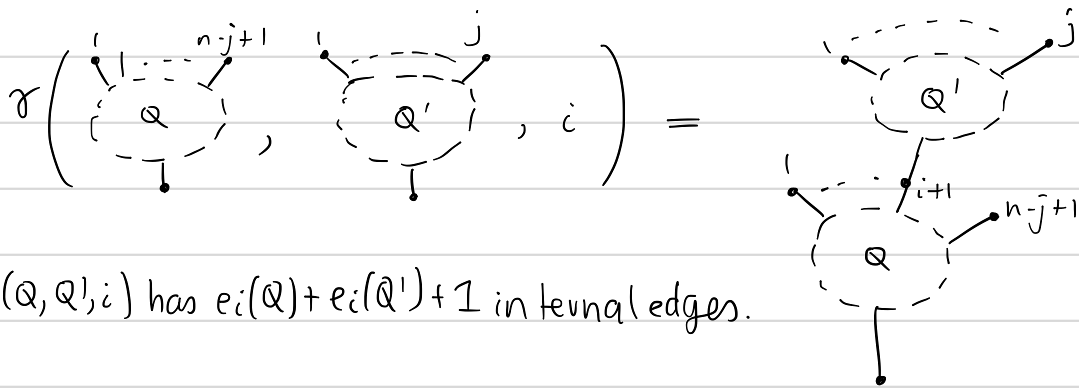
<u>Proof of claim</u>: By definition

$$\rho_n^P = \sum_{T\in \mathcal{T}_n} \sum_{e\in E_i(T)} (-1)^{e_i(T)} \langle D_{P,e} \rangle \qquad (16.2)$$

For n fixed there is a bijection

$$\gamma : \coprod_{2\leq j\leq n-1} \mathcal{T}_{n-j+1} \times \mathcal{T}_j \times \{0,\dots,n-j\} \longrightarrow \mathcal{T}_n^+ \qquad (16.3)$$

which is defined on (Q, Q', i) by attaching Q' to Q at the $(i+1)$st leaf, via a new internal edge which is marked, i.e.



Note that $\gamma(Q,Q',i)$ has $e_i(Q)+e_i(Q')+1$ internal edges.

Hence (16.2) may be rewritten as

$$\rho_n^P = \sum_{2\leq j\leq n-1} \sum_{\substack{Q\in J_{n-j+1}\\ Q'\in J_j}} \sum_{0\leq i\leq n-j} (-1)^{e_i(Q)+e_i(Q')+1} \langle D_Q\rangle \circ (id^{\otimes i}\otimes \langle D_{Q'}\rangle \otimes id^{\otimes(n-j-i)})$$

$$= \sum_{\substack{i\geq 0,\, j\geq 2\\ i+j\leq n,\, j\leq n-1}} (-1)^1 \left\{ \sum_{Q\in J_{n-j+1}} (-1)^{e_i(Q)} \langle D_Q\rangle \right\} \circ \left(id^{\otimes i}\otimes \left\{\sum_{Q'\in J_j} \langle D_{Q'}\rangle\right\} \otimes id^{\otimes(n-j-i)}\right)$$

$$= -\sum_{\substack{i\geq 0,\, j\geq 2\\ i+j\leq n,\, j\leq n-1}} \rho_{n-j+1} \circ \left(id_{B[1]}^{\otimes i} \otimes \rho_j \otimes id_{\mathcal{H}[1]}^{\otimes(n-j-i)}\right).$$

as claimed. □

which completes the proof of the Theorem. □

Appendix (Height vs Branch denotation)

