Notes on A_∞ min models for MFs

① (ainfmf)

Let $(A, m_n)_{n\geq 1}$ be a nonunital A_∞ algebra. $\Pi : A \to A$ degree zero, closed, with $\Pi^2 = \Pi$. Suppose we are given a homotopy $H : A \to A[-1]$, s.t.

$$1 - \Pi = dH + Hd \qquad (\text{also } H^2 = 0 ?)$$

We let B be a splitting of Π in complexes

$$\begin{tikzcd} A \arrow[r, shift left, "p"] & B \arrow[l, shift left, "i"] \end{tikzcd} \qquad (1.1)$$

$$pi = 1 \qquad ip = \Pi$$

There is a sequence of linear operators

$$m_n^B : B^{\otimes n} \to B[2-n] \qquad (1.2)$$

$$m_1^B = d^B = p m_1 i$$

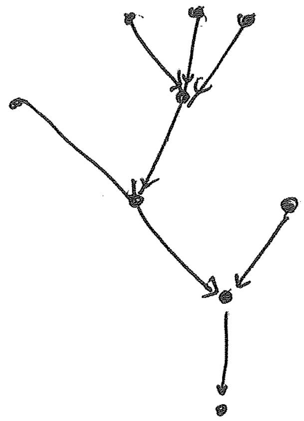
$$m_2^B = p m_2 (i \otimes i)$$

$$m_n^B = \sum_T (-1)^s m_{n,T} \qquad n \geq 3$$

where the summation is taken over all oriented planar trees T with $n+1$ tail vertices (including the root), such that the (oriented) valency $|v|$ (ingoing edges) of every internal vertex T is at least 2. In order to describe the linear map $m_{n,T} : B^{\otimes n} \to B[2-n]$ we need some preparations: $\bar{T}$ is another tree obtained by the insertion of a new vertex into every internal edge. To every tail we assign i, to an old vertex m_k and to every new edge H.

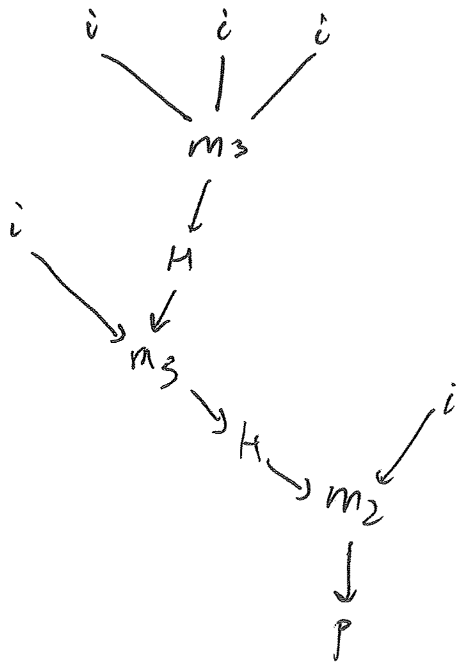
e.g.

T :



(2.1)

$\overline{T}$



The number of internal edges in T is 2, this is the numbers that given in (1.2). (see Lazaroiu for the sign).

<u>Theorem</u> The sequence m_n^B, $n \geq 1$ defines an A_∞-structure.

The stabilisation of k

Let k be a field of char. 0 and $W \in k[x_1, \dots, x_n] =: R$ a potential. We define

$$k^{stab} = \left(\Lambda(R\psi_1 \oplus \cdots \oplus R\psi_n), \textstyle\sum_i x_i \psi_i^* + \sum_i W_i \psi_i \right) \tag{3.1}$$

where $W \in m^2$ and $W = \sum_i x_i W_i$ for $W_i \in m$. Then the odd operator $\psi_j = \psi_j \wedge -$ on k^{stab} satisfies

$$[\psi_j, d_{k^{stab}}] = x_j \cdot 1 \tag{3.2}$$

That is, the x_j act null-homotopically on k^{stab}. With

$$S = \Lambda(k\theta_1 \oplus \cdots \oplus k\theta_n)$$

we have a diagram of $\mathbb{Z}_2$-graded cpxs over k.

