ainfcat3

2/10/16

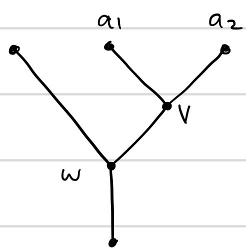
Trees and signs for A_∞-categories (checked)

We revisit some of the sign issues from ainfmf14 which have come up again in ainfcat2. Throughout S is a commutative, associative but not necessarily unital ring. We begin with basic material on trees.

Definitions Here a tree is a connected acyclic (unoriented) graph. A rooted tree is a tree in which one vertex has been designated the root (for us, the root is always a leaf vertex, i.e. a vertex of valency 1). We make a rooted tree an oriented graph by orienting all edges towards the root (i.e. in the direction of any path to the root). In a rooted tree the parent of a vertex is the vertex connected to it on the path to the root. Every vertex but the root has a parent. A child of a vertex v is a vertex of which v is the parent. A plane tree is a rooted tree together with an ordering for the children of each vertex. A morphism of plane trees is a morphism of oriented graphs which preserves the root vertex and the ordering on children, i.e. if $w \le w'$ are children of v then $f(w) \le f(w')$ as children of $f(v)$. A plane tree is valid if it has $n+1$ leaves (including the root) for some $n \geq 2$ and all non-leaves have valency at least three. We call leaves external vertices, non-leaves internal vertices, the edges meeting a leaf are external edges, the others are internal edges.

Def^n Given a vertex v in a rooted tree T the depth of v is the length of the unique path from v to the root. The height of v is the length of the longest path from v to a leaf. The height of the tree is the height of the root.

Example



T has height 3, v has depth 2 and height 1, while w has depth 1 and height 2.

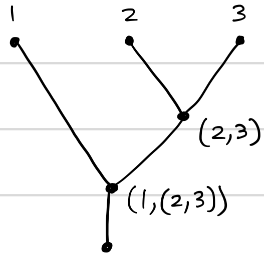
Note that our embeddings in $\mathbb{R}^2$ (= the page) assign the clockwise order (i.e. the children of v are, in order, (a_1, a_2)).

<u>Def^n</u> Let T be a plane tree and define a relation $<$ on the vertices as follows: $w < w'$ if there is a vertex v in $P_w \cap P_{w'}$ (here $P_w, P_{w'}$ are respectively the paths from w, w' to the root) and children c, c' of v with $c < c'$ according to the chosen order, and $c \in P_w$, $c' \in P_{w'}$.

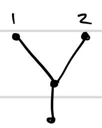
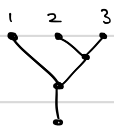
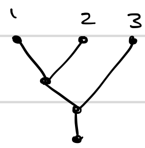
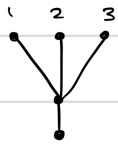
<u>Lemma</u> If we restrict $<$ to (a) all vertices at a fixed depth or (b) non-root leaves then it is a total order.

<u>Def^n</u> Let T be a valid plane tree with $n+1$ leaves. The <u>combinatorial tree</u> of T, denoted $c(T)$, is an ordered set defined as follows. Let the non-root leaves be $v_1, \ldots, v_k$ (in order, according to $<$ above) and define $\alpha(v_i) = i$. We define α recursively on non-leaf vertices vertices by the condition that if v has ordered children $w_1, \ldots, w_n$ then $\alpha(v) = (\alpha(w_1), \ldots, \alpha(w_n))$.

<u>Example</u>



Clearly the set of isomorphism classes of valid plane trees with $n+1$ leaves is in bijection with a set of (suitably defined) combinatorial trees. <u>This set we denote by $\mathcal{T}_n$.</u>

<u>Example</u> $\mathcal{T}_2 = \{$  $\}$, $\mathcal{T}_3 = \{$ , ,  $\}$.

Note that the two trees in $\mathcal{T}_3$ are isomorphic as oriented graphs, but not as plane trees.

