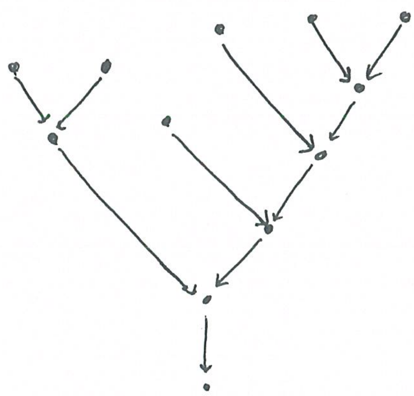
Minimal models 16 — non vacuum II (checked)

The signs at the end of ainfmf12 are ridiculous. Here we try to do better, by sticking with the suspended forward multiplications as long as possible. As usual k is a char. 0 field and $W \in k[x_1, \dots, x_n]$ with $W \in m^3$, $\mathcal{A} = \Lambda(k\psi_1^* \oplus \cdots \oplus k\psi_n^*)$ and the minimal model is $(\mathcal{A}, \{\rho_q\}_{q \geq 2})$ with

$$\rho_q : \mathcal{A}[1]^{\otimes q} \longrightarrow \mathcal{A}[1] \tag{1.1}$$

$$\rho_q(\Lambda_1 \otimes \cdots \otimes \Lambda_q) = \sum_T \rho_T(\Lambda_1 \otimes \cdots \otimes \Lambda_q)$$

as described on p. 3.5 of ainfmf2. For a tree T, e.g.

T:  (1.2)

We attach Φ^{-1} at inputs, $\hat{H}$ at internal edges and r_2 at internal vertices. Both $\hat{H}$ and r_2 have degree $+1$, but they come in pairs with the exception of the final vertex (r_2) so $|\rho_T| = 1$. For a sequence $J = \{j_1 < \cdots < j_t\}$ in $\{1, \dots, n\}$ we would like to compute the $\psi_J^* = \psi_{j_1}^* \cdots \psi_{j_t}^*$ component of $\rho_T(\Lambda_1 \otimes \cdots \otimes \Lambda_q)$, i.e.

$$\rho_T(\Lambda_1 \otimes \cdots \otimes \Lambda_q)_J \in k \tag{1.3}$$

For the same reasons explained in ainfmf12

$$\rho_T(\Lambda_1 \otimes \cdots \otimes \Lambda_q)_J = \sum_{\substack{\text{fittings} \\ F}} \rho_T(\Lambda_1 \otimes \cdots \otimes \Lambda_q)_J^F \qquad (2.1)$$

where $\rho_T(\Lambda_1 \otimes \cdots \otimes \Lambda_q)_J^F$ denotes the number obtained by moving the selected "survivor" fermions in the input to the far left and then computing with vacuum boundary conditions. To make sense of this note that

$$\mathbb{1} := 1 \otimes 1 \in (S \otimes \mathrm{End}_R(k^{stab}))[1] \text{ is } \underline{\text{odd}},$$

Hence, identifying ψ_i^* with the operator of left mult on $S \otimes \mathrm{End}R$,

$$\underset{\text{as an element}}{\Lambda_1 \otimes \cdots \otimes \Lambda_q} = \underset{\text{as operator}}{(\Lambda_1 \otimes \cdots \otimes 1)} \circ \cdots \circ (1 \otimes \cdots \otimes \Lambda_q)(1 \otimes \cdots \otimes 1) \cdot (-1)^{(q-1)|\Lambda_q| + \cdots + |\Lambda_2|} \qquad (2.2)$$

where $|\Lambda_i|$ denotes the usual number of fermions in Λ_i. Then

$$\rho_T(\Lambda_1 \otimes \cdots \otimes \Lambda_q)_J = (-1)^{\sum_{i=1}^q (i-1)|\Lambda_i|} \underbrace{P_J}_{\text{projection onto } \psi_J^*} \circ \rho_T \qquad (2.3)$$
$$\circ (\Lambda_1 \otimes \cdots \otimes 1) \circ \cdots \circ (1 \otimes \cdots \otimes \Lambda_q)(1 \otimes \cdots \otimes 1)$$