$$\begin{tikzcd} S \otimes_k \mathrm{End}_R(k^{stab}) \arrow[r, shift left, "\exp(-\delta)"] & K \otimes_R \mathrm{End}_R(k^{stab}) \arrow[l, shift left, "\exp(\delta)"] \arrow[r, shift left, "\pi"] \arrow[loop below, "H"] & \mathrm{End}_R(k^{stab}) \otimes_R R/m \arrow[l, shift left, "\sigma_\infty"] \\ & & \underline{\mathrm{End}}_R(k^{stab}) \arrow[u, phantom, "\shortparallel"] \end{tikzcd}$$

where $\underline{\mathrm{End}}_R(k^{stab})$ has differential zero, and we set

$$\Phi := \pi \exp(-\delta) \qquad \pi \sigma_\infty = 1$$

$$\Phi^{-1} := \exp(\delta) \sigma_\infty \qquad H = [d_K, \nabla]^{-1} \nabla$$

$$\delta = \textstyle\sum_i \psi_i \theta_i^* \qquad 1 - [d_K + d_{End}, H] = \sigma_\infty f.$$

Hence

$$\Phi^{-1}\Phi = \exp(\delta)\sigma_\infty \pi \exp(-\delta)$$
$$= 1 - \exp(\delta)[d_k + d_{End}, H]\exp(-\delta)$$
$$= 1 - [d_{End}, \underbrace{\exp(\delta)H\exp(-\delta)}_{\text{call this } \hat{H}}]$$

So overall we have

$$\begin{tikzcd} S \otimes_k \mathrm{End}_R(k^{stab}) \arrow[r, shift left, "\Phi"] \arrow[loop left, "\hat{H}"] & \underline{\mathrm{End}}_R(k^{stab}) \arrow[l, shift left, "\Phi^{-1}"] \end{tikzcd} \qquad (4.1)$$

$$\Phi\Phi^{-1} = 1, \quad \Phi^{-1}\Phi = 1 - [d_{End}, \hat{H}]$$
$$\text{i.e. } 1 - \Phi^{-1}\Phi = [d_{End}, \hat{H}]$$

Now of course

$$\underline{\mathrm{End}}_R(k^{stab}) = \Lambda(k\psi_1 \oplus \cdots \oplus k\psi_n)$$

is a finite-dimensional $\mathbb{Z}_2$-graded k-vector space, to which we may apply the minimal model algorithm. It has operations which we denote $\sigma_n, n \geq 1$, where the DG-alg structure on $S \otimes \mathrm{End}_R(k^{stab})$ is denoted

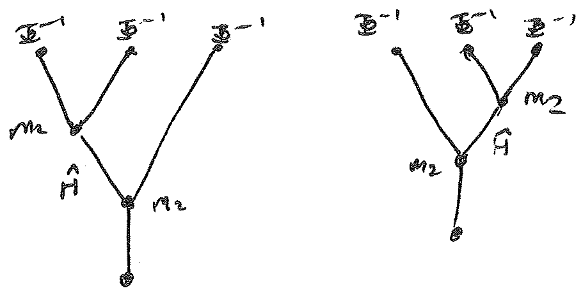
$$m_1 = 1 \otimes d_{End} = 1 \otimes [d_k^{stab}, -].$$
$$m_2 = (S \otimes End) \otimes (S \otimes End)$$
$$\cong (S \otimes S) \otimes (End \otimes End) \longrightarrow S \otimes End$$

where we use the exterior algebra product on S.
All m_n, $n \geq 3$ are zero. Then (1.2) gives

$$
\begin{aligned}
b_1 &= 0 \\
b_2 &= \Phi\, m_2(\Phi^{-1} \otimes \Phi^{-1}) \qquad\qquad (5.1) \\
b_n &= \textstyle\sum_T (-1)^{s(T)}\, b_{n,T} \qquad n \geq 3
\end{aligned}
$$

where the trees only have m_2's. Thus we have only trees with n incoming leaves, with two incoming edges at every vertex, e.g.

