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# Minimal models for MFs 26 - Feynman diagrams III

In this note we study the coefficients which appear in the Feynman diagrams computing the $A_\infty$-products in minimal models associated to Landau-Ginzburg models, of the form

$$\sum_{\sigma \in S_n} \frac{1}{a+a_{\sigma(1)}} \cdot \frac{1}{a+a_{\sigma(1)}+a_{\sigma(2)}} \cdots \frac{1}{a+a_{\sigma(1)}+\cdots+a_{\sigma(n)}} \, . \qquad (1.1)$$

At first we thought these terms should be understood in terms of the elementary calculations of correlation functions in 1D scalar field theories, which boil down to integrals like

$$\int ds\,dt\, e^{-ia|s|} e^{-ib|t|} e^{-ic|s-t|} = \frac{-2}{(a+b)(b+c)} + \frac{-2}{(a+b)(a+c)} + \frac{-2}{(a+c)(b+c)} \, .$$

See e.g. Timothy G. Abbott's "Feynman diagrams in Quantum Mechanics" and Appendix A of this note. However such integrals will not naturally give the precise form of (1.1) as far as we can tell.

Instead the correct context appears to be soft photon amplitudes and infrared divergences, where the $a=0$ case of (1.1) appears explicitly in

[W] Weinberg "The Quantum Theory of Fields I"
[PS] Peskin & Shroeder "An introduction to QFT"

respectively as [W, p.538] and [PS, p.204]. The coefficients of the form (1.1) probably appear in any situation like the one described in [L, §4.1.3], where

[L] Lazaroiu "String field theory and brane superpotentials"

That is, where the propagator involves inverting a commutar $[d_K, \nabla]$ where $d_K$ is some sort of Koszul differential and $\nabla$ is a connection

<u>Example</u> Set $\mathcal{H} = k[x_1, \dots, x_n] \otimes_k \Lambda(k\theta_1 \oplus \cdots \oplus k\theta_n)$ and

$$d_K = \sum_i x_i \otimes \theta_i^* \qquad (\theta_i^* = \theta_i^* \lrcorner (-),$$

$$\nabla = \sum_i \partial_{x_i} \otimes \theta_i \qquad \theta_i = \theta_i \wedge (-) \,)$$

Then we have

$$\begin{aligned} d_K \nabla (x^\gamma \otimes \omega) &= d_K \left( \textstyle\sum_i \partial_i (x^\gamma) \theta_i \omega \right) \\ &= \textstyle\sum_{i,j} x_j \partial_i (x^\gamma) \theta_j^* \theta_i \omega \end{aligned}$$

$$\begin{aligned} \nabla d_K (x^\gamma \otimes \omega) &= \nabla \left( \textstyle\sum_j x_j x^\gamma \otimes \theta_j^* \omega \right) \\ &= \textstyle\sum_{i,j} \partial_i (x_j x^\gamma) \otimes \theta_i \theta_j^* \omega \\ &= \textstyle\sum_{i,j} (\delta_{ij} + x_j \partial_i)(x^\gamma) \otimes (\delta_{ij} - \theta_j^* \theta_i)(\omega) \end{aligned}$$

$$\begin{aligned} [d_K, \nabla](x^\gamma \otimes \omega) &= \textstyle\sum_{i,j} \delta_{ij} x^\gamma \otimes \delta_{ij} \omega - \sum_{i,j} \delta_{ij} x^\gamma \otimes \theta_j^* \theta_i (\omega) \\ &\quad + \textstyle\sum_{i,j} x_j \partial_i (x^\gamma) \otimes \delta_{ij} \omega \\ &= \textstyle\sum_i x^\gamma \otimes \omega - \sum_i x^\gamma \otimes \theta_i^* \theta_i (\omega) \\ &\quad + \textstyle\sum_i x_i \partial_i (x^\gamma) \otimes \omega \\ &= n \cdot x^\gamma \otimes \omega + x^\gamma \otimes \left( \textstyle\sum_i \theta_i^* \theta_i \right)(\omega) \\ &\quad + \left( \textstyle\sum_i x_i \partial_i \right)(x^\gamma) \otimes \omega \\ &= n \cdot x^\gamma \otimes \omega - (n - |\omega|) x^\gamma \otimes \omega \\ &\quad + |\gamma| x^\gamma \otimes \omega \\ &= (|\omega| + |\gamma|) x^\gamma \otimes \omega \end{aligned}$$

Hence on every $k$-basis element for $\mathcal{H}$ other than $1 \otimes 1$, $[d_K, \nabla]$ is invertible. The homotopy $H$ in (ainfmf9) is $H = [d_K, \nabla]^{-1} \nabla$ and this generates the coefficients (1.1).

