Minimal models for MFs 19 - Feynman (checked)

Our aim here is to reformulate the contributions to the total vacuum amplitudes defining ρ_T (as in p. 11 ainfmf16) in a way that is both easier to understand and compute. The total vacuum amplitudes are

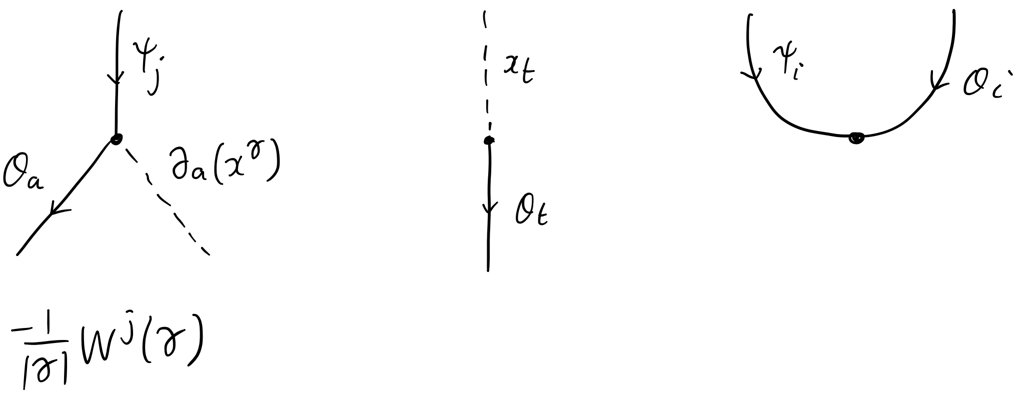
$$\sum_{\mathscr{C} \in \mathrm{Con}(T)} \mathcal{O}(T, \mathscr{C})(\psi_{A_1}^* \otimes \cdots \otimes \psi_{A_q}^*)_{const} \qquad (1.1)$$

with this contributing to $\rho_{\hat{T}}(\psi_{A_q}^* \otimes \cdots \otimes \psi_{A_1}^*)$ under the usual reversals. Each configuration $\mathscr{C}$ determines an operator, which can be understood in terms of Feynman diagrams. Our purpose in this note is to derive these operators, which take as input fermion states

$$\Psi_{in} := \psi_{A_1}^* \otimes \cdots \otimes \psi_{A_q}^* \in \mathcal{A} = \bigwedge (k\psi_1^* \oplus \cdots \oplus k\psi_n^*). \qquad (1.2)$$

Now, the Feynman rules are

(1.3)



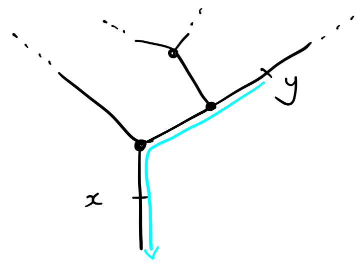
$\boxed{\theta x \psi\text{-interaction}}$ $\quad$ $\boxed{\theta x\text{-interaction}}$ $\quad$ $\boxed{\psi\theta\text{-interaction}}$

Our conventions are as in ainfmf9, i.e. k is a char. 0 field, $W \in m^3$, all trees are connected with a chosen planar embedding.

<u>Def</u>

Def^{N} Given locations x, y in a tree T, we write $y \leq x$ and say y is <u>above</u> x if x occurs on the unique path between y and the root of the tree. We say y is <u>strictly above</u> x, written $y < x$, if $y \leq x$ and $y \neq x$. We say y is <u>above to the right</u> (resp. <u>left</u>) of x if $y < x$ and on the unique path from y to the root, the internal vertex immediately preceding x (which may be x itself) appears after the internal edge or input above and to the right of the vertex.

e.g.



$y < x$ and y is above to the right of x (obviously uses the planar embedding)

Def^{N} Let T be a tree with $q \geq 2$ inputs and $\mathcal{C}$ a configuration, as defined on p. 18 ainfmf9. An <u>internal Feynman diagram</u> F of type $\mathcal{C}$ is a labelled oriented graph, whose set of vertices consists precisely of:

- for each input vertex or internal edge x, a vertex of F $v(x, j)$ for every $j \in J^{\mathcal{C}}(x)$, called a <u>$\mathcal{O}x\Psi$-vertex</u>.

- for each internal vertex x in T, a vertex of F $u(x, j)$ for every $j \in J^{\mathcal{C}}(x)$, called a <u>$\Psi\mathcal{O}$-vertex</u>.

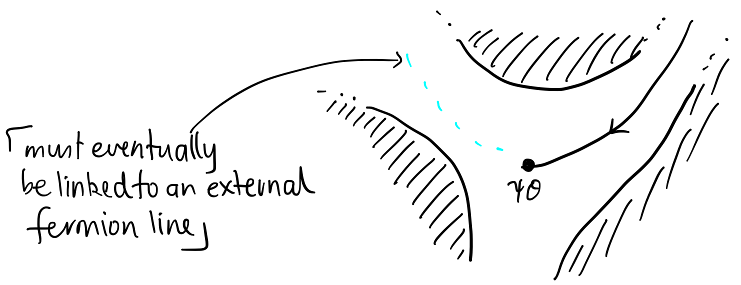
- for each internal edge x in T, a single vertex $w(x)$ of F called a <u>$\mathcal{O}x$-vertex</u>.

We say a vertex $v(x,j)$, $u(x,j)$ or $w(x)$ is <u>located</u> at x in T.

The edges of F are subject to the conditions:

- Edges $v \longrightarrow v'$ only connect vertices v located strictly above vertices v'!

- Any $\psi\mathcal{O}$-vertex $u(x,j)$ has <u>one</u> incoming edge and <u>no</u> outgoing edges, with the incoming edge originating in a $\mathcal{O}x$-vertex $w(y)$ or $\mathcal{O}x\psi$-vertex $v(y,k)$ with y above and to the right of x, and resp. $j = t^{\mathcal{C}}_{\cdot}(y)$ or $j = a^{\mathcal{C}}_{k}(y)$.

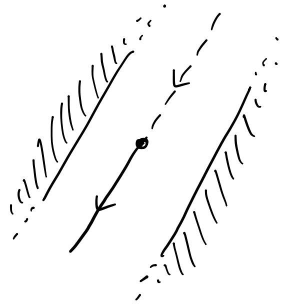


(3.1)

[We label such an edge $\mathcal{O}_j$]

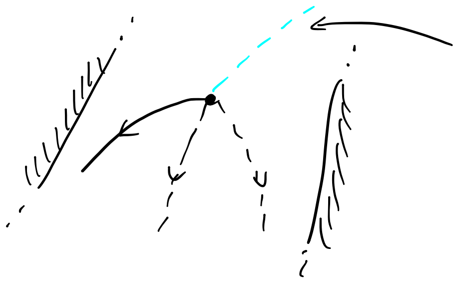
- Any $\mathcal{O}x$-vertex $w(x)$ has <u>one</u> incoming edge which originates in a $\mathcal{O}x\psi$-vertex $v(y,j)$, and <u>one</u> outgoing edge which terminates in a $\psi\mathcal{O}$-vertex $u(z,k)$, with $k=j$:

(3.2)



[We label the incoming edge x_j and the outgoing edge $\mathcal{O}_j$]

- Any $\mathcal{O}x\Psi$-vertex $v(x,j)$ has <u>no</u> incoming edges and $|\gamma_j^{\mathcal{E}}(x)|$ outgoing edges, one of which terminates at a $\Psi\mathcal{O}$-vertex and the rest of which terminate at $\mathcal{O}x$-vertices:



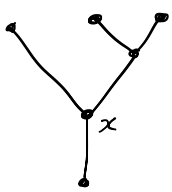
⌜again, will connect to an external fermion⌟

There are two consistency constraints:

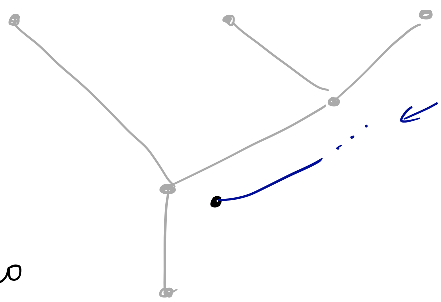
- the $\Psi\mathcal{O}$-vertex $u(y,k)$ involved has $a_j^{\mathcal{E}}(x)=k$, and
- the product of x_k (for the k just defined) with x_t, as x_t ranges over all labels assigned to the incident edges terminating at $\mathcal{O}x$-vertices, equals $x^{\gamma_j^{\mathcal{E}}(x)}$.

Thus every internal Feynman diagram has the same vertices, but many possible configurations of edges.

<u>Example</u> There exist configurations with <u>no</u> Feynman diagrams, e.g.



$m(x)=1$, all other locations zero



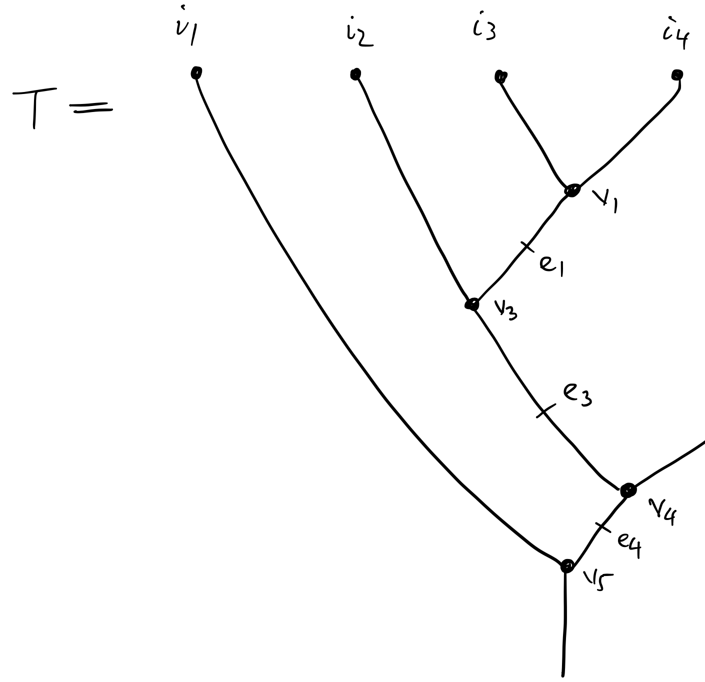
← nothing to connect to.