$\boxed{n=3}$



(5.2)

Thus

$$
\begin{aligned}
b_3 = &- \Phi\, m_2\big(\hat{H}\, m_2(\Phi^{-1} \otimes \Phi^{-1}) \otimes \Phi^{-1}\big) \qquad (5.3) \\
&- \Phi\, m_2\big(\Phi^{-1} \otimes \hat{H}\, m_2(\Phi^{-1} \otimes \Phi^{-1})\big)
\end{aligned}
$$

let us compute b_2, using $\nabla = \sum_i \partial_{x_i} \theta_i$ and for the usual reason $J = (\delta\nabla + \delta\nabla)$ on $K \otimes_R \mathrm{End}_R(k^{stab})$ acts by p on a form of θ-degree p, so mod $\underline{x}$ we may treat

$$
\begin{aligned}
\hat{H} &= \exp(\delta)\, H \exp(-\delta) \\
&= \exp(\delta)\, J^{-1} \textstyle\sum_j \partial_{x_j} \theta_j \cdot \exp\left(-\sum_i \psi_i \theta_i^*\right)
\end{aligned}
$$

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$$b_2 = \Phi m_2 (\Phi^{-1} \otimes \Phi^{-1})$$
$$= \pi \exp(-\delta) m_2 \Big(\exp(\delta) \mathcal{E}_\infty \otimes \exp(\delta) \mathcal{E}_\infty \Big) \qquad (6.1)$$

where

$$\mathcal{E}_\infty = \sum_{s \geq 0} \sum_{p_1, \dots, p_s} (-1)^{\binom{s+1}{2}} \frac{1}{s!} At_{p_1} \cdots At_{p_s} \Theta_{p_1} \cdots \Theta_{p_s} \qquad (6.2)$$

where $At_i = [d_{End}, \partial_{x_i}]$. Now $\delta = \sum_i \psi_i \Theta_i^*$

$$\exp(\delta) \mathcal{E}_\infty = \sum_{s \geq 0} \sum_{p_1, \dots, p_s} (-1)^{\binom{s+1}{2}} \sum_{q \geq 0} \frac{1}{q!} \delta^q \frac{1}{s!} At_{p_1} \cdots At_{p_s} \Theta_{p_1} \cdots \Theta_{p_s} \qquad (6.3)$$

We may write

$1+2+\dots+q-1$

$$\delta^q = \sum_{i_1, \dots, i_q} \psi_{i_1} \Theta_{i_1}^* \cdots \psi_{i_q} \Theta_{i_q}^*$$
$$= \sum_{i_1, \dots, i_q} \psi_{i_1} \cdots \psi_{i_q} \Theta_{i_1}^* \cdots \Theta_{i_q}^* (-1)^{\binom{q}{2}} \qquad (6.4)$$
$$= \sum_{i_1 < \dots < i_q} \sum_{\sigma \in S_q} \cancel{\psi_{\sigma} \cdots \psi_{\sigma}} (-1)^{\binom{q}{2}} \psi \cdots$$
$$= \sum_{i_1 < \dots < i_q} q! (-1)^{\binom{q}{2}} \psi_{i_1 < \dots < i_q} \Theta^*_{i_1 < \dots < i_q}$$

Hence

$$\exp(\delta) = \sum_{q \geq 0} \frac{1}{q!} \delta^q = \sum_{q \geq 0} \sum_{i_1 < \dots < i_q} (-1)^{\binom{q}{2}} \psi_{i_1} \cdots \psi_{i_q} \Theta_{i_1}^* \cdots \Theta_{i_q}^*$$

$$\exp(-\delta) = \sum_{q \geq 0} \sum_{i_1 < \dots < i_q} (-1)^{\binom{q+1}{2}} \psi_{i_1} \cdots \psi_{i_q} \Theta_{i_1}^* \cdots \Theta_{i_q}^* \qquad (6.5)$$

Now we write $\psi_{\underline{i}}$ for $\underline{i} = (i_1, \dots, i_q)$, $i_1 < \dots < i_q$.
The numbers we want to compute are

$$(b_2)^{\underline{k}}_{\underline{i},\underline{j}} = \langle \underline{k} | b_2 | \underline{i} \otimes \underline{j} \rangle$$

— no, because m_2 acts on operators,

$$= \langle \psi_{\underline{k}} | b_2 | \psi_{\underline{i}} \otimes \psi_{\underline{j}} \rangle$$

which include ψ_i^*.

