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Minimal models for MFs 28 (checked)

Our aim in this note is to write down the theory we have developed in its full generality, namely as a model of the DG-category $mf(k[x], W)$ extended by the exterior algebra $\Lambda(k\theta_1 \oplus \cdots \oplus k\theta_n)$. This has several components. Fixing a potential $(k[x], W)$ over a commutative $\mathbb{Q}$-algebra k, and writing $\mathcal{C}$ for the DG-category $\mathcal{C} = mf(k[x], W)$, they are $(\,|\underline{x}| =: n\,)$ (see ainfmf34 for why we do not need to assume noetherian)

1. Putting strict homotopy retracts on $\mathcal{C}(X,Y) \otimes_k \Lambda(\oplus_{i=1}^n k\theta_i)$ for all $X, Y \in \mathcal{C}$
2. Running the minimal model construction
3. Extracting the Feynman rules

The result of this is an A_∞-structure on the collection of $\mathbb{Z}_2$-graded k-modules underlying $\mathcal{C} \otimes_{k[x]} k[x]/(\partial W)$. Note that this A_∞-category is <u>not</u> minimal, although some of its endomorphism A_∞-algebras will be. Nonetheless, this A_∞-category is quasi-isomorphic (in fact homotopy equivalent) over k to $\mathcal{C} \otimes_k \Lambda(\oplus_{i=1}^n k\theta_i)$.

<u>Notation</u>

- $R = k[x_1, \ldots, x_n]$
- $J_W = R/(\partial_{x_1} W, \ldots, \partial_{x_n} W)$.
- $F_\theta = \oplus_{i=1}^n k\theta_i$ as a $\mathbb{Z}_2$-graded module with $|\theta_i| = 1$.
- $\mathcal{C}$ = DG-category of finite-rank free MFs of W (note we do not add summands at this stage)

($\mathcal{C}$ can in fact be any full-sub-DG-category of $mf(R,W)$)

<u>Note</u> We will later put some restriction on the $\lambda_i^{Y,X}$ (see p. 8) and we will see that in order to form a DG-category we are forced to make $t^{Y,X}$ indept. of Y, X (see p. 4.5). (in reference to material overleaf)

① Strict homotopy retracts

⌜note: here t_i is not necessarily a partial derivative, so there is a slight mismatch with [Cut], but we have paid attention to this and it is OK⌟

Here we primarily refer to our cut operation paper, hereby denoted [Cut]. Let X, Y be (f.rank, free) matrix factorisations, and assume that

- $t_1^{Y,X}, \ldots, t_n^{Y,X}$ is a quasi-regular sequence in R, such that
- $R/(t_1^{Y,X}, \ldots, t_n^{Y,X})$ is a finitely generated free k-module, (2.1)
- each $t_i^{Y,X}$ acts null-homotopically on $\mathrm{Hom}_R(Y,X)$ with null-homotopy $\lambda_i^{Y,X}$. (R-linear)
- the Koszul complex over R of $t_1^{Y,X}, \ldots, t_n^{Y,X}$ is exact except in degree zero

(pre 10-3-19 we had removed that k was noetherian, but forgot to add this)

Set $I = (t_1^{Y,X}, \ldots, t_n^{Y,X})$. The I-adic completion $\hat{R}$ of R has $\underline{t}^{Y,X}$ as a quasi-regular sequence and $\hat{R}/(\underline{t}^{Y,X}) \cong R/(\underline{t}^{Y,X})$. Then by the push forward paper there is a standard flat k-linear connection (writing $t_i = t_i^{Y,X}$)

$$\nabla^\circ : \hat{R} \longrightarrow \hat{R} \otimes_{k[\underline{t}]} \Omega^1_{k[\underline{t}]/k} \tag{2.2}$$

We run through the steps of [Cut, §4.3], with ΛF_θ our new notation for S_m, and $\mathrm{Hom}_R(Y,X) \cong Y^\vee | X$ for $Y|X$ (see [Cut, §4.5]), and $R = k[\underline{x}]$ in place of the $k[\underline{y}]$ there.

<u>First step</u> ∇° extends to a k-linear operator ∇ on $\hat{R} \otimes_{k[\underline{t}]} \Omega^*_{k[\underline{t}]/k}$. Choosing a homogeneous basis for X, Y and taking the induced basis on $\mathrm{Hom}_R(Y,X)$ over R, and extending ∇, we get a k-linear splitting homotopy

$$H = [d_K, \nabla]^{-1} \nabla \circlearrowright \left(K \otimes_R \mathrm{Hom}_R(Y,X) \otimes_R \hat{R}, \; d_K \right), \tag{2.3}$$

where $(K, d_K) = \left(\Lambda(F_\theta) \otimes_k R, \sum_{i=1}^n t_i^{Y,X} \theta_i^* \right)$, and we identify θ_i with dt_i.

⌜<u>NOTE</u> Constructing ∇ depends on a choice of k-basis for $R/(\underline{t}^{Y,X})$⌟

which corresponds to the strong deformation retract (3.1)

$$\left(\hat{R}/(\underline{t}^{Y,X}) \otimes_R \mathrm{Hom}_R(Y,X), 0 \right) \underset{\sigma}{\overset{\pi}{\rightleftarrows}} \left(K \otimes_R \mathrm{Hom}_R(Y,X) \otimes_R \hat{R}, d_K \right), H$$

<u>Second step</u> We now view d_{Hom}, the standard differential on $\mathrm{Hom}_R(Y,X)$, as a perturbation, and we learn that

$$\phi_\infty = \sum_{m \geq 0} (-1)^m (H d_{Hom})^m H \qquad (3.2)$$

is a k-linear splitting homotopy, to which is associated the following k-linear strong deformation retract of complexes

$$\begin{tikzcd} \left(\hat{R}/(\underline{t}^{Y,X}) \otimes_R \mathrm{Hom}_R(Y,X), \overline{d_{Hom}} \right) \arrow[d, bend left, "\sigma_\infty"] \\ \left(K \otimes_R \mathrm{Hom}_R(Y,X) \otimes_R \hat{R}, d_K + d_{Hom} \right) \arrow[u, bend right, "\pi"'] \arrow[loop below, "\phi_\infty"] \end{tikzcd} \qquad (3.3)$$

where (3.3.5)

$$\phi_\infty = \sum_{m \geq 0} (-1)^m (H d_{Hom})^m H, \quad \sigma_\infty = \sum_{m \geq 0} (-1)^m (H d_{Hom})^m \sigma$$

<u>Third step</u> Since each $t_i^{Y,X}$ acts null-homotopically on $\mathrm{Hom}_R(Y,X)$ we have an isomorphism of complexes (over R, in fact) (3.4)

$$\left(K \otimes_R \mathrm{Hom}_R(Y,X) \otimes_R \hat{R}, d_K + d_{Hom} \right) \underset{\exp(\delta)}{\overset{\exp(-\delta)}{\rightleftarrows}} \left(\wedge F_{\theta} \otimes_k \mathrm{Hom}_R(Y,X) \otimes_R \hat{R}, d_{Hom} \right)$$

where $\delta = \sum_i \lambda_i \theta_i^*$. Note here we do <u>not</u> assume λ_i is a partial derivative of e.g. d_X. This is not necessary.

Fourth step The canonical map $\varepsilon : \mathrm{Hom}_R(Y,X) \longrightarrow \mathrm{Hom}_R(Y,X) \otimes_R \hat{R}$ is by Remark 7.7 of the pushforward paper, a homotopy equivalence over k (see ainfmf34). Hence we have homotopy equivalences of k-complexes, combining (3.3), (3.4)

see Remark below

$$
\begin{array}{c}
\left(\bigwedge F_\theta \otimes_k \mathrm{Hom}_R(Y,X),\ d_{\mathrm{Hom}} \right) \\
\Big\downarrow 1 \otimes \varepsilon \\
\left(\bigwedge F_\theta \otimes_k \mathrm{Hom}_R(Y,X) \otimes_R \hat{R},\ d_{\mathrm{Hom}} \right) \\
\exp(-\delta) \Big\downarrow \quad \Big\uparrow \exp(\delta) \\
\left(K \otimes_R \mathrm{Hom}_R(Y,X) \otimes_R \hat{R},\ d_K + d_{\mathrm{Hom}} \right) \circlearrowleft \phi_\infty \\
\pi \Big\downarrow \quad \Big\uparrow \sigma_\infty \\
\left(\hat{R}/(\underline{t}^{Y,X}) \otimes_R \mathrm{Hom}_R(Y,X),\ \overline{d_{\mathrm{Hom}}} \right) \\
\Big\uparrow \cong \\
\left(R/(\underline{t}^{Y,X}) \otimes_R \mathrm{Hom}_R(Y,X),\ \overline{d_{\mathrm{Hom}}} \right)
\end{array}
\tag{4.1}
$$

Note that $\mathrm{Hom}_R(Y,X) \otimes_R R/(\underline{t}^{Y,X})$ is by our hypotheses a $\mathbb{Z}_2$-graded complex of finite free k-modules. Keeping in mind that all our DG-categories are $\mathbb{Z}_2$-graded, we have the DG-category $\mathcal{C}$ with $\mathcal{C}(Y,X) := (\mathrm{Hom}_R(Y,X), d_{\mathrm{Hom}})$ and we now introduce

Remark p. 4 of ainfmf34 shows there is a deformation retract over k between $K_R(\underline{t})$ and R/I, and so we may apply p. 6 to $X = \mathrm{Hom}_R(X,Y)$, and $\varphi : k \longrightarrow R$ (so $W=0$), to get ε is a k-linear homotopy equivalence.

<u>DG-category $\hat{\mathcal{C}}$</u> Next we define a DG-category $\hat{\mathcal{C}}$ with $\hat{\mathcal{C}}(Y,X) = \mathrm{Hom}_R(Y,X) \otimes_R \hat{R}$. The subtlety here is that the topology with respect to which we take the completion varies as the pair Y, X varies. To be more precise we now write

$$I_{Y,X} = (t_1^{Y,X}, \ldots, t_n^{Y,X}) \subseteq R$$

and for the associated completion we write

$$\hat{R}_{Y,X} := \varprojlim_r R / I_{Y,X}^r .$$

In order for the DG-category $\hat{\mathcal{C}}$ to make sense we need to have k-linear maps, for each triple Z, Y, X of matrix factorisations

$$m_{Z,Y,X} : \hat{R}_{Z,Y} \otimes_k \hat{R}_{Y,X} \longrightarrow \hat{R}_{Z,X} .$$

For this to exist we would need, given Cauchy sequences (a_n), (b_n) in R for the $I_{Z,Y}$-adic and $I_{Y,X}$-adic topologies respectively, to show that $(a_n b_n)$ is Cauchy for the $I_{Z,Y}$-adic topology. We would show this by calculating

$$\begin{aligned} a_m b_m - a_n b_n &= a_m b_m - a_m b_n + a_m b_n - a_n b_n \\ &= a_m (b_m - b_n) + (a_m - a_n) b_n , \end{aligned}$$

and using that for any $N \geq 1$ there are $a, b \geq 1$ with $I_{Z,Y}^a \subseteq I_{Z,X}^N$ and $I_{Y,X}^b \subseteq I_{Z,X}^N$. But given Z, Y, X are arbitrary, this shows:

<u>Upshot</u> For products $m_{Z,Y,X}$ to exist (and thus for $\hat{\mathcal{C}}$ to exist) we need the $I_{Y,X}$-adic topology on R to be <u>independent of Y, X</u>.

② DG-categories

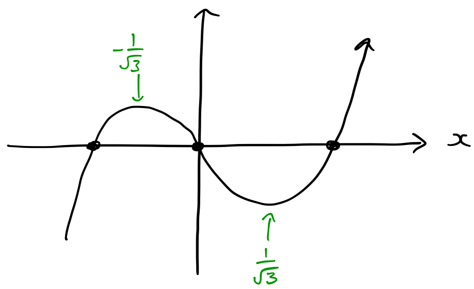
⌈ $t_i = \partial_{x_i} W$ is a choice which works, but see the Remark overleaf ⌋

In light of the previous page, we now fix $t_i^{Y,X} = t_i$ to be independent of Y, X and let $\hat{R}$ denote the $I = (t_1, \dots, t_n)$-adic completion. We continue to let $\lambda_i^{Y,X}$ be arbitrary, although see p. ⑧. Using the usual R-algebra structure on $\hat{R}$ we define

<u>Def^n</u> The DG-category $\hat{\mathcal{C}} = \mathcal{C} \otimes_R \hat{R}$, has $\hat{\mathcal{C}}(Y,X) = \mathrm{Hom}_R(Y,X) \otimes_R \hat{R}$.

As we see in (4.1), our methods produce an A_∞-minimal model of $\hat{\mathcal{C}}$ not $\mathcal{C}$. As an example of what can go wrong if we try different sequences $\underline{t}$, we have:

<u>Example</u> Let k be an alg. closed field and $W = x^3 - x = x(x-1)(x+1)$.



This is a potential, and in the DG-category $\mathcal{C}$ we may consider objects

$$P_{-1} = \begin{pmatrix} 0 & x+1 \\ x(x-1) & 0 \end{pmatrix}, \quad P_0 = \begin{pmatrix} 0 & x \\ x^2-1 & 0 \end{pmatrix}, \quad P_1 = \begin{pmatrix} 0 & x-1 \\ x(x+1) & 0 \end{pmatrix}.$$

We know $\partial_x W$ acts null-homotopically on all mapping complexes, and completing along the ideal $(\partial_x W)$ corresponds to taking the formal scheme around the critical points i.e. since $\partial_x W = 3x^2 - 1$, at $\{\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\}$.

Note e.g. that $d_{P_{-1}} = (x+1)\theta^* + x(x-1)\theta$, and $\partial_x(d_{P_{-1}}) = \theta^* + (2x-1)\theta$, which clearly satisfy $[d_{P_{-1}}, \partial_x(d_{P_{-1}})] = \partial_x W \cdot 1$.

Now it is true that $\{x\}$ is a quasi-regular sequence in $R = k[x]$ acting null-homotopically on $\mathrm{Hom}_R(P_0, P_0)$, but the (x)-adic topology on R is certainly not equal to the $(\partial_x W)$-adic topology (or the $(x-1)$-adic topology for that matter). Whereas we know $\mathcal{C}$ is homotopy equivalent to

$$\hat{\mathcal{C}} = \mathcal{C} \otimes_R \varprojlim_r R/(\partial_x W)^r.$$

So this completion preserves P_{-1}, P_0, P_1.

<u>Remark</u> We do not want to assume $t_i = \partial_{x_i} W$ because we want to preserve the freedom to replace $\mathcal{C}$ by a sub-DG-category (say just k^{stab}) in which case we may be able to use a sequence $\underline{t}$ which is not the partial derivatives (e.g. $x_1, \dots, x_n$).

<u>Def^n</u> The DG-category $\wedge F_\theta \otimes_k \hat{\mathcal{C}}$ is the tensor product of $\wedge F_\theta$, viewed as a one object DG-category (with zero differential) and $\hat{\mathcal{C}}$, as in p. 6 ect, so

$$(\wedge F_\theta \otimes_k \hat{\mathcal{C}})(Y,X) := \left(\wedge F_\theta \otimes_k \mathrm{Hom}_R(Y,X),\ 1 \otimes d_{\mathrm{Hom}} \right)$$

and the composition is (using graded swaps)

$$\begin{array}{c}
(\wedge F_\theta \otimes_k \hat{\mathcal{C}})(Y,X) \otimes_k (\wedge F_\theta \otimes_k \hat{\mathcal{C}})(Z,Y) \\
\| \\
\wedge F_\theta \otimes_k \hat{\mathcal{C}}(Y,X) \otimes_k \wedge F_\theta \otimes_k \hat{\mathcal{C}}(Z,Y) \\
\cong \quad 1 \otimes \mathrm{swap} \otimes 1 \\
\wedge F_\theta \otimes_k \wedge F_\theta \otimes_k \hat{\mathcal{C}}(Y,X) \otimes_k \hat{\mathcal{C}}(Z,Y) \\
\big\downarrow m \otimes m \\
\wedge F_\theta \otimes_k \hat{\mathcal{C}}(Z,X)
\end{array} \tag{5.1}$$

Next we argue that (4.1) gives a strict homotopy retraction of $\mathcal{A} = \wedge F_\theta \otimes_k \hat{\mathcal{C}}$ viewed as an A_∞-category, according to p. 5 ainfcal2. To this end we ignore the bottommost step of (4.1) and simply identify $R/(t^{Y,X})$ with $\hat{R}/(t^{Y,X})$. With this in mind, the overall content of (4.1) is a k-linear homotopy equivalence

$$\begin{tikzcd}
\left(\wedge F_\theta \otimes_k \mathrm{Hom}_R(Y,X) \otimes_R \hat{R},\ d_{\mathrm{Hom}}\right) \arrow[r, shift left, "\Phi"] \arrow[loop below, "\hat{H}"] & \left(R/(t^{Y,X}) \otimes_R \mathrm{Hom}_R(Y,X),\ \overline{d_{\mathrm{Hom}}}\right) \arrow[l, shift left, "\Phi^{-1}"]
\end{tikzcd} \tag{5.2}$$

where (cf. p. 2 ainfmf2) we have $\Phi \circ \Phi^{-1} = 1$ and $\Phi^{-1} \circ \Phi = 1 - [d_{\mathrm{Hom}}, \hat{H}]$

$$\Phi = \pi \circ \exp(-\delta), \qquad \Phi^{-1} = \exp(\delta) \circ \sigma_\infty \tag{5.3}$$
$$\hat{H} = \exp(\delta) \circ \phi_\infty \circ \exp(-\delta)$$

<u>Lemma</u> The data (P, G) consisting of

$$P_{Y,X} := \Phi^{-1} \circ \Phi \circlearrowright \mathcal{A}(Y,X)$$
$$(6.1)$$
$$G_{Y,X} := \hat{H} \circlearrowright \mathcal{A}(Y,X)$$

form a strict homotopy retract on the DG-category $\mathcal{A}$ in the sense of p. 5 ainfcat2, with base ring k.