To understand how (1.1) appears in the study of infrared divergences we recall some terminology:

- The four-momentum $p^\mu$ of a physical particle of mass $m$ satisfies (with $p = (p^0; \underline{p})$)

$$p^2 = p^\mu p_\mu = E^2 - |\underline{p}|^2 = m^2. \qquad (3.1)$$

← called the Einstein energy-momentum relation

Given a fixed mass $m$, a four-momenta $p$ with $p^2 = m^2$ is called <u>on the mass-shell</u> (or simply <u>on-shell</u>), otherwise it is <u>off-shell</u>. Virtual particles corresponding to internal propagators in a Feynman diagram are allowed to be off-shell, although the further off-shell they are, the more the amplitude is suppressed.

More precisely, an internal line in a Feynman diagram corresponds to a commutator of two operators, which are labelled by precise momenta, say $q$, and other quantum numbers, and represent a particular type of particle, a "real" example of which has mass $m$ say. But we allow Feynman diagrams (recall we integrate over $q$) in which $q^2 < m^2$.

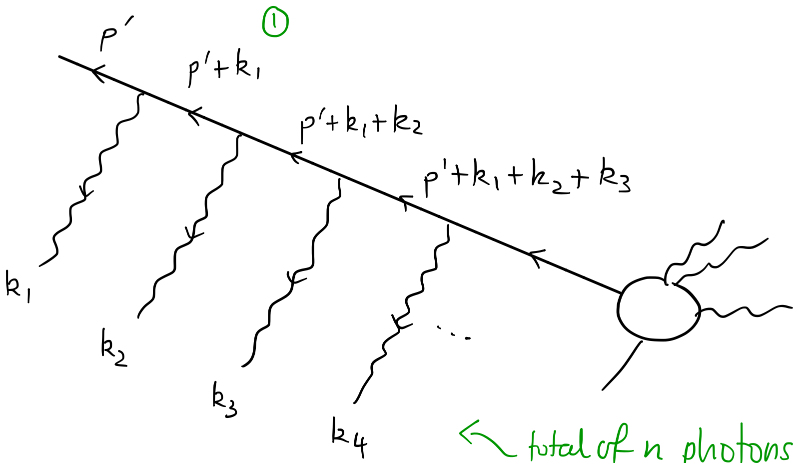
- Recall the propagator $\overset{p}{\text{- - - -}\!\!\succ\text{- -}}$ $\quad D_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i e^{-ip\cdot(x-y)}}{p^2 - m^2 + i\varepsilon}$
(in the position space picture) which gives in the momentum space picture a fraction

$$\frac{i}{p^2 - m^2 + i\varepsilon} \qquad (3.2)$$

For fermions, we get a different factor in the numerator but the same denominator (this all being for <u>internal</u> lines).

In [PS, §6.5] in a discussion of infrared divergences the authors consider photons with "soft" momenta. Firstly, photons are massless, so for a real photon (i.e. on-shell) with 4-momentum $k$ we have $k^2 = 0$. Obviously for a virtual photon we may have $k^2 \neq 0$. A real photon is soft if $k < E_\ell$ where $E_\ell$ is some energy cutoff, while a virtual photon is soft if $k^2 < E_\ell$.