<u>Defⁿ</u> Let S be a commutative, associative but not-necessarily unital ring, and Q a plane tree. A <u>decoration</u> of Q is the following data, where "graded" means either $\mathbb{Z}$ or $\mathbb{Z}_2$-graded: (we assume Q has at least one edge)

(i) a graded S-module L_v for each leaf v (inc. the root),

(ii) a graded S-module M_e for every edge e,

(iii) for each internal vertex v with incoming edges (in order) $e_1, \dots, e_k$ and outgoing edge e an integer N_v (in $\mathbb{Z}$ or $\mathbb{Z}_2$) and a degree N_v S-linear map

$$\phi_v : M_{e_1} \otimes_S \cdots \otimes_S M_{e_k} \longrightarrow M_e. \qquad (3.1)$$

(iv) for each non-root leaf vertex ℓ a degree zero S-linear map

$$\phi_\ell : L_\ell \longrightarrow M_e$$

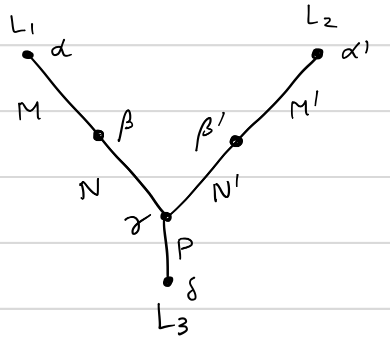
where e is incident at ℓ.

(v) a degree zero S-linear map

$$\phi_r : M_e \longrightarrow L_r$$

where e is incident at r, which is the root.

<u>Example</u>



$$\begin{aligned} &\alpha : L_1 \to M, \ \alpha' : L_2 \to M', \\ &\beta : M \to N, \ \beta' : M' \to N', \\ &\gamma : N \otimes N' \to P, \\ &\delta : P \to L_3 \end{aligned} \qquad (3.2)$$

<u>Def</u>

Def^n Suppose in Q that every non-root leaf has equal depth. The <u>height denotation</u> $\langle D \rangle_H$ of D is a homogeneous S-linear map $L_{e_1} \otimes_S \cdots \otimes_S L_{e_n} \longrightarrow L_r$ where $e_1, \ldots, e_n$ denote the non-root leaves (in order), defined as follows:

- Let h be the height of Q (recall every non-root leaf has depth h). For $1 \le d \le h$ let Q_d be the set of vertices of depth d (so Q_h is the set of non-root leaves). Make Q_d a totally ordered set via (6.1), say $Q_d = (v_1, \ldots, v_m)$ and for $1 \le i \le m$ let $(e_i^1, \ldots, e_i^{a_i})$ be the incoming edges at v_i, in their assigned order, and f_i the outgoing edge at v_i. Define a homogeneous S-linear map for $d < h$ by

$$\Phi_d := \phi_{v_1} \otimes \cdots \otimes \phi_{v_m} : \Big(\bigotimes_j M_{e_1^j} \Big) \otimes_S \cdots \otimes_S \Big(\bigotimes_j M_{e_m^j} \Big) \longrightarrow M_{f_1} \otimes_S \cdots \otimes_S M_{f_m}. \qquad (4.1)$$

using the <u>Koszul sign convention</u> (see p. 6 ainfcat2).

- Define Φ_h separately to be $\Phi_h = \bigotimes_{v \in Q_h} \phi_v$ and $\Phi_0 = \phi_r$ where r is the root. Finally,

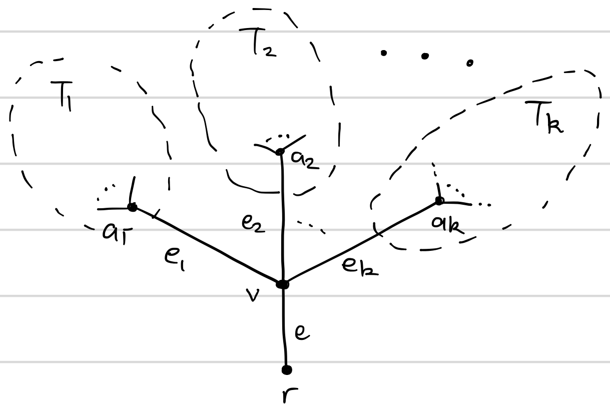
$$\langle D \rangle_H := \Phi_0 \circ \Phi_1 \circ \cdots \circ \Phi_h \qquad (4.2)$$