<u>Proof</u> Clear by construction. $\square$

By the minimal model theorem (p. 10 ainfcat2) there are maps $\{\rho_k\}_{k \geq 1}$ making the k-modules $\mathrm{Im}(P_{Y,X})$ into an A_∞-category over k. But there is an isomorphism

$$\mathrm{Im}(P_{Y,X}) = \mathrm{Im}(\Phi^{-1}) \xleftarrow[\Phi^{-1}]{\cong} R/(t^{Y,X}) \otimes_R \mathrm{Hom}_R(Y,X) \qquad (6.2)$$

and we interpret the minimal model construction as putting an A_∞-structure on the set of objects $\mathrm{ob}(\mathcal{C})$ and $\mathbb{Z}_2$-graded free k-modules $R/(t^{Y,X}) \otimes_R \mathrm{Hom}_R(Y,X)$.

<u>Def^n</u> Let $\mathcal{B}$ denote the $\mathbb{Z}_2$-graded A_∞-category over k with

- objects $\mathrm{ob}(\mathcal{B}) = \mathrm{ob}(\mathcal{C})$
- $\mathcal{B}(Y,X) := R/(t^{Y,X}) \otimes_R \mathrm{Hom}_R(Y,X)$

and higher products $\{\rho_k\}_{k \geq 1}$ as produced by the minimal model theorem applied to the above strict homotopy retract on $\mathcal{A}$.

Calculating Feynman Rules

Following ainfcat2, let Q be the commutative associative k-algebra denoted $R_{\hat{\mathcal{A}}}$ on p.3 ainfcat2, and let us write (recall $\mathcal{A} = \Lambda F_\theta \otimes_k \hat{\mathcal{C}}$)

$$\mathcal{H} = \bigoplus_{Y,X \in \mathrm{ob}\mathcal{A}} \mathrm{Hom}_{\mathcal{A}}(Y,X) = \bigoplus_{Y,X \in \mathrm{ob}\mathcal{A}} \Lambda F_\theta \otimes_k \mathrm{Hom}_R(Y,X) \otimes_R \hat{R} \qquad (7.1)$$

with its Q-bimodule structure, with forward suspended operations

$$r_n : \mathcal{H}[1]^{\otimes_Q n} \longrightarrow \mathcal{H}[1]. \qquad (7.2)$$

Then p.9 ainfcat2 produces the k-linear Q-bilinear maps

$$\rho_n := \sum_{T \in \mathcal{J}_n} \rho_T \in \mathrm{Hom}_{Q\mathrm{Mod}_Q}\left(B[1]^{\otimes_Q n}, B[1]\right). \qquad (7.3)$$

where $B := \mathrm{Im}(P) \subseteq \mathcal{H}$ identified with the relevant direct sum of quotients as in (6.2).

<u>Def^n</u> For $1 \le i \le n$ we define the odd $\hat{R}$-linear, Q-bilinear operators

$$\lambda_i, \mathcal{O}_i^* \circlearrowright \mathcal{H} \qquad \lambda_i = \sum_{Y,X} \lambda_i^{Y,X}, \quad \mathcal{O}_i^* = \sum_{Y,X} \mathcal{O}_i^*$$

and we set $\delta = \sum_i \delta_i$, with $\delta_i = \lambda_i \mathcal{O}_i^* = \sum_{Y,X} \lambda_i^{Y,X} \mathcal{O}_i^*$.

<u>Remark</u> Given a matrix factorisation X set $\lambda_i^X := \lambda_i^{X,X}(1)$, which is an odd R-linear operator on X, with $[d_X, \lambda_i^X] = t_i \cdot 1_X$, since

$$t_i \cdot 1_{\mathrm{Hom}(X,X)} = \left[d_{\mathrm{Hom}(X,X)}, \lambda_i^{X,X}\right] \xRightarrow{ev_{1_X}} t_i \cdot 1_X = d_{\mathrm{Hom}(X,X)}\left(\lambda_i^{X,X}(1)\right) = [d_X, \lambda_i^X].$$

⌜Note: r_2 uses the tilde grading $\widetilde{\omega \otimes \alpha} = |\omega| + |\alpha| + 1$ for $\omega \in \Lambda F_\theta$, $\alpha \in \mathrm{Hom}(Y,X)$.⌟

Then let us write $\widetilde{\lambda_i^{Y,X}}$ for the odd R-linear operator on $\mathrm{Hom}_R(Y,X)$ given by

$$\widetilde{\lambda_i^{Y,X}}(\alpha) = \lambda_i^X \circ \alpha$$

Then

$$\begin{aligned}
\left[d_{\mathrm{Hom}(Y,X)}, \widetilde{\lambda_i^{Y,X}}\right](\alpha) &= d_X \circ \widetilde{\lambda_i^{Y,X}}(\alpha) + (-1)^{|\alpha|} \widetilde{\lambda_i^{Y,X}}(\alpha) \circ d_Y \\
&\quad + \widetilde{\lambda_i^{Y,X}}\, d_{\mathrm{Hom}}(\alpha) \\
&= d_X \lambda_i^X \alpha + \cancel{(-1)^{|\alpha|} \lambda_i^X \alpha\, d_Y} \\
&\quad + \lambda_i^X d_X \alpha - \cancel{(-1)^{|\alpha|} \lambda_i^X \alpha\, d_Y} \\
&= t_i \cdot \alpha
\end{aligned}$$

Hence $\left[d_{\mathrm{Hom}(Y,X)}, \widetilde{\lambda_i^{Y,X}}\right] = t_i \cdot 1_{\mathrm{Hom}(Y,X)}$, and so the operators $\widetilde{\lambda_i^{Y,X}}$ are just as good as $\lambda_i^{Y,X}$ for the purposes of the foregoing. Moreover, this type of homotopy is much <u>better</u> suited to the calculations we are about to do.

<u>Hypothesis</u> We have for each X a fixed choice of null-homotopy $\lambda_i^X : X \longrightarrow X$ (odd, R-linear) for t_i, that is, with $[d_X, \lambda_i^X] = t_i \cdot 1_X$, and for all Y, X we use the homotopies

$$\lambda_i^{Y,X} \circlearrowleft \mathrm{Hom}_R(Y,X) \qquad \lambda_i^{Y,X}(\alpha) = \lambda_i^X \circ \alpha. \tag{8.1}$$

<u>Note</u> This is not a constraint: as we have shown, it is possible to arrange under our setup.

<u>Def^n</u> For $1 \le i \le n$ we define the odd $\hat{R}$-linear, Q-bilinear operators $[\lambda_i, -] \circlearrowleft \mathcal{H}$ defined on $\omega \otimes \alpha \in \bigwedge F_\theta \otimes_R \mathrm{Hom}_R(Y,X) \otimes_R \hat{R}$ by

$$[\lambda_i, -](\omega \otimes \alpha) = (-1)^{|\omega|} \omega \otimes \left\{ \lambda_i^X \circ \alpha - (-1)^{|\alpha|} \alpha \circ \lambda_i^Y \right\} \tag{8.2}$$

Lemma The following diagram commutes

$$\begin{tikzcd} \mathcal{H}[1]\otimes_Q \mathcal{H}[1] \arrow[r, "r_2"] \arrow[d, "\delta_i\otimes 1 + 1\otimes\delta_i - \mathcal{O}_i^*\otimes[\lambda_i,-]"'] & \mathcal{H}[1] \arrow[d, "\delta_i"] \\ \mathcal{H}[1]\otimes_Q \mathcal{H}[1] \arrow[r, "r_2"'] & \mathcal{H}[1] \end{tikzcd} \qquad (9.1)$$

from (8.2)

Proof Note that $r_2(x_2\otimes x_1) = (-1)^{\tilde{x}_1\tilde{x}_2+\tilde{x}_1+1}\, x_1\circ x_2$ where $\circ$ is the composition in $\vec{\mathcal{C}}$.
To check (9.1) commutes we may choose objects X, Y, Z and check commutativity of

$$\begin{tikzcd} {[\Lambda F_\theta\otimes_k \mathrm{Hom}_R(Z,Y)\otimes_R\hat{R}][1]\otimes_k[\Lambda F_\theta\otimes_k \mathrm{Hom}_R(Y,X)\otimes_R\hat{R}][1]} \arrow[r, "r_2"] \arrow[d, "\delta_i\otimes 1+1\otimes\delta_i-\mathcal{O}_i^*\otimes[\lambda_i,-]"'] & {[\Lambda F_\theta\otimes_k \mathrm{Hom}_R(Z,X)\otimes_R\hat{R}][1]} \arrow[d, "\delta_i"] \\ {[\Lambda F_\theta\otimes_k \mathrm{Hom}_R(Z,Y)\otimes_R\hat{R}][1]\otimes_k[\Lambda F_\theta\otimes_k \mathrm{Hom}_R(Y,X)\otimes_R\hat{R}][1]} \arrow[r, "r_2"'] & {[\Lambda F_\theta\otimes_k \mathrm{Hom}_R(Z,X)\otimes_R\hat{R}][1]} \end{tikzcd}$$

To this end let $w_1, w_2\in\Lambda F_\theta$ be homogeneous and take $x_1\in\mathrm{Hom}_R(Y,X)$, $x_2\in\mathrm{Hom}_R(Z,Y)$.
Then

$$\begin{aligned} r_2(\delta_i\otimes 1+1\otimes\delta_i)\big((w_2\otimes x_2)\otimes(w_1\otimes x_1)\big) &= r_2\big(\delta_i(w_2\otimes x_2)\otimes w_1\otimes x_1 \\ &\qquad + w_2\otimes x_2\otimes\delta_i(w_1\otimes x_1)\big) \\ &= r_2\big(\lambda_i(\mathcal{O}_1^* w_2\otimes x_2)\otimes w_1\otimes x_1 + w_2\otimes x_2\otimes\lambda_i(\mathcal{O}_1^* w_1\otimes x_1)\big) \\ &= r_2\big((-1)^{|w_2|+1}\mathcal{O}_1^* w_2\otimes\lambda_i(x_2)\otimes w_1\otimes x_1 + (-1)^{|w_1|+1} w_2\otimes x_2\otimes\mathcal{O}_1^* w_1\otimes\lambda_i(x_1)\big) \end{aligned}$$

$$= (-1)^{|w_2|+1+(\theta_1 w_2 \otimes \lambda_i(x_2))^\sim (w_1 \otimes x_1)^\sim + (w_1 \otimes x_1)^\sim + 1} \mu\left(w_1 \otimes x_1 \otimes \theta_1^* w_2 \otimes \lambda_i(x_2)\right)$$

$$+ (-1)^{|w_1|+1+(w_2 \otimes x_2)^\sim (\theta_1 w_1 \otimes \lambda_i(x_1))^\sim + (\theta_1 w_1 \otimes \lambda_i(x_1))^\sim + 1} \mu\left(\theta_1^* w_1 \otimes \lambda_i(x_1) \otimes w_2 \otimes x_2\right)$$

$$= (-1)^{|w_2|+1+(|w_2|+1+|x_2|+1\cancel{+1})(|w_1|+|x_1|+1)+|w_1|+|x_1|+1+1} (-1)^{|x_1|(|w_1|+1)} w_1\, \theta_1^*(w_2) \otimes x_1 \circ \lambda_i(x_2)$$

$$+ (-1)^{|w_1|+1+(|w_2|+|x_2|+1)(|w_1|+1+|x_1|+\cancel{1}+1)+(|w_1|+\cancel{1}+|x_1|+\cancel{1}+1)\cancel{+1}} (-1)^{|w_1|(|x_1|+1)}$$
$$\theta_1^*(w_1) w_2 \otimes \lambda_i(x_1) \circ x_2 .$$

$$= (-1)^{|w_2|+|w_1|+\cancel{|x_1|}+\cancel{1}+|w_1||w_2|+\cancel{|w_2||x_1|}+|w_2|+|w_1|+|x_1|\cancel{+1}+|x_2||w_1|+|x_2||x_1|+|x_2| + \cancel{|x_1||w_2|}+\cancel{|x_1|}}$$
$$w_1\, \theta_1^*(w_2) \otimes x_1 \circ \lambda_i(x_2)$$

$$+ (-1)^{|w_1|+|w_1|+\cancel{|x_1|}\cancel{+1}+|w_1||w_2|+\cancel{|w_2|}+\cancel{|w_2||x_1|}+|x_2||w_1|+|x_2|+|x_2||x_1|+|w_1|\cancel{+1}+\cancel{|x_1|} + |w_2||x_1|+|w_2|}$$
$$\theta_1^*(w_1) w_2 \otimes \lambda_i(x_1) \circ x_2 .$$

$$= (-1)^{|w_1||w_2|+|x_2||w_1|+|x_2||x_1|+|x_2|+|x_1|} w_1\, \theta_1^*(w_2) \otimes x_1 \circ \lambda_i(x_2) \qquad (10.1)$$

$$+ (-1)^{|w_1||w_2|+|x_2||w_1|+|x_2|+|x_2||x_1|+|w_1|} \theta_1^*(w_1) w_2 \otimes \lambda_i(x_1) \circ x_2 .$$

$$= (-1)^{|w_1||w_2|+|x_2||w_1|+|x_2|+|x_2||x_1|+|w_1|} \Big\{ (-1)^{|w_1|+|x_1|} w_1\, \theta_1^*(w_2) \otimes x_1 \circ \lambda_i(x_2)$$
$$+ \theta_1^*(w_1) w_2 \otimes \lambda_i(x_1) \circ x_2 \Big\}$$

Then we also compute

$$\delta_i\ r_2\left((w_2 \otimes x_2) \otimes (w_1 \otimes x_1)\right) = (-1)^{(w_2 \otimes x_2)^\sim (w_1 \otimes x_1)^\sim + (w_1 \otimes x_1)^\sim + 1} \delta_i \mu\left((w_1 \otimes x_1) \otimes (w_2 \otimes x_2)\right)$$

$$= (-1)^{(|w_2|+|x_2|+1)(|w_1|+|x_1|+1)+(|w_1|+|x_1|+1)+1}$$
$$\cdot (-1)^{|x_1||w_2|} \delta_i\left(w_1 w_2 \otimes x_1 \circ x_2\right)$$

$$= (-1)^{|w_2||w_1|+\cancel{|w_2||x_1|}+\cancel{|w_2|}+|x_2||w_1|+|x_2||x_1|+|x_2|+\cancel{|w_1|}+\cancel{|x_1|}\cancel{+1} + \cancel{|w_1|}+\cancel{|x_1|}+\cancel{|x_1||w_2|}} \cdot (-1)^{|w_1|+\cancel{|w_2|}\cancel{+1}} \theta_i^*(w_1 w_2) \otimes \lambda_i(x_1 \circ x_2).$$

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10.5

$$= (-1)^{|w_2||w_1| + |x_2||w_1| + |x_2||x_1| + |x_2| + |w_1|} \, \theta_i^*(w_1 w_2) \otimes \lambda_i(x_1 \circ x_2)$$

Hence

$$\left\{ \delta_i r_2 - r_2(1 \otimes \delta_i + \delta_i \otimes 1) \right\} \left((w_2 \otimes x_2) \otimes (w_1 \otimes x_1) \right)$$

$$= (-1)^{|w_1||w_2| + |x_2||w_1| + |x_2| + |x_2||x_1| + |w_1|} \left\{ \theta_i^*(w_1 w_2) \otimes \lambda_i(x_1 \circ x_2) - (-1)^{|w_1| + |x_1|} w_1 \theta_i^*(w_2) \otimes x_1 \circ \lambda_i(x_2) - \theta_i^*(w_1) w_2 \otimes \lambda_i(x_1) \circ x_2 \right\}$$

But

$$\theta_i^*(w_1 w_2) \otimes \lambda_i(x_1 \circ x_2) - (-1)^{|w_1| + |x_1|} w_1 \theta_i^*(w_2) \otimes x_1 \circ \lambda_i(x_2) - \theta_i^*(w_1) w_2 \otimes \lambda_i(x_1) \circ x_2$$

$$= \theta_i^*(w_1) w_2 \otimes \lambda_i^X \cancel{\circ} x_1 \circ x_2 - (-1)^{|w_1| + |x_1|} w_1 \theta_i^*(w_2) \otimes x_1 \circ \lambda_i^Y \circ x_2 + (-1)^{|w_1|} w_1 \theta_i^*(w_2) \otimes \lambda_i^X \circ x_1 \circ x_2 - \theta_i^*(w_1) w_2 \otimes \lambda_i^X \cancel{\circ} x_1 \circ x_2$$

$$= (-1)^{|w_1|} w_1 \theta_i^*(w_2) \otimes \left\{ \lambda_i^X \circ x_1 - (-1)^{|x_1|} x_1 \circ \lambda_i^Y \right\} \circ x_2$$

$$= (-1)^{|w_1|} w_1 \theta_i^*(w_2) \otimes [\lambda_i, -](x_1) \circ x_2$$

So finally

$$\left\{ \delta_i r_2 - r_2(1 \otimes \delta_i + \delta_i \otimes 1) \right\} \left((w_2 \otimes x_2) \otimes (w_1 \otimes x_1) \right)$$

$$= (-1)^{|w_1||w_2| + |x_2||w_1| + |x_2| + |x_2||x_1|} \, w_1 \theta_i^*(w_2) \otimes [\lambda_i, -](x_1) \circ x_2$$