(7.1)

Clearly by (6.5), computing $\langle \underline{k} | b_2 |$ means computing

$$\sum$$

$$At_i = \left[\left[\sum_j x_j \psi_j^* + \sum_j W_j \psi_j, -\right], \partial_{x_i}\right]$$

(7.2)

act on couplings between ψ_i, ψ_j

$$= \left[\psi_i^* + \sum_j \partial_{x_i}(W_j)\psi_j, -\right]$$

$W = xW_x$

$\partial_x W = W_x + x\partial_x W_x$

$$= [\psi_i^*, -] + \sum_j \partial_{x_i}(W_j)[\psi_j, -].$$

has the effect of cancelling with ψ_i in a operator $\psi_{a_1}\cdots\psi_{a_t}\psi_{b_1}^*\cdots\psi_{b_s}^*$

Say $n=1$, so we have only ψ, θ.

$$\delta_\infty = 1 - At\,\theta$$
$$= 1 - [\psi^*, -]\theta - \partial_x(W_1)[\psi, -]\theta$$

$$\exp(\delta) = 1 + \psi\theta^*$$
$$\exp(-\delta) = 1 - \psi\theta^*$$

$\delta_\infty(1) = 1$ $\quad$ $\delta_\infty(\psi) = \psi - \theta$

$\delta_\infty(\psi^*) = \psi^* - \partial_x(W_x)\cdot\theta$

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$$m_2\Big([1+\psi\theta^*](1-A\otimes\theta)\otimes[1+\psi\theta^*](1-A\otimes\theta) \Big)$$

~~$\psi \otimes \psi$~~

End

$\mathrm{End}_k(\Lambda\psi)$

woops.
The θ^* act on θ's
in S, operators
only compose in End

$$\cancel{[1+\psi\theta^*](\psi) = \psi}$$
$$\cancel{[1+\psi\theta^*](\psi^*) = \psi^* - \psi\psi^*\theta^*}$$
$$\cancel{[1+\psi\theta^*](\theta) = \theta + \psi\theta^*\theta}$$

$\psi\otimes\psi^* \longmapsto$

$$\begin{aligned}
\cancel{m_2(\psi\otimes\psi^*)} &= m_2\Big([1+\psi\theta^*](\psi-\theta)\otimes[1+\psi\theta^*](\psi^*-\partial_x(W_x)\theta) \Big)\\
&= m_2\Big([\psi-\theta-\psi\cancel{\theta^*\theta}]\otimes[\psi^*\cancel{\psi\psi^*\theta^*}\\
&\qquad -\partial_x(W_x)\theta-\partial_x(W_x)\psi\cancel{\theta^*\theta}] \Big)\\
&= \cancel{\psi\psi^*-\partial_x(W_x)\psi\theta}\\
&\quad \cancel{-\theta\psi^*+\theta\psi\psi^*\theta^*+\partial_x(W_x)\theta\psi\theta^*\theta}\\
&\quad \cancel{-\psi\theta^*\theta\psi^*+\psi\theta^*\theta\psi\psi^*\theta^*+\partial_x(W_x)\psi\theta^*\theta\psi\theta^*\theta}
\end{aligned}$$

$$= m_2\Big([-\theta]\otimes[\psi^*-\partial_x(W_x)\theta - \partial_x(W_x)\psi] \Big)$$
$$= -\theta\psi^* + \partial_x(W_x)\theta\psi$$

$$\therefore \pi\exp(-\delta)m_2(\uparrow) = \pi(1-\psi\theta^*)(\uparrow)$$

$$\cancel{=\pi\psi\ldots} \qquad = -\pi\psi\theta^*\big(-\theta\psi^*+\partial_x(W_x)\theta\psi\big)$$

$$= \psi\psi^* \cancel{-\partial_x(W_x)}$$

9 ainPwf

$$\begin{array}{c} \quad a \qquad\qquad\qquad b_1 \quad c_1 \qquad\qquad b_2 \quad c_2 \\ b_2 = \Pi \exp(-\delta)\, m_2 \big(\exp(\delta)\, b_\infty \otimes \exp(\delta)\, b_\infty \big) \end{array}$$