<u>Example</u> For the decoration of (3.2)

$$\langle D \rangle_H = \underbrace{\delta}_{\Phi_0} \circ \underbrace{\gamma}_{\Phi_1} \circ \underbrace{(\beta \otimes \beta')}_{\Phi_2} \circ \underbrace{(\alpha \otimes \alpha')}_{\Phi_3} \qquad (4.3)$$

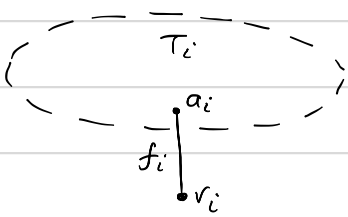
Note that there appears to be a typo in [L] first display after Figure 1, which should read $\mathrm{id}_B \otimes r_2 \otimes r_3 \otimes \mathrm{id}_B$.

Def^n Let Q be any plane tree and D a decoration. Let r be the root and v the vertex adjacent to r. Suppose v has valence $k+1$ (possibly $k=0$). Consider the diagram (as usual we assume Q has at least one edge)



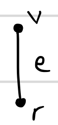
(5.1)

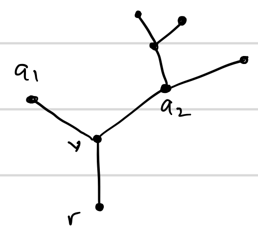
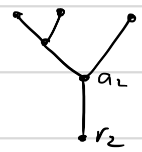
We define plane trees $\widetilde{T}_1, \ldots, \widetilde{T}_k$ to be (note e_i is <u>not</u> in $\widetilde{T}_i$)

$\widetilde{T}_i :=$ 

(5.2)

where r_i is some new vertex, which we declare the root of $\widetilde{T}_i$. We make $\widetilde{T}_i$ a plane tree using the ordering from Q, and decorate $\widetilde{T}_i$ according to a decoration D_i which agrees with D plus assigns M_{e_i} from D to f_i and also $L_{r_i} = M_{e_i}$, $\phi_{r_i} = id$, ϕ_{a_i} as in D.

<u>Example</u> (a) $Q = \Big\{$  $\Big\}$ so there are no $\widetilde{T}_i$.

(b) $Q = \Big\{$  $\Big\}$ then $\widetilde{T}_1 = \Big\{$  $\Big\}$, $\widetilde{T}_2 = \Big\{$  $\Big\}$

<u>Def^n</u> Let Q, D be as above (Q any plane tree with ≥ 1 edge). The <u>branch denotation</u> $\langle D \rangle_B$ is a homogeneous S-linear map $L_{\ell_1} \otimes_S \cdots \otimes_S L_{\ell_n} \longrightarrow L_r$ where $\ell_1, \ldots, \ell_n$ denote the non-root leaves (in order), defined as follows:

- if Q has one edge, so

$$Q = \left\{ \begin{tikzcd} \overset{L}{v} \arrow[d, "e"] \\ \underset{L_r}{r} \end{tikzcd} M_e \;, \quad \phi_v : L \longrightarrow M_e, \ \phi_r : M_e \longrightarrow L_r \right\} \qquad (6.1)$$

then $\langle D \rangle_B := \phi_r \circ \phi_v$.

- otherwise we define recursively using (5.1)

$$\langle D \rangle_B := \phi_r \circ \phi_v \circ \left(\langle D_1 \rangle_B \otimes_S \cdots \otimes_S \langle D_k \rangle_B \right). \qquad (6.2)$$

where D_i is the decoration of $\tilde{T}_i$ defined above.