For each fitting we get the usual sign $|F|$ plus $|K_1| + \cdots + |K_q|$ to move things past ρ, as in

$$\rho\, I_1 \cdots I_q$$
$$\rho\, K_1 \overline{K_1} \cdots K_q \overline{K_q}$$
$$K_1 \cdots K_q\, \rho\, \overline{K_1} \cdots \overline{K_q} \qquad (3.1)$$
$$J\, \rho\, \overline{K_1} \cdots \overline{K_q}$$

Note we do not use tilde degrees here, as these are the Koszul signs for operaton of multiplication. Hence

$$\rho_T(\Lambda_1 \otimes \cdots \otimes \Lambda_q)_J = \sum_F (-1)^{\sum_{i=1}^q (i-1)|\Lambda_i| + |F| + \sum_{i=1}^q |K_i|} \qquad (3.2)$$

$$P_1 \circ \rho_T \circ (\psi^*_{\overline{K_1}} \otimes \cdots \otimes 1) \circ \cdots \circ (1 \otimes \cdots \otimes \psi^*_{\overline{K_q}})(1 \otimes \cdots \otimes 1)$$

$$= \sum_F (-1)^{\sum_{i=1}^q (i-1)(|\Lambda_i| + |\overline{K_i}|) + |F| + \sum_{i=1}^q |K_i|} \rho_T(\psi^*_{\overline{K_1}} \otimes \cdots \otimes \psi^*_{\overline{K_q}})_{const}$$

But mod 2, $|\Lambda_i| + |\overline{K_i}| = |K_i|$ so

Hence

$$\rho_T(\Delta_1 \otimes \cdots \otimes \Delta_q)_J = \sum_F (-1)^{\sum_{i=1}^{q} i|K_i| + |F|} \rho_T(\psi^*_{\bar{K}_1} \otimes \cdots \otimes \psi^*_{\bar{K}_q})_{const} \tag{4.1}$$

Now by p. 20 ainfmf2 and p. 20.5 ainfmf9,

$$\rho_T(\psi^*_{\bar{K}_1} \otimes \cdots \otimes \psi^*_{\bar{K}_q})_{const} = (-1)^{e_i(T)+q+1 \cancel{1 + \sum_{j \geq 1} \binom{M_j}{2}} + \sum_{i<j} (|\bar{K}_i|+1)(|\bar{K}_j|+1) + \sum_{i=1}^{q} (|\bar{K}_i|+1) P_i} \tag{4.2}$$

$$\mathrm{eval}_{\hat{T}}(\psi^*_{\bar{K}_q} \otimes \cdots \otimes \psi^*_{\bar{K}_1})_{const}$$

where M_j, P_i have the usual meanings. By (20.5.2) ainfmf9

$$\mathrm{eval}_{\hat{T}}(\psi^*_{\bar{K}_q} \otimes \cdots \otimes \psi^*_{\bar{K}_1})_{const} \tag{4.3}$$

$$= \sum_{\mathcal{C} \in \mathrm{Con}(\hat{T})} O(\hat{T}, \mathcal{C})(\psi^*_{\bar{K}_q} \otimes \cdots \otimes \psi^*_{\bar{K}_1})_{const}$$

Hence

Proposition For a sequence $J = \{j_1 < \cdots < j_i\}$ in $\{1, \ldots, n\}$

$$\rho_T(\Delta_1 \otimes \cdots \otimes \Delta_q)_J = \sum_{\substack{\text{fittings } F \\ \text{for } J}} (-1)^{\sum_{i=1}^{q} i|K_i| + |F| + e_i(T)+q+1 \cancel{1 + \sum_{j \geq 1} \binom{M_j}{2}} + \sum_{i<j} (|\bar{K}_i|+1)(|\bar{K}_j|+1) + \sum_i (|\bar{K}_i|+1) P_i} \tag{4.4}$$

$$\cdot \sum_{\mathcal{C} \in \mathrm{Con}(\hat{T})} O(\hat{T}, \mathcal{C})(\psi^*_{\bar{K}_q} \otimes \cdots \otimes \psi^*_{\bar{K}_1})_{const}$$