Whereas

[Note that the sign here from moving $[\lambda_i,-]$ past $w_2\otimes x_2$ is $\widetilde{w_2\otimes x_2}$]

$$r_2\left(\mathcal{O}_i^*\otimes[\lambda_i,-]\right)\left((w_2\otimes x_2)\otimes(w_1\otimes x_1)\right)$$

$$= r_2\left((-1)^{|w_2|+|x_2|+1}\,\mathcal{O}_i^*(w_2\otimes x_2)\otimes[\lambda_i,-](w_1\otimes x_1)\right)$$

$$= (-1)^{|w_2|+|x_2|+1+|w_1|}\,r_2\left(\mathcal{O}_i^*(w_2)\otimes x_2\otimes w_1\otimes[\lambda_i,-](x_1)\right)$$

$$= (-1)^{|w_2|+|x_2|+|w_1|+1}(-1)^{(\mathcal{O}_i^*(w_2)\otimes x_2)^\sim(w_1\otimes[\lambda_i,-](x_1))^\sim+(w_1\otimes[\lambda_i,-](x_1))^\sim+1}$$
$$\mu\left(w_1\otimes[\lambda_i,-](x_1)\otimes\mathcal{O}_i^*(w_2)\otimes x_2\right)$$

$$= (-1)^{|w_2|+|x_2|+|w_1|+1+(|w_2|+|x_2|)(|w_1|+|x_1|)+|w_1|+|x_1|+1}$$
$$\cdot(-1)^{(|x_1|+1)(|w_2|+1)}\,w_1\,\mathcal{O}_i^*(w_2)\otimes[\lambda_i,-](x_1)\circ x_2$$

$$= (-1)^{\cancel{|w_2|}+|x_2|+\cancel{|w_1|}\cancel{+1}+|w_2||w_1|+\cancel{|w_2|}|x_1|+|x_2||w_1|+|x_2||x_1|+\cancel{|w_1|}+\cancel{|x_1|}+1}$$
$$^{+|x_1|\cancel{|w_2|}+\cancel{|x_1|}+\cancel{|w_2|}\cancel{+1}}\quad w_1\mathcal{O}_i^*(w_2)\otimes[\lambda_i,-](x_1)\circ x_2$$

$$= -(-1)^{|x_2|+|w_2||w_1|+|x_2||w_1|+|x_1||x_2|}\,w_1\mathcal{O}_i^*(w_2)\otimes[\lambda_i,-](x_1)\circ x_2$$

$$= -\left\{\delta_i r_2 - r_2(1\otimes\delta_i+\delta_i\otimes 1)\right\}\left((w_2\otimes x_2)\otimes(w_1\otimes x_1)\right)$$

This proves that

$$\delta_i r_2 = r_2\left(1\otimes\delta_i+\delta_i\otimes 1-\mathcal{O}_i^*\otimes[\lambda_i,-]\right).\ \square$$

Def^n Set $\Xi_i = \Theta_i^* \otimes [\lambda_i, -]$, an $\hat{R}$-linear $\mathbb{Q}$-bilinear operator on $\mathcal{H}[1] \otimes_{\mathbb{Q}} \mathcal{H}[1]$, and write $\Xi = \sum_i \Xi_i$. (note the $\otimes$ in Ξ_i sees the tilde grading)

Lemma As operators on $\mathcal{H}[1] \otimes_{\mathbb{Q}} \mathcal{H}[1]$ we have, for $1 \leq i, j \leq n$

$$\begin{aligned} &(a) \quad [\delta_i \otimes 1, 1 \otimes \delta_j] = 0, \\ &(b) \quad [\delta_i \otimes 1, \Xi_j] = 0, \\ &(c) \quad [\Xi_j, 1 \otimes \delta_i] = \Theta_j^* \otimes [\lambda_j, \lambda_i] \Theta_i^*. \end{aligned} \tag{12.1}$$

where $[\lambda_i, \lambda_j]$ means $\lambda_i \lambda_j + \lambda_j \lambda_i$ with each λ_i as in (8.1).

Proof (a) is trivial, and (b) also since $\delta_i = \lambda_i \Theta_i^*$. For (c) we have to check

$$[1 \otimes \delta_i, \Xi_j](\omega_1 \otimes \alpha_1 \otimes \omega_2 \otimes \alpha_2)$$

⌜don't forget $\otimes$ is wrt the $\sim$ grading!⌟

$$= (-1)^{|\omega_1|+|\alpha_1|+1+|\omega_2|}(1 \otimes \delta_i)(\Theta_j^*(\omega_1) \otimes \alpha_1 \otimes \omega_2 \otimes [\lambda_j, \alpha_2])$$

$$- (-1)^{|\omega_2|+1} \Xi_j(\omega_1 \otimes \alpha_1 \otimes \Theta_i^*(\omega_2) \otimes \lambda_i \alpha_2)$$

$$= (-1)^{|\omega_1|+|\alpha_1|} \Theta_j^*(\omega_1) \otimes \alpha_1 \otimes \Theta_i^*(\omega_2) \otimes \lambda_i [\lambda_j, \alpha_2]$$

$$- (-1)^{\cancel{|\omega_2|}+\cancel{1}+|\omega_1|+|\alpha_1|+\cancel{1}+\cancel{|\omega_2|}+1} \Theta_j^*(\omega_1) \otimes \alpha_1 \otimes \Theta_i^*(\omega_2) \otimes [\lambda_j, \lambda_i \alpha_2]$$

$$= (-1)^{|\omega_1|+|\alpha_1|} \Theta_j^*(\omega_1) \otimes \alpha_1 \otimes \Theta_i^*(\omega_2) \otimes \{ \lambda_i \lambda_j \alpha_2 - (-1)^{|\alpha_2|} \lambda_i \alpha_2 \lambda_j + \lambda_j \lambda_i \alpha_2 - (-1)^{|\alpha_2|+1} \lambda_i \alpha_2 \lambda_j \}$$

$$= -(\Theta_j^* \otimes [\lambda_i, \lambda_j] \Theta_i^*)(\omega_1 \otimes \alpha_1 \otimes \omega_2 \otimes \alpha_2). \quad \square$$

<u>Lemma</u> The following diagram commutes:

$$\begin{tikzcd}
\mathcal{H}[1] \otimes_{\mathbb{Q}} \mathcal{H}[1] \arrow[r, "r_2"] \arrow[d, "\exp(-\delta)\otimes\exp(-\delta)"'] & \mathcal{H}[1] \arrow[dd, "\exp(-\delta)"] \\
\mathcal{H}[1] \otimes_{\mathbb{Q}} \mathcal{H}[1] \arrow[d, "\exp(\Xi - 1\otimes\delta)\exp(1\otimes\delta)"'] & \\
\mathcal{H}[1] \otimes_{\mathbb{Q}} \mathcal{H}[1] \arrow[r, "r_2"'] & \mathcal{H}[1]
\end{tikzcd}$$

<u>Proof</u> By (12.1) we have

$$\exp\left(-[\delta\otimes 1 + 1\otimes\delta - \Xi]\right) = \exp(\Xi - 1\otimes\delta)\circ\{\exp(-\delta)\otimes 1\}$$

Hence by p. 9

$$\begin{aligned}
\exp(-\delta)\, r_2 &= \sum_{m\geq 0} (-1)^m \frac{1}{m!}\, \delta^m r_2 \\
&= \sum_{m\geq 0} (-1)^m \frac{1}{m!}\, r_2\left(\delta\otimes 1 + 1\otimes\delta - \Xi\right)^m \\
&= r_2 \exp\left(-[\delta\otimes 1 + 1\otimes\delta - \Xi]\right) \\
&= r_2 \exp(\Xi - 1\otimes\delta)\circ\{\exp(-\delta)\otimes 1\} \\
&= r_2 \exp(\Xi - 1\otimes\delta)\exp(1\otimes\delta)\circ\{\exp(-\delta)\otimes\exp(-\delta)\}. \quad \square
\end{aligned}$$

We can calculate that

$$\exp(-1\otimes\delta+\Xi) = \sum_{n\geq 0}\frac{1}{n!}\left(-\textstyle\sum_j (1\otimes \lambda_j \theta_j^*) + \sum_i \theta_i^* \otimes [\lambda_i, -]\right)^n$$

$$= \sum_{r,s\geq 0}\ \sum_{\underline{i}=i_1<\cdots<i_r}\ \sum_{\underline{j}=j_1<\cdots<j_s} \frac{1}{(r+s)!}(-1)^{s+\binom{r}{2}+\binom{r+s}{2}} \tag{13.1.1}$$

$$\cdot \sum_{\sigma\in S_{s+r}} (-1)^{|\sigma|}\, \theta_{\underline{i}}^* \otimes \sigma_\bullet\big(\lambda_{j_1},\ldots,\lambda_{j_s},[\lambda_{i_1},-],\ldots,[\lambda_{i_r},-]\big)\,\theta_{\underline{j}}^*$$

where $\sigma_\bullet$ means we arrange the $r+s$ operators in the order indicated by σ, including an additional Koszul sign, and $\theta_{\underline{i}}^* = \theta_{i_1}^*\cdots\theta_{i_r}^*$, $\theta_{\underline{j}}^* = \theta_{j_1}^*\cdots\theta_{j_s}^*$. For example if $\sigma=(1\,2)$ then $\sigma_\bullet(\lambda_1,\lambda_2) = -\lambda_2\lambda_1$. So the overall sign of $(-1)^{|\sigma|}\sigma_\bullet$ is the sign of separately ordering $\underline{i}$, $\underline{j}$.

<u>Defn</u> Let $\mathcal{K}$ denote the total category of the A_∞-category $\mathcal{B}$ from p. 6, just as $\mathcal{H}$ is the total category of $\mathcal{A}$ (see (7.1)) so that

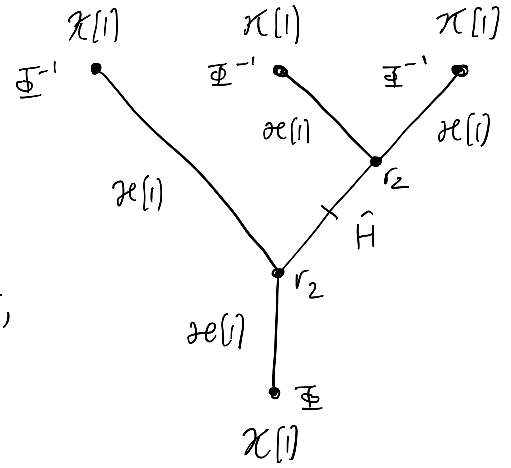
$$\mathcal{K} = \bigoplus_{Y, X \in ob(\mathcal{B})} \operatorname{Hom}_{\mathcal{B}}(Y, X). \qquad (14.1)$$

Now, returning to the definition of the operations ρ_n from (7.3), as a sum for $n \geq 2$ of ρ_T as T ranges over valid plane trees $T \in \mathcal{T}_n$ with internal vertices of valency 3 only, and

$$\rho_T = (-1)^{e_i(T)} \langle D \rangle_B$$

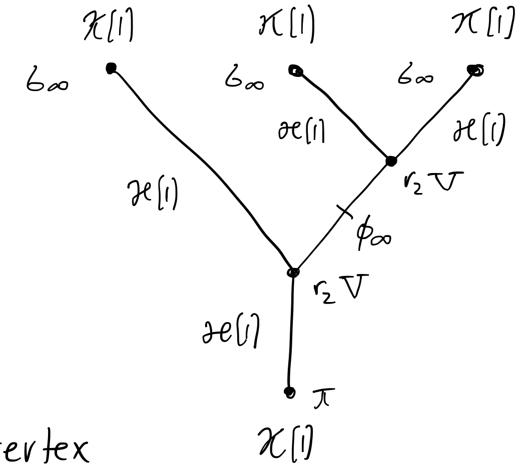
is, up to a sign, the branch denotation of the decoration D of T which places

- $\mathcal{K}[1]$ at every leaf, including the root,
- $\mathcal{H}[1]$ to every edge,
- the morphism $\Phi^{-1} : \mathcal{K}[1] \longrightarrow \mathcal{H}[1]$ to each input vertex,
- the morphism $\Phi : \mathcal{H}[1] \longrightarrow \mathcal{K}[1]$ to the root vertex,
- to vertices of $A(T)$ coming from an internal edge of T, we assign $\hat{H}$
- to internal vertices we assign r_2.



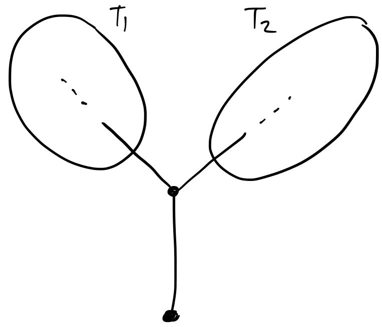
Lemma ρ_T is equal to $(-1)^{e_i(T)}$ times the branch denotation of the decoration assigning

- $\sigma_\infty : \mathcal{X}[1] \longrightarrow \mathcal{H}[1]$ to each input,
- $\pi : \mathcal{H}[1] \longrightarrow \mathcal{X}[1]$ to the root,
- ϕ_∞ to each internal edge of T,
- $\underbrace{r_2 \circ \exp(\Xi - 1\otimes\delta)\exp(1\otimes\delta)}_{\text{we denote this by } \nabla \text{ in the tree}}$ to each internal vertex



Proof Fix a valid plane tree $T \in \mathcal{T}_n$ with only internal vertices of valency 3. Then by the definition of branch denotation of (ainfcat3) p. 5, if

$T =$



Then for some trees T_1, T_2 with induced decorations D_1, D_2 we have

$$\langle D\rangle_B = \Phi \circ r_2 \circ \left(\langle D_1\rangle_B \otimes_{\mathbb{Q}} \langle D_2\rangle_B \right)$$

$$= \pi \exp(-\delta) r_2 \left(\langle D_1\rangle_B \otimes_{\mathbb{Q}} \langle D_2\rangle_B \right)$$

by (12.2) $$= \pi\, r_2 \exp(\Xi - 1\otimes\delta)\exp(1\otimes\delta) \circ \left(\exp(-\delta) \otimes_{\mathbb{Q}} \exp(-\delta) \right) \circ \left(\langle D_1\rangle_B \otimes \langle D_2\rangle_B \right)$$

$$= \pi\, r_2 \nabla \left(\exp(-\delta)\langle D_1\rangle_B \otimes \exp(-\delta)\langle D_2\rangle_B \right)$$

Now each T_i is either just a leaf, or begins with an internal edge, and in the former case

$$\exp(-\delta)\langle D_i\rangle_B = \exp(-\delta)\,\Phi^{-1} = z_\infty$$

while in the latter case

$$\begin{aligned}\exp(-\delta)\langle D_i\rangle_B &= \exp(-\delta)\,\hat{H}\,\langle D_i'\rangle_B \\ &= \exp(-\delta)\exp(\delta)\,\phi_\infty \exp(-\delta)\langle D_i'\rangle_B \\ &= \phi_\infty \exp(-\delta)\langle D_i'\rangle_B\end{aligned}$$

and so we see by induction on n that the claim holds. $\square$

<u>Def^n</u> Just to save on rewriting, set

$$\nabla := \exp(\Xi - 1\otimes\delta)\exp(1\otimes\delta).$$

See (13.1.1) for explicit formulas.

Aside on connections

In the context of Appendix B of the pushforward paper, and with the notation there, let ∇° be a connection induced by a section $\mathfrak{z}$, so that we get

$$\nabla^\circ : R \longrightarrow R \otimes_{S[\underline{t}]} \Omega^1_{S[\underline{t}]/S}$$

$$\nabla^\circ(r) = \sum_{j=1}^{n} \sum_{M \in \mathbb{N}^n} M_j \, \mathfrak{z}(r_M) \, t^{M-e_j} \otimes dt_j .$$

It follows that, with $\delta = \sum_{i=1}^n t_i (dt_i)^* \lrcorner (-)$ the Koszul operator on $R \otimes_{S[\underline{t}]} \Omega^*_{S[\underline{t}]/S}$ we have

$$\delta \nabla^\circ(r) = \sum_{j=1}^n \sum_{M \in \mathbb{N}^n} M_j \, \mathfrak{z}(r_M) t^M$$

$$= \sum_{M \in \mathbb{N}^n} |M| \cdot \mathfrak{z}(r_M) t^M$$

So that $\delta \nabla^\circ$ is the S-linear operator acting on R by sending $r \in R$ with its unique representation $\sum_M \mathfrak{z}(r_M) t^M$ to the element $\delta \nabla^\circ(r)$ with components

$$\delta \nabla^\circ(r)_M = |M| r_M, \qquad M \in \mathbb{N}^n.$$

Example Let $R = S[[x]]$ and take $t = x^d$, so that $P = R/(t)$ is the free S-module on $1, x, \ldots, x^{d-1}$ and given (for $d > 1$) (we choose $\mathfrak{z} : P \longrightarrow R$, $\mathfrak{z}(x^i) = x^i$)

$$r = \sum_{i=0}^\infty r^i x^i \qquad r^i \in S$$

$$= \sum_{M=0}^\infty \underbrace{\left(\sum_{i=0}^{d-1} r^{Md+i} x^i \right)}_{\text{call this } r_M} t^M$$

we see that this $r_M = \sum_{i=0}^{d-1} (r^{Md+i} x^i)$ gives the relevant components. Hence

$$\begin{aligned} \delta \nabla^\circ (r) &= \delta \nabla^\circ \left(\sum_M r_M t^M \right) \\ &= \sum_M M \cdot r_M t^M \end{aligned}$$

So in particular for $M \geqslant 0$ and $0 \leq i < d$

$$\delta \nabla^\circ (x^{Md+i}) = M \cdot x^{Md+i}.$$

Passing to power series in t

By Appendix B of the pushforward paper we have a $k[[\underline{t}^{Y,X}]]$-linear isomorphism for each pair $Y, X \in \mathrm{ob}(\mathcal{A})$

$$\mathcal{T}_{Y,X} : R/(\underline{t}^{Y,X}) \otimes_k k[[\underline{t}^{Y,X}]] \longrightarrow \hat{R} \qquad (19.1)$$

which is induced by the section of $\hat{R} \longrightarrow \hat{R}/(\underline{t}^{Y,X}) \cong R/(\underline{t}^{Y,X})$ associated with our chosen connection, as explained above. Already in the perturbation step we have chosen homogeneous bases of Y, X, so that we may write

$$Y = \tilde{Y} \otimes_k R, \quad X = \tilde{X} \otimes_k R \qquad (19.2)$$

for $\mathbb{Z}_2$-graded free k-modules $\tilde{X}, \tilde{Y}$, in which case

$$\mathrm{Hom}_R(Y, X) \cong \mathrm{Hom}_k(\tilde{Y}, \tilde{X}) \otimes_k R \qquad (19.3)$$