<u>Remark</u> We may write $(\ell_1, \ldots, \ell_n) = (\ell_1^1, \ldots \ell_1^{m_1}, \ell_2^1, \ldots, \ell_2^{m_2}, \ldots)$ where ℓ_i^j are the non-root leaves in T_i, so that $\langle D_i \rangle_B : L_{\ell_i^1} \otimes_S \cdots \otimes_S L_{\ell_i^{m_i}} \longrightarrow M_{e_i}$. Hence $\langle D_1 \rangle_B \otimes \cdots \otimes \langle D_k \rangle_B : L_{\ell_1} \otimes \cdots \otimes L_{\ell_n} \longrightarrow M_{e_1} \otimes \cdots \otimes M_{e_k}$. Again, the Koszul sign rule is used to define (6.2).

<u>Def^n</u> Let Q be a plane tree, D a decoration, and $L_1, \ldots, L_n, L_r$ the modules assigned to the leaves so that $\langle D \rangle_B : L_1 \otimes_S \cdots \otimes_S L_n \longrightarrow L_r$. We define

$$\mathrm{eval}_D : L_1 \otimes_S \cdots \otimes_S L_n \longrightarrow L_r$$

to be the S-linear map associated to D as a diagram in S<u>Mod</u>, ignoring the grading. Recursively, if Q has one edge $\mathrm{eval}_D = \phi_r \circ \phi_v$ and in the case of (6.2) we use the same formula but as plain S-linear maps (i.e. <u>no</u> Koszul signs).

<u>Lemma</u> With the above notation

$$\langle D\rangle_B(a_1\otimes\cdots\otimes a_n) = (-1)^s\, \mathrm{eval}_D(a_1\otimes\cdots\otimes a_n) \qquad (6.5.1)$$

where we write $\ell_1,\dots,\ell_n$ for the non-root leaves in their order, and

$$s = \sum_{i=1}^{n}\sum_{v>\ell_i} |a_i|\cdot|\phi_v| \qquad (6.5.2)$$

where v ranges over <u>all</u> vertices, including leaves.

<u>Proof</u> By induction on the height of Q. If the height is 1 we are in the situation of (6.1), and the claim is clear. Otherwise

$$\begin{aligned}\langle D\rangle_B(a_1\otimes\cdots\otimes a_n) &= \phi_r\circ\phi_v\circ(\langle D_1\rangle_B\otimes_s\cdots\otimes_s\langle D_k\rangle_B)(a_1\otimes\cdots\otimes a_n)\\ &= (-1)^t\phi_r\circ\phi_v\big(\langle D_1\rangle_B(a_1\otimes\cdots\otimes a_{m_1})\\ &\qquad\otimes\langle D_2\rangle_B(a_{m_1+1}\otimes\cdots\otimes a_{m_2})\\ &\qquad\otimes\cdots\\ &\qquad\otimes\langle D_k\rangle_B(a_{m_{k-1}+1}\otimes\cdots\otimes a_n)\big)\end{aligned}$$

where (set $m_0=0$)

$$t = \sum_{i=1}^{k}\Big(\Sigma_{j=m_{i-1}+1}^{m_i}|a_j|\Big)\Big(\Sigma_{\ell=i+1}^{k}|\langle D_\ell\rangle_B|\Big)$$

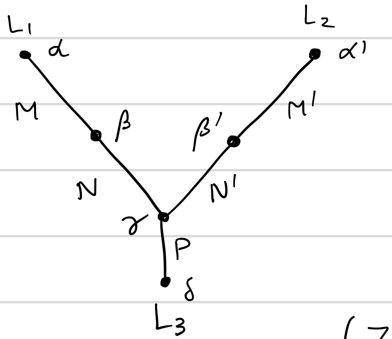
By the inductive hypothesis each $\langle D_i\rangle_B(\cdots)$ can be written as a sign times $\mathrm{eval}_{D_i}(\cdots)$, and is easy to see the signs match. □

Comparison of height and branch denotations

Let Q be a plane tree with at least one edge, D a decoration of Q, and let $\ell_1, \dots, \ell_n$ be the non-root leaves. Suppose all non-root leaves have the same depth, then we have defined two homogeneous S-linear maps

$$\langle D \rangle_H, \langle D \rangle_B : L_{\ell_1} \otimes_S \cdots \otimes_S L_{\ell_n} \longrightarrow L_r,$$

which differ by signs. We wish to calculate these signs.