$\mathrm{End}^{\alpha s}$

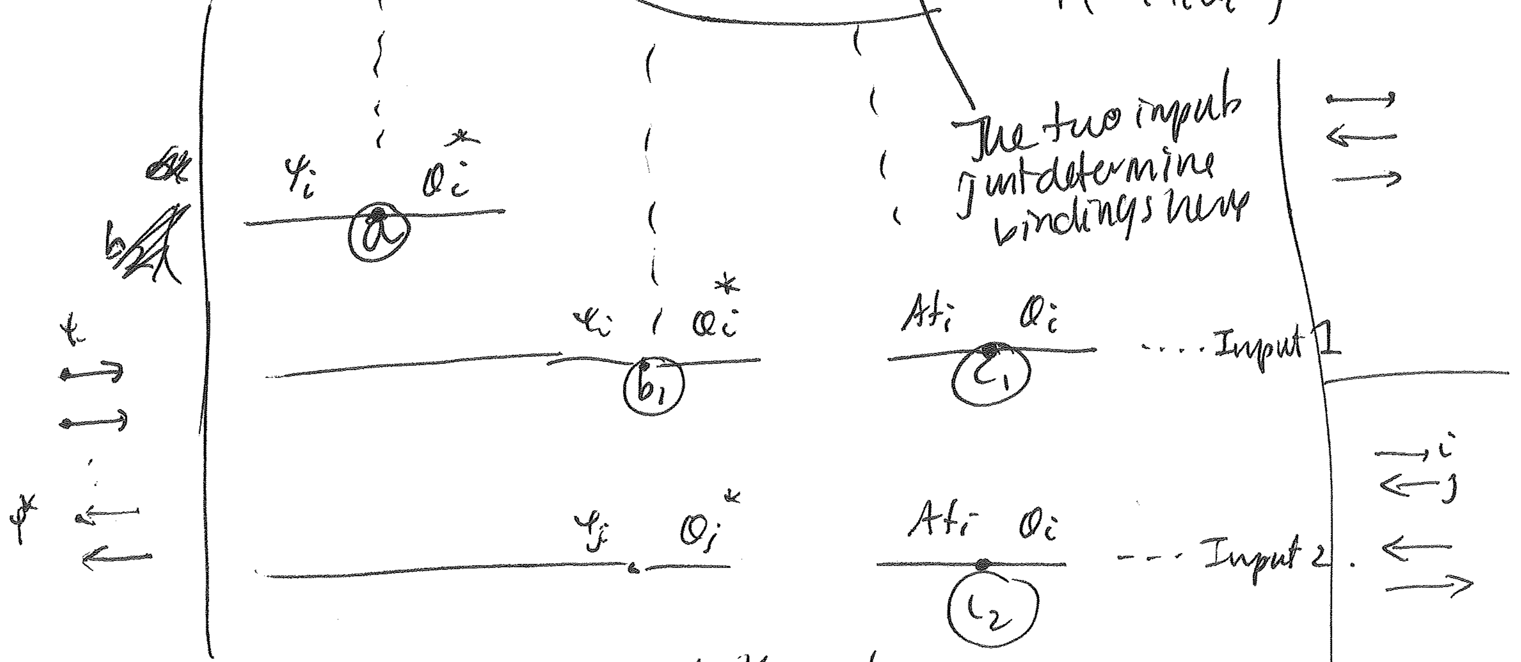
$\underline{\mathrm{End}}$.

