Minimal models for MFs 25 (checked)

Let A denote for $d>2$ the minimal A_∞-algebra with underlying $\mathbb{Z}_2$-graded algebra the exterior algebra $\Lambda(k\varepsilon)$ where $|\varepsilon|=1$, and A_∞-products $\{m_n\}_{n\geq 1}$ with only m_2, m_d nonzero, and $m_d(\varepsilon\otimes\cdots\otimes\varepsilon)=1$.

Given $2\leq i\leq d-2$ with $i<d-i$ define

$$M_{(i)} = \Lambda(k\xi) \qquad\qquad |\xi|=1$$

$$\alpha_n : A^{\otimes(n-1)}\otimes M_{(i)} \longrightarrow M_{(i)}$$

⌈h, k to be determined⌋

$$\begin{aligned} &\alpha_1 = 0, \quad \alpha_2(1,-) = \mathrm{id} \\ &\alpha_{i+1}(\varepsilon,\ldots,\varepsilon,-) = (-1)^h \xi^* \lrcorner (-) \\ &\alpha_{d-i+1}(\varepsilon,\ldots,\varepsilon,-) = (-1)^k \xi \wedge (-) \end{aligned} \qquad (1.1)$$

We check (some of) the A_∞-constraints making $\{\alpha_n\}_{n\geq 1}$ into an A_∞-module (using $(-1)^{r+st}$ signs). The possible contributors are $m_2, m_d, \alpha_2, \alpha_{i+1}, \alpha_{d-i+1}$. The interesting one is where $\alpha_{i+1}, \alpha_{d-i+1}$ both contribute, which is the constraint for $n=d+1$. Schematically, this will have terms

$$A^{\otimes d}\otimes M_{(i)} \longrightarrow M_{(i)}$$

$$\underset{\substack{r=0\\ s=d\\ t=1}}{\underset{(1)}{\alpha_2 m_d}} + \underset{\substack{r=i\\ s=d-i+1\\ t=0}}{\underset{(2)}{\alpha_{i+1}\alpha_{d-i+1}}} + \underset{\substack{r=d-i\\ s=i+1\\ t=0}}{\underset{(3)}{\alpha_{d-i+1}\alpha_{i+1}}} \qquad (1.2)$$

which we compute separately. It is clear all three terms are nonvanishing only on (among basis elts of. $\Lambda(k\varepsilon)^{\otimes d}$) the basis element $\varepsilon^{\otimes d}$, and

$$\alpha_2 m_d(\varepsilon, \ldots, \varepsilon, x) = \alpha_2(1, x) = x \qquad (2.1)$$

$$\alpha_{i+1}\alpha_{d-i+1}(\varepsilon, \ldots, \varepsilon, x) = \alpha_{i+1}(\varepsilon, \ldots, \varepsilon, \xi \wedge x) = (-1)^{h+k} \xi^* \lrcorner (\xi \wedge x)$$

$$\alpha_{d-i+1}\alpha_{i+1}(\varepsilon, \ldots, \varepsilon, x) = \alpha_{d-i+1}(\varepsilon, \ldots, \varepsilon, \xi^* \lrcorner x) = (-1)^{h+k} \xi \wedge (\xi^* \lrcorner x)$$

The signs in (1.2) are ① $(-1)^d$ ② $(-1)^i$ and ③ $(-1)^{d-i}$, so the constraint for $n = d+1$ reads (in its only nontrivial values) for $x \in M(i)$ homogeneous

$$0 = \Big[(-1)^d \alpha_2(m_d \otimes \mathbb{1}) + (-1)^i \alpha_{i+1}(\mathbb{1}^{\otimes i} \otimes \alpha_{d-i+1}) \qquad (2.2)$$
$$+ (-1)^{d-i} \alpha_{d-i+1}(\mathbb{1}^{\otimes d-i} \otimes \alpha_{i+1}) \Big](\varepsilon, \ldots, \varepsilon, x)$$

Now the Koszul sign rule says (since $|\alpha_n| = 2-n = n$)

$$(\mathbb{1}^{\otimes i} \otimes \alpha_{d-i+1})(\varepsilon, \ldots, \varepsilon, x) = (-1)^{i(d-i+1)} \varepsilon^{\otimes i} \otimes \alpha_{d-i+1}(\varepsilon^{\otimes d-i}, x)$$
$$= (-1)^{i(d-i+1)} \varepsilon^{\otimes i} \otimes (\xi \wedge x) \cdot (-1)^k$$

$$(\mathbb{1}^{\otimes (d-i)} \otimes \alpha_{i+1})(\varepsilon, \ldots, \varepsilon, x) = (-1)^{(d-i)(i+1)} \varepsilon^{\otimes d-i} \otimes (\xi^* \lrcorner x) \cdot (-1)^h$$

Hence (2.2) expands to

$$0 = (-1)^d x + (-1)^{i + i(d-i+1) + h+k} \xi^* \lrcorner (\xi \wedge x) \qquad \overset{\text{graded commutator}}{[\xi^* \lrcorner(-), \xi \wedge (-)] = id}$$
$$+ (-1)^{d-i+(d-i)(i+1)+h+k} \xi \wedge (\xi^* \lrcorner x)$$

$$= (-1)^d x + (-1)^{i(d-i)+h+k} x$$

<u>Upshot</u> provided we choose h, k such that $h+k+i(d-i) \equiv d+1 \pmod 2$, the $n = d+1$ A_∞-constraint will be satisfied.