Example Recall the example from (3.2),

$$\langle D \rangle_H = \delta \circ \gamma \circ (\beta \otimes \beta') \circ (\alpha \otimes \alpha')$$

$$\begin{aligned} \langle D \rangle_B &= \delta \circ \gamma \circ (\beta \circ \alpha \otimes \beta' \circ \alpha') \\ &= (-1)^{|\alpha||\beta'|} \delta \circ \gamma \circ (\beta \otimes \beta') \circ (\alpha \otimes \alpha') \\ &= (-1)^{|\alpha||\beta'|} \langle D \rangle_H . \end{aligned} \tag{7.1}$$

Recall the relation $<$ on vertices from p. 2.

Lemma $\langle D \rangle_B = (-1)^{J(Q,D)} \langle D \rangle_H$ where

$$J(Q,D) = \sum_{\substack{w < w' \\ \mathrm{depth}(w') < \mathrm{depth}(w)}} |\phi_w| |\phi_{w'}| \tag{7.2}$$

here the sum ranges over all pairs $w < w'$ with w, w' any non-root vertices.

<u>Proof</u> By induction on h, the height of Q. The base case is $h=1$, i.e.

$$Q = \left\{ \begin{tikzcd} \bullet^{v} \arrow[d, "e"] \\ \bullet_{r} \end{tikzcd} \right\}$$

in which case the claim is trivially true. Now suppose $h > 1$ and that we decompose Q as in (5.1). Then by def^N of $\langle D \rangle_B$ and the inductive hypothesis

$$\begin{aligned} \langle D \rangle_B &= \phi_r \circ \phi_v \circ \left(\langle \tilde{T}_1 \rangle_B \otimes_S \cdots \otimes_S \langle \tilde{T}_k \rangle_B \right) \\ &= \phi_r \circ \phi_v \circ \left((-1)^{\mathcal{J}(\tilde{T}_1)} \langle \tilde{T}_1 \rangle_H \otimes \cdots \otimes (-1)^{\mathcal{J}(\tilde{T}_k)} \langle \tilde{T}_k \rangle_H \right) \qquad (8.2) \\ &= (-1)^{\sum_i \mathcal{J}(\tilde{T}_i)} \phi_r \circ \phi_v \circ \left(\langle \tilde{T}_1 \rangle_H \otimes \cdots \otimes \langle \tilde{T}_k \rangle_H \right) \end{aligned}$$

where $\mathcal{J}(\tilde{T}_i)$ denotes the sign (7.2) for the induced decoration of $\tilde{T}_i$. Note that by our hypothesis on Q, all the $\tilde{T}_i$ have height $h-1$, and so by def^N

$$\langle \tilde{T}_i \rangle_H = \Phi^i_2 \circ \Phi^i_3 \circ \cdots \circ \Phi^i_h \quad \text{where} \quad \Phi^i_d = \bigotimes_v \phi_v$$

for v ranging over vertices in $\tilde{T}_i$ of depth d in Q (note that $\Phi^i_2 = \phi_{a_i}$ in Q). Observe that $|\Phi^i_d| = \sum_v |\phi_v|$ where v ranges over the contributing vertices in $\tilde{T}_i$.