Combining this with (12.2) of airfmf12 we have for any fitting F with associated $K_i, \overline{K_i}, J$

(5.1)

$$\psi^*_J = m\{(\psi_{\overline{K_1}} \lrcorner -) \otimes \cdots \otimes (\psi_{\overline{K_q}} \lrcorner -)\}(\Lambda_1 \otimes \cdots \otimes \Lambda_q)$$
$$\cdot (-1)^{|F| + \sum_{i<j} |\overline{K_i}||\overline{K_j}| + |J| \cdot \sum_i |\overline{K_i}|}$$

We try combining this sign into the one in (4.4). The overall sign is

$$\left.\begin{array}{l}\sum_{i=1}^q i|K_i| + \cancel{|F|} + \cancel{1 + \sum_{j\geq 1}\binom{M_j}{2}}^{\,e_i(T)+q+1} + \sum_{i<j}(|\overline{K_i}|+1)(|\overline{K_j}|+1) \\ \quad + \sum_{i=1}^q (|\overline{K_i}|+1)P_i\end{array}\right\}(4.4)\text{ sign}$$

$$+ \cancel{|F|} + \sum_{i<j} |\overline{K_i}||\overline{K_j}| + |J| \cdot \sum_i |\overline{K_i}| \Big\} (5.1)\text{ sign} \qquad (5.2)$$

$$= \sum_{i=1}^q i(|\Lambda_i| + |\overline{K_i}|) + \cancel{1 + \sum_{j\geq 1}\binom{M_j}{2}}^{\,e_i(T)+q+1} + \cancel{\sum_{i<j}|\overline{K_i}||\overline{K_j}|}$$
$$+ \sum_{i<j} |\overline{K_i}| + \sum_{i<j}|\overline{K_j}| + \sum_{i<j} 1 + \sum_{i=1}^q P_i$$
$$+ \sum_{i=1}^q (|\Lambda_i| + |K_i|)P_i + \cancel{\sum_{i<j}|\overline{K_i}||\overline{K_j}|} + \sum_i (|\Lambda_i| + |\overline{K_i}|) \cdot \sum_j |\overline{K_j}|$$

$$= \cancel{1} + \binom{q}{2} + \sum_i P_i + (q-1)\sum_i |\overline{K_i}| + \sum_i |\overline{K_i}|P_i + \cancel{\sum_{j\geq 1}\binom{M_j}{2}}_{\,e_i(T)+q+1}$$
$$+ \sum_i (i)(|\Lambda_i| + |\overline{K_i}|)$$

$$+ \sum_i (|\Lambda_i| + |\overline{K_i}|) \cdot \sum_j |\overline{K_j}|$$

But this is hardly an improvement

I don't think we can plausibly say this is better than (4.4). So let us try a formulation where instead of (5.1) we use, for a fitting F of J,

$$\psi^*_J = (-1)^Q (\psi_{\bar{K}_1} \lrcorner \Lambda_1) \cdot \cdots \cdot (\psi_{\bar{K}_q} \lrcorner \Lambda_q) \tag{6.1}$$

what is Q? well it is the sign of

$$I_1 \cdots I_q$$
$$\bar{K}_1 K_1 \cdots \bar{K}_q K_q$$

plus the sign of

$$K_1 \cdots K_q \longrightarrow J.$$

[$|F|$ is the sign of

$$I_1, \dots, I_q$$
$$K_1 \bar{K}_1, \dots, K_q \bar{K}_q$$
$$K_1 \cdots K_q \bar{K}_1 \cdots \bar{K}_q$$
$$J \bar{K}_1, \dots, \bar{K}_q \ ]$$