Combining (19.1) and (19.3) we have a $k[[\underline{t}^{Y,X}]]$-linear isomorphism

$$\begin{aligned} \mathrm{Hom}_R(Y,X) \otimes_R \hat{R} &\cong \mathrm{Hom}_k(\tilde{Y}, \tilde{X}) \otimes_k R \otimes_R \hat{R} \\ &\cong \mathrm{Hom}_k(\tilde{Y}, \tilde{X}) \otimes_k R/(\underline{t}^{Y,X}) \otimes_k k[[\underline{t}^{Y,X}]] \end{aligned}$$

these are formal variables

Hence finally (for $\mathcal{H}$ see (7.1), for $\mathcal{N}$ see (14.1)) we have isomorphisms of Q-bimodules

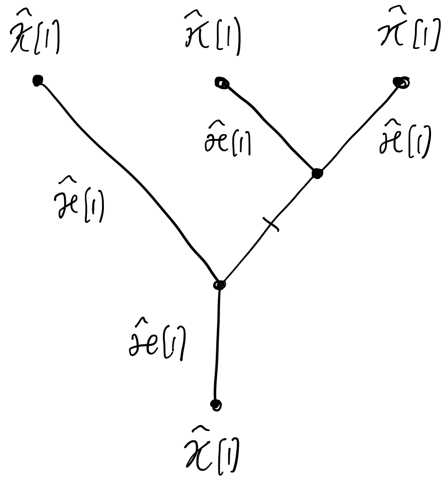
$$\mathcal{H} \xrightarrow[\cong]{\omega_{\mathcal{H}}} \bigoplus_{Y,X \in \mathrm{ob}(\mathcal{A})} \wedge F_\theta \otimes_k R/(\underline{t}^{Y,X}) \otimes_k \mathrm{Hom}_k(\tilde{Y}, \tilde{X}) \otimes_k k[[\underline{t}^{Y,X}]]$$

$$\mathcal{N} \xrightarrow[\cong]{\omega_{\mathcal{N}}} \bigoplus_{Y,X \in \mathrm{ob}(\mathcal{A})} R/(\underline{t}^{Y,X}) \otimes_k \mathrm{Hom}_k(\tilde{Y}, \tilde{X}) \qquad (19.4)$$

Denote the RHS of (19.4) by $\hat{\mathcal{H}}$ and $\hat{\pi}$ respectively. Let $n \geq 2$ and $T \in \mathcal{J}_n$ have internal vertices of valency 3 only.

<u>Def^n</u> $\hat{\rho}_T$ is equal to $(-1)^{ei(T)}$ times the branch denotation of the decoration assigning

- $w_{\mathcal{H}} \, \mathcal{G}_\infty \, w_\pi^{-1} : \hat{\pi}[1] \longrightarrow \hat{\mathcal{H}}[1]$ to each input,
- $w_\pi \, \pi \, w_{\mathcal{H}}^{-1} : \hat{\mathcal{H}}[1] \longrightarrow \hat{\pi}[1]$ to the root,
- $w_{\mathcal{H}} \, \phi_\infty \, w_{\mathcal{H}}^{-1}$ to each internal edge of T,
- $w_{\mathcal{H}} \circ r_2 \circ \nabla \circ (w_{\mathcal{H}}^{-1} \otimes w_{\mathcal{H}}^{-1})$ to each internal vertex.

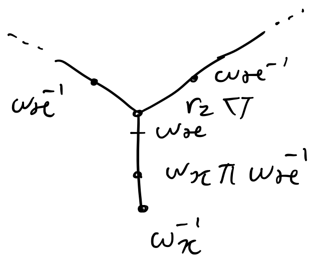


<u>Lemma</u> There is a commutative diagram

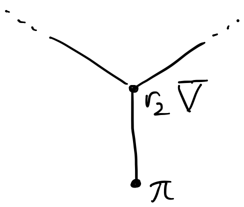
$$\begin{tikzcd}
\hat{\pi}[1]^{\otimes_{\mathbb{Q}} n} \arrow[rr, "\hat{\rho}_T"] & & \hat{\pi}[1] \\
\pi[1] \arrow[u, "\cong"', "w_\pi^{\otimes n}"] \arrow[rr, "\rho_T"'] & & \pi[1] \arrow[u, "w_\pi", "\cong"']
\end{tikzcd}$$

<u>Proof</u> We have e.g.

$$w_\pi^{-1} \circ \hat{\rho}_T =$$



$$=$$



$$= \cdots = \rho_T \circ (w_\pi^{-1})^{\otimes n} . \quad \square$$

We deduce that the isomorphism of Q-bimodules $\mathcal{H} \xrightarrow{\cong} \hat{\mathcal{H}}$ identifies the A_∞-algebra structure on $\mathcal{H}$ given by $\rho_n = \sum_T \rho_T$ with an A_∞-structure on $\hat{\mathcal{H}}[1]$ given by $\hat{\rho}_n := \sum_T \hat{\rho}_T$.

It remains now to understand the operators

- $\hat{\delta}_\infty := \omega_{\mathcal{X}} \delta_\infty \omega_{\mathcal{H}}^{-1}$
- $\hat{\pi} := \omega_{\mathcal{H}} \pi \omega_{\mathcal{X}}^{-1}$ $\qquad\qquad$ (21.1)
- $\hat{\phi}_\infty := \omega_{\mathcal{X}} \phi_\infty \omega_{\mathcal{X}}^{-1}$
- $\omega_{\mathcal{X}} r_2 \nabla (\omega_{\mathcal{X}}^{-1} \otimes \omega_{\mathcal{X}}^{-1})$

in the definition of $\hat{\rho}_T$. By (3.3.5) we have

$$\begin{aligned}\hat{\phi}_\infty &= \omega_{\mathcal{X}} \phi_\infty \omega_{\mathcal{X}}^{-1} = \sum_{m \geq 0} (-1)^m \omega_{\mathcal{X}} (H d_{Hom})^m H \omega_{\mathcal{X}}^{-1} \\ &= \sum_{m \geq 0} (-1)^m (\omega_{\mathcal{X}} H \omega_{\mathcal{X}}^{-1} \omega_{\mathcal{X}} d_{Hom} \omega_{\mathcal{X}}^{-1})^m \omega_{\mathcal{X}} H \omega_{\mathcal{X}}^{-1} \\ &= \sum_{m \geq 0} (-1)^m (\hat{H} \hat{d}_{Hom})^m \hat{H}\end{aligned} \tag{21.2}$$

where $\hat{H} := \omega_{\mathcal{X}} H \omega_{\mathcal{X}}^{-1}$, $\hat{d}_{Hom} := \omega_{\mathcal{X}} d_{Hom} \omega_{\mathcal{X}}^{-1}$. Similarly, with $\hat{\delta} = \omega_{\mathcal{X}} \delta \omega_{\mathcal{H}}^{-1}$,

$$\hat{\delta}_\infty = \sum_{m \geq 0} (-1)^m (\hat{H} \hat{d}_{Hom})^m \hat{\delta} \tag{21.3}$$

<u>Lemma</u> $\hat{\pi} : \hat{\mathcal{X}} \to \hat{\mathcal{H}}$ is given on components by the k-linear map

$$\begin{gathered}\bigwedge F_\theta \otimes_k R/(\underline{t}^{Y,X}) \otimes_k \mathrm{Hom}_k(\tilde{Y}, \tilde{X}) \otimes_k k[[\underline{t}^{Y,X}]] \\ \downarrow \\ R/(\underline{t}^{Y,X}) \otimes_k \mathrm{Hom}_k(\tilde{Y}, \tilde{X})\end{gathered} \tag{21.4}$$

induced by $\bigwedge(F_\theta \xrightarrow{0} 0) : \bigwedge F_\theta \to k$ and $k[[\underline{t}^{Y,X}]] \longrightarrow \dfrac{k[[\underline{t}^{Y,Y}]]}{(\underline{t}^{Y,X})} \cong k$.

Proof This amounts to checking commutativity of (see (19.1) for $\mathcal{J}$)

$$\begin{tikzcd} R/(\underline{t}^{Y,X}) \otimes_k k[[\underline{t}^{Y,X}]] \arrow[r, "\mathcal{J}"] \arrow[d, "\pi"] & \hat{R} \arrow[d, "\pi"] \\ R/(\underline{t}^{Y,X}) \otimes_k k \arrow[r, "can", "\cong"'] & \hat{R}/(\underline{t}^{Y,X}) \end{tikzcd} \qquad (19.1)$$

where both vertical maps are quotients by the ideals generated by $\underline{t}^{Y,X}$. But $\mathcal{J}$ is $k[[\underline{t}^{Y,X}]]$ linear, so $\pi\mathcal{J}$ vanishes on the kernel of the other π, and thus induces a horizontal map on the bottom row, which is clearly the canonical one arising from $R \longrightarrow \hat{R}$, since by constuction $\mathcal{J}$ restricted to $R/(\underline{t}^{Y,X}) \otimes_k k$ is a section of π. $\square$.

Lemma $\hat{\sigma} : \hat{\mathcal{H}} \longrightarrow \hat{\mathcal{H}}$ is given on components by the k-linear map

$$\begin{array}{c} \wedge F_\theta \otimes_k R/(\underline{t}^{Y,X}) \otimes_k \operatorname{Hom}_k(\tilde{Y}, \tilde{X}) \otimes_k k[[\underline{t}^{Y,X}]] \\ \uparrow \\ R/(\underline{t}^{Y,X}) \otimes_k \operatorname{Hom}_k(\tilde{Y}, \tilde{X}) \end{array} \qquad (22.1)$$

which is induced by the inclusions $k = k \cdot 1 \hookrightarrow \wedge F_\theta$ and $k \hookrightarrow k[[\underline{t}^{Y,X}]]$.

Proof This amounts to commutativity of

$$\begin{tikzcd} R/(\underline{t}^{Y,X}) \otimes_k k[[\underline{t}^{Y,X}]] \arrow[r, "\mathcal{J}"] & \hat{R} \\ R/(\underline{t}^{Y,X}) \otimes_k k \arrow[r, "\cong"'] \arrow[u, "1 \otimes inc"] & \hat{R}/(\underline{t}^{Y,X}) \arrow[u, "\sigma"'] \end{tikzcd}$$

which commutes by definition of $\mathcal{J}$. $\square$

<u>Lemma</u> $\hat{H} : \hat{\mathcal{H}} \longrightarrow \hat{\mathcal{H}}$ is given on components by the k-linear operator

$$\hat{H} \circlearrowright \wedge F_\theta \otimes_k R/(\underline{t}^{Y,X}) \otimes_k \operatorname{Hom}_k(\tilde{Y}, \tilde{X}) \otimes_k k[[\underline{t}^{Y,X}]] \qquad (23.1)$$

which is $\hat{H} = [\hat{d}_K, \nabla_t]^{-1} \nabla_t$ where $\hat{d}_K = \sum_{i=1}^n t_i^{Y,X} \theta_i^*$ and $\nabla_t = \sum_{i=1}^n \frac{\partial}{\partial t_i} \theta_i$.

<u>Proof</u> Since $\omega_{\mathcal{H}}$ is $k[[\underline{t}]]$-linear, $\omega_{\mathcal{H}} d_K \omega_{\mathcal{H}}^{-1} = \hat{d}_K$ as described. Moreover by definition of the connection ∇° on $\hat{R}$ the diagram

$$\begin{array}{ccc} \hat{R} & \xrightarrow{\nabla^\circ} & \hat{R} \otimes_{k[\underline{t}]} \Omega^1_{k[\underline{t}]/k} \\ J \,\wr\wr & & \wr\wr\, J \otimes 1 \\ R/(\underline{t}^{Y,X}) \otimes_k k[[\underline{t}^{Y,X}]] & \longrightarrow & R/(\underline{t}^{Y,X}) \otimes_k k[[\underline{t}^{Y,X}]] \otimes_{k[\underline{t}]} \Omega^1_{k[\underline{t}]/k} \end{array} \qquad (23.2)$$

commutes, where we choose a k-basis for $R/(\underline{t}^{Y,X})$ and use it to extend the canonical connection ∇° on $k[[\underline{t}^{Y,X}]]$. It follows that the operator ∇ on $\hat{R} \otimes_k \wedge(F_\theta)$ in the definition of H on $\mathcal{H}$ is identified under

$$\hat{R} \otimes_k \wedge F_\theta \overset{J}{\cong} R/(\underline{t}^{Y,X}) \otimes_k k[[\underline{t}^{Y,X}]] \otimes_k \wedge F_\theta$$

with the bottom row of (23.2). The operator ∇ in H is also extended using the chosen homogeneous basis for X, Y. Thus, if we let ∇_t denote the connection extended using these choices to the space in (23.1), we have $\omega_{\mathcal{H}} \nabla \omega_{\mathcal{H}}^{-1} = \nabla_t$ and hence

$$\begin{aligned} \hat{H} = \omega_{\mathcal{H}} H \omega_{\mathcal{H}}^{-1} &= \omega_{\mathcal{H}} [d_K, \nabla]^{-1} \nabla \omega_{\mathcal{H}}^{-1} \\ &= \omega_{\mathcal{H}} [d_K, \nabla]^{-1} \omega_{\mathcal{H}}^{-1} \omega_{\mathcal{H}} \nabla \omega_{\mathcal{H}}^{-1} \\ &= [\hat{d}_K, \nabla_t]^{-1} \nabla_t . \end{aligned}$$

$\square$

<u>Lemma</u> Given $\omega \in \Lambda F_\Theta$ of Θ-degree $|\omega| \in \mathbb{Z}$ and a power series $f \in k[[t^{\gamma,\gamma}]]$ written as $f = \sum_{\alpha \in \mathbb{N}^n} f_\alpha t^\alpha$ with $f_\alpha \in k$, we have (assuming $|\omega| > 0$)

$$[\hat{d}_K, \nabla_t]^{-1}(\omega \otimes f) = \omega \otimes \sum_{\alpha \in \mathbb{N}^n} \frac{1}{|\omega| + |\alpha|} f_\alpha t^\alpha \qquad (24.1)$$

<u>Proof</u> By Lemma 8.7 of the pushforward paper

$$[\hat{d}_K, \nabla_t]\Big(\omega \otimes \sum_\alpha \frac{1}{|\omega|+|\alpha|} f_\alpha t^\alpha\Big)$$

$$= \omega \otimes \Big[|\omega| + \hat{d}_K \nabla_t^0\Big]\Big(\sum_\alpha \frac{1}{|\omega|+|\alpha|} f_\alpha t^\alpha\Big)$$

$$= \omega \otimes \Big[\sum_\alpha \frac{|\omega|}{|\omega|+|\alpha|} f_\alpha t^\alpha + \hat{d}_K\Big(\sum_j \sum_\alpha \frac{1}{|\omega|+|\alpha|} f_\alpha \alpha_j t^{\alpha - e_j} \otimes \Theta_j\Big)\Big]$$

$$= \omega \otimes \sum_\alpha \frac{|\omega|+|\alpha|}{|\omega|+|\alpha|} f_\alpha t^\alpha$$

$$= \omega \otimes \sum_\alpha f_\alpha t^\alpha . \square$$

<u>Def^n</u> Let $[\lambda_i, -]^\wedge := w_{\mathcal{H}} [\lambda_i, -] w_{\mathcal{H}}^{-1}$ denote the Q-bilinear operator on $\hat{\mathcal{H}}$.

<u>Def^n</u> Set $\hat{\Xi}_i = \Theta_i^* \otimes [\lambda_i, -]^\wedge$, a k-linear Q-bilinear operator on $\hat{\mathcal{H}}[1] \otimes_Q \hat{\mathcal{H}}[1]$ and write $\hat{\Xi} = \sum_i \hat{\Xi}_i$, and $\hat{\nabla} = \exp(\hat{\Xi} - 1 \otimes \hat{\delta}) \exp(1 \otimes \hat{\delta})$.

where $\hat{\delta} = \sum_i \Theta_i^* \hat{\lambda}_i$, and $\hat{\lambda}_i$ is defined as overleaf

<u>Def^n</u> Let $\hat{r}_2 : \hat{\mathcal{H}}[1] \otimes_Q \hat{\mathcal{H}}[1] \longrightarrow \hat{\mathcal{H}}[1]$ denote the Q-bilinear operator $w_{\mathcal{H}} r_2 (w_{\mathcal{H}}^{-1} \otimes w_{\mathcal{H}}^{-1})$.

<u>Lemma</u> $w_{\mathcal{H}} r_2 \nabla (w_{\mathcal{H}}^{-1} \otimes w_{\mathcal{H}}^{-1})$ is equal to $\hat{r}_2 \hat{\nabla}$.

<u>Proof</u> Clear. $\square$

The ingredients remaining to be understood are

- $\hat{r}_2 = w_{\mathfrak{X}} r_2 (w_{\mathfrak{X}}^{-1} \otimes w_{\mathfrak{X}}^{-1})$.
- $\hat{d}_{Hom} = w_{\mathfrak{X}} d_{Hom} w_{\mathfrak{X}}^{-1}$
- $[\lambda_i, -]^\wedge = w_{\mathfrak{X}} [\lambda_i, -] w_{\mathfrak{X}}^{-1}$

These all hinge on the following considerations.