Hence we calculate

$$\begin{aligned} \langle \tilde{T}_1 \rangle_H \otimes \cdots \otimes \langle \tilde{T}_k \rangle_H &= \left(\Phi^1_2 \circ \cdots \circ \Phi^1_h \right) \otimes \left(\Phi^2_2 \circ \cdots \circ \Phi^2_h \right) \otimes \cdots \\ &\qquad \cdots \otimes \left(\Phi^k_2 \circ \cdots \circ \Phi^k_{h-1} \right) \\ &= (-1)^s \left(\Phi^1_2 \otimes \cdots \otimes \Phi^k_2 \right) \circ \left(\Phi^1_3 \otimes \cdots \otimes \Phi^k_3 \right) \circ \cdots \qquad (8.3) \\ &\qquad \cdots \circ \left(\Phi^1_h \otimes \cdots \otimes \Phi^k_h \right) \end{aligned}$$

where $s = \sum_{a<b,\, i<j} |\Phi^a_j| |\Phi^b_i|$ where $1 \le a < b \le k$ and $2 \le i < j \le h$

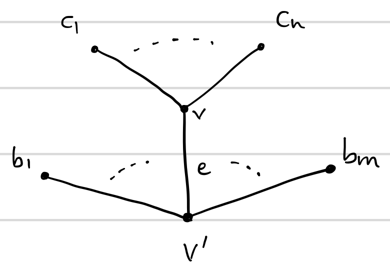
It follows that, combining (8.2) and (8.3),

$$\langle D \rangle_B = \sum_i (-1)^{\mathcal{T}(\widetilde{T}_i) + s} \langle D \rangle_H. \qquad (9.1)$$

Given a pair of vertices $w < w'$ in Q with $\mathrm{depth}(w') < \mathrm{depth}(w)$ we have $w \in T_i$ and $w' \in T_j$ for some i, j. In fact we must have $i \leq j$. If $i = j$ then $|\phi_w||\phi_{w'}|$ contributes to $\mathcal{T}(\widetilde{T}_i)$ and if $i < j$ then $|\phi_w||\phi_{w'}|$ contributes to s via $|\Phi^i_{\mathrm{depth}(w)}||\Phi^j_{\mathrm{depth}(w')}|$, which proves that the sign in (9.1) agrees with (7.2). $\square$

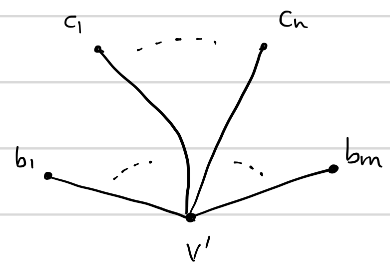
Edge contraction (write $V(T)$ for the vertex set)

Def^n Let T be a plane tree and e an internal edge of T, which connects vertices v and v'. Suppose v' is closer to the root and that v has children $c_1, \dots, c_n$ (note $n \geq 1$ since e is internal) while v' has children $b_1, \dots, b_m$ with $b_j = v$.



(10.1)

We define the plane tree $T \lrcorner e$ to have vertices $V(T) \setminus \{v\}$. We connect $c_1, \dots, c_n$ to v' and order the children of v' as follows: $b_1, \dots, b_{j-1}, c_1, \dots, c_n, b_{j+1}, \dots, b_m$.

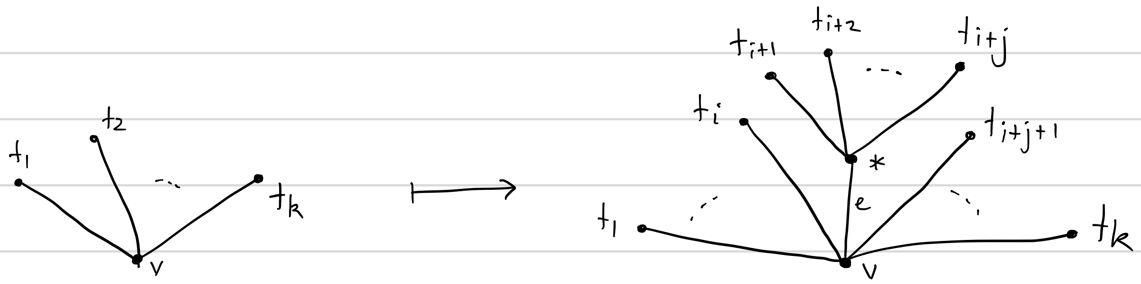


(10.2)

Clearly $T \lrcorner e$ is a plane tree with the same root as T and one less internal edge.