That is,

$$Q = |F| + \sum_{i<j} |\bar{K}_i||K_j| + \sum_i |K_i||\bar{K}_i|$$

Hence (4.4) becomes

$$\rho_T(\Lambda_1 \otimes \cdots \otimes \Lambda_q) = \sum_J \sum_{\substack{\text{fittings } F \\ \text{for } J}} (-1)^{\substack{\sum_{i=1}^q i|K_i| + 1 + \sum_j \binom{M_j}{2} \\ + \sum_{i<j} (|\bar{K}_i|+1)(|\bar{K}_j|+1) + \sum_i (|\bar{K}_i|+1)p_i \\ + \sum_{i<j} |\bar{K}_i||K_j| \\ + \sum_i |K_i||\bar{K}_i|}}$$

$$\sum_{\mathscr{C} \in \mathrm{Con}(\hat{T})} \mathcal{O}(T, \mathscr{C}) (\psi^*_{\bar{K}_q} \otimes \cdots \otimes \psi^*_{\bar{K}_1})_{\mathrm{const}} \tag{6.2}$$

$$\cdot (\psi_{\bar{K}_1} \lrcorner \Lambda_1) \cdot \cdots \cdot (\psi_{\bar{K}_q} \lrcorner \Lambda_q)$$

The sign here is susceptible to

$$\sum_i i|K_i| + \sum_{i<j}(|\bar{K}_i|+1)(|\bar{K}_j|+1) + \sum_{i<j}|\bar{K}_i||K_j|$$

$$= \sum_i i|K_i| + \sum_{i<j}|\bar{K}_i||\bar{K}_j| + \sum_{i<j}|\bar{K}_i| + \sum_{i<j}|\bar{K}_j| + \binom{q}{2} + \sum_{i<j}|\bar{K}_i||K_j|$$

$$= \sum_i i|\bar{K}_i| + (q-1)\sum_i|\bar{K}_i| + \binom{q}{2} + \sum_{i<j}|\bar{K}_i||\Lambda_j| \quad \text{mod } 2$$
$$+ \sum_i i|\Lambda_i|$$

$$= \sum_i (q+i-1)|\bar{K}_i| + \binom{q}{2} + \sum_{i<j}|\bar{K}_i||\Lambda_j| + \sum_i i|\Lambda_i| \qquad (7.1)$$

The somewhat horrendous final formula is

<u>Proposition</u> For any tree T, (writing $|K_i| = |\Lambda_i| + |A_i|$)

$$\rho_T(\Lambda_1\otimes\cdots\otimes\Lambda_q) = \sum_{A_1,\dots,A_q\subseteq\{1,\dots,n\}} (-1)^{\sum_i(q+i-1)|\bar{K}_i| + \binom{q}{2} + \sum_{i<j}|\bar{K}_i||\Lambda_j| + 1 + \sum_j\binom{M_j}{2} + \sum_i(|\bar{K}_i|+1)P_i + \sum_i i|\Lambda_i| + \sum_i|K_i||\bar{K}_i|}$$

$$\sum_{\mathscr{C}\in\mathrm{Con}(\hat{T})} \Theta(\hat{T},\mathscr{C})(\psi^*_{A_q}\otimes\cdots\otimes\psi^*_{A_1})_{\mathrm{const}}$$

$$\cdot(\psi_{A_1}\lrcorner\Lambda_1)\cdot\ \cdots\ \cdot(\psi_{A_q}\lrcorner\Lambda_q)$$

(7.2)

To check the signs we can compare with a case we computed by hand,

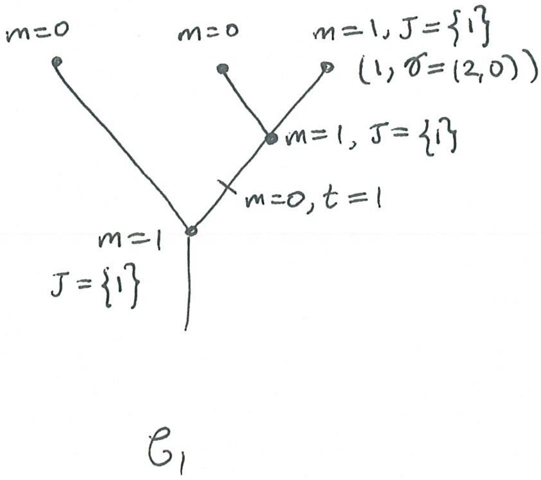
<u>Example</u> For $W = y^3 - x^3$ we computed ρ_3 on p. 19 ainfmf5