Def^N Given a tree T, configuration $\mathcal{C}$ and internal Feynman diagram F as above, and an input $\mathcal{Y}_{in}$ (which amounts to sets $A_i \subseteq \{1, \dots, n\}$ for $1 \leq i \leq q$) a Feynman diagram F^{tot} extending F is a graph obtained from F by

- adding one new vertex for each element of $\coprod_{i=1}^{q} A_i$
- adding one new oriented edge for each $j \in \coprod_{i=1}^{q} A_i$, which originates in the corresponding new vertex and terminates in either
    - a $Y\Theta$-vertex of F $u(x,k)$ with $k=j$, OR
    - a $\Theta \times Y$-vertex of F $v(x,k)$ with $k=j$

such that every $Y\Theta$-vertex and $\Theta \times Y$-vertex of F is the endpoint of exactly one such new edge.

<u>Example</u> From p. 11 ainfmf11, with $q=6$,



(6.1)

(locations i, v, e marked)

The configuration assigns (only giving nonzero values)

$$m(i_4)=2,\ m(i_6)=2,\ m(v_1)=2,\ m(v_2)=2,$$
$$m(v_3)=1,\ m(v_4)=1,\ m(v_5)=2.$$

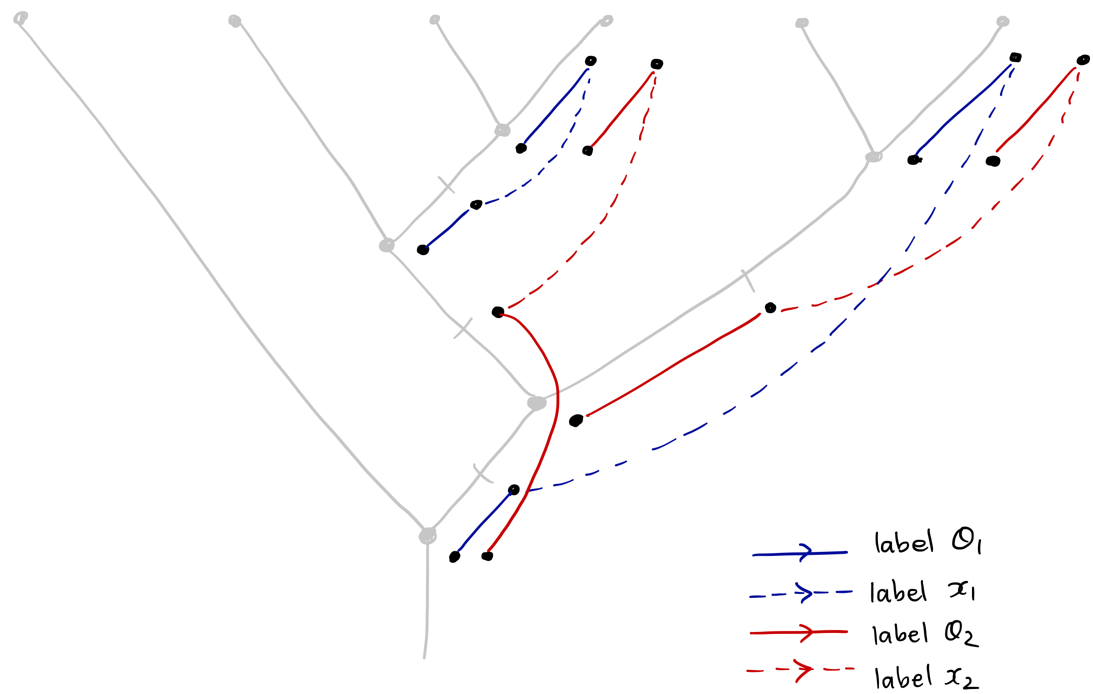
With the convention blue $=\Upsilon_1^*$, red $=\Upsilon_2^*$,

$$J(i_4)=J(i_6)=J(v_1)=J(v_2)=J(v_5)=\{1,2\}$$
$$J(v_3)=\{1\},\ J(v_4)=\{2\}$$

and always $a_j(x)=j$, $\gamma_j(x)=\begin{cases}(2,0) & j=1\\(0,2) & j=2\end{cases}$. for any location x.

(as $W=y^3-x^3$
$$=\underbrace{x\cdot(-x^2)}_{W^1}+\underbrace{y\cdot(y^2)}_{W^2}$$
$\therefore\ W^1(\gamma)=-1$ for $\gamma=(2,0)$ $W^2(\gamma)=1$ for $\gamma=(0,2)$)

The internal Feynman diagram corresponding to p. 11 ainfmf11 is

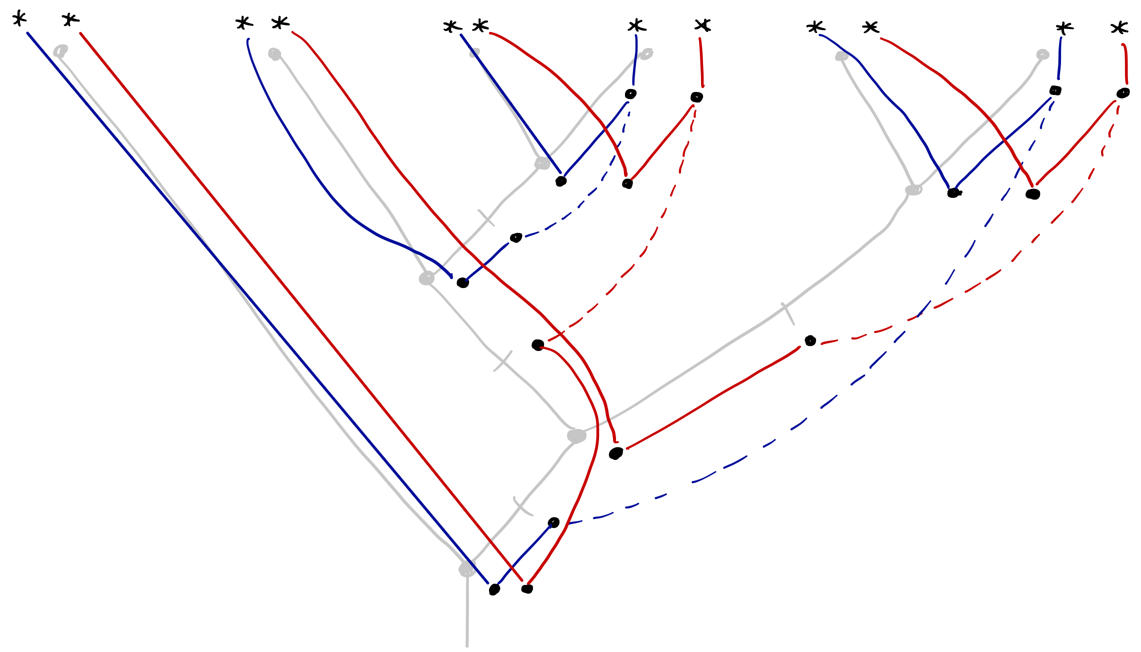


(7.1)

One total Feynman diagram extending this for the input

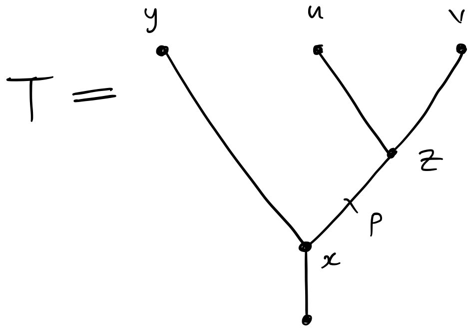
$$\Psi_{in} = (\varphi_1^* \varphi_2^*)^{\otimes 6}$$

is the following (this is the one on p. 11 ainfmf11)



(7.2)

A Feynman diagram for a configuration $\mathscr{C}$ of T and input Ψ_{in} describes a choice of contractions in the evaluation of the operator $\mathcal{O}^{pre}(T,\mathscr{C})(\Psi_{in})$ of p. 19 ainfmf9. For example, using the tree from p. 2 ainfmf10 for $n=1$, $W=x^3$,

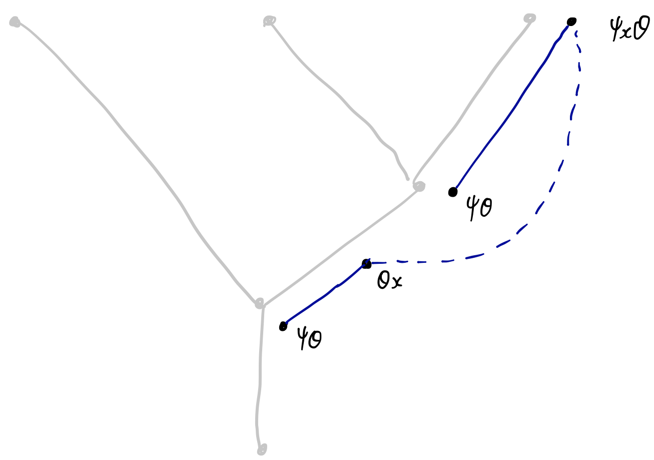


(8.1)