The relevant diagram is




(4.1)

← total of $n$ photons $k_1, \dots, k_n$

where the wavy lines are soft virtual photons and $p'$ the 4-momentum of an external fermion line. The propagator at 1 has denominator

$$(p'+k_1)^2 - m^2 + i\varepsilon = (p')^2 + 2p' \cdot k_1 + k_1^2 - m^2 + i\varepsilon. \qquad (4.2)$$

Now, if $p'$ is on-shell, so $(p')^2 = m^2$, and we ignore $k_1^2$ since it is below our cutoff (i.e. the photon is soft) this is $2p' \cdot k_1$. Ignoring also the $i\varepsilon$ factor, the amplitude for (4.1) involves the ratio

$$\frac{1}{p' \cdot k_1} \cdot \frac{1}{p' \cdot (k_1+k_2)} \cdot \dots \cdot \frac{1}{p' \cdot (k_1 + \dots + k_n)} \qquad (4.3)$$

Moreover we have to sum over all permutations, yielding

$$\sum_{\sigma \in S_n} \frac{1}{p' \cdot k_{\sigma(1)}} \cdot \frac{1}{p' \cdot (k_{\sigma(1)} + k_{\sigma(2)})} \cdots \cdot \frac{1}{p' \cdot (k_{\sigma(1)} + \cdots + k_{\sigma(n)})} \qquad (5.1)$$

$$= \frac{1}{p' \cdot k_{\sigma(1)}} \cdots \cdot \frac{1}{p' \cdot k_{\sigma(n)}}$$

⌜sometimes called the eikonal identity, see p.9 of arXiv:1507.02167⌟

By the calculation of p. 5 ainfmf9, which is the $a=0$ case of (1.1).

<u>Upshot</u> If we now consider the case where the photons are still soft, but $p'$ is <u>not</u> necessarily on-shell, and set

$$a = (p')^2 - m^2$$
$$a_i = 2p' \cdot k_i$$

then instead of (4.3) we see

$$\frac{1}{(p')^2 + 2p' \cdot k_1 - m^2} \cdot \frac{1}{(p')^2 + 2p' \cdot (k_1 + k_2) - m^2} \cdots$$

$$= \frac{1}{a + a_1} \; \frac{1}{a + a_1 + a_2} \cdots \frac{1}{a + a_1 + \cdots + a_n}$$

then in place of (5.1) the amplitude involves (1.1).

<u>Question</u> Is this case ($p'$ off-shell) discussed somewhere in the physics literature? Are there any non-obvious things to be said about this kind of sum along the lines of (5.1) in the $a=0$ case?

## Appendix A

**Lemma** $\int_{-\infty}^{\infty} e^{-|x|}dx = 2.$

⌜ by a change of variable, for $a>0$ $\int_{-\infty}^{\infty} e^{-a|x|}dx = \frac{2}{a}$ ⌟

**Proof**
$$\int_{-\infty}^{\infty} e^{-|x|}dx = \int_{-\infty}^{0} e^{x}dx + \int_{0}^{\infty} e^{-x}dx$$
$$= \lim_{a\to\infty}\left( \int_{-a}^{0} e^{x}dx + \int_{0}^{a} e^{-x}dx \right)$$
$$= \lim_{a\to\infty}\left( [e^{x}]_{-a}^{0} + [-e^{-x}]_{0}^{a} \right)$$
$$= \lim_{a\to\infty}\left( 1 - e^{-a} + (-e^{-a}+1) \right) = 2. \square$$

**Lemma** For $b, c, \lambda > 0$, $\int_{-\infty}^{\infty} e^{-b|x|-c|x-\lambda|}dx = \frac{e^{-b\lambda}+e^{-c\lambda}}{b+c} + \frac{e^{-b\lambda}-e^{-c\lambda}}{c-b}$

**Proof** We have

$$\int_{-\infty}^{\infty} e^{-b|x|-c|x-\lambda|}dx = \int_{-\infty}^{0} e^{bx+c(x-\lambda)}dx + \int_{0}^{\lambda} e^{-bx+c(x-\lambda)}dx + \int_{\lambda}^{\infty} e^{-bx-c(x-\lambda)}dx$$

$$= \int_{-\infty}^{0} e^{(b+c)x-c\lambda}dx + \int_{0}^{\lambda} e^{(c-b)x-c\lambda}dx + \int_{\lambda}^{\infty} e^{-(b+c)x+c\lambda}dx$$

$$= \lim_{a\to\infty}\left[ \frac{1}{b+c} e^{(b+c)x-c\lambda} \right]_{-a}^{0} + \lim_{a\to\infty}\left[ \frac{-1}{b+c} e^{-(b+c)x+c\lambda} \right]_{\lambda}^{a} + \left[ \frac{1}{c-b} e^{(c-b)x-c\lambda} \right]_{0}^{\lambda}$$

$$= \frac{e^{-c\lambda}}{b+c} + \frac{e^{-b\lambda}}{b+c} + \frac{e^{-b\lambda}}{c-b} - \frac{e^{-c\lambda}}{c-b}. \square$$