$$\rho_3(\Lambda_1 \otimes \Lambda_2 \otimes \Lambda_3) = (-1)^{\sum_{i<j} \tilde{\Lambda}_i \tilde{\Lambda}_j} \Big(-[\psi_1, \Lambda_3] \cdot [\psi_1, \Lambda_2] \cdot [\psi_1, \Lambda_1]$$
$$[\psi_2, \Lambda_3] \cdot [\psi_2, \Lambda_2] \cdot [\psi_2, \Lambda_1] \Big) \qquad (8.1)$$

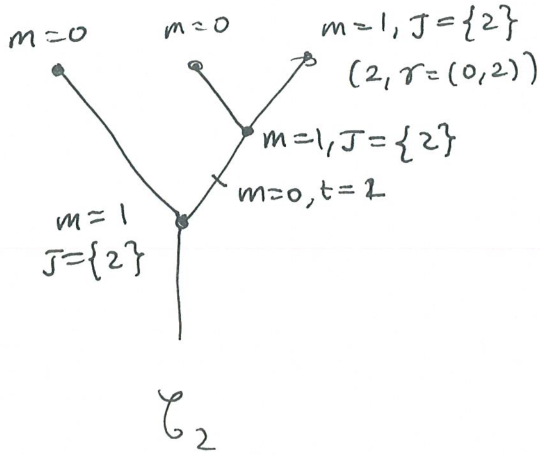
In this case (9.2) says

$$\rho_3(\Lambda_1 \otimes \Lambda_2 \otimes \Lambda_3)$$
$$= \rho_T(\Lambda_1 \otimes \Lambda_2 \otimes \Lambda_3) \quad \text{for} \quad T =$$

We computed on p. 4 ainfmf10 the only configurations which (for $\hat{T}$) contribute. There are two, namely



$\mathcal{C}_1$



$\mathcal{C}_2$

The only subsets $\underline{A} = (A_1, A_2, A_3)$ with $\mathcal{O}(\hat{T}, \mathcal{C})(\psi^*_{A_1} \otimes \psi^*_{A_2} \otimes \psi^*_{A_3})_{\text{const}} \neq 0$ are

$\boxed{\mathcal{C}_1}$ $\quad \underline{A} = (\{1\}, \{1\}, \{1\})$ amplitude 1 (not counting int. edge sign)

$\boxed{\mathcal{C}_2}$ $\quad \underline{A} = (\{2\}, \{2\}, \{2\})$ " -1

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In both cases $|A_i| = 1$ so $(q=3)$ by (7.2)

$$\rho_3(\Lambda_1 \otimes \Lambda_2 \otimes \Lambda_3) = \sum_{p=1,2} (-1)^{\Sigma_i (i)(1 \cdots) + 3 + 1 + \Sigma_{i<j} 1(|\Lambda_j| \ ) + \Sigma_j \binom{M_j}{2} + \Sigma_i i|\Lambda_i| + \Sigma_i |\Lambda_i| + \Sigma_i 1}$$

$$\mathcal{O}(\tilde{T}, \mathcal{C}_p)(\psi_p^* \otimes \psi_p^* \otimes \psi_p^*)_{const}$$

$$\cdot (\psi_p \lrcorner \Lambda_1) \cdot (\psi_p \lrcorner \Lambda_2) \cdot (\psi_p \lrcorner \Lambda_3) \quad (9.1)$$

Now for this T, $M_1 = 1$, $M_2 = 1$ so $\Sigma_j \binom{M_j}{2} = 0$ and $P_1 = 0, P_2 = 1, P_3 = 1$

$$\mathcal{O}(\tilde{T}, \mathcal{C}_1)(\psi_1^* \otimes \psi_1^* \otimes \psi_1^*)_{const} = 1$$

$$\mathcal{O}(\tilde{T}, \mathcal{C}_2)(\psi_2^* \otimes \psi_2^* \otimes \psi_2^*)_{const} = -1$$