The configuration $\mathscr{C}$ described there has (only nonzero entries given)

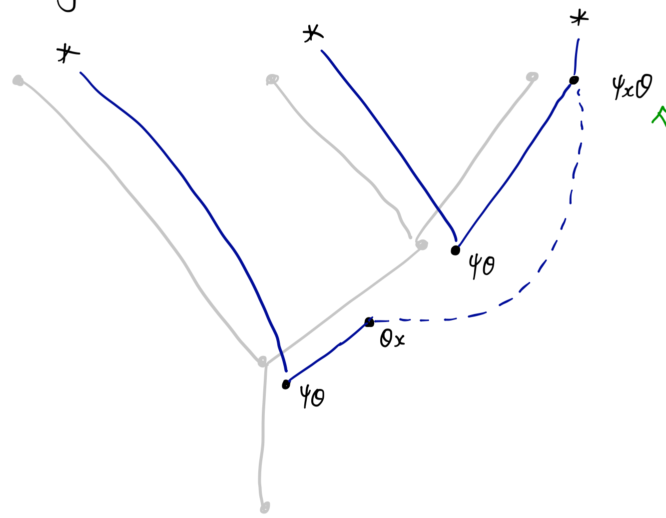
$$m(x) = m(v) = m(z) = 1 \qquad (\text{so } J \text{ is } \{1\} \text{ at all locations})$$

with $\gamma(v) = 2$. There is only one possible internal Feynman diagram:



(8.2)

On input $\Psi_{in} = \psi_1^* \otimes \psi_1^* \otimes \psi_1^*$ the only possible completion to a full Feynman diagram is



(9.1)

$(\gamma = 2,\ w' = x^2)$
Note: the insertion here has
$$(-1)^m \frac{1}{|\gamma|} W^1(\gamma)\,\partial_x(x^\gamma) = (-1)^1 \cdot \tfrac{1}{2} \cdot 1 \cdot 2x = (-1)^1 x$$

Note: this operator is correct according to ainfmf14

which corresponds to the contractions

(9.2)

$$\mathcal{O}^{pre}(T, \mathcal{C})(\Psi_{in}) =$$

$$\pi(-1)^1 m_2([\psi_1,-]\otimes \theta_1^*)\Big(\psi_1^* \otimes \theta_1 \partial_x (-1)^1 m_2([\psi_1,-]\otimes\theta_1^*)\big(\psi_1^* \otimes (-1)^1 x\, \theta_1 [\psi_1,-](\psi_1^*)\big)\Big)$$

Performing them in the labelled order gives

(9.3)

$$= (-1)^4 \pi m_2([\psi_1,-]\otimes 1)\Big(\psi_1^* \otimes \partial_x m_2([\psi_1,-]\otimes\theta_1^*)\big(\psi_1^*\otimes x\theta_1[\psi_1,-](\psi_1^*)\big)\Big)$$

$$= (-1)^4 \pi m_2([\psi_1,-]\otimes 1)\Big(\psi_1^* \otimes m_2([\psi_1,-]\otimes\theta_1^*)\big(\psi_1^*\otimes \theta_1[\psi_1,-](\psi_1^*)\big)\Big)$$

$$= (-1)^5 \pi m_2([\psi_1,-]\otimes 1)\Big(\psi_1^* \otimes m_2([\psi_1,-]\otimes 1)\big(\psi_1^*\otimes [\psi_1,-](\psi_1^*)\big)\Big)$$

$$= (-1)^5.$$

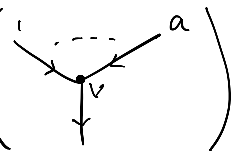
but we still get signs from e.g. $\theta_1^*(\psi_1^* \otimes \theta_1, \cdots) = -\psi_1^* \otimes \cdots$ because that is how $[\psi_1,-]\otimes\theta_1^*$ is defined (not the tree's fault).

Note that up to the penultimate step, the only role of the inputs was to contribute signs, so that modulo signs the same calculation shows that as a functional on $\mathcal{A}^{\otimes 3}$,

$$\vartheta^{pre}(T,\mathcal{C})(-) = \pm[\psi_1,-]\otimes[\psi_1,-]\otimes[\psi_1,-].$$

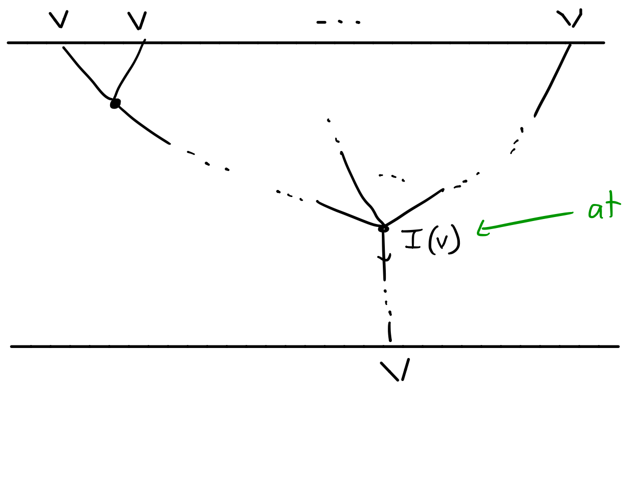
Contraction on trees

We detour for a moment to carefully define contraction. Let k be a commutative ring, $\mathcal{V}$ the monoidal category of k-modules. For this subsection, a tree T is a connected tree whose $q+1$ leaves have one designated the root, and if we orient edges towards this root there is a chosen linear ordering on the incoming edges at each vertex (this yields a planar embedding). We allow vertices of any valency. Let V be a fixed $\mathbb{Z}_2$-graded k-module and suppose there is an assignment I of homogeneous linear maps to each internal vertex of T, i.e.

$I\Bigg($ 

$\Bigg) \in \mathrm{Hom}_k(V^{\otimes a}, V).$ (10.1)

This data determines a string diagram for $\mathcal{V}$ (note: ungraded)



⌜leaves do not appear⌟ (10.2)

at each vertex v

The denotation of which is a linear map $V^{\otimes q} \longrightarrow V$.

<u>Example</u> Recall that $\mathcal{O}^{pre}(T, \mathcal{C}) : \mathcal{A}^{\otimes q} \longrightarrow \mathcal{A}$ is defined on p. 19 ainfmf9 by assigning operators to vertices and then defining the resulting operator algorithmically: i.e. feed ingredients in the top and compute on them, with no interference of signs. Contrast this to ainfmf14 applied to the same input of operators-at-vertices, which produces the same linear map $\mathcal{A}^{\otimes q} \to \mathcal{A}$ with possibly <u>different signs</u>.

This "algorithmic" def^N of $\mathcal{O}^{pre}(T, \mathcal{C})$ is simply the linear map assigned to the operator labelled tree viewed as a diagram in the category of (ungraded) vector spaces. That is,

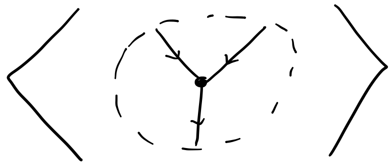
1. From the "operator" tree T with internal vertices of only valency 3, we construct T' with new vertices of valency 2 inserted at internal edges and as immediate descendants of every input vertex.

2. With $V = S \otimes_k \mathrm{End}_k(k^{stab})$ associate to the internal vertices of T' the operators prescribed by ainfmf9 p. 18, 19.

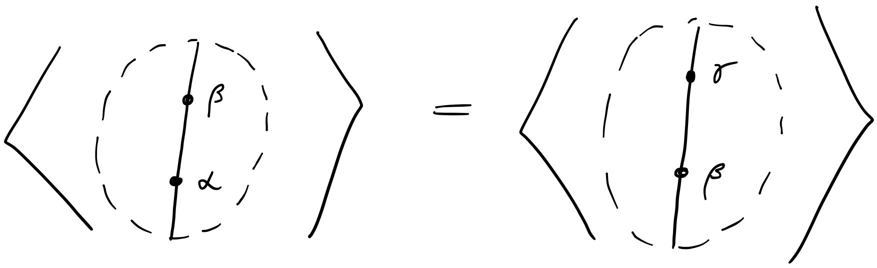
3. Let $\beta : (S \otimes_k \mathrm{End}_k(k^{stab}))^{\otimes q} \longrightarrow S \otimes_k \mathrm{End}_k(k^{stab})$ be the resulting linear map from the denotation of (10.2). Then by def^N

$$\mathcal{O}^{pre}(T, \mathcal{C}) := \pi \circ \beta \circ \zeta^{\otimes q} : \mathcal{A}^{\otimes q} \longrightarrow \mathcal{A}.$$

Given a general operator-decorated tree T whose denotation is $\langle T \rangle : V^{\otimes q} \longrightarrow V$, we draw manipulations locally on T using diagrams:

$\langle T \rangle =$ 

For example, if $\alpha\beta = \beta\gamma$ as maps $V \longrightarrow V$,


(12.1)

With $S = \Lambda(k\mathcal{O}_1 \oplus \cdots \oplus k\mathcal{O}_n)$, $k^{stab} = \Lambda(k\psi_1 \oplus \cdots \oplus k\psi_n)$, $R = k[x_1, \ldots, x_n]$,

$$
\begin{aligned}
V = S \otimes_k \mathrm{End}_R(k^{stab}) &\cong \Lambda(k\mathcal{O}_1 \oplus \cdots \oplus k\mathcal{O}_n) \\
&\quad \otimes_k \mathrm{End}_k(\Lambda(k\psi_1 \oplus \cdots \oplus k\psi_n)) \\
&\quad \otimes_k k[x_1, \ldots, x_n]
\end{aligned} \tag{12.2}
$$