Lemma For $b, c > 0$, $\lambda \leq 0$,

$$\int_{-\infty}^{\infty} e^{-b|x| - c|x-\lambda|} dx = \frac{e^{b\lambda}}{b+c} + \frac{e^{c\lambda}}{b+c} + \frac{e^{c\lambda}}{b-c} - \frac{e^{b\lambda}}{b-c}.$$

Proof We have

$$\int_{-\infty}^{\infty} e^{-b|x| - c|x-\lambda|} dx = \int_{-\infty}^{\lambda} e^{bx + c(x-\lambda)} dx + \int_{\lambda}^{0} e^{bx - c(x-\lambda)} dx + \int_{0}^{\infty} e^{-bx - c(x-\lambda)} dx$$

$$= \int_{-\infty}^{\lambda} e^{(b+c)x - c\lambda} dx + \int_{\lambda}^{0} e^{(b-c)x + c\lambda} dx + \int_{0}^{\infty} e^{-(b+c)x + c\lambda} dx$$

$$= \lim_{a \to \infty} \left[ \frac{1}{b+c} e^{(b+c)x - c\lambda} \right]_{-a}^{\lambda} + \lim_{a \to \infty} \left[ \frac{-1}{b+c} e^{-(b+c)x + c\lambda} \right]_{0}^{a} + \left[ \frac{1}{b-c} e^{(b-c)x + c\lambda} \right]_{\lambda}^{0}$$

$$= \frac{e^{b\lambda}}{b+c} + \frac{e^{c\lambda}}{b+c} + \frac{e^{c\lambda}}{b-c} - \frac{e^{b\lambda}}{b-c}. \quad \square$$

Lemma For $b, c > 0$ and any $\lambda \in \mathbb{R}$,

$$\int_{-\infty}^{\infty} e^{-b|x| - c|x-\lambda|} dx = \frac{1}{b+c}\left(e^{-b|\lambda|} + e^{-c|\lambda|}\right) + \frac{1}{b-c}\left(e^{-c|\lambda|} - e^{-b|\lambda|}\right)$$

Lemma With $a,b,c > 0$

$$\int ds \int dt \, e^{-a|s|} e^{-b|t|} e^{-c|s-t|} = \frac{2}{(a+b)(b+c)} + \frac{2}{(a+c)(b+c)} + \frac{2}{(a+b)(a+c)}$$

Proof We have by the above

$$\int dt \, e^{-b|t|} e^{-c|s-t|} = \frac{1}{b+c}\left( e^{-b|s|} + e^{-c|s|} \right) + \frac{1}{b-c}\left( e^{-c|s|} - e^{-b|s|} \right)$$

$$\therefore \quad \int ds \int dt \, e^{-a|s|} e^{-b|t|} e^{-c|s-t|} = \frac{1}{b+c} \int ds \left( e^{-(a+b)|s|} + e^{-(a+c)|s|} \right) + \frac{1}{b-c} \int ds \left( e^{-(a+c)|s|} - e^{-(a+b)|s|} \right)$$

$$= \frac{1}{b+c}\left( \frac{2}{a+b} + \frac{2}{a+c} \right) + \frac{1}{b-c}\left( \frac{2}{a+c} - \frac{2}{a+b} \right)$$

$$= \quad " \quad \quad " \quad + \frac{2}{b-c}\left( \frac{a+b-(a+c)}{(a+c)(a+b)} \right)$$

$$= \frac{2}{(a+b)(b+c)} + \frac{2}{(a+c)(b+c)} + \frac{2}{(a+b)(a+c)} . \quad \square$$