$\exp(\sum_i At_i \Theta_i)$ $\exp(\sum_i At_i \Theta_i)$

$\exp(\sum_i \psi_i \Theta_i^*)$

Θ's are totally internal

The two inputs just determine bindings here



ψ_i's and Atiyah's can survive to the end.

on $\mathrm{End}(\Lambda\psi)$ ~~$[\psi_i, -]$ acts as ψ_i^*~~ ~~$[\psi_i^*, -]$ acts as ψ_i~~

ψ_i, ψ_i^* here are left mult on $\mathrm{End}(k^{stab})$

We need to pair on either end

$$[At_i, \psi_j] = \big[\, [\psi_i^*, -] + \textstyle\sum_q \partial_{x_i}(W_q)[\psi_q, -],\ \psi_j \big]$$

$$= \big[[\psi_i^*, -], \psi_j\big] + \textstyle\sum_q \partial_{x_i}(W_q)\big[[\psi_q, -], \psi_j\big]$$

~~$[\psi_i, -](ab)$~~
~~$= \psi_i ab - (-1)^{|a|+|b|} ab\psi_i$~~
~~$= a[\psi_i, -](b)$~~
~~$a\psi_i b$~~

$[d[a,c],d]$

$[[d,a],c]$
$[[c,d],a]$

Atiyah classes can bind to either end.

A_∞ Let A be an $\mathbb{Z}_2$-graded algebra $a, b, c \in A$ (10)

a infinf

$$[a, bc] = abc - (-1)^{|a||b|+|a||c|} bca$$

$$(-1)^{|a||b|} b[a,c] + [a,b]c$$

$$= \cancel{(-1)^{|a||b|} bac} - (-1)^{|a||b|+|a||c|} bca$$

$$+ abc - \cancel{(-1)^{|a||b|} bac}$$

$$= [a, bc].$$

In this sense $[a, -]$ is a derivation.

$$[[a,-], b_L](c) = [a,-](bc)$$

$$|a| \quad |b| \qquad - (-1)^{|a||b|} b[a,c]$$

$$= [a, bc] - (-1)^{|a||b|} b[a,c]$$

$$= [a,b]c$$

$$\therefore \boxed{[[a,-], b_L] = [a,b]_L}$$

$$[At_i, \psi_j] = [[\psi_i^*, -] + \Sigma_q \partial_{x_i}(W_q)[\psi_q, -], \psi_j]$$

$$= [[\psi_i^*, -], \psi_j] = \delta_{ij} \cdot 1$$

$$[At_i, \psi_j^*] = \Sigma_q \partial_{x_i}(W_q)\delta_{qj} = \partial_{x_i}(W_j) \cdot 1$$

A_2 can operate on $\mathrm{End}_A(k^{stab})$?

$\cancel{[\psi_i^*, -]}$

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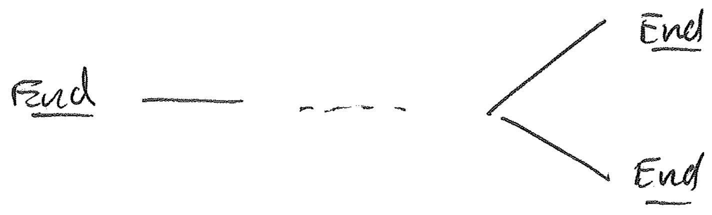
An operator $\mathcal{J}$ on $\mathrm{End}_k(\wedge \psi_1 \cdots \psi_n)$ can be recovered from



We view the input operators as being

$$\underbrace{\phi_{i_1}^* \cdots \phi_{i_t}^* \, \phi_{i_1} \cdots \phi_{i_t}}_{\substack{\cancel{\text{meaning left mult by}} \\ \text{these means mult in the ring.}}} \cdot (\overset{id}{\cancel{\phi}})$$

idempotent.