The <u>standard operators</u> on V are the homogeneous operators $V \longrightarrow V$,

$$
G, \lambda, \mathcal{O}_i, \mathcal{O}_i^*, [\psi_i, -], x_i, \partial_{x_i} \qquad \begin{array}{l} 1 \le i \le n, \lambda \in k \\ G(\alpha) = (-1)^{|\alpha|}\alpha \end{array} \tag{12.3}
$$

<u>Def^N</u> We call an operator-decorated tree as in (10.2) <u>standard</u> if (V as above)

(i) all non-leaf vertices have valency 2 or 3,
(ii) the operators at each valency 2 vertex are standard,
(iii) the operator at each valency 3 vertex is $m_2 : V^{\otimes 2} \longrightarrow V$.

in the sense of ainfmf9

Def^N Given a tree T with configuration $\mathscr{C}$, $D_{T,\mathscr{C}}$ is the operator-decorated tree defined above whose denotation $\langle D_{T,\mathscr{C}} \rangle$ is related to $\mathcal{O}^{pre}(T,\mathscr{C})$ via

$$\mathcal{O}^{pre}(T,\mathscr{C}) = \pi \circ \langle D_{T,\mathscr{C}} \rangle \circ \delta^{\otimes q}. \qquad (13.1)$$

Def^N We now define a <u>standard</u> operator-decorated tree $S_{T,\mathscr{C}}$ such that

$$\langle S_{T,\mathscr{C}} \rangle = \langle D_{T,\mathscr{C}} \rangle \qquad (13.2)$$

This construction is based on the following calculation, with $m = |J|$

$$\prod_{j\in J}\big([\psi_j,-]\otimes \mathcal{O}_j^{*}\big) = (-1)^{\binom{m}{2}} \prod_{j\in J}\big([\psi_j,-]\otimes 1\big)\prod_{j\in J}\big(1\otimes \mathcal{O}_j^{*}\big). \qquad (13.3)$$

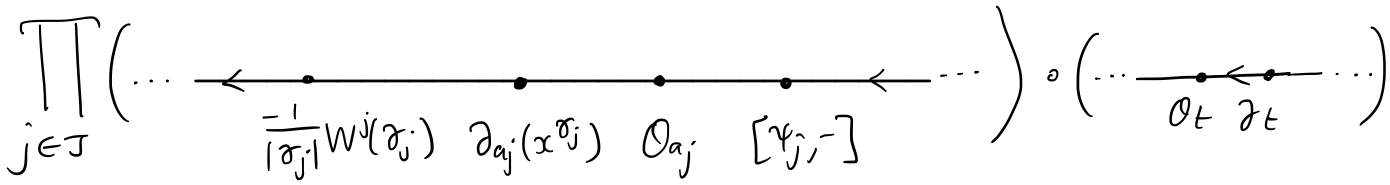
We define $S_{T,\mathscr{C}}$ using a method similar to p. (11). Beginning with T we

(1) given an input vertex assigned m, $J\subseteq\{1,\dots,n\}$ by $\mathscr{C}$ we insert in T immediately below the input the following decorated vertices

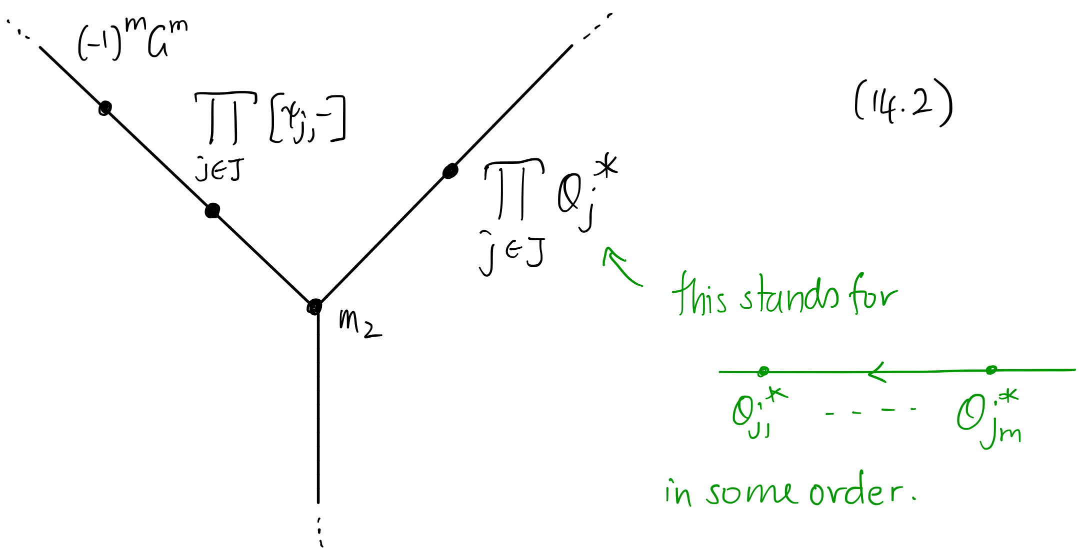
$$\prod_{j\in J}\Big(\cdots \longleftarrow \underset{\frac{-1}{|\gamma_j|}W^j(\gamma_j)}{\bullet} \text{———} \underset{\partial_{a_j}(x^{\gamma_j})}{\bullet} \text{———} \underset{\mathcal{O}_{a_j}}{\bullet} \text{———} \underset{[\psi_j,-]}{\bullet} \longleftarrow \cdots\Big) \qquad (13.4)$$

where the product means we connect these segments (orientation as shown towards the root), and $\partial_{a_j}(x^{\gamma_j})$ stands for numerous x-type vertices with single variables, and a λ-type vertex with the rest.

(2) To an internal edge we assign in the same fashion (14.1)



(3) To an internal vertex we assign the decorated subtree (using (13.3))



(14.2)

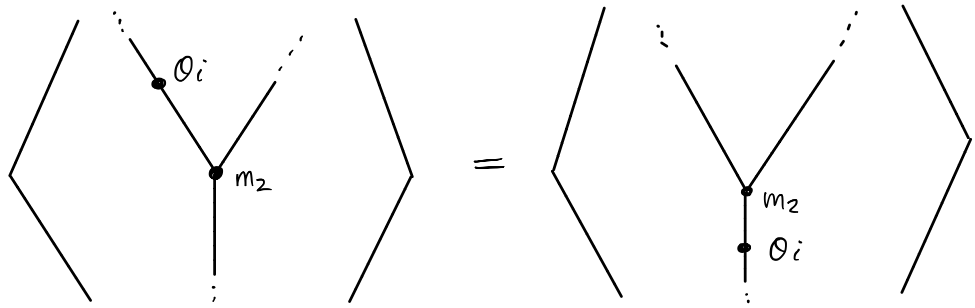
where the products are in any order (as long as they match).

<u>Lemma</u> $S_{T,\mathcal{E}}$ is a standard operator-decorated tree and $\langle S_{T,\mathcal{E}}\rangle = \langle D_{T,\mathcal{E}}\rangle$.

<u>Proof</u> Clear. $\square$

<u>Lemma</u> Let K be a standard operator-decorated tree, so $\langle K\rangle : V^{\otimes q} \to V$.
We say K is <u>vacuum-trivial</u> if $\pi \circ \langle K\rangle \circ \delta^{\otimes q} = 0$.

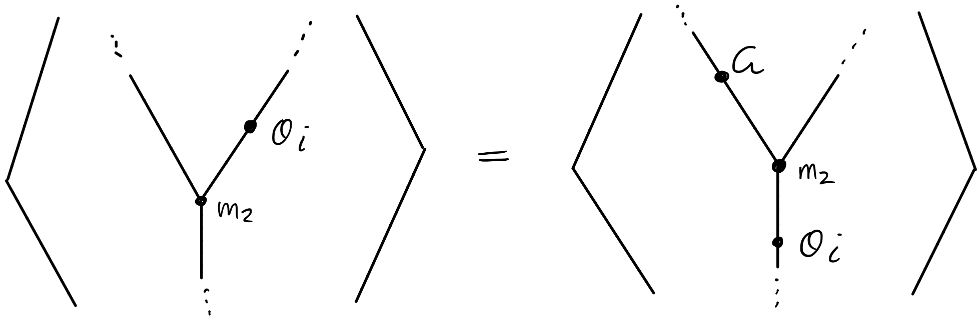
<u>From now on</u> K denotes a standard operator-decorated tree. Locally in such a tree we have an obvious relation



(15.1)

$\alpha \otimes \beta \mapsto O_i \alpha \beta$ $\qquad\qquad$ $\alpha \otimes \beta \mapsto O_i \alpha \beta$

and since O_i anticommutes with everything in $V = S \otimes_R \mathrm{End}_R(k^{stab})$, also

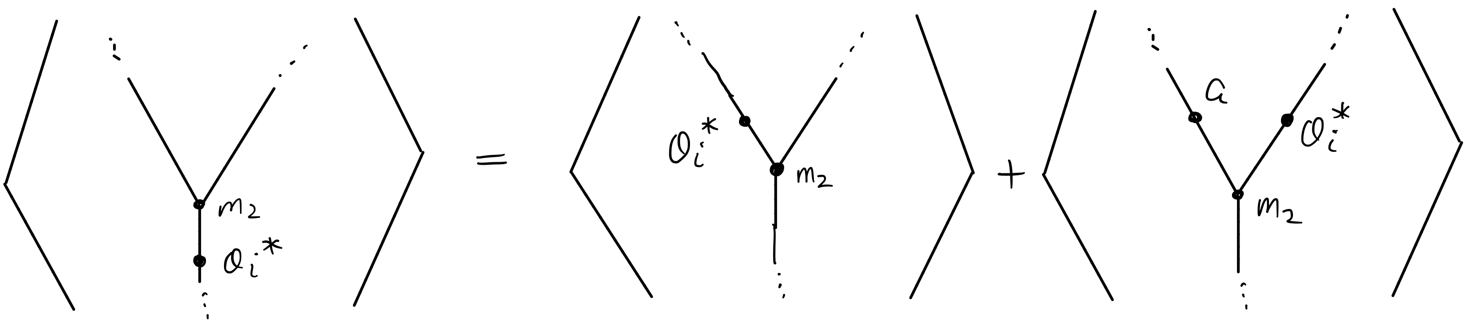


(15.2)

$\alpha \otimes \beta \mapsto \alpha O_i \beta$ $\qquad\qquad$ $\alpha \otimes \beta \mapsto O_i (-1)^{|\alpha|} \alpha \beta$

$= (-1)^{|\alpha|} O_i \alpha \beta$

And finally $\qquad$ (15.3)



$$O_i^*(\alpha\beta) = O_i^*(\alpha)\beta + (-1)^{|\alpha|}\alpha O_i^*(\beta)$$

<u>Lemma</u> If there are distinct vertices in K labelled $\mathcal{O}_i$, and the path between these vertices does <u>not</u> contain a vertex labelled $\mathcal{O}_i^*$, then $\langle K \rangle = 0$.