<u>Defn</u> Let $\gamma^{Y,X}$ denote the k-linear map (25.1)

$$R/(\underline{t}^{Y,X}) \otimes_k R/(\underline{t}^{Y,X}) \xrightarrow{\mathfrak{z} \otimes \mathfrak{z}} \hat{R} \otimes_k \hat{R} \xrightarrow{\text{mult.}} \hat{R} \xrightarrow[\cong]{J^{-1}} R/(\underline{t}^{Y,X}) \otimes_k k[[\underline{t}^{Y,X}]],$$

which we denote γ if it will not cause confusion. Since $R/(\underline{t}^{Y,X})$ is a finite free k-module we may identify γ as a tensor

$$\gamma \in (R/(\underline{t}^{Y,X}))^* \otimes_k (R/(\underline{t}^{Y,X}))^* \otimes_k R/(\underline{t}^{Y,X}) \otimes_k k[[\underline{t}^{Y,X}]].$$

and with respect to a k-basis $z_1, \ldots, z_\mu$ of $R/(\underline{t}^{Y,X})$ we may write

$$\gamma = \sum_{i,j,k} \sum_{\alpha \in \mathbb{N}^n} \gamma^{ij}{}_{k\alpha}\, z_i^* \otimes z_j^* \otimes z_k \otimes t^\alpha \qquad (25.2)$$

where $\gamma^{ij}{}_{k\alpha} \in k$ is the coefficient of t^α in $J^{-1}(\mathfrak{z}(z_i)\mathfrak{z}(z_j))$. Note that $\gamma^{ij}{}_{k\alpha} = \gamma^{ji}{}_{k\alpha}$, and by definition

$$\mathfrak{z}(z_i)\mathfrak{z}(z_j) = \sum_k \sum_{\alpha \in \mathbb{N}^n} \gamma^{ij}{}_{k\alpha}\, \mathfrak{z}(z_k) t^\alpha \qquad (25.3)$$

Lemma Let $r \in R$ and denote by $r^{\#}$ the $k[\![\underline{t}^{y,x}]\!]$-linear operator

$$R/(\underline{t}^{y,x}) \otimes_k k[\![\underline{t}^{y,x}]\!] \xrightarrow{J} \hat{R} \xrightarrow{r\cdot(-)} \hat{R} \xrightarrow{J^{-1}} R/(\underline{t}^{y,x}) \otimes_k k[\![\underline{t}^{y,x}]\!]. \qquad (26.1)$$

Write $r = \sum_{k=1}^{\mu} \sum_{\alpha \in \mathbb{N}^n} r_{k\alpha}\, \mathfrak{z}(z_k) t^\alpha$ for $r_{k\alpha} \in k$, then

$$r^{\#}(z_i \otimes 1) = \sum_{\ell=1}^{\mu} z_\ell \otimes \sum_{\delta \in \mathbb{N}^n} \Big[\sum_{\alpha+\beta=\delta} \sum_{k=1}^{\mu} r_{k\alpha} \gamma^{ki}_{\ell\beta} \Big] t^\delta$$

Proof This is a simple calculation

$$\begin{aligned}
J r^{\#}(z_i \otimes 1) &= r \cdot \mathfrak{z}(z_i) \\
&= \sum_{k=1}^{\mu} \sum_{\alpha \in \mathbb{N}^n} r_{k\alpha}\, \mathfrak{z}(z_k) t^\alpha \cdot \mathfrak{z}(z_i) \\
&= \sum_{k,\alpha} r_{k\alpha} \big[\mathfrak{z}(z_k)\mathfrak{z}(z_i) \big] t^\alpha \\
&= \sum_{k,\alpha,\ell,\beta} r_{k\alpha} \gamma^{ki}_{\ell\beta}\, \mathfrak{z}(z_\ell) t^{\alpha+\beta} \\
&= \sum_\delta \sum_{k,\ell} \sum_{\alpha+\beta=\delta} r_{k\alpha} \gamma^{ki}_{\ell\beta}\, \mathfrak{z}(z_\ell) t^d \\
&= \sum_\delta \sum_\ell \Big\{ \sum_k \sum_{\alpha+\beta=\delta} r_{k\alpha} \gamma^{ki}_{\ell\beta} \Big\} \mathfrak{z}(z_\ell) t^d . \qquad \square
\end{aligned}$$

Thus as a tensor in $(R/(\underline{t}^{y,x}))^* \otimes R/(\underline{t}^{y,x}) \otimes k[\![\underline{t}^{y,x}]\!]$ (ignoring the power series on the left due to linearity)

$$(r^{\#})^i_{\ell\delta} = \sum_{\alpha+\beta=\delta} \sum_{k=1}^{\mu} r_{k\alpha} \gamma^{ki}_{\ell\beta} \qquad (26.2)$$

Aside For $\delta \in \mathbb{N}^n$ we write $r^{\#}_\delta : R/(\underline{t}) \longrightarrow R/(\underline{t})$ for the k-linear map

$$r^{\#}_\delta(z_i) = \sum_{\alpha+\beta=\delta} \sum_{\ell,k=1}^{\mu} r_{k\alpha} \gamma^{ki}_{\ell\beta} \cdot z_\ell \qquad (26.3$$

so that $r^{\#}(z_i) = \sum_{\delta \in \mathbb{N}^n} r^{\#}_\delta(z_i) t^\delta$

Let $\mathcal{X}$ be an odd R-linear operator on $\mathrm{Hom}_R(Y,X)$ (e.g. d_{Hom} or $[\lambda_i,-]$). We define the $k[[\underline{t}^{Y,X}]]$-linear operator $\hat{\mathcal{X}} := \omega_{\mathscr{X}} \mathcal{X} \omega_{\mathscr{X}}^{-1}$, i.e. the composite

$$\begin{array}{c} R/(t^{Y,X}) \otimes_k \mathrm{Hom}_k(\tilde{Y},\tilde{X}) \otimes_k k[[\underline{t}^{Y,X}]] \\ \downarrow \omega_{\mathscr{X}}^{-1} \\ \mathrm{Hom}_k(\tilde{Y},\tilde{X}) \otimes_k \hat{R} \\ \downarrow \mathcal{X} \\ \mathrm{Hom}_k(\tilde{Y},\tilde{X}) \otimes_k \hat{R} \\ \downarrow \omega_{\mathscr{X}} \\ R/(t^{Y,X}) \otimes_k \mathrm{Hom}_k(\tilde{Y},\tilde{X}) \otimes_k k[[\underline{t}^{Y,X}]] \end{array} \tag{27.1}$$

Let us write $\{\alpha_u\}_{u=1}^r$ for our k-basis of $\mathrm{Hom}_k(\tilde{Y},\tilde{X})$ based on the chosen k-basis of X, Y. Then we have $\mathcal{X}(\alpha_u) = \sum_v \mathcal{X}_{vu}\alpha_v$, with $\mathcal{X}_{vu} \in R$. Then

$$\begin{aligned} \hat{\mathcal{X}}(z_i \otimes \alpha_u \otimes 1) &= \omega_{\mathscr{X}}(\mathcal{X}(\alpha_u \otimes \sigma(z_i))) \\ &= \omega_{\mathscr{X}}(\mathcal{X}(\alpha_u)\cdot \sigma(z_i)) \\ &= \textstyle\sum_v \omega_{\mathscr{X}}(\alpha_v \otimes \mathcal{X}_{vu}\sigma(z_i)) \\ &= \textstyle\sum_v \alpha_v \otimes \omega_{\mathscr{X}}(\mathcal{X}_{vu}\cdot \sigma(z_i)) \\ &= \textstyle\sum_v \alpha_v \otimes \mathcal{X}^{\#}_{vu}(z_i) \end{aligned} \tag{27.2}$$

Or, using (26.2), for $1 \le \ell \le \mu$, $\delta \in \mathbb{N}^n$,

$$\hat{\mathcal{X}}^{iu}_{v\ell\delta} = (\mathcal{X}^{\#}_{vu})^{i}_{\ell\delta} = \sum_{\alpha+\beta=\delta}\sum_{k=1}^{\mu} (\mathcal{X}_{vu})_{k\alpha}\,\gamma^{ki}_{\ell\beta} \tag{27.3}$$

So as a tensor in $(R/(\underline{t}^{Y,X}))^* \otimes \mathrm{Hom}_k(\tilde{Y},\tilde{X})^* \otimes (R/(\underline{t}^{Y,X})) \otimes \mathrm{Hom}_k(\tilde{Y},\tilde{X}) \otimes k[[\underline{t}^{Y,X}]]$,

$$\hat{\mathcal{X}} = \sum_{i,u,\ell,v,\delta} \hat{\mathcal{X}}^{iu}_{v\ell\delta}\; z_i^* \otimes \alpha_u^* \otimes z_\ell^* \otimes \alpha_v \otimes t^\delta \tag{27.4}$$

$\boxed{\hat{r}_2}$ By definition $\hat{r}_2$ is the composite

$$\hat{\mathcal{H}}[1] \otimes_Q \hat{\mathcal{H}}[1] \xrightarrow{\omega_{\mathcal{H}}^{-1} \otimes \omega_{\mathcal{H}}^{-1}} \mathcal{H}[1] \otimes_Q \mathcal{H}[1] \xrightarrow{r_2} \mathcal{H}[1] \xrightarrow{\omega_{\mathcal{H}}} \hat{\mathcal{H}}[1]$$

On components, say for $X, Y, Z \in ob(\mathcal{A})$, this is

$$\begin{array}{c}
\Big(\bigwedge F_\theta \otimes_k R/(\underline{t}^{Z,Y}) \otimes_k \operatorname{Hom}_k(\tilde{Z}, \tilde{Y}) \otimes_k k[[\underline{t}^{Z,Y}]] \Big)[1] \\
\otimes_k \\
\Big(\bigwedge F_\theta \otimes_k R/(\underline{t}^{Y,X}) \otimes_k \operatorname{Hom}_k(\tilde{Y}, \tilde{X}) \otimes_k k[[\underline{t}^{Y,X}]] \Big)[1] \\
\Big\downarrow \omega_{\mathcal{H}}^{-1} \otimes \omega_{\mathcal{H}}^{-1} \\
\big(\bigwedge F_\theta \otimes_k \operatorname{Hom}_R(Z,Y) \otimes_R \hat{R} \big)[1] \\
\otimes_k \\
\big(\bigwedge F_\theta \otimes_k \operatorname{Hom}_R(Y,X) \otimes_R \hat{R} \big)[1] \\
\Big\downarrow r_2 \\
\big(\bigwedge F_\theta \otimes_k \operatorname{Hom}_R(Z,X) \otimes_R \hat{R} \big)[1] \\
\Big\downarrow \omega_{\mathcal{H}} \\
\Big(\bigwedge F_\theta \otimes_k R/(\underline{t}^{Z,X}) \otimes_k \operatorname{Hom}_k(\tilde{Z}, \tilde{X}) \otimes_k k[[\underline{t}^{Z,X}]] \Big)[1]
\end{array} \tag{28.1}$$

Since $\underline{t}^{Y,X} = \underline{t}$ is independent of Y, X, the map (28.1) is actually $k[[\underline{t}]]$-linear, and so it can be described easily using the tensor γ from earlier, as follows. Given $\omega, \omega' \in \bigwedge F_\theta$ and $\alpha \in \operatorname{Hom}_k(\tilde{Z}, \tilde{Y})$, $\beta \in \operatorname{Hom}_k(\tilde{Y}, \tilde{X})$, and $1 \le i, j \le \mu$

$$\hat{r}_2\big([\omega \otimes z_i \otimes \alpha] \otimes [\omega' \otimes z_j \otimes \beta] \big) \tag{28.2}$$

$$= \omega_{\mathcal{H}} r_2\Big(\big[\omega \otimes \gamma(z_i) \otimes \alpha\big] \otimes \big[\omega' \otimes \gamma(z_j) \otimes \beta\big] \Big)$$

$$= W_{\mathcal{H}} (-1)^{\widetilde{\omega\otimes\alpha}\,\widetilde{\omega'\otimes\beta} + \widetilde{\omega'\otimes\beta} + 1} \left[\omega'\otimes \delta(z_j)\otimes\beta\right] \overset{\text{usual product}}{\cdot} \left[\omega\otimes\delta(z_i)\otimes\alpha\right]$$

$$= (-1)^{\widetilde{\omega\otimes\alpha}\,\widetilde{\omega'\otimes\beta} + \widetilde{\omega'\otimes\beta} + 1} (-1)^{|\omega||\beta|} W_{\mathcal{H}}\left(\omega'\omega\otimes\delta(z_i)\delta(z_j)\otimes\beta\alpha\right)$$

$$= (-1)^{(|\omega|+|\alpha|+1)(|\omega'|+|\beta|+1)+|\omega'|+|\beta|+|\omega||\beta|} \sum_k\sum_\delta \gamma^{ij}_{k\delta}\, \omega'\omega\otimes z_k\otimes\beta\alpha\otimes t^\delta \tag{29.1}$$

The sign here is

$$|\omega||\omega'| + \cancel{|\omega||\beta|} + |\omega| + |\alpha||\omega'| + |\alpha||\beta| + |\alpha| + \cancel{|\omega'|} + \cancel{|\beta|} + 1 + \cancel{|\omega'|} + \cancel{|\beta|} + \cancel{|\omega||\beta|}$$

$$= |\omega||\omega'| + |\omega| + |\alpha||\omega'| + |\alpha||\beta| + |\alpha| + 1 \tag{29.2}$$

Now

$$\widetilde{\omega'}\widetilde{\omega} + \widetilde{\omega'} + 1 + \widetilde{\alpha}\widetilde{\beta} + \widetilde{\beta} + 1 = |\omega'||\omega| + (\cancel{|\omega'|} + |\omega| + \cancel{1}) + \cancel{|\omega'|} + \cancel{1} + \cancel{1}$$
$$+ |\alpha||\beta| + |\alpha| + \cancel{|\beta|} + \cancel{1} + \cancel{|\beta|} + \cancel{1} + \cancel{1}$$

$$= (29.2) + |\alpha||\omega'| + 1$$

Let us state this as a Lemma:

Lemma We have

$$\hat{r}_2\left([\omega\otimes z_i\otimes\alpha]\otimes[\omega'\otimes z_j\otimes\beta]\right) \tag{29.3}$$
$$\|$$
$$(-1)^{|\alpha||\omega'|+1} \sum_k\sum_\delta \gamma^{ij}_{k\delta}\, r_2(\omega,\omega')\otimes z_k\otimes r_2(\alpha,\beta)\otimes t^\delta$$

where $r_2 : (\Lambda F_0)[1]\otimes(\Lambda F_0)[1] \longrightarrow (\Lambda F_0)[1]$ and similarly on Hom spaces has the expected meaning.

Formally, we have

$$
\begin{aligned}
r_2(r_2(x,y),z) &= r_2\left((-1)^{\tilde x\tilde y+\tilde y+1}yx,\ z\right)\\
&= (-1)^{\tilde x\tilde y+\tilde y+1+\widetilde{yx}\tilde z\underset{+1}{+}\tilde z+1}\, z(yx)\\
&= (-1)^{|x||y|+\cancel{|x|}+\cancel{|y|}+\cancel{1}+\cancel{|y|}+|y||z|+|y|+|x||z|+\cancel{|x|}+\cancel{|z|}+\cancel{1}+\cancel{|z|}+\cancel{1}+\cancel{1}}\, z(yx)\\
&= (-1)^{|x||y|+|y||z|+|y|+\underset{+1}{|x||z|}}\, z(yx)
\end{aligned}
$$

$$
\begin{aligned}
r_2(x,r_2(y,z)) &= (-1)^{\tilde y\tilde z+\tilde z+1}\, r_2(x,zy)\\
&= (-1)^{\tilde y\tilde z+\tilde z+1+\tilde x\widetilde{zy}+\widetilde{zy}+1}\,(zy)x\\
&= (-1)^{|y||z|+\cancel{|y|}+\cancel{|z|}+\cancel{1}+\cancel{|z|}+\cancel{1}\underset{\cancel{+1}}{+}|x||z|+|x||y|+|x|+\cancel{|z|}+\cancel{|y|}+\cancel{1}+\cancel{|z|}+|y|+\cancel{1}+\cancel{1}}\,(zy)x\\
&= (-1)^{|x||z|+|x||y|+|x|+|y|}\,(zy)x
\end{aligned}
$$

so that

$$
r_2(r_2(x,y),z) = (-1)^{|x|+|y||z|+1}\, r_2(x,r_2(y,z)). \qquad (30.1)
$$

What the hell! Who wants to deal with that.

<u>Conclusion</u> We switch to using the usual product μ_2 rather than r_2.