$\therefore$ The sign in (9.1) is

$$\Sigma_i i(1 \quad) + \Sigma_{i<j} 1(|\Lambda_j| \quad) + \Sigma_i |\Lambda_i| + \Sigma_i i|\Lambda_i| + 3$$

$$= 1+2+3 + \Sigma_j (j-1)|\Lambda_j| + \Sigma_j j|\Lambda_j| + \Sigma_i |\Lambda_i|$$

$$= \Sigma_j |\Lambda_j| + \Sigma_i |\Lambda_i| + 3 = 1$$

The conversion also needs

$$(\psi_p \lrcorner \Lambda_1) \cdot (\psi_p \lrcorner \Lambda_2) \cdot (\psi_p \lrcorner \Lambda_3) = (-1)^{\Sigma_{i<j} \tilde{\Lambda}_i \tilde{\Lambda}_j} (\psi_p \lrcorner \Lambda_3) \cdot (\psi_p \lrcorner \Lambda_2) \cdot (\psi_p \lrcorner \Lambda_1)$$

with this (9.1) matches (8.1) precisely.

To cleanup (7.2) slightly

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$$\rho_T(\Lambda_1 \otimes \cdots \otimes \Lambda_q)$$

$$= \sum_{A_1,\dots,A_q \subseteq \{1,\dots,n\}} (-1)^{\substack{1 + \binom{q}{2} + \sum_{j \geq 1} \binom{M_j}{2} + \sum_i (|A_i|+1) P_i \\ + \sum_i (q+i)|A_i| + \sum_{i \leq j} |A_i||\Lambda_j| \\ + \sum_i i|\Lambda_i|}} \qquad (10.1)$$

$$\cdot \sum_{\mathcal{C} \in \mathrm{Con}(\bar{T})} \Theta(\bar{T}, \mathcal{C}) (\Upsilon^*_{A_q} \otimes \cdots \otimes \Upsilon^*_{A_1})_{const}$$

$$\cdot (\Upsilon_{A_1} \lrcorner \Lambda_1) \cdot \cdots \cdot (\Upsilon_{A_q} \lrcorner \Lambda_q)$$

where $A_1, \dots, A_q$ ranges over all subsets (although the product at the end guarantees $I_i \setminus A_i$ all disjoint)

We conclude

Proposition We have

$$\rho_q(\Lambda_1 \otimes \cdots \otimes \Lambda_q) = (-1)^{1+\binom{q}{2}+\sum_i i|\Lambda_i|} \sum_T (-1)^{\sum_{j\geq 1}\binom{M_j}{2}+\sum_i P_i}$$

$$\sum_{A_1,\dots,A_q \subseteq \{1,\dots,n\}} (-1)^{\sum_i |A_i| P_i + \sum_i (q+i)|A_i| + \sum_{i\leq j} |A_i||\Lambda_j|}$$

$$\sum_{\mathscr{C}\in \mathrm{Con}(\hat{T})} \mathcal{O}(\hat{T},\mathscr{C})(\psi^*_{A_q}\otimes\cdots\otimes\psi^*_{A_1})_{const} \cdot (\psi_{A_1} \lrcorner \Lambda_1)\cdots(\psi_{A_q} \lrcorner \Lambda_q) \tag{11.1}$$

Note that nothing here depends on $\mathscr{C}$, so configurations only contribute via the total amplitude

$$\begin{aligned}\mathcal{O}(\hat{T})(\psi^*_{A_q}\otimes\cdots\otimes\psi^*_{A_1})_{const} \\ := \sum_{\mathscr{C}\in\mathrm{Con}(\hat{T})} \mathcal{O}(\hat{T},\mathscr{C})(\psi^*_{A_q}\otimes\cdots\otimes\psi^*_{A_1})_{const}\end{aligned} \tag{11.2}$$