Given $T \in \mathcal{T}_n$ we have defined

$$\rho_T = (-1)^{e_i(T)} \langle D \rangle_H .$$

Now, by (7.2) of ainfcat3 we have

$$\langle D \rangle_H = (-1)^{J(A(T),D)} \langle D \rangle_B$$

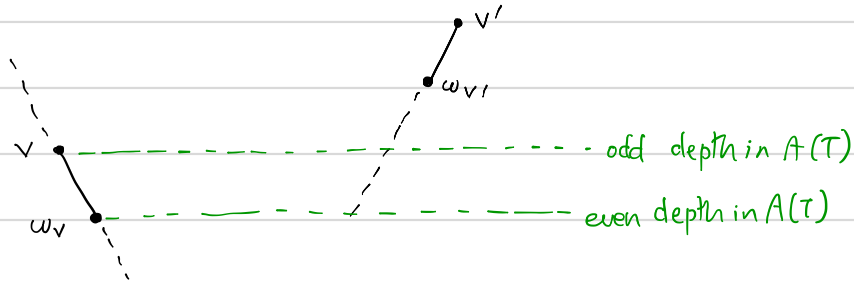
where $J(A(T),D) = \sum_{w<w', \mathrm{depth}(w') < \mathrm{depth}(w)} |\phi_w||\phi_{w'}|$ and ϕ_w is the morphism assigned to w by D. The only decorations of nonzero degree in D are r_k's (degree $+1$) on internal vertices and a's (degree -1) on vertices created on midpoints of edges of T. Note the depth is computed <u>in $A(T)$</u>.

<u>Lemma</u> $J(A(T),D) = \sum_{d>1} \binom{N_d}{2}$ where N_d is the number of internal vertices at depth d in T.

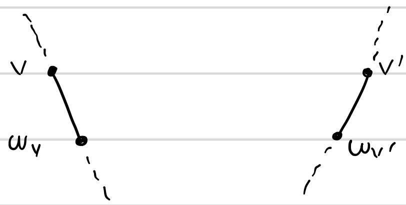
$\ulcorner$ as usual $\binom{0}{2} = \binom{1}{2} = 0 \lrcorner$

<u>Proof</u> If r_k decorates a vertex adjacent to the root it does not contribute to $J(A(T),D)$. Let v be an internal vertex of T, viewed as a vertex in $A(T)$, and suppose v is <u>not</u> adjacent to the root, so in T the edge emanating from v acquires a midpoint vertex w_v. As v varies over all internal vertices of T not adjacent to the root the v, w_v enumerate <u>all</u> the contributing vertices to $J(A(T),D)$. Moreover all the v's have odd depth in $A(T)$, as $\mathrm{depth}_{A(T)}(v) = 1 + 2(\mathrm{depth}_T v - 1)$, and consequently $\mathrm{depth}_{A(T)}(w_v) = 2(\mathrm{depth}_T v - 1)$ is always even. Let V denote the set of internal vertices of T not adjacent to the root. Then ($v < v'$ in T or $A(T)$ is the same) given $v, v' \in V$ with $v < v'$ there are three possible relationships between (v, w_v) and $(v', w_{v'})$ indicated in the following diagram

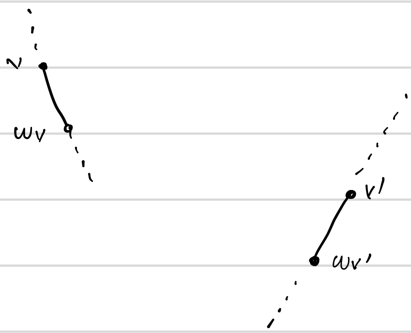
(A) $\mathrm{depth}_T\, v < \mathrm{depth}_T\, v'$



(B) $\mathrm{depth}_T\, v = \mathrm{depth}_T\, v'$



(C) $\mathrm{depth}_T\, v > \mathrm{depth}_T\, v'$



Now a pair in configuration (A) does not contribute to $\mathcal{J}(A(T), D)$, a pair in config. (B) contributes -1 and a pair in config. (C) contributes zero. So we have

$$\mathcal{J}(A(T), D) = \sum_{\substack{v, v' \in V \\ v < v' \\ \mathrm{depth}_T(v) = \mathrm{depth}_T(v')}} -1$$

Among the vertices at a fixed depth $<$ is a total order, so we conclude that

$$\mathcal{J}(A(T), D) = \sum_{d \geq 1} \binom{N_d}{2}$$

as claimed. $\square$