For this we need to revisit the algebra Q. We can put a <u>different</u> Q-bimodule structure on $\mathcal{H}$ to the one described in p.③ (ainfcat2) where $\epsilon_a \in Q$ acts on the <u>left</u> as the projector onto $\mathcal{H}_a$ (i.e. how ϵ_a used to act on the <u>right</u>) and ϵ_b acts on the <u>right</u> as the projector onto ${}_b\mathcal{H}$ (i.e. how it used to act on the <u>left</u>). When we write $\tilde\otimes_Q$ we mean $\otimes_Q$ using this bimodule structure on the tensor factors. Hence

$$
\mathcal{H}^{\tilde\otimes_Q k} = \bigoplus_{a_0,\ldots,a_k} \mathrm{Hom}_{\mathcal{A}}(a_k,a_{k-1}) \otimes \mathrm{Hom}_{\mathcal{A}}(a_{k-1},a_{k-2}) \otimes \cdots \otimes \mathrm{Hom}_{\mathcal{A}}(a_1,a_0) \qquad (30.2)
$$

Returning now to the broader point

We started with the DG-category $\mathcal{C} = mf(k[x], W)$, which we extended to the DG-category $\hat{\mathcal{C}} = \mathcal{C} \otimes_k \hat{R}$ (see p. 4.75) and then to $\mathcal{A} = \Lambda F_\theta \otimes_k \hat{\mathcal{C}}$, (see p. 5). On this we defined a strict homotopy retract (p. 6) and denoted by $\mathcal{B}$ the associated minimal model with

$$\mathcal{B}(Y,X) = R/(\underline{t}) \otimes_R \mathrm{Hom}_R(Y,X) \qquad (31.1)$$

and higher products $\{\rho_k\}_{k\geq 1}$. We write $\mathcal{H}$ for the total A_∞-algebra of $\mathcal{A}$ and $\mathcal{K}$ for the total A_∞-algebra of $\mathcal{B}$, and then in (19.4) we found isomorphic models of $\mathcal{H}, \mathcal{K}$ resp.

$$\hat{\mathcal{H}} = \bigoplus_{Y,X \in ob(\mathcal{A})} \Lambda F_\theta \otimes_k R/(\underline{t}) \otimes_k \mathrm{Hom}_k(\tilde{Y}, \tilde{X}) \otimes_k k[[\underline{t}]]$$

$$\hat{\mathcal{K}} = \bigoplus_{Y,X \in ob(\mathcal{A})} R/(\underline{t}) \otimes_k \mathrm{Hom}_k(\tilde{Y}, \tilde{X}) \qquad (31.2)$$

We defined $\hat{\rho}_T$ to be the higher product on $\hat{\mathcal{K}}$ corresponding to ρ_T on $\mathcal{K}$ under the iso between $\mathcal{K}$ and $\hat{\mathcal{K}}$, and p. 21 - 30 have been analysing the constituents of this higher product $\hat{\rho}_T$. Now, following ainfmf2 p. 14 and ainfmf9 p. 20 we would like to write the value of (for $T \in \mathcal{J}_k$)

$$\hat{\rho}_T : \hat{\mathcal{K}}[1]^{\otimes_a k} \longrightarrow \hat{\mathcal{K}}[1] \qquad (31.3)$$

on a tensor $\psi_1 \otimes \cdots \otimes \psi_k$ in terms of the evaluation of a tree involving no Koszul signs and with μ_2 (meaning ordinary multiplication in ΛF_θ and $\mathcal{C}$) replacing r_2 at all trivalent vertices.

⌜see (30.2)⌟

$\underline{\text{Def}^n}$ Let $\hat{\mu}_2$ denote the Q-bilinear map $\hat{\mathcal{H}} \tilde{\otimes}_Q \hat{\mathcal{H}} \longrightarrow \hat{\mathcal{H}}$ given by

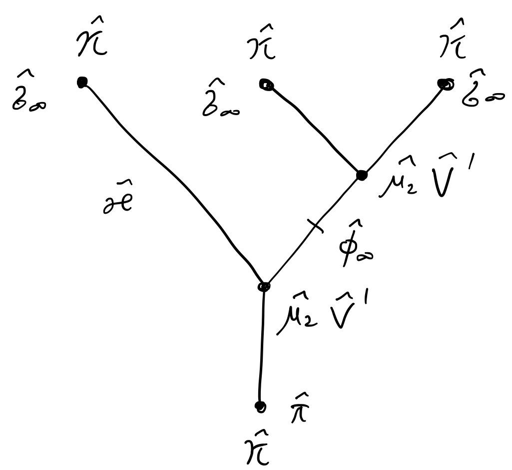
$$\hat{\mathcal{H}} \tilde{\otimes}_Q \hat{\mathcal{H}} \xrightarrow{w_{\mathcal{H}}^{-1} \otimes w_{\mathcal{H}}^{-1}} \mathcal{H} \tilde{\otimes}_Q \mathcal{H} \xrightarrow{\mu_2} \mathcal{H} \xrightarrow{w_{\mathcal{H}}} \hat{\mathcal{H}} \qquad (32.1)$$

where μ_2 is the product. By p. 29 we have, for $\omega, \omega' \in \wedge F_0$ and $1 \le i, j \le \mu$ and α, β composable morphisms,

$$\hat{\mu}_2\big([\omega \otimes z_i \otimes \alpha] \otimes [\omega' \otimes z_j \otimes \beta] \big)$$
$$\|$$
$$(-1)^{|\alpha||\omega'|} \sum_{k=1}^{\mu} \sum_{\delta} \gamma_{k\delta}^{ij} \; \omega \wedge \omega' \otimes z_k \otimes \alpha\beta \otimes t^{\delta} \qquad (32.2)$$

Let $k \geq 2$ and $T \in \mathcal{J}_k$ have only internal vertices of valency 3. In the following we give $\hat{\mathcal{H}}$ and $\hat{\mathcal{K}}$ the modified Q-bimodule structure of (30.2), when interpreting diagrams.

$\underline{\text{Def}^n}$ Consider the following decoration of $A(T)$ by Q-bimodules



- $\hat{\mathcal{H}}$ to each internal edge of T and $\hat{\mathcal{K}}$ to each leaf of T (with the (30.2) bimodule structure)
- $\hat{\zeta}_\infty : \hat{\mathcal{K}} \longrightarrow \hat{\mathcal{H}}$ to each input,
- $\hat{\pi} : \hat{\mathcal{H}} \longrightarrow \hat{\mathcal{K}}$ to the root,
- $\hat{\phi}_\infty : \hat{\mathcal{H}} \longrightarrow \hat{\mathcal{H}}$ to each internal edge of T,
- $\hat{\mu}_2 \hat{V}'$ to each interval vertex, where $\hat{V}' := \exp(\hat{\Xi}' - \hat{\delta} \otimes 1)\exp(\hat{\delta} \otimes 1)$

Here we take

Def^n Set $\hat{\Xi}_i' = [\lambda_i, -]^{\wedge} \otimes O_i^*$, a k-linear Q-bilinear operator on $\hat{\mathcal{H}} \tilde{\otimes}_Q \hat{\mathcal{H}}$ and write $\hat{\Xi}' = \sum_i \hat{\Xi}_i'$.

Def^n Set $\hat{\delta}_i' = \hat{\lambda}_i O_i^* \otimes 1$ and write $\hat{\delta}' = \sum_i \hat{\delta}_i'$, operator on $\hat{\mathcal{H}} \tilde{\otimes}_Q \hat{\mathcal{H}}$.

Def^n Let eval_T denote the Q-bilinear map $\hat{\mathcal{K}}^{\tilde{\otimes}_Q k} \longrightarrow \hat{\mathcal{K}}$ obtained by evaluating the above decorated tree, without Koszul signs. That is, we simply feed in inputs at the top of the tree and evaluate each operator in succession. More precisely, we take the branch denotation in the category of ungraded Q-bimodules.

Let $\hat{T}$ be the tree obtained by mirroring T, as in p.14 ainfmf2.

Proposition Given $x_1, \dots, x_k \in \hat{\mathcal{K}}$ and a tree T as above, (33.1)

$$\hat{\rho}_T(x_1, \dots, x_k) = (-1)^{e_i(T) + \sum_{i<j} \tilde{x}_i \tilde{x}_j + \sum_i \tilde{x}_i p_i + k + 1} \mathrm{eval}_{\hat{T}}(x_k, \dots, x_1)$$

where p_i is the parity of the ith leaf (see p.15 ainfmf2).

Proof $\hat{\rho}_T$ is the branch denotation of the decoration with $\hat{\mathcal{H}}[1], \hat{\mathcal{K}}[1], \hat{\sigma}_\infty, \hat{\phi}_\infty, \hat{\pi}$ and $\hat{r}_2 \hat{V}$, times $(-1)^{e_i(T)}$. By definition

$$\begin{aligned} \hat{r}_2(x_1, x_2) &= (-1)^{\tilde{x}_1 \tilde{x}_2 + \tilde{x}_2 + 1} \hat{\mu}_2(x_2, x_1), \\ &= \hat{\mu}_2 W(x_1, x_2) \end{aligned} \tag{33.3}$$

where $W(x_1, x_2) = (-1)^{\tilde{x}_1 \tilde{x}_2 + \tilde{x}_2 + 1} x_2 \otimes x_1$, so $\hat{r}_2 = \hat{\mu}_2 W$.

$$
\begin{aligned}
W\hat{\Xi}(x_1,x_2) &= \sum_i W\left(\mathcal{O}_i^* \otimes [\lambda_i,-]^{\wedge}\right)(x_1,x_2) \\
&= \sum_i (-1)^{\tilde{x}_1} W\left(\mathcal{O}_i^*(x_1) \otimes [\lambda_i,-]^{\wedge}(x_2)\right) \\
&= \sum_i (-1)^{\tilde{x}_1 + (\tilde{x}_1+1)(\tilde{x}_2+1) + \tilde{x}_2+1+1} [\lambda_i,-]^{\wedge}(x_2) \otimes \mathcal{O}_i^*(x_1) \\
&= \sum_i (-1)^{\tilde{x}_1\tilde{x}_2+1} [\lambda_i,-]^{\wedge}(x_2) \otimes \mathcal{O}_i^*(x_1) \\
&= \sum_i (-1)^{\tilde{x}_1\tilde{x}_2+\tilde{x}_2+1} \left([\lambda_i,-]^{\wedge} \otimes \mathcal{O}_i^*\right)(x_2 \otimes x_1) \\
&= \hat{\Xi}' W(x_1,x_2).
\end{aligned}
$$

and

$$
\begin{aligned}
W(1\otimes\delta)(x_1,x_2) &= W\left(x_1 \otimes \delta(x_2)\right) \\
&= (-1)^{\tilde{x}_1\tilde{x}_2+\tilde{x}_2+1} \delta(x_2)\otimes x_1 \\
&= (\delta\otimes 1) W(x_1,x_2)
\end{aligned}
$$

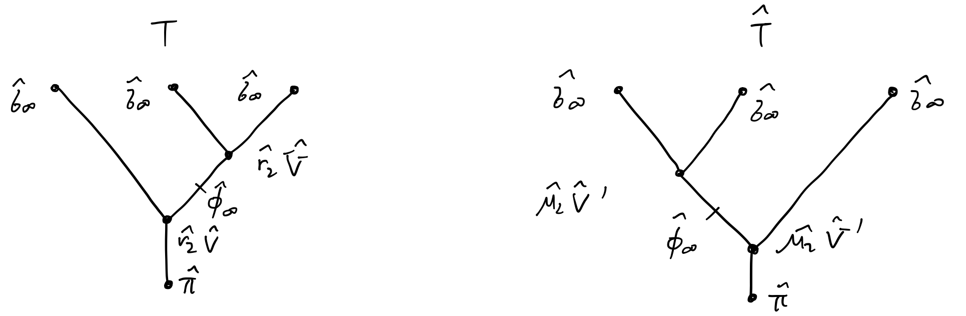
Hence

$$
\begin{aligned}
\hat{r}_2\hat{V} &= \hat{r}_2 \exp(\hat{\Xi} - 1\otimes\hat{\delta})\exp(1\otimes\hat{\delta}) \\
&= \hat{\mu}_2 W \exp(\hat{\Xi} - 1\otimes\hat{\delta})\exp(1\otimes\hat{\delta}) \\
&= \hat{\mu}_2 \exp(\hat{\Xi}' - \hat{\delta}\otimes 1)\exp(\hat{\delta}\otimes 1) W
\end{aligned}
$$

Let us write (this is terrible)

$$\hat{V}' := \exp(\hat{\Xi}' - \hat{\delta} \otimes 1) \exp(\hat{\delta} \otimes 1)$$

Let us now consider an example:



(35.2)

We have (using that with respect to the $\sim$-degree, $\hat{r}_2, \hat{\phi}_\infty$ are odd, $\hat{\Xi}, \hat{\mathcal{E}}_\infty, \hat{\pi}$ are even,

$$\hat{\rho}_T(x_1, x_2, x_3) = \hat{\pi}\hat{r}_2\hat{V}\Big(\underbrace{\hat{\mathcal{E}}_\infty(x_1)}_{\text{tilde degree } \tilde{x}_1} \otimes \underbrace{\hat{\phi}_\infty \hat{r}_2 \hat{V}\big(\hat{\mathcal{E}}_\infty(x_2) \otimes \hat{\mathcal{E}}_\infty(x_3)\big)}_{\text{tilde degree } \tilde{x}_2 + \tilde{x}_3}\Big)$$

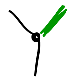
$$= \hat{\pi}\hat{\mu}_2\hat{V}'\Big(\hat{\phi}_\infty \hat{r}_2 \hat{V}\big(\hat{\mathcal{E}}_\infty(x_2) \otimes \hat{\mathcal{E}}_\infty(x_3)\big) \otimes \hat{\mathcal{E}}_\infty(x_1)\Big) \cdot (-1)^{\tilde{x}_1(\tilde{x}_2 + \tilde{x}_3) + \tilde{x}_2 + \tilde{x}_3 + 1}$$

$$= (-1)^{\tilde{x}_1(\tilde{x}_2 + \tilde{x}_3) + \tilde{x}_2 + \tilde{x}_3 + 1} \cdot (-1)^{\tilde{x}_2\tilde{x}_3 + \tilde{x}_3 + 1} \hat{\pi}\hat{\mu}_2\hat{V}'\Big(\hat{\phi}_\infty \hat{\mu}_2\hat{V}'\big(\hat{\mathcal{E}}_\infty(x_3) \otimes \hat{\mathcal{E}}_\infty(x_2)\big) \otimes \hat{\mathcal{E}}_\infty(x_1)\Big)$$

$$= (-1)^{\tilde{x}_1(\tilde{x}_2 + \tilde{x}_3) + \tilde{x}_2 + \tilde{x}_3 + 1} \cdot (-1)^{\tilde{x}_2\tilde{x}_3 + \tilde{x}_3 + 1} \operatorname{eval}_{\hat{T}}(x_3, x_2, x_1)$$

The upshot is that the signs arise entirely from the W, and the tilde degree arriving at an input to a $\hat{r}_2 \hat{V}$ is the sum of all $\tilde{x}_j$ feeding into that input. We deduce easily from this that in general

$$\hat{\rho}_T(x_1,\dots,x_k) = (-1)^{\sum_{i<j}\tilde{x}_i\tilde{x}_j + \sum_{i=1}^k \tilde{x}_i p_i + k+1+e_i(T)} \, \mathrm{eval}_{\hat{T}}(x_k,\dots,x_1)$$

where p_i is the number of times the path from the ith leaf in T (counting from the left) enters a trivalent vertex as the right-hand branch (i.e. ) on its way to the root, and $k+1$ is the number of internal vertices in T. $\square$

The next task is to describe the Feynman rules governing the expansion of eval_T, but we leave this for ainfmf29.

<u>Note</u> (14/1/2019) We corrected a serious error stemming from a mistaken calculation of $[\Xi_i, 1\otimes\delta_j]$ (see p.12), which we used to think was zero.

Transfer of Clifford operators

We take as our starting point the homotopy equivalence of (5.2) over k

$$\left(\bigwedge F_{\theta} \otimes_k \mathrm{Hom}_R(Y,X) \otimes_R \hat{R}, \; d_{\mathrm{Hom}} \right) \underset{\Phi^{-1}}{\overset{\Phi}{\rightleftarrows}} \left(R/(\underline{t}^{Y,X}) \otimes_R \mathrm{Hom}_R(Y,X), \; \overline{d_{\mathrm{Hom}}} \right)$$

which is derived from (4.1). Recall $\Phi = \pi \circ \exp(-\delta)$, $\Phi^{-1} = \exp(\delta) \circ \sigma_\infty$. Set $I = (\underline{t}^{Y,X})$ and write $\underline{t}^{Y,X} = (t_1, \ldots, t_n)$. In the cut operation paper [Cut] we studied the transferred operators $\Phi \mathcal{O}_i^* \Phi^{-1}$ and $\Phi \mathcal{O}_i \Phi^{-1}$ and here we recapitulate that discussion. This is necessary because in some relevant parts of [Cut] it is assumed $t_i = \partial_{x_i} W$ and we are not imposing such a hypothesis here. It is worth inserting a copy of (4.1) here (making the identification $R/I \cong \hat{R}/I\hat{R}$):

$$\begin{array}{ccc}
\mathcal{O}_i, \mathcal{O}_i^* & \circlearrowright & \left(\bigwedge F_\theta \otimes_k \mathrm{Hom}_R(Y,X) \otimes_R \hat{R}, \; d_{\mathrm{Hom}} \right) \\
 & & \exp(-\delta) \Big\downarrow \; \Big\uparrow \exp(\delta) \\
\begin{array}{l} \exp(-\delta)\mathcal{O}_i \exp(\delta) \\ \exp(-\delta)\mathcal{O}_i^* \exp(\delta) \end{array} & \circlearrowright & \left(K \otimes_R \mathrm{Hom}_R(Y,X) \otimes_R \hat{R}, \; d_K + d_{\mathrm{Hom}} \right) \\
 & & \pi \Big\downarrow \; \Big\uparrow \sigma_\infty \\
\begin{array}{l} \Phi \mathcal{O}_i \Phi^{-1} = \pi \exp(-\delta) \mathcal{O}_i \exp(\delta) \sigma_\infty \\ \Phi \mathcal{O}_i^* \Phi^{-1} = \pi \exp(-\delta) \mathcal{O}_i^* \exp(\delta) \sigma_\infty \end{array} & \circlearrowright & \left(R/I \otimes_R \mathrm{Hom}_R(Y,X), \; \overline{d_{\mathrm{Hom}}} \right)
\end{array} \tag{37.1}$$

Now [Cut, §4.2] handles the transfer from the first line to the second, and [Cut, p.38] (ultimately the pushforward paper) handles the transfer from the second to the third.