<u>Proof</u> We can commute the two $\mathcal{O}_i$'s towards the unique vertex where the paths meet, via the relations (15.1), (15.2), where they annihilate. $\square$

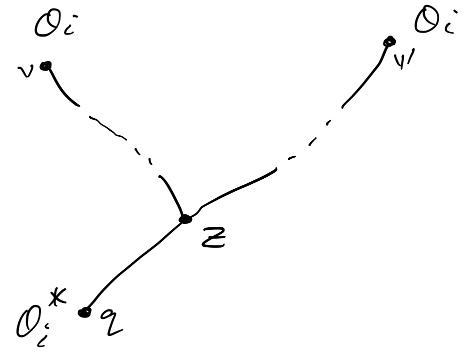
<u>Lemma</u> Suppose that K is <u>not</u> vacuum-trivial and let Θ be the set of all vertices of K, $\Theta_i \subseteq \Theta$ the set of all vertices labelled $\mathcal{O}_i$, and $\Theta_i^* \subseteq \Theta$ the vertices labelled $\mathcal{O}_i^*$. The function

$$f_i : \Theta_i \longrightarrow \Theta_i^* \tag{16.1}$$

$$f_i(v) = \text{the first vertex } v' \text{ on the path } v \longrightarrow \text{root with } v' \in \Theta_i^*$$

is well-defined and <u>bijective</u>.

<u>Proof</u> Suppose the path $v \longrightarrow$ root contains no $\mathcal{O}_i^*$. Then since $\mathcal{O}_i$ (anti)commutes with the other operators in (12.3) we can use (15.1), (15.2) to see that K is vacuum-trivial, contradicting our hypothesis. So f is well-defined. If there is a path



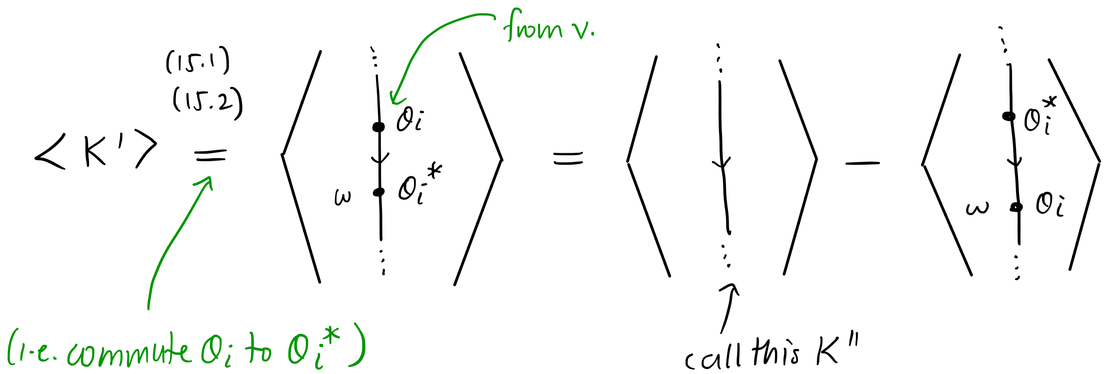
(16.2)

with two distinct $\mathcal{O}_i$ vertices having $f_i(v) = f_i(v') = q$, and with z denoting the first common vertex of the two paths, then by the previous lemma we must have $z \neq q$. But then (15.1), (15.2) allow us to annihilate the $\mathcal{O}_i$'s between q, z. This proves f_i is <u>injective</u>.

To prove f is surjective let $w \in \Theta_i^*$ be given. We may without loss of generality assume all $w' \in \Theta_i^*$ above w are in the image of f_i (possibly there are no such w'). Suppose $w \notin f_i(\Theta_i)$. We begin commuting the $\mathcal{O}_i^*$ at w up the tree using (15.3). In each summand we must encounter either an input or another $\mathcal{O}_i^*$ before a $\mathcal{O}_i$ (by hypothesis). But $\mathcal{O}_i^* \delta = 0$ and $\mathcal{O}_i^* \mathcal{O}_i^* = 0$, so we would have $\pi \circ \langle K \rangle \circ \delta^{\otimes q} = 0$, a contradiction. Hence $w \in f_i(\Theta_i)$, and f_i is a bijection. $\square$

<u>Def^N</u> Now let $T, \mathcal{C}$ be a tree (as in ainfmf9) with configuration, take $K = S_{T,\mathcal{C}}$, so that $\mathcal{O}^{pre}(T, \mathcal{C}) = \pi \circ \langle S_{T,\mathcal{C}} \rangle \circ \delta^{\otimes q}$. We can compute $\mathcal{O}^{pre}(T, \mathcal{C})$ as follows. Firstly, we may assume $f_i : \Theta_i \longrightarrow \Theta_i^*$ is bijective for each i (otherwise $\mathcal{O}^{pre}(T, \mathcal{C}) = 0$ by the lemma).

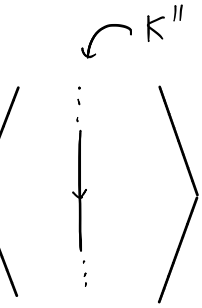
1. Set $K' = K$.
2. For $1 \leq i \leq n$ DO until no $\mathcal{O}_i^*$-vertices in K':
   - Choose a $\mathcal{O}_i^*$-vertex w in K' which is "maximal" in the sense that there is no other $\mathcal{O}_i^*$-vertex between our chosen one and the root. Let v be the matching $\mathcal{O}_i$-vertex in K' via f_i. Then



(17.1)

Since w was chosen maximal,

$$\pi\circ\langle K'\rangle\circ\mathfrak{z}^{\otimes q} = \pi\circ\Bigg\langle$$



$$\Bigg\rangle\circ\mathfrak{z}^{\otimes q} \qquad (18.1)$$

where K'' is as in (17.1). We now replace $K' := K''$ and continue. So at each step $\pi\circ\langle K'\rangle\circ\mathfrak{z}^{\otimes q}$ is unchanged, but pairs $(v, f_i(v))$ are removed (contracted) and some G's from (15.2) are introduced.

3 Since all f_i are bijective step 2 terminates with a standard-operator-decorated tree K' such that

(i) K' contains no $\mathcal{O}_i$ or $\mathcal{O}_i^*$-vertices
(ii) $\mathcal{O}^{pre}(T,\mathcal{C}) = \pi\circ\langle K'\rangle\circ\mathfrak{z}^{\otimes q}$.

Similarly we apply all ∂_{x_i}'s to the x_i's, and the result is either zero or a standard operator-decorated tree K_f with

(i) K_f contains no $\mathcal{O}_i, \mathcal{O}_i^*, x_i, \partial_{x_i}$ vertices
(i.e. only λ, G and $[\Psi_i, -]$ vertices)
(ii) $\mathcal{O}^{pre}(T,\mathcal{C}) = \pi\circ\langle K_f\rangle\circ\mathfrak{z}^{\otimes q}$.

Call the above the contraction algorithm.