Anyway, $\mathcal{J}$ can be recovered from its postcomp with the usual idempotents,

$$\phi_{i_1}^* \cdots \phi_{i_t}^* \, \phi_{i_1} \cdots \phi_{i_t}$$

__Lemma__ The projector on $\wedge(k\psi_1 \oplus \cdots \oplus k\psi_n)$ onto the factor $\psi_{i_1} \cdots \psi_{i_k}$ is (set $\underline{j} = \underline{i}^c$) $\quad i_1 < \cdots < i_k$

$$\cancel{\psi_{i_1} \cdots \psi_{j_k} \psi_{i_k}^* \cdots \psi_{i_1}^*}$$

$$\cancel{\psi_{j_t}^* \cdots \psi_{j_1}^* \psi_{j_1} \cdots \psi_{j_t}}$$

$$\pm \, \psi_{j_1}^* \cdots \psi_{j_t}^* \, \psi_1 \cdots \psi_n \, \psi_{i_k}^* \cdots \psi_{i_1}^* =: e(\underline{i})$$

which encodes the numbers

$$\therefore \quad \mathcal{J} = \sum_{\underline{i}} e(\underline{i}) \mathcal{J}$$

$$P_1 \, \psi_{i_k}^* \cdots \psi_{i_1}^* \, \mathcal{J}$$

↑ project $k \cdot 1 \subseteq S$

Some calculations

$$b_2 = \pi \exp(-\delta)\, m_2\big(\exp(\delta)\,\delta_\infty \otimes \exp(\delta)\,\delta_\infty \big) : \underline{End}^{\otimes 2} \to \underline{End}$$

$$\delta_\infty = \sum_{s\geq 0} \sum_{p} (-1)^{\binom{s+1}{2}} \frac{1}{s!} At_p\, \theta_p$$

$$At_i = [\psi_i^*, -] + \sum_q \partial_{x_i}(W_q)[\psi_q, -]$$

$$\delta = \sum_i \psi_i \theta_i^*$$

Now the ψ_i, ψ_i^* generate the ring $\underline{End}$, so any operator may be made as a linear sum of $\psi_{i_1} \cdots \psi_{i_a} \psi_{j_1}^* \cdots \psi_{j_b}^*$'s, so we may evaluate b_2 on such terms. Let us write Ψ for such a product (also Ψ'). Then viewing ψ_i as ψ_i^L (the operator of left mult on $\underline{End}$)

$$\underline{End} \ni \Psi = \underbrace{\psi_{i_1}^L \circ \cdots \circ \psi_{i_a}^L\, \psi_{j_1}^{*L} \circ \cdots \circ \psi_{j_b}^{*L}}_{\text{this is } \Psi^L}(id)$$

The ψ's occurring in δ are all to be read as ψ^L, whereas At_i contains e.g. $[\psi_i^*, -] \circlearrowright End_R(k^{stab})$, but we may view

$$[\psi_i^*, -]\big(\Psi\big) = [\psi_i^*, -] \circ \psi_{i_1}^L \circ \cdots \circ \psi_{j_b}^{*L}(id)$$

and in this way deal only with operators on $End_R(k^{stab})$, and use their commutation rel^{n}s

$$[At_i, \psi_j] = \delta_{ij} \cdot 1 \qquad [At_i, \psi_j^*] = \partial_{x_i}(W_j) \cdot 1$$

means ψ_j^L

The whole calculation can be pushed to $\underline{End}$, so e.g. $[At_i, \psi_j^*]$ vanishes if $\partial_{x_i}(W_j) \in m$. (so if $W \in m^3$).

airfmt

Now $\underline{\mathrm{End}}(\Lambda\psi)$ is a Frobenius algebra under

$$\underline{\mathrm{End}} \otimes_k \underline{\mathrm{End}} \longrightarrow \cancel{\mathrm{End}}\ k.$$
$$\varphi \otimes \psi \longmapsto \mathrm{str}(\varphi\psi)$$

$$\therefore \quad \underline{\mathrm{End}} \xrightarrow{\cong} \underline{\mathrm{End}}^*$$
$$\varphi \longmapsto \mathrm{str}(\varphi -)$$