Now [Cut, Lemma 4.17] gives $\exp(-\delta)\mathcal{O}_i^* \exp(\delta) = \mathcal{O}_i^*$ and [Cut, Theorem 4.28] says

$$\begin{aligned} \exp(-\delta)\,\mathcal{O}_i \exp(\delta) &= \mathcal{O}_i - \sum_{p \geq 0} \sum_{i_1,\dots,i_p} \frac{1}{(p+1)!} [\lambda_{i_p}, [\lambda_{i_{p-1}}, \dots [\lambda_{i_1}, \lambda_i] \cdots] \\ &\qquad \circ\, \mathcal{O}_{i_1}^* \cdots \mathcal{O}_{i_p}^* \\ &= \mathcal{O}_i - \lambda_i - \tfrac{1}{2}\sum_j [\lambda_j, \lambda_i]\mathcal{O}_j^* + \cdots \end{aligned} \tag{38.1}$$

Next we copy elements of the proof of [Cut, Prop 4.35]. This is the first time [Cut] is assuming $t_i = \partial_{x_i} W$ but we are not, so we must now take care. The first thing to note is that [Cut, Lemma 4.36] still holds, that is, there is a k-linear homotopy

$$\pi\, \mathcal{O}_{j_1}^* \cdots \mathcal{O}_{j_p}^*\, \sigma_\infty \simeq At_{j_1} \cdots At_{j_p} \tag{38.2}$$

in reference to the second and third rows of (37.1). Here as usual At_j means

$$At_j = [d_{Hom}, \partial_{t_j}] \circlearrowright R/I \otimes_R \operatorname{Hom}_R(Y, X)$$

as defined in [Pushforward, §9] with respect to $k[\underline{t}] \longrightarrow R$. The proof of (38.2) in the current generality is the same as in [Cut], as indeed there we cite the pushforward paper which is working in the present generality. That is, writing $\mathcal{J}$ for $[d_K, \nabla]$,

$$\begin{aligned} \sigma_\infty &= \sum_{m \geq 0} (-1)^m (H\, d_{Hom})^m \sigma \\ &= \sum_{m \geq 0} (-1)^m (\mathcal{J}^{-1} \nabla d_{Hom}) \cdots (\mathcal{J}^{-1} \nabla d_{Hom}) \sigma \end{aligned}$$

Now $\nabla \mathcal{J} = \mathcal{J} \nabla$ and so on inputs of nonzero $\mathcal{O}$-degree, $\nabla \mathcal{J}^{-1} = \mathcal{J}^{-1}\nabla$, hence

$$\mathcal{Z}_\infty = \sum_{m\geq 0} (-1)^m J^{-1}[\nabla, d_{Hom}] J^{-1}[\nabla, d_{Hom}] \cdots J^{-1}[\nabla, d_{Hom}]\mathcal{Z}$$

$$= \sum_{m\geq 0} (-1)^m \{ J^{-1}[\nabla, d_{Hom}] \}^m \mathcal{Z}. \tag{39.1}$$

(see also p. 9 of ainfmf29)

By the same calculation, incidentally,

$$\phi_\infty = \sum_{m\geq 0} (-1)^m (H d_{Hom})^m H \tag{39.2}$$

$$= \sum_{m\geq 0} (-1)^m \overbrace{J^{-1}\nabla d_{Hom} \cdots J^{-1}\nabla d_{Hom}}^{m \text{ copies}} J^{-1}\nabla$$

$$= \sum_{m\geq 0} (-1)^m J^{-1}[\nabla, d_{Hom}] \cdots J^{-1}[\nabla, d_{Hom}] J^{-1}\nabla$$

$$= \sum_{m\geq 0} (-1)^m \{ J^{-1}[\nabla, d_{Hom}] \}^m J^{-1}\nabla$$

(see also p. 11 of ainfmf29)

Note that as an operator on $\wedge F_\theta \otimes R/I \otimes k[\underline{t}]$, we have modulo $(\underline{t})$ (see (24.1))

$$J^{-1}(\omega \otimes z \otimes f) \equiv \frac{1}{|\omega|}\, \omega \otimes z \otimes f.$$

Since $[\nabla, d_{Hom}]$ is $k[\underline{t}]$-linear it follows that

$$\pi\, \theta_{j_1}^* \cdots \theta_{j_p}^* \mathcal{Z}_\infty = \sum_{m\geq 0} (-1)^m \pi \theta_{j_1}^* \cdots \theta_{j_p}^* \{ J^{-1}[\nabla, d_{Hom}]\}^m \mathcal{Z}$$

$$= \sum_{m\geq 0} (-1)^m \frac{1}{m!} \pi\, \theta_{j_1}^* \cdots \theta_{j_p}^* \{ \textstyle\sum_{k=1}^n \theta_k [\partial_{t_k}, d_{Hom}] \}^m \mathcal{Z}$$

$$= \sum_{m\geq 0} (-1)^m \frac{1}{m!} \pi\, \theta_{j_1}^* \cdots \theta_{j_p}^* \{ \textstyle\sum_{k=1}^n \theta_k (-At_k) \}^m \mathcal{Z}$$

$$= (-1)^{p+p} \frac{1}{p!} \pi\, \theta_{j_1}^* \cdots \theta_{j_p}^* \{ \sum_{\delta\in S_p} [\theta_{j_{\delta 1}} At_{j_{\delta 1}}] \cdots [\theta_{j_{\delta p}} At_{j_{\delta p}}] \} \mathcal{Z}$$

$$= (-1)^{p+p} \frac{1}{p!} \pi \, \theta_{j_1}^* \cdots \theta_{j_p}^* \Big\{ \sum_{\delta \in S_p} [\theta_{j_{\delta 1}}, At_{j_{\delta 1}}] \cdots [\theta_{j_{\delta p}}, At_{j_{\delta p}}] \Big\} 6$$

$$= \frac{1}{p!} \pi \, \theta_{j_1}^* \cdots \theta_{j_p}^* \Big\{ \sum_{\delta \in S_p} (-1)^{\binom{p}{2}} \theta_{j_{\delta 1}} \cdots \theta_{j_{\delta p}} At_{j_{\delta 1}} \cdots At_{j_{\delta p}} \Big\}$$

$$= \frac{1}{p!} \pi \, \theta_{j_1}^* \cdots \theta_{j_p}^* \Big\{ \sum_{\delta \in S_p} (-1)^{|\delta|} \theta_{j_p} \cdots \theta_{j_1} At_{j_{\delta 1}} \cdots At_{j_{\delta p}} \Big\}$$

$$= \frac{1}{p!} \sum_{\delta \in S_p} (-1)^{|\delta|} At_{j_{\delta 1}} \cdots At_{j_{\delta p}}$$

So far this has been equality of operators on $(R/I \otimes_R \mathrm{Hom}_R(Y, X), d_{\mathrm{Hom}})$, but now we can apply the fact that the Atiyah classes anti-commute up to homotopy (see Theorem 3.11 of [Cut], or rather its proof, where notice we are using that $[\partial_{t_j}, At_i]$ is $k[\underline{t}]$-linear so that it passes to the quotient and serves as a k-linear homotopy there) to see that there is a homotopy (38.2). So finally

<u>Lemma</u> We have for $1 \le i \le n$

$$\Phi \, \theta_i^* \, \Phi^{-1} = \pi \theta_i^* 6_\infty = At_i$$

($\simeq$ meaning k-linear htpy)

$$\Phi \, \theta_i \, \Phi^{-1} \simeq -\lambda_i - \sum_{p \ge 1} \sum_{i_1, \ldots, i_p} \frac{1}{(p+1)!} [\lambda_{i_p}, [\lambda_{i_{p-1}}, \ldots [\lambda_{i_1}, \lambda_i] \cdots] At_{i_1} \cdots At_{i_p}$$

<u>Proof</u> Applying (38.2) to (38.1) we have

$$\Phi \, \theta_i \, \Phi^{-1} = \underbrace{\pi \, \theta_i 6_\infty}_{\text{zero}}$$

$$- \sum_{p \ge 0} \sum_{i_1, \ldots, i_p} \frac{1}{(p+1)!} [\lambda_{i_p}, [\lambda_{i_{p-1}}, \ldots [\lambda_{i_1}, \lambda_i] \cdots] \circ \pi \, \theta_{i_1}^* \cdots \theta_{i_p}^* 6_\infty$$

$$= -\sum_{p\geq 0}\sum_{i_1,\dots,i_p}\frac{1}{(p+1)!}\left[\lambda_{i_p},\left[\lambda_{i_{p-1}},\dots\left[\lambda_{i_1},\lambda_i\right]\cdots\right]\right.$$

$$\circ\ \frac{1}{p!}\sum_{\sigma\in S_p}(-1)^{|\sigma|}At_{i_{\sigma 1}}\cdots At_{i_{\sigma p}}$$

$$\simeq -\sum_{p\geq 0}\sum_{i_1,\dots,i_p}\frac{1}{(p+1)!}\left[\lambda_{i_p},\left[\lambda_{i_{p-1}},\dots\left[\lambda_{i_1},\lambda_i\right]\cdots\right]At_{i_1}\cdots At_{i_p}\right.$$

$$= -\lambda_i-\sum_{p\geq 1}\sum_{i_1,\dots,i_p}\frac{1}{(p+1)!}\left[\lambda_{i_p},\left[\lambda_{i_{p-1}},\dots\left[\lambda_{i_1},\lambda_i\right]\cdots\right]At_{i_1}\cdots At_{i_p}\right. .\ \square$$

Appendix A : alternative homotopy

Note The $\delta_i = \lambda_i \mathcal{O}_i^* = \sum_{Y,X} \lambda_i^{Y,X} \mathcal{O}_i^*$ are defined using the conventional homotopy

$$\lambda_i^{Y,X}(\alpha) = \lambda_i^X \circ \alpha .$$

Suppose instead we use $\tilde{\delta}_i = \tilde{\lambda}_i \mathcal{O}_i^* = \sum_{Y,X} \tilde{\lambda}_i^{Y,X} \mathcal{O}_i^*$ where

$$\tilde{\lambda}_i^{Y,X}(\alpha) = (-1)^{|\alpha|} \alpha \circ \lambda_i^Y .$$

Then the above calculation yields

$$r_2(\tilde{\delta}_i \otimes 1 + 1 \otimes \tilde{\delta}_i)((w_2 \otimes x_2) \otimes (w_1 \otimes x_1))$$

$$= (-1)^{|w_1||w_2| + |x_2||w_1| + |x_2| + |x_2||x_1| + |w_1|} \Big\{ (-1)^{|w_1| + |x_1|} w_1 \mathcal{O}_1^*(w_2) \otimes x_1 \circ \tilde{\lambda}_i(x_2) + \mathcal{O}_1^*(w_1) w_2 \otimes \tilde{\lambda}_i(x_1) \circ x_2 \Big\}$$

$$\tilde{\delta}_i \; r_2 \; ((w_2 \otimes x_2) \otimes (w_1 \otimes x_1))$$

$$= (-1)^{|w_2||w_1| + |x_2||w_1| + |x_2||x_1| + |x_2| + |w_1|} \mathcal{O}_i^*(w_1 w_2) \otimes \tilde{\lambda}_i(x_1 \circ x_2)$$

$$\Big\{ \tilde{\delta}_i r_2 - r_2(1 \otimes \tilde{\delta}_i + \tilde{\delta}_i \otimes 1) \Big\} \big((w_2 \otimes x_2) \otimes (w_1 \otimes x_1) \big)$$

$$= (-1)^{|w_1||w_2| + |x_2||w_1| + |x_2| + |x_2||x_1| + |w_1|} \Big\{ \mathcal{O}_i^*(w_1 w_2) \otimes \tilde{\lambda}_i(x_1 \circ x_2) - (-1)^{|w_1| + |x_1|} w_1 \mathcal{O}_1^*(w_2) \otimes x_1 \circ \tilde{\lambda}_i(x_2) - \mathcal{O}_1^*(w_1) w_2 \otimes \tilde{\lambda}_i(x_1) \circ x_2 \Big\}$$

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a.2

$$\theta_i^*(w_1 w_2) \otimes \tilde{\lambda}_i(x_1 \circ x_2) - (-1)^{|w_1|+|x_1|} w_1 \theta_i^*(w_2) \otimes x_1 \circ \tilde{\lambda}_i(x_2)$$
$$- \theta_i^*(w_1) w_2 \otimes \tilde{\lambda}_i(x_1) \circ x_2$$

$$= \theta_i^*(w_1) w_2 \otimes (-1)^{|x_1|+|x_2|} x_1 \circ x_2 \circ \lambda_i - (-1)^{|w_1|+|x_1|} w_1 \cancel{\theta_i^*(w_2)} \otimes (-1)^{|x_2|} x_1 x_2 \lambda_i$$
$$+ (-1)^{|w_1|} w_1 \cancel{\theta_i^*(w_2) \otimes} (-1)^{|x_1|+|x_2|} x_1 \circ x_2 \circ \lambda_i - (-1)^{|x_1|} \theta_i^*(w_1) w_2 \otimes x_1 \lambda_i x_2$$

$$= -(-1)^{|x_1|} \theta_i^*(w_1) w_2 \otimes \{ x_1 \circ \lambda_i \circ x_2 - (-1)^{|x_2|} x_1 \circ x_2 \circ \lambda_i \}$$

$$= - \theta_i^*(w_1) w_2 \otimes (-1)^{|x_1|} x_1 \circ [\lambda_i, -](x_2)$$

Hence

$$\{ \tilde{\delta}_i r_2 - r_2 (1 \otimes \tilde{\delta}_i + \tilde{\delta}_i \otimes 1) \} ((w_2 \otimes x_2) \otimes (w_1 \otimes x_1))$$

$$= (-1)^{|w_1||w_2| + |x_2||w_1| + |x_2| + |x_2||x_1| + |w_1| + 1} \theta_i^*(w_1) w_2 \otimes (-1)^{|x_1|} x_1 \circ [\lambda_i, -](x_2)$$

whereas

$$r_2([\lambda_i, -] \otimes \theta_i^*)((w_2 \otimes x_2) \otimes (w_1 \otimes x_1))$$

$$= (-1)^{|w_2|+|x_2|+1} r_2([\lambda_i, -](w_2 \otimes x_2) \otimes \theta_i^*(w_1 \otimes x_1))$$

$$= (-1)^{|w_2|+|x_2|+1} r_2((-1)^{|w_2|} w_2 \otimes \{ \lambda_i x_2 - (-1)^{|x_2|} x_2 \lambda_i \} \otimes \theta_i^*(w_1) \otimes x_1)$$

$$= (-1)^{|x_2|+1} r_2(w_2 \otimes \lambda_i x_2 \otimes \theta_i^*(w_1) \otimes x_1)$$
$$+ r_2(w_2 \otimes x_2 \lambda_i \otimes \theta_i^*(w_1) \otimes x_1)$$

$$= (-1)^{|x_2|+1+(|w_2|+|x_2|+1+1)(|w_1|+1+|x_1|+1)+(|w_1|+1+|x_1|+1)+1} \mu\left(\mathcal{O}_i^*(w_1)\otimes x_1 \otimes w_2 \otimes \lambda_i x_2\right)$$

$$+ (-1)^{(|w_2|+|x_2|+1+1)(1+|w_1|+|x_1|+1)+(1+|w_1|+|x_1|+1)+1} \mu\left(\mathcal{O}_i^*(w_1)\otimes x_1 \otimes w_2 \otimes x_2 \lambda_i\right)$$

$$= (-1)^{|x_2|+1+(|w_2|+|x_2|)(|w_1|+|x_1|)+|w_1|+|x_1|+1+|x_1||w_2|} \mathcal{O}_i^*(w_1) w_2 \otimes x_1 \lambda_i x_2$$

$$+ (-1)^{(|w_2|+|x_2|)(|w_1|+|x_1|)+|w_1|+|x_1|+|x_1||w_2|+1} \mathcal{O}_i^*(w_1) w_2 \otimes x_1 x_2 \lambda_i$$

$$= (-1)^{|x_2|+\cancel{1}+|w_2||w_1|+\cancel{|w_2||x_1|}+|x_2||w_1|+|x_2||x_1|+|w_1|+|x_1|+\cancel{1}+\cancel{|x_1||w_2|}} \mathcal{O}_i^*(w_1) w_2 \otimes x_1 \lambda_i x_2$$
$$+ (-1)^{|w_2||w_1|+\cancel{|w_2||x_1|}+|x_2||w_1|+|x_2||x_1|+|w_1|+|x_1|+\cancel{|x_1||w_2|}+1} \mathcal{O}_i^*(w_1) w_2 \otimes x_1 x_2 \lambda_i$$

$$= (-1)^{|w_2||w_1|+|x_2||w_1|+|x_2||x_1|+|x_2|+|w_1|+|x_1|} \mathcal{O}_i^*(w_1) w_2 \otimes \left\{ x_1 \lambda_i x_2 - (-1)^{|x_2|} x_1 x_2 \lambda_i \right\}$$

$$= (-1)^{|w_2||w_1|+|x_2||w_1|+|x_2||x_1|+|x_2|+|w_1|} \mathcal{O}_i^*(w_1) w_2 \otimes (-1)^{|x_1|} x_1 \circ [\lambda_i, -](x_2).$$

This proves $\tilde{\delta}_i r_2 - r_2(1\otimes\tilde{\delta}_i + \tilde{\delta}_i \otimes 1) = - r_2([\lambda_i,-]\otimes \mathcal{O}_i^*)$ or written differently,

<u>Lemma</u> The following diagram commutes

$$\begin{tikzcd}
\mathcal{H}[1]\otimes_Q \mathcal{H}[1] \arrow[r, "r_2"] \arrow[d, "\tilde{\delta}_i\otimes 1 + 1\otimes\tilde{\delta}_i - [\lambda_i,-]\otimes \mathcal{O}_i^*"'] & \mathcal{H}[1] \arrow[d, "\tilde{\delta}_i"] \\
\mathcal{H}[1]\otimes_Q \mathcal{H}[1] \arrow[r, "r_2"'] & \mathcal{H}[1]
\end{tikzcd}$$

that is,

$$\tilde{\delta}_i r_2 = r_2\left(1\otimes\tilde{\delta}_i + \tilde{\delta}_i\otimes 1 - [\lambda_i,-]\otimes\mathcal{O}_i^*\right).$$