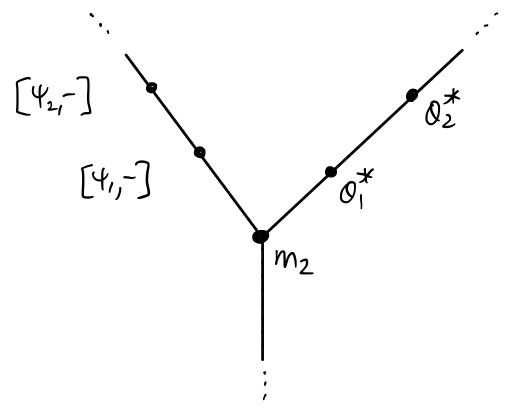
<u>Example</u> Consider the tree/configuration pair $T, \mathcal{C}$ presented on p. 6. We construct the standard operator-decorated tree $K = S_{T,\mathcal{C}}$ defined on p. 13. The operator we need to insert at i_4 is $(W = y^3 - x^3,\ W^1 = -x^3,\ W^2 = y^3)$

$$\prod_{j\in J}\Big(\cdots \longleftarrow \tfrac{-1}{|\sigma_j|}W^j(\sigma_j)\;\;\partial_{a_j}(x^{\sigma_j})\;\;\mathcal{O}_{a_j}\;\;[\psi_j,-] \longleftarrow \cdots\Big)$$

$$= \cdots\ \tfrac{1}{2}\;\;\partial_x(x^2)\;\;\mathcal{O}_1\;\;[\psi_1,-] \longleftarrow -\tfrac{1}{2}\;\;\partial_y(y^2)\;\;\mathcal{O}_2\;\;[\psi_2,-]\ \cdots$$

$$= \cdots\ x\;\;\mathcal{O}_1\;\;[\psi_1,-] \longleftarrow -1\;\;y\;\;\mathcal{O}_2\;\;[\psi_2,-]\ \cdots \qquad (19.1)$$

And the same at i_6. At v_1, v_2 we have the insertions $(c^2 = \mathrm{id})$



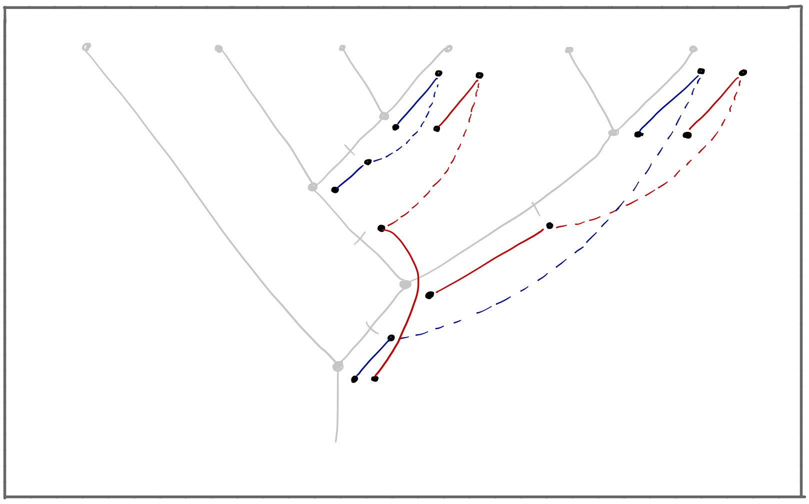
(19.2)

while at e_1 for example we insert ($t(e_1) = 1$)

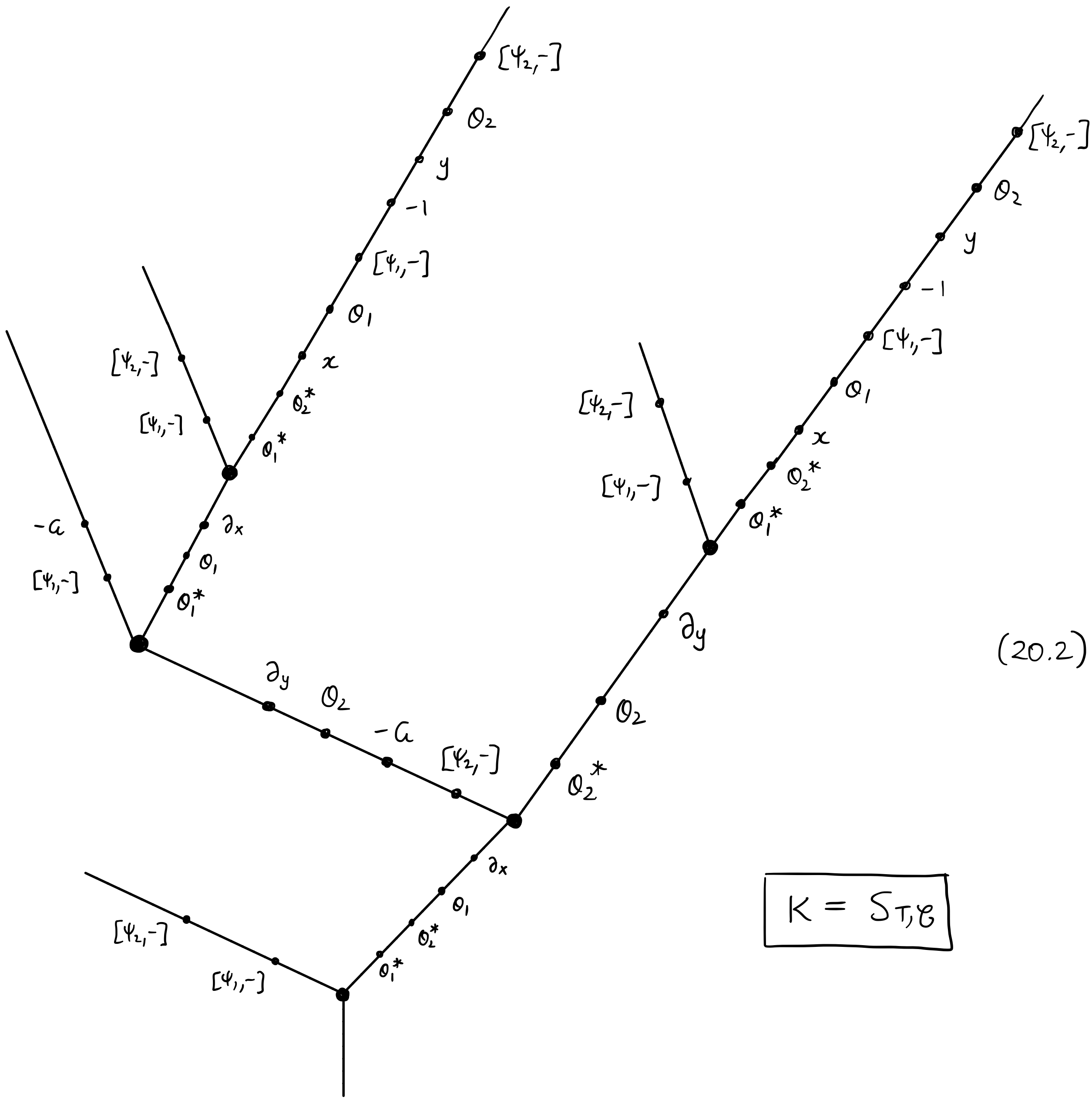
$$\cdots\ \mathcal{O}_1 \;\;\; \partial_x\ \cdots \qquad (19.3)$$

The final result is shown overleaf:

(from p. 7)



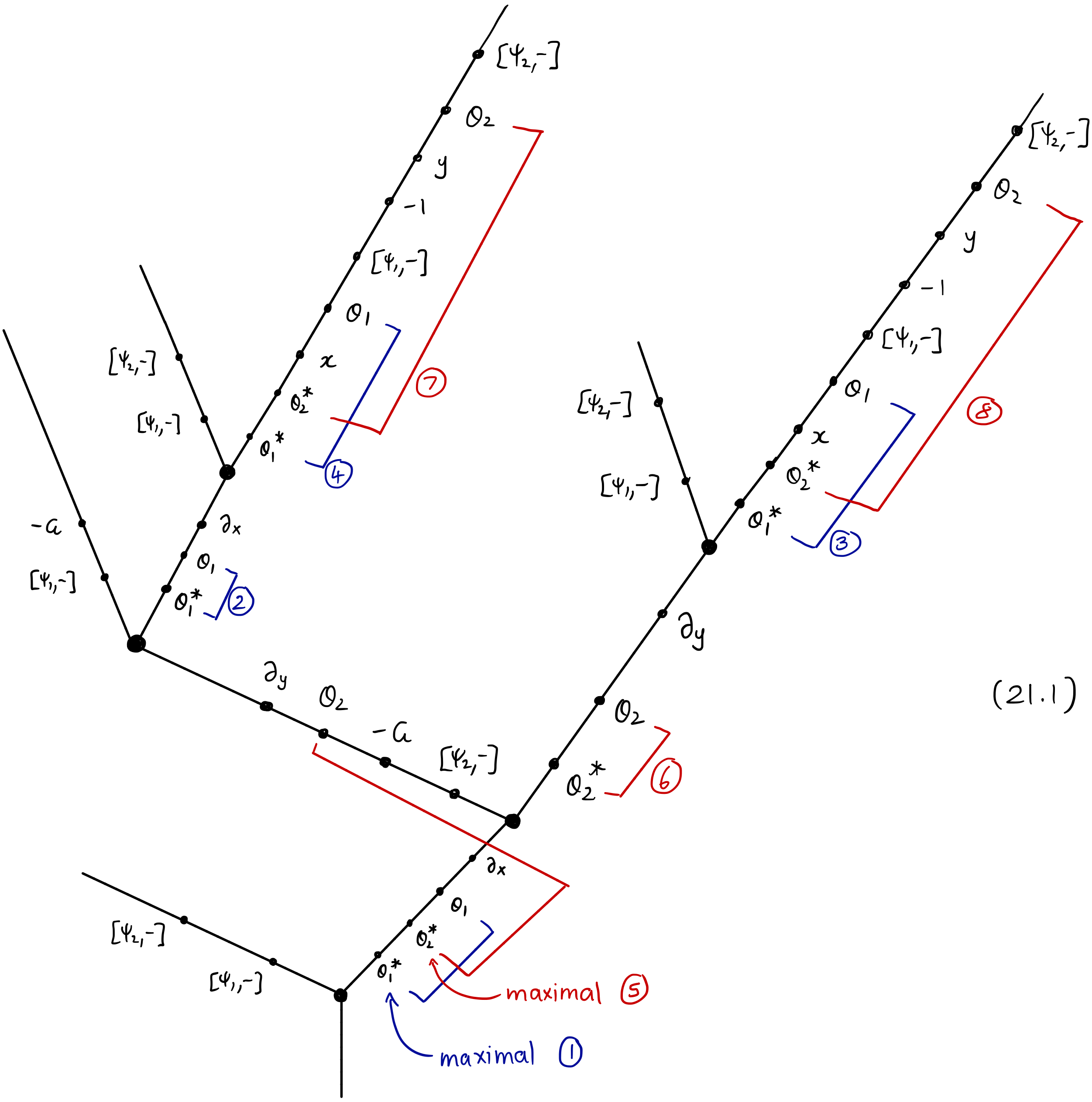
(20.1)