The inverse is defined by sending a functional χ to an expression involving the idempotents $e(\underline{i})$ from earlier. Since

$$\Lambda(k\psi_1 \oplus \cdots \oplus k\psi_n) = \bigoplus_{\underline{i}} k\,\psi_{i_1}\cdots\psi_{i_q}$$
$$\underline{i} \longleftarrow i_1 < \cdots < i_q \quad 0 \le q \le n.$$

$$\begin{array}{cc} & \cdots\cdots\ k\psi_{i_1}\cdots\psi_{i_q}\ \cdots \\ \begin{array}{c}\vdots \\ k\psi_{p_1}\cdots\psi_{p_b} \\ \vdots\end{array} & \begin{pmatrix} & & 0 & & \\ & & \vdots & & \\ & & 0 & & \\ 0 & \cdots & 0 \quad 1 & 0 & \cdots \\ & & 0 & & \\ & & \vdots & & \end{pmatrix} \end{array} =: E_{\underline{p},\underline{i}}$$

$$E_{\underline{p},\underline{i}} = \pm\, \psi^*_{\underline{p}^c}\, \psi_1 \cdots \psi_n\, \psi^*_{\underline{i}}$$

Then as usual $\mathrm{str}(\varphi E_{\underline{p},\underline{i}}) \overset{\pm}{=}$ coeff of φ on $E_{\underline{p},\underline{i}}$ so

$$\underline{\mathrm{End}}^* \longrightarrow \underline{\mathrm{End}}$$
$$\chi \longmapsto \sum_{\underline{p},\underline{i}} \chi(E_{\underline{p},\underline{i}})\, E_{\underline{p},\underline{i}}$$

We have higher operations (contint)

$$b_n : \underline{End}^{\otimes n} \longrightarrow \underline{End}$$

$$\underline{End}^{\otimes n} \otimes \underline{End}^{*} \longrightarrow k \qquad \varphi_1 \otimes \cdots \otimes \varphi_n \otimes x \mapsto x(b_n(\varphi_1 \otimes \cdots \otimes \varphi_n))$$

$$\tilde{b}_n : \underline{End}^{\otimes n} \otimes \underline{End} \longrightarrow k$$

$$\tilde{b}_n(\varphi_1 \otimes \cdots \otimes \varphi_n \otimes \beta) = str(\beta\, b_n(\varphi_1 \otimes \cdots \otimes \varphi_n))$$

naive product

We will try to compute $\tilde{b}_n$ rather than b_n.

<u>Example</u> ~~Since str(At...)~~ On $\underline{End} \otimes S$

$$b_2 = \pi \exp(-\delta) m_2 (\exp(\delta) \partial_\infty \otimes \exp(\delta) \partial_\infty)$$

$$b_2(\varphi_1 \otimes \varphi_2) = \pi \exp(-\delta) m_2 (\exp(\delta) \partial_\infty(\varphi_1) \otimes \exp(\delta) \partial_\infty(\varphi_2))$$

$$= \pi \exp(-\delta) \cancel{\exp(\delta)}\, \partial_\infty(\varphi_1) \exp(\delta) \partial_\infty(\varphi_2).$$

$$= \pi\, \partial_\infty(\varphi_1) \exp(\delta) \partial_\infty(\varphi_2)$$

$$= \varphi_1 \underset{\substack{\uparrow \\ \text{naive}}}{\cdot} \pi \exp(\delta) \partial_\infty(\varphi_2)$$

$$= \varphi_1 \cdot \sum_{s \geq 0} \sum_{|f| = s} \pm \frac{1}{s!} At_f \psi_f (\varphi_2)$$

When $n=1$ this is $\qquad At = [\psi_i^*, -] + \partial_{x_i}(W_x)[\psi_j, -]$

$$= \varphi_1 \cdot [\varphi_2 \pm At\, \psi(\varphi_2)]$$

cinfmt

So forexample

$$\Psi \otimes \Psi^* \longmapsto \Psi \cdot \left[\Psi^* \pm At\, \Psi(\Psi^*) \right]$$

$$At(\Psi \Psi^*) = \Psi^* + \partial_x (W_x) \Psi^*$$