Appendix B: Notes on commutators

Lemma Let ξ, η be homogeneous k-linear operators on $\wedge F_\theta$ and Λ a homogeneous k-linear operator on $\bigoplus_{Y,X} \mathrm{Hom}_R(Y,X)$ such that $|\xi| + |\eta| = |\Lambda|$, so

$$\xi \otimes \Lambda \eta \circlearrowleft \mathcal{H}[1] \otimes_Q \mathcal{H}[1]$$

is homogeneous. Then

$$[\Xi_j, \xi \otimes \Lambda \eta] = (-1)^{|\xi|} \theta_j^* \xi \otimes [[\lambda_j, -], \Lambda] \eta$$

Proof We calculate

$$[\Xi_j, \xi \otimes \Lambda \eta](w_1 \otimes \alpha_1 \otimes w_2 \otimes \alpha_2)$$

$$= \Xi_j\Big((-1)^{|\xi|(|w_1| + |\alpha_1| + 1) + |\Lambda|(|w_2| + |\eta|)} \xi(w_1) \otimes \alpha_1 \otimes \eta(w_2) \otimes \Lambda(\alpha_2)\Big)$$

$$- (-1)^{|\xi|}(\xi \otimes \Lambda \eta)\Big((-1)^{|w_1| + |\alpha_1| + 1 + |w_2|} \theta_j^*(w_1) \otimes \alpha_1 \otimes w_2 \otimes [\lambda_j, \alpha_2]\Big)$$

$$= (-1)^{|\xi||w_1| + |\xi||\alpha_1| + \cancel{|\xi|} + |\Lambda||w_2| + |\Lambda||\eta| + |\alpha_1| + |w_1| + \cancel{|\xi|} + 1 + |w_2| + |\eta|}$$

$$\theta_j^* \xi(w_1) \otimes \alpha_1 \otimes \eta(w_2) \otimes [\lambda_j, \Lambda(\alpha_2)]$$

$$+ (-1)^{1 + |\xi| + |w_1| + |\alpha_1| + 1 + |w_2| + |\xi|(|w_1| + |\alpha_1| + \cancel{1} + \cancel{1}) + |\Lambda|(|w_2| + |\eta|)} \xi \theta_j^*(w_1) \otimes \alpha_1 \otimes \eta(w_2) \otimes \Lambda[\lambda_j, \alpha_2]$$

$$= (-1)^{|\xi||w_1| + |\xi||\alpha_1| + |\Lambda||w_2| + |\Lambda||\eta| + |\alpha_1| + |w_1| + 1 + |w_2| + |\eta|}$$

$$\theta_j^* \xi(w_1) \otimes \alpha_1 \otimes \eta(w_2) \otimes \Big\{ [\lambda_j, -]\Lambda + (-1)^{|\eta| + 1 + |\xi|} \Lambda[\lambda_j, -]\Big\}(\alpha_2)$$

whereas

$$\{ O_j^* \xi \otimes [[\lambda_j, -], \Lambda]\eta \}(\omega_1 \otimes \alpha_1 \otimes \omega_2 \otimes \alpha_2)$$

$$= (-1)^{(|\omega_1|+|\alpha_1|+1)(|\xi|+1)+(|\Lambda|+1)(|\omega_2|+|\eta|)} O_j^*\xi(\omega_1)\otimes\alpha_1\otimes\eta(\omega_2)\otimes[[\lambda_j,-],\Lambda](\alpha_2)$$

$$= (-1)^{|\xi||\omega_1|+|\xi||\alpha_1|+|\xi|+|\omega_1|+|\alpha_1|+1+|\Lambda||\omega_2|+|\omega_2|+|\eta||\Lambda|+|\eta|} O_j^*\xi(\omega_1)\otimes\alpha_1\otimes\eta(\omega_2)\otimes[[\lambda_j,-],\Lambda](\alpha_2). \quad \square$$

<u>Lemma</u> With the same setup as the previous lemma,

$$[1\otimes\delta_j, \xi\otimes\Lambda\eta] = (-1)^{|\Lambda|}\xi\otimes[\lambda_j,\Lambda]O_j^*\eta \qquad (14.1)$$

<u>Proof</u> We compute

$$[1\otimes\delta_j,\xi\otimes\Lambda\eta](\omega_1\otimes\alpha_1\otimes\omega_2\otimes\alpha_2)$$

$$= (1\otimes\delta_j)\Big((-1)^{|\xi|(|\omega_1|+|\alpha_1|+1)+|\Lambda|(|\omega_2|+|\eta|)}\xi(\omega_1)\otimes\alpha_1\otimes\eta(\omega_2)\otimes\Lambda(\alpha_2)\Big)$$

$$- (\xi\otimes\Lambda\eta)\Big((-1)^{|\omega_2|+1}\omega_1\otimes\alpha_1\otimes O_j^*(\omega_2)\otimes\lambda_j\alpha_2\Big)$$

$$= (-1)^{\substack{|\xi||\omega_1|+|\xi||\alpha_1|+|\xi|+|\Lambda||\omega_2|+|\Lambda||\eta| \\ |\omega_2|+|\eta|+1}}\xi(\omega_1)\otimes\alpha_1\otimes O_j^*\eta(\omega_2)\otimes\lambda_j\Lambda(\alpha_2)$$

$$+ (-1)^{\substack{|\omega_2|+|\xi|(|\omega_1|+|\alpha_1|+1) \\ +|\Lambda|(|\omega_2|+1+|\eta|)+|\eta|}}\xi(\omega_1)\otimes\alpha_1\otimes O_j^*\eta(\omega_2)\otimes\Lambda\lambda_j(\alpha_2)$$

$$= (-1)^{|\xi||\omega_1|+|\xi||\alpha_1|+|\xi|+|\Lambda||\omega_2|+|\Lambda||\eta|+|\omega_2|+|\eta|+1}$$
$$\xi(\omega_1)\otimes\alpha_1\otimes O_j^*\eta(\omega_2)\otimes\{\lambda_j\Lambda(\alpha_2)-(-1)^{|\Lambda|}\Lambda\lambda_j(\alpha_2)\}$$

whereas

$$\{\xi\otimes[\lambda_j,\Lambda]\,\mathcal{O}_j^*\eta\}(\omega_1\otimes\alpha_1\otimes\omega_2\otimes\alpha_2)$$

$$=(-1)^{(|\omega_1|+|\alpha_1|+1)|\xi|+(|\Lambda|+1)(|\omega_2|+|\eta|+1)}\,\xi(\omega_1)\otimes\alpha_1\otimes\mathcal{O}_j^*\eta(\omega_2)\otimes[\lambda_j,\Lambda](\alpha_2)$$

. □

Now we can for example calculate that

$$(1\otimes\delta)(1\otimes\delta)\Xi=(1\otimes\delta)\{\Xi(1\otimes\delta)+[1\otimes\delta,\Xi]\}$$

$$=(1\otimes\delta)\Xi(1\otimes\delta)-(1\otimes\delta)\sum_{i,j}\mathcal{O}_j^*\otimes[\lambda_j,\lambda_i]\mathcal{O}_i^*.$$

$$=\{\Xi(1\otimes\delta)+[1\otimes\delta,\Xi]\}(1\otimes\delta)$$

$$-\sum_{i,j}(\mathcal{O}_j^*\otimes[\lambda_j,\lambda_i]\mathcal{O}_i^*)(1\otimes\delta)$$

$$-\sum_{i,j}[1\otimes\delta,\mathcal{O}_j^*\otimes[\lambda_j,\lambda_i]\mathcal{O}_i^*]$$

$$=\Xi(1\otimes\delta)^2-2\sum_{i,j}(\mathcal{O}_j^*\otimes[\lambda_j,\lambda_i]\mathcal{O}_i^*)(1\otimes\delta)$$

$$-\sum_{i,j,k}\mathcal{O}_j^*\otimes[\lambda_k,[\lambda_j,\lambda_i]]\mathcal{O}_k^*\mathcal{O}_i^*$$

$$=\Xi(1\otimes\delta)^2+2\sum_{i,j,k}(\mathcal{O}_j^*\otimes[\lambda_j,\lambda_i]\lambda_k\mathcal{O}_i^*\mathcal{O}_k^*$$

$$+\sum_{i,j,k}(\mathcal{O}_j^*\otimes[\lambda_k,[\lambda_j,\lambda_i]]\mathcal{O}_i^*\mathcal{O}_k^*$$

$$=\Xi(1\otimes\delta)^2+\sum_{i,j,k}\mathcal{O}_j^*\otimes\{\lambda_k[\lambda_j,\lambda_i]+[\lambda_j,\lambda_i]\lambda_k\}\mathcal{O}_i^*\mathcal{O}_k^*$$

$$(1\otimes\delta)^2\Xi^2 = \Xi(1\otimes\delta)^2\Xi + [1\otimes\delta^2,\Xi]\Xi$$

$$= \Xi\{\Xi(1\otimes\delta^2) + [1\otimes\delta^2,\Xi]\} + [1\otimes\delta^2,\Xi]\Xi$$

$$= \Xi^2(1\otimes\delta^2) + [\Xi,[1\otimes\delta^2,\Xi]]$$

$$= \Xi^2(1\otimes\delta^2) + \sum_{i,j,k,l}[\Xi_l,\ \theta_j^*\otimes\{\lambda_k[\lambda_j,\lambda_i] + [\lambda_j,\lambda_i]\lambda_k\}\theta_i^*\theta_k^*]$$

$$= \Xi^2(1\otimes\delta^2) - \sum_{i,j,k,l}\theta_l^*\theta_j^*\otimes\big[[\lambda_l,-],\ \lambda_k[\lambda_j,\lambda_i] + [\lambda_j,\lambda_i]\lambda_k\big]\theta_i^*\theta_k^*$$

$$= \Xi^2(1\otimes\delta^2) - \sum_{i,j,k,l}\theta_l^*\theta_j^*\otimes[\lambda_l,\lambda_k[\lambda_j,\lambda_i] + [\lambda_j,\lambda_i]\lambda_k]\theta_i^*\theta_k^*$$

Lemma Let Λ be a homogeneous operator on $\bigoplus_{Y,X}\mathrm{Hom}_R(Y,X)$ of the form $\Lambda(\alpha) = \lambda\circ\alpha$ for some family $\{\lambda^Y\}_Y$ all of the same degree. Then as operators on $\mathcal{H}$,

$$[[\lambda_j,-],\Lambda] = [\lambda_j,\lambda] \qquad (16.1)$$

where the RHS means post-composition with $[\lambda_j^Y,\lambda^Y]$.

Proof We calculate

$$\begin{aligned}[[\lambda_j,-],\Lambda](\alpha) &= [\lambda_j,-](\lambda\circ\alpha) - (-1)^{|\Lambda|}\Lambda([\lambda_j,\alpha])\\ &= \lambda_j\lambda\alpha - (-1)^{|\alpha|+|\lambda|}\lambda\alpha\lambda_j\\ &\quad -(-1)^{|\Lambda|}\lambda\lambda_j\alpha + (-1)^{|\Lambda|+|\alpha|}\lambda\alpha\lambda_j\\ &= \{\lambda_j\lambda - (-1)^{|\Lambda|}\lambda\lambda_j\}\circ\alpha\ .\ \square\end{aligned}$$

In general for $a, b \geq 1$ we have by the cut systems paper, since $[-, \Xi^b]$ is a graded derivation

$$[(1 \otimes \delta)^a, \Xi^b] = \sum_{z=0}^{a-1} (1 \otimes \delta)^z [1 \otimes \delta, \Xi^b] (1 \otimes \delta)^{a-z-1}$$

and

$$\begin{aligned} \Xi^b &= \Big(\sum_{i=1}^n \theta_i^* \otimes [\lambda_i, -] \Big)^b \\ &= \sum_{i_1, \ldots, i_b} \big(\theta_{i_1}^* \otimes [\lambda_{i_1}, -] \big) \circ \cdots \circ \big(\theta_{i_b}^* \otimes [\lambda_{i_b}, -] \big) \\ &= \sum_{i_1, \ldots, i_b} (-1)^{\binom{b}{2}} \theta_{i_1}^* \cdots \theta_{i_b}^* \otimes [\lambda_{i_1}, -] \circ \cdots \circ [\lambda_{i_b}, -] \end{aligned}$$

Hence by (14.1)

$$[1 \otimes \delta, \Xi^b] = \sum_{j, i_1, \ldots, i_b} (-1)^{\binom{b}{2} + b} \theta_{i_1}^* \cdots \theta_{i_b}^* \otimes \big[\lambda_j, [\lambda_{i_1}, -] \circ \cdots \circ [\lambda_{i_b}, -] \big] \theta_j^*$$

where $\big[\lambda_j, [\lambda_{i_1}, -] \circ \cdots \circ [\lambda_{i_b}, -]\big]$ sends $\alpha \in \operatorname{Hom}_R(Y, X)$ to

$$\lambda_j^Y \circ [\lambda_{i_1}, [\lambda_{i_2}, [\cdots [\lambda_{i_b}, \alpha] \cdots]$$
$$- (-1)^{|b|} [\lambda_{i_1}, \cdots [\lambda_{i_b}, \lambda_j^Y \circ \alpha] \cdots].$$

Note that this is <u>not</u> $[\lambda_j, -] \circ [\lambda_{i_1}, -] \circ \cdots \circ [\lambda_{i_b}, -]$ applied to α

In fact, since the commutator is a graded differential

$$\left[\lambda_j, [\lambda_{i_1},-]\circ\cdots\circ[\lambda_{i_b},-]\right] = \sum_{q=0}^{b} (-1)^q [\lambda_{i_1},-]\circ\cdots\circ[\lambda_{i_q},-] \circ [\lambda_j,[\lambda_{i_{q+1}},-]] \circ [\lambda_{i_{q+2}},-]\circ\cdots\circ[\lambda_{i_b},-].$$

By the earlier calculations, see (16.1), we know $[\lambda_j,[\lambda_{i_{q+1}},-]]$ acts by post-composition with $[\lambda_j,\lambda_{i_{q+1}}]$, and to avoid confusion we will write that operator as $[\lambda_j,\lambda_{i_{q+1}}]^{\bullet}$. So

$$[1\otimes\delta, \Xi^b] = \sum_{j,i_1,\ldots,i_b} (-1)^{\binom{b}{2}+b} \theta_{i_1}^* \cdots \theta_{i_b}^* \otimes \sum_{q=0}^{b} (-1)^q [\lambda_{i_1},-]\circ\cdots\circ[\lambda_{i_q},-]\circ[\lambda_j,\lambda_{i_{q+1}}]^{\bullet} \circ [\lambda_{i_{q+2}},-]\circ\cdots\circ[\lambda_{i_b},-]\,\theta_j^*$$

$$= \sum_{j,i_1,\ldots,i_b} (-1)^{\binom{b+1}{2}} \theta_{i_1}^* \cdots \theta_{i_b}^* \otimes \sum_{q=0}^{b} (-1)^q \Big[\lambda_{i_1},[\lambda_{i_2},[\cdots,[\lambda_{i_q}, [\lambda_j,\lambda_{i_{q+1}}]\circ[\lambda_{i_{q+2}},[\cdots \cdots[\lambda_{i_b},-]\cdots]\theta_j^*$$

old

$$\hat{\Xi}(x_1,x_2) = \sum_i \left(O_i^* \otimes [\lambda_i,-]^\wedge\right)(x_1,x_2)$$
$$= \sum_i (-1)^{\tilde{x}_1} O_i^*(x_1)\otimes[\lambda_i,-]^\wedge(x_2)$$

Hence

$$\hat\Xi^\ell(x_1,x_2) = \sum_{i_1,\dots,i_\ell} (-1)^{\tilde x_1 + (\tilde x_1+1)+\cdots+(\tilde x_1+\ell-1)} O_{i_\ell}^*\cdots O_{i_1}^*(x_1) \otimes [\lambda_{i_\ell},-]^\wedge\cdots[\lambda_{i_1},-]^\wedge(x_2)$$

and so

$$\hat r_2 \hat\Xi^\ell(x_1,x_2) = \sum_{i_1,\dots,i_\ell} (-1)^{\ell\tilde x_1 + \binom{\ell}{2}} \hat r_2\left(O^*_{i_\ell}\cdots O^*_{i_1}(x_1), [\lambda_{i_\ell},-]^\wedge\cdots(x_2)\right)$$
$$= \sum_{i_1,\dots,i_\ell} (-1)^{\ell\tilde x_1+\binom{\ell}{2}+(\tilde x_1+\ell)(\tilde x_2+\ell)+(\tilde x_2+\ell)+1} \hat\mu_2\left([\lambda_{i_\ell},-]^\wedge\cdots(x_2), O^*_{i_\ell}\cdots O^*_{i_1}(x_1)\right)$$
$$= \sum_{i_1,\dots,i_\ell} (-1)^{\cancel{\ell\tilde x_1}+\tilde x_1\tilde x_2+\cancel{\ell\tilde x_1}+\ell\tilde x_2+\cancel{\ell}+\tilde x_2+\cancel{\ell}+1+\binom{\ell}{2}} \hat\mu_2 (-1)^{|x_2|+(|x_2|+1)+\cdots+(|x_2|+\ell-1)}\left([\lambda_{i_\ell},-]^\wedge\otimes O^*_{i_\ell}\right)\cdots\left([\lambda_{i_1},-]^\wedge\otimes O^*_{i_1}\right)(x_2\otimes x_1)$$
$$= (-1)^{\tilde x_1\tilde x_2+\ell\tilde x_2+\tilde x_2+1+\cancel{\binom{\ell}{2}}+\ell|x_2|+\cancel{\binom{\ell}{2}}}\hat\mu_2\hat\Xi'^\ell(x_2,x_1)$$
$$= (-1)^{\tilde x_1\tilde x_2+\cancel{\ell|x_2|}+\ell+|x_2|+\cancel{1}+\cancel{1}+\cancel{\ell|x_2|}}$$
$$= (-1)^{\tilde x_1\tilde x_2+\tilde x_2+\ell+1}\hat\mu_2\hat\Xi'^\ell(x_2,x_1) \qquad (34.1)$$
$$= (-1)^{\tilde x_1\tilde x_2+\tilde x_2+1}\hat\mu_2(-\hat\Xi')^\ell(x_2,x_1)$$