(20.2)

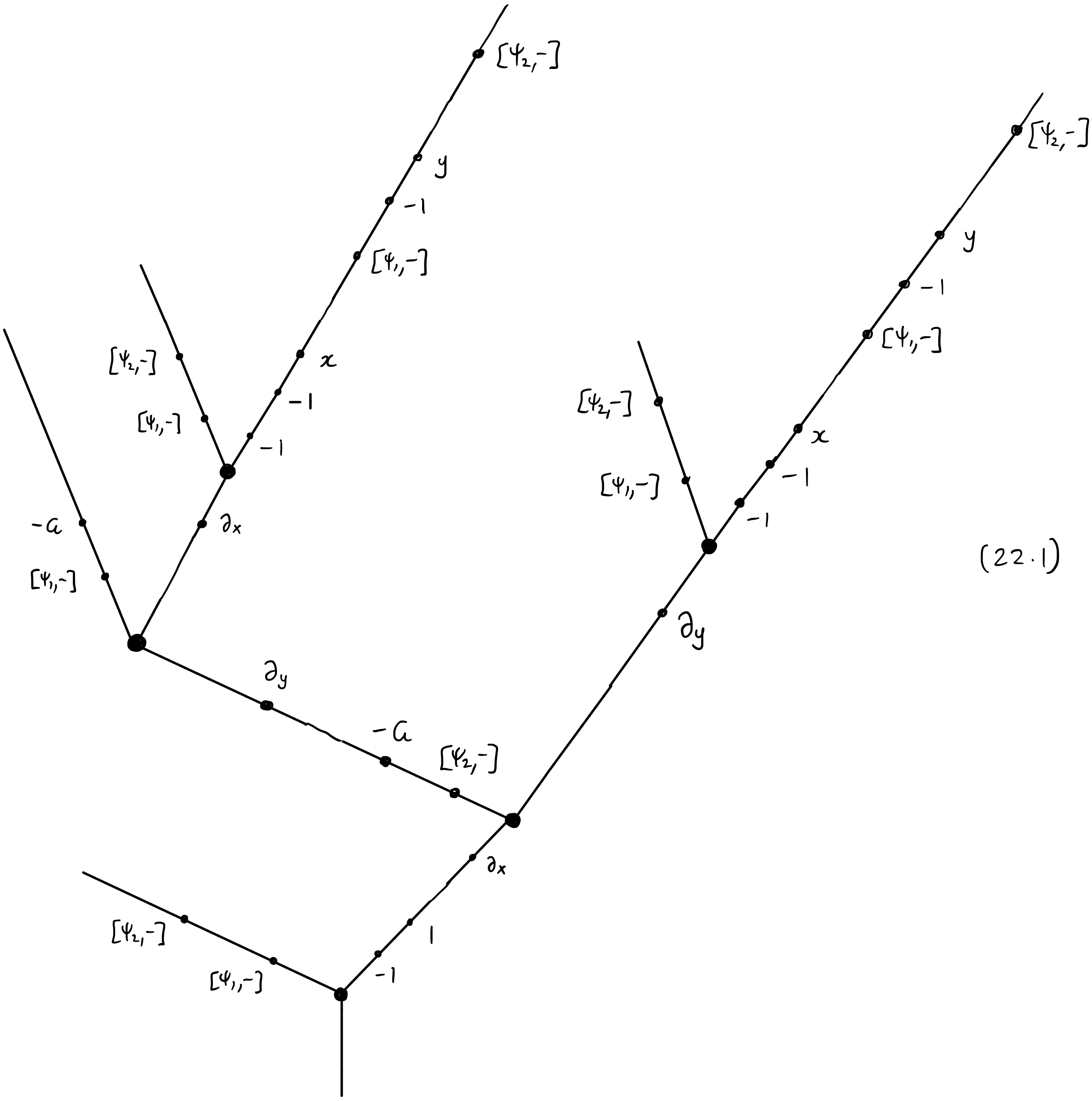
$$\boxed{K = S_{T,\mathcal{C}}}$$

The mappings f_1, f_2 are shown below (resp. in blue and red)



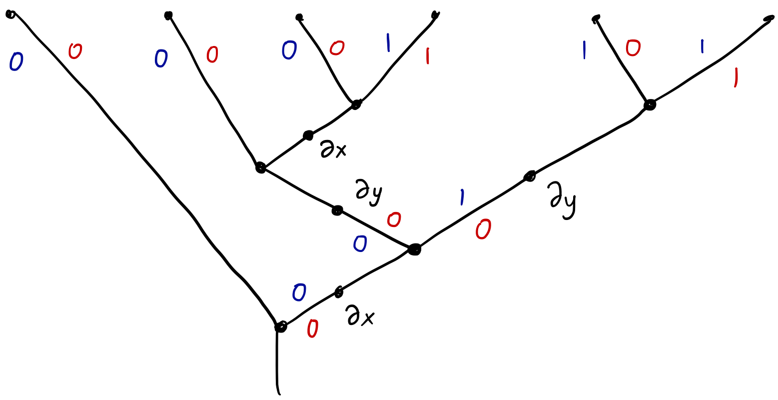
(21.1)

According to our contraction algorithm, the contractions are performed in the order shown, yielding (note the new a's and signs). Note sometimes there are multiple maximal θ_i^* vertices and we have arbitrarily chosen one.



(22.1)

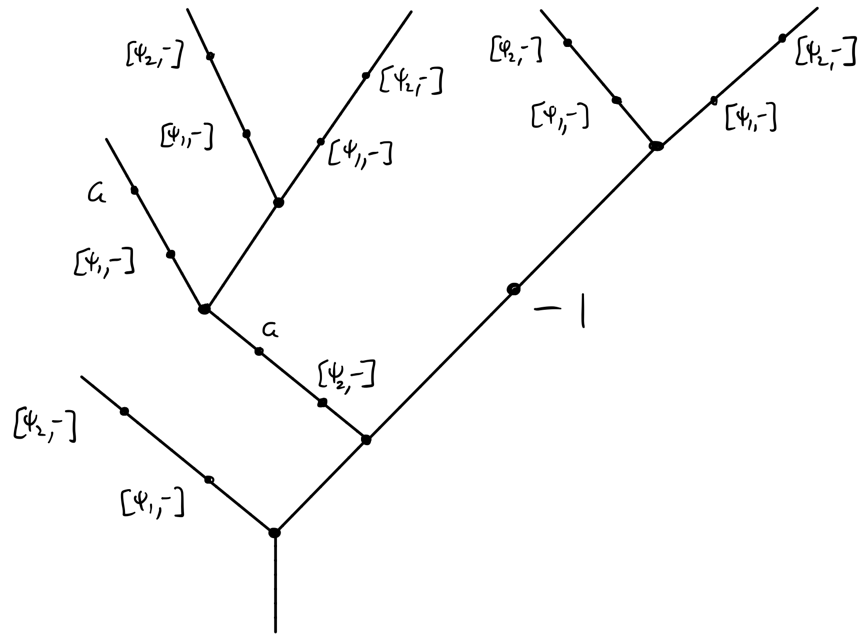
The next step is to apply ∂_x, ∂_y, which schematically comes down to the following count (blue marks x-degree flowing along an edge, red y-degree)



(22.2)

We conclude that contracting the x_i, ∂_i pairs does not contribute any constants. Performing these contractions and cancelling signs yields

$K_f =$



(23.1)

Hence we conclude

$$\begin{aligned}\mathcal{O}^{pre}(T,\mathcal{C}) &= \pi \circ \langle K_f \rangle \circ \zeta^{\otimes 6} \\ &= (-1)\,[\psi_1,[\psi_2,-]] \cdot \big[\psi_2, a\big([\psi_1, a(-)] \cdot [\psi_1,[\psi_2,-]] \cdot [\psi_1,[\psi_2,-]]\big)\big] \\ &\quad \cdot [\psi_1,[\psi_2,-]] \cdot [\psi_1,[\psi_2,-]]\end{aligned}$$

Of course this may be simplified further (the $[\psi_2,-]$ in the second term can be moved to input 2), to obtain ($[\psi_{12},-] := [\psi_1,[\psi_2,-]]$)

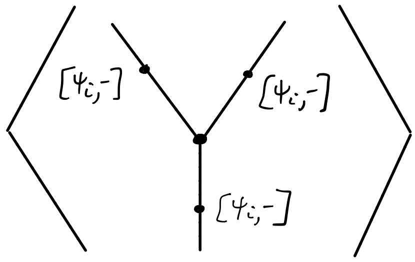
$$\begin{aligned}\mathcal{O}^{pre}(T,\mathcal{C}) &= [\psi_{12},-] \cdot \big[\psi_2, [\psi_1,-] \cdot [\psi_1,[\psi_2, a(-)]] \cdot [\psi_1,[\psi_2,a(-)]]\big] \\ &\quad \cdot [\psi_{12},-] \cdot [\psi_{12},-] \\ &= -[\psi_{12},-] \cdot [\psi_{12},-] \cdot [\psi_{12}, a(-)] \cdot [\psi_{12}, a(-)] \\ &\quad \cdot [\psi_{12},-] \cdot [\psi_{12},-].\end{aligned}$$

As explained on p. 11 ainfmf11 the product of the F-factors (i.e. the coefficients in (20.1) of ainfmf9) in this case is $1/8$. So we conclude

$$\mathcal{O}(T, \mathcal{C}) = -\frac{1}{8}[\psi_{12}, -] \cdot [\psi_{12}, -] \cdot [\psi_{12}, G(-)] \cdot [\psi_{12}, G(-)] \cdot [\psi_{12}, -] \cdot [\psi_{12}, -].$$

Observations about enumeration

We have now shown that for any $T, \mathcal{C}$ the operator $\mathcal{O}(T, \mathcal{C})$ is, up to scalars and G's, constructed from $[\psi_i, -]$ insertions on the edges of T, both internal and external. The possible patterns of these insertions are heavily constrained, since for example



$$= 0 \qquad (24.1)$$

We attempt to describe these constraints using boolean formulas, and thus equations over $\mathbb{Z}_2[q_1, \ldots, q_N]$, for some N. Let E be the set of locations in T (meaning: input vertices, internal vertices and internal edges). We introduce a family of boolean variables

$$Q = \{ q_i(x) \}_{1 \le i \le n,\, x \in E} \qquad (24.2)$$

where $q_i(x) = 1$ means "$[\psi_i, -]$ is inserted at location x", or more precisely $i \in J(x)$.