Edge insertion

Def^n Let T be a plane tree and v an internal vertex, with children $t_1, \dots, t_k$. Given integers $i \geq 0$ and $j \geq 1$ with $i + j \leq k$ we define the <u>edge insertion</u> $\mathrm{ins}(T, v, i, j)$ to be the plane tree with vertices $V(T) \cup \{*\}$. The parent/child relations (which clearly determine the edges) between vertices in $V(T)$ are preserved in $\mathrm{ins}(T, i, j)$ with the exception of $t_{i+1}, \dots, t_{i+j}$ which now have parent $*$. The parent of $*$ is v, and it has $t_{i+1}, \dots, t_{i+j}$ as its only children. Denote the $* - v$ edge by e.



(11.1)

The ordering on children is as in T, with the exception of v and $*$ where (11.1) gives the ordering. Clearly $\operatorname{ins}(T, v, i, j)$ is a plane tree with one more internal edge than T.

<u>Lemma</u> If T is a valid plane tree and e an internal edge then $T \lrcorner\, e$ is valid. If Q is a valid plane tree, v an internal vertex with k children, and $2 \le j \le k-1$ then $\operatorname{ins}(Q, v, i, j)$ is valid.

<u>Proof</u> Note that in $\operatorname{ins}(Q, v, i, j)$ the vertex v has $k-j+1$ children, so for the tree to be valid we need both $j \geq 2$ and $j \le k-1$. $\square$

Let $\mathcal{C}_n$ be the category of valid plane trees with $n+1$ leaves, so that $\mathcal{T}_n$ is the set of isomorphism classes of objects of $\mathcal{C}_n$. We can write $\mathcal{T}_n = \coprod_{c \geq 0} \mathcal{T}_n^{(c)}$ where $\mathcal{T}_n^{(c)}$ denotes trees with c internal edges, i.e. $e_i(T) = c$.

<u>Def^n</u> $\mathcal{T}_n^+$ is the set of pairs (T, e) where $T \in \mathcal{T}_n$ and e is an internal edge of T. (note that $\mathcal{T}_2^+ = \phi$ since $\mathcal{T}_2 = \{ Y \}$).

Let us write $|v|$ for the valency of v (so if v has k children, $|v| = k+1$).

FALSE

<u>Lemma</u> For $n > 2$ the map $\mathcal{T}_n^+ \xrightarrow{\pi} \mathcal{T}_n$, $\pi(T, e) = T \lrcorner\, e$ is surjective, and

$$\pi^{-1}(Q) = \{ (\operatorname{ins}(Q, v, i, j), e) \mid v \text{ an internal vertex}, i \geq 0, i+j \le |v|-1, \; 2 \le j \le |v|-2 \}$$

<u>Proof</u> If $Q \in \mathcal{T}_n$ then since $n \geq 2$ there is at least one internal vertex. Clearly $ins(Q, v, i, j) \lrcorner e = Q$ so π is surjective. Suppose that $T \lrcorner e = Q$. In the situation of (10.1), (10.2) we have

$$T = ins(Q, v', j-1, n)$$

proving the second statement. $\square$

This implies there is a bijection between pairs $(T, e) \in \mathcal{T}_n^+$ and tuples (Q, v, i, j) with $Q \in \mathcal{T}_n$, v an internal vertex, $i \geq 0$, $i + j \leq |v| - 1$, $2 \leq j \leq |v| - 2$ for any $n > 2$. More precisely: (noting $|v| = k+1$)

<u>Lemma</u> For $n > 2$ and $c \geq 0$ there is a bijection

$$\left(\mathcal{T}_n^{(c+1)}\right)^+ \longrightarrow \{(Q, v, i, j) \mid Q \in \mathcal{T}_n^{(c)}, \ v \text{ an internal vertex}, \ i \geq 0, \ i + j \leq |v| - 1, \ 2 \leq j \leq |v| - 2\}. \tag{12.1}$$

<u>Note</u> We restrict to $n > 2$ because $\mathcal{T}_2^+ = \phi$.