The set E is partially ordered by the relation $y \le x$ (y is above x). The idea is that Q depends on T, and we write down a set F of boolean formulas over Q. Any configuration $\mathcal{C}$ determines values of all the variables $q_i(x) \in \mathbb{Z}_2$, and we design F so that

$$\exists f \in F \text{ with } f(\mathcal{C}) = 0 \implies \mathcal{O}(T, \mathcal{C}) = 0. \qquad (25.1).$$

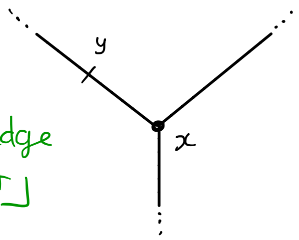
where $f(\mathcal{C})$ has the obvious meaning: f evaluated with the $q_i(x)$ corresp. to $\mathcal{C}$. (so $q_i^{\mathcal{C}}(x) = \delta_{i \in J^{\mathcal{C}}(x)}$).

1. The generalisation of (24.1) is the following: if $q_i(x) = 1$ then there must exist some path from x to an input (taking the left branch at x, if x is an internal vertex) with the property that $q_i(y) = 0$ for all $y \neq x$ on the path. Otherwise, using (15.3) for $[\Psi_i, -]$ we will have somewhere $[\Psi_i, -]^2 = 0$. For each location x:

(1a) x is an input : no constraint

(1b) x is an internal vertex :

$$\boxed{f_{x,i} := q_i(x) \supset \bigvee_{p \in \mathcal{P}(y)} \bigwedge_{z \in p} \neg q_i(z)}$$

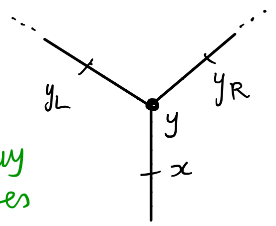


⌜y is an int. edge or an input⌟

$\mathcal{P}(y) = \{$ set of paths from inputs to y, inclusive, given as sequences of locations, excluding int. vertices we enter from the right $\}$

(1c) x is an internal edge :

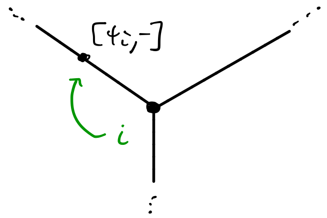
$$\boxed{g_{x,i} := q_i(x) \supset \Big(\Big(\bigvee_{p \in \mathcal{P}(y_R)} \bigwedge_{z \in p} \neg q_i(z) \Big) \vee \Big(\neg q_i(y) \wedge \bigvee_{p \in \mathcal{P}(y_L)} \bigwedge_{z \in p} \neg q_i(z) \Big) \Big)}$$



⌜y_R, y_L may be int. edges or inputs⌟

Note For x an interval vertex,

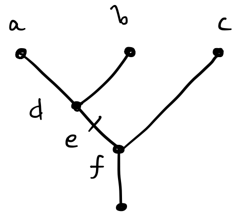
$q_i(x) = 1 \iff i \in J(x) \implies$ in the standard operator decorated tree we have $[\psi_i, -]$ on the edge shown below:



(26.1)

This is why $P(y)$ excludes int. vertices we enter from the right. A path in T is determined by its starting point, so we could write $P(y) = \{$ inputs a s.t. there is a path $a \longrightarrow y$ in $T\}$ and then restrict $\bigwedge_z$.

Example $T =$



with $n=1$, so we have variables $q(a), q(b), \ldots, q(f)$.

The constraints are:

$$\boxed{d} \qquad q(d) \supset \neg q(a) \tag{26.2}$$

$$\boxed{e} \qquad q(e) \supset \big(\neg q(b) \vee (\neg q(d) \wedge \neg q(a))\big)$$

$$\boxed{f} \qquad q(f) \supset \bigvee_{p \in P(e)} \bigwedge_{z \in p} \neg q(z) \qquad P(e) = \{a \to e, b \to e\}$$

$$= q(f) \supset \big[\big(\neg q(a) \wedge \neg q(d) \wedge \neg q(e) \big) \vee \big(\neg q(b) \wedge \neg q(e) \big) \big]$$

note no $\neg q(d)$ here, as explained above

Continuing this example, we convert these constraints to polynomial equations in $\mathbb{Z}_2[\{r_i(x)\}_{1\leq i\leq n, x\in E}]$ as follows. Given a boolean formula F we define the <u>set</u> of polynomials P_F via the recursive def^N given below, such that there is a bijection for any F,

$$\{\text{solutions of } P_F \text{ in } \mathbb{Z}_2\} \longleftrightarrow \{\text{satisfying assignments of } F\}$$

<u>Def^N</u> We define (assume only atoms are negated in F) ⌜we also write $P(F)$ for P_F⌟

- $P_{q_i(x)} = \{r_i(x) - 1\}$ for $1 \leq i \leq n$, $x \in E$.
- $P_{\neg q_i(x)} = \{r_i(x)\}$ for $1 \leq i \leq n$, $x \in E$.
- $P_{F \wedge G} = P_F \cup P_G$.
- $P_{F \vee G} = \{fg \mid f \in P_F, g \in P_G\}$.

<u>Example</u> In the situation of (26.2) we have

$\boxed{d}$ $$P_{q(d) \supset \neg q(a)} = P_{\neg q(d) \vee \neg q(a)}$$
$$= \{r(d)r(a)\}$$

$\boxed{e}$ $$P\Big(q(e) \supset \big(\neg q(b) \vee (\neg q(d) \wedge \neg q(a))\big)\Big)$$
$$= \{r(e)g\}_{g \in P(\neg q(b) \vee (\neg q(d) \wedge \neg q(a)))}$$
$$= \{r(e)r(b)g\}_{g \in P(\neg q(d) \wedge \neg q(a))}$$
$$= \{r(e)r(b)r(d),\ r(e)r(b)r(a)\}$$

$\boxed{f}$ $P\Big(q(f) \supset \big[(\neg q(a) \wedge \neg q(d) \wedge \neg q(e)) \vee (\neg q(b) \wedge \neg q(e)) \big] \Big)$

$= \{ r(f)g \mid g \in P((\neg q(a) \wedge \neg q(d) \wedge \neg q(e)) \vee (\neg q(b) \wedge \neg q(e))) \}$

$= \{ r(f) g_1 g_2 \mid g_1 \in P(\neg q(a) \wedge \neg q(d) \wedge \neg q(e)),$
$\qquad g_2 \in P(\neg q(b) \wedge \neg q(e)) \}$

$= \{ r(f) g_1 g_2 \mid g_1 \in \{ r(a), r(d), r(e) \}, g_2 \in \{ r(b), r(e) \} \}.$

Using the mapping $r(a) \longleftrightarrow r(1), \ldots, r(f) \longleftrightarrow r(6)$ we can find the Gröbner basis of the above constraints, together with $r(i)^2 + r(i)$, in $\mathbb{Z}_2[r(1), \ldots, r(6)]$. It is:

$$\begin{bmatrix} a & b & c & d & e & f \\ 1 & 2 & 3 & 4 & 5 & 6 \end{bmatrix}$$

```
> ring rr=2,(r(1..6)),dp;
> ideal I = r(4)r(1), r(5)r(2)r(4), r(5)r(2)r(1),
            r(6)r(1)r(2), r(6)r(1)r(5),
            r(6)r(4)r(2), r(6)r(4)r(5),
            r(6)r(5)r(2), r(6)r(5)r(5),
            r(1)^2+r(1), ..., r(6)^2+r(6);
```

The Gröbner basis of I contains (in addition to $r(i)^2 + r(i)$),

$$\{ r(5)r(6),\ r(1)r(4),\ r(2)r(4)r(6),\ r(1)r(2)r(6),$$
$$r(2)r(4)r(5),\ r(1)r(2)r(5) \}$$

So $r(5)=0,\ r(1)=0 \Rightarrow$ either $r(2)$ or $r(6)=0$
$r(5)=0,\ r(1)=1 \Rightarrow r(4)=0$, either $r(2)$ or $r(6)=0$
$r(5)=1,\ r(1)=0 \Rightarrow r(6)=0$, either $r(2)$ or $r(4)=0$
$r(5)=1,\ r(1)=1 \Rightarrow r(4)=r(6)=0$, and $r(2)=0$.

<u>many sol^{N}s</u>

Without additional constraints this is not very useful.

Summary

Above, we made progress on the conversion of the operator decorated trees of ainfmf9 into "normal forms" involving only $[\psi_i, -]$ operators (and G, and constants). We left unresolved the signs, and also the symmetry factors (both the $F(x)$ from ainfmf9 and the factors from ∂x, x annihilation in the contraction-to-normal form).