Minimal models for MFs 29 (checked)

Continuing from ainfmf28 we write down the Feynman rules governing the higher products in an A_∞-category quasi-isomorphic to $\bigwedge F_\theta \otimes_k mf(R,W) \otimes_R \hat{R}$. We begin by recollecting the setup and hypotheses from ainfmf28.

- k is a commutative $\mathbb{Q}$-algebra and $W \in R = k[x_1,\ldots,x_n]$ is a potential.
- $\mathcal{C}$ is a full sub-DG-category of the DG-category $mf(R,W)$ of finite-rank free matrix factorisations of W.
- $F_\theta = \bigoplus_{i=1}^n k\theta_i$ as a $\mathbb{Z}_2$-graded module with $|\theta_i| = 1$.
- $t_1,\ldots,t_n$ is a quasi-regular sequence in R such that
  - $R/(\underline{t})$ is a f.g. free k-module, and
  - t_i acts null-homotopically on $\mathrm{Hom}_R(Y,X)$ for all $X,Y \in ob(\mathcal{C})$, $1 \le i \le n$
  - the Koszul complex of $t_1,\ldots,t_n$ over R is exact except in degree zero.
- for each $X \in ob(\mathcal{C})$ and $1 \le i \le n$ we choose a null-homotopy λ_i^X for the action of t_i on $\mathrm{Hom}_R(X,X)$ and for $Y \in ob(\mathcal{C})$ define the null-homotopy $\lambda_i^{Y,X}$ on $\mathrm{Hom}_R(Y,X)$ for the action of t_i to be $\lambda_i^{Y,X}(\alpha) = \lambda_i^X \circ \alpha$ (see p. 8 ainfmf28).
- we choose a k-basis $\{z_i\}_{i=1}^{\mu}$ of $R/(\underline{t})$ and homogeneous R-bases for every $X \in ob(\mathcal{C})$, together with a k-linear section $\mathfrak{z}$ of $\hat{R} \longrightarrow \hat{R}/(\underline{t})$, where $\hat{R}$ denotes the $(\underline{t})$-adic completion.
- Q is the grading algebra generated by orthogonal idempotents $\{\epsilon_a\}_{a \in ob(\mathcal{C})}$.

NOTE Errors were found in ainfmf28 which mean we replace $\exp(\tilde{\delta})$ throughout with $\mathcal{V}$, this has not yet been done.

- We use two $\mathbb{Z}_2$-graded Q-bimodules

$$\hat{\mathcal{H}} = \bigoplus_{Y,X \in ob(\mathcal{C})} \wedge F_\theta \otimes_k R/(\underline{t}) \otimes_k \mathrm{Hom}_k(\tilde{Y}, \tilde{X}) \otimes_k k[[\underline{t}]]$$
$$\hat{\mathcal{K}} = \bigoplus_{Y,X \in ob(\mathcal{C})} R/(\underline{t}) \otimes_k \mathrm{Hom}_k(\tilde{Y}, \tilde{X}) \qquad (2.1)$$

where $Y \cong R \otimes_k \tilde{Y}$ encodes the chosen homogeneous R-basis in terms of a chosen k-basis for $\tilde{Y}$. As discussed on p. 30 ainfmf28 there are actually two natural Q-bimodule structures on $\hat{\mathcal{H}}, \hat{\mathcal{K}}$. The "usual" one which we use in ainfmf28 and which follows the definitions in ainfcat2 is appropriate for forward suspended products. The "alternative" bimodule structure (i.e. the one on p. 30 ainfmf28) is appropriate for ordinary products. We write $\tilde{\otimes}_Q$ when tensoring using this other bimodule structure (which is the only time this issue manifests itself).

- Given $k \geq 2$ and $T \in \mathcal{T}_k$ (all our trees have only internal vertices of valency 3) we define a Q-bilinear map (w.r.t. the alternative bimodule structure)

$$\mathrm{eval}_T : \hat{\mathcal{K}}^{\tilde{\otimes}_Q k} \longrightarrow \hat{\mathcal{K}} \qquad (2.2)$$

as on p. 33 ainfmf28. Then the Q-bilinear map $\hat{\rho}_T : \hat{\mathcal{K}}[1]^{\otimes_Q k} \longrightarrow \hat{\mathcal{K}}[1]$ giving the A_∞-structure on $\hat{\mathcal{K}}[1]$ is related to eval_T by (33.1) of ainfmf28.

The purpose of this note is to unpack the definition of eval_T and describe it in terms of Feynman diagrams. The relevant ingredients are $\hat{\sigma}_\infty, \hat{\pi}, \hat{\phi}_\infty, \hat{\mu}_2, \hat{\Xi}'$ which we now recall in turn.

$\boxed{\hat{\sigma}_\infty}$ is the $\mathbb{Q}$-bilinear map $\hat{\mathcal{K}} \longrightarrow \hat{\mathcal{H}}$ given by the formula

$$\hat{\sigma}_\infty = \sum_{m \geq 0} (-1)^m \left(\hat{H} \hat{d}_{Hom} \right)^m \hat{\mathfrak{z}} \qquad (3.1)$$

which is (21.3) of ainfmf28. Here $\hat{H} : \hat{\mathcal{H}} \longrightarrow \hat{\mathcal{H}}$ is the odd map given by

$$\hat{H} = [\hat{d}_K, \nabla_t]^{-1} \nabla_t, \qquad \hat{d}_K = \sum_{i=1}^n t_i^{Y,X} \theta_i^*, \qquad \nabla_t = \sum_{i=1}^n \frac{\partial}{\partial t_i} \theta_i. \qquad (3.2)$$

see p.23 ainfmf28. The map $\hat{\mathfrak{z}} : \hat{\mathcal{K}} \longrightarrow \hat{\mathcal{H}}$ is the obvious inclusion, see p.22 ainfmf28. Finally, $\hat{d}_{Hom}$ is given as an odd operator on $\hat{\mathcal{H}}$, or rather the Y, X summand thereof, by the tensor from (27.4) ainfmf28, which we describe by naming our chosen homogeneous basis of $\tilde{X}$ by $\{e_u^X\}_u$ for any $X \in ob(\mathcal{C})$, so that $\alpha_{u_1 u_2} = e_{u_1}^X \circ (e_{u_2}^Y)^*$ is a k-basis for $Hom_k(\tilde{Y}, \tilde{X})$.

In terms of this basis the operator $\phi \longmapsto d_X \circ \phi$ has the coordinates

$$\begin{aligned}
\alpha_{u_1 u_2} \longmapsto d_X \circ \alpha_{u_1 u_2} &= d_X \circ e_{u_1}^X \circ (e_{u_2}^Y)^* \\
&= d_X(e_{u_1}^X) \circ (e_{u_2}^Y)^* \\
&= \sum_{u_3} (d_X)_{u_3 u_1} e_{u_3}^X \circ (e_{u_2}^Y)^* \qquad (d_X)_{u_3 u_1} \in R \\
&= \sum_{u_3} (d_X)_{u_3 u_1} \alpha_{u_3 u_2}
\end{aligned}$$

That is, as a matrix w.r.t. the basis $\{\alpha_{u_1 u_2}\}_{u_1, u_2}$ the operator $d_X \circ (-)$ has the matrix with entry $(d_X \circ -)_{ab,cd} = \delta_{bd} \cdot (d_X)_{ac}$ where δ is the Dirac delta.
Similarly $\phi \longmapsto (-1)^{|\phi|} \phi \circ d_Y$ has action

$$\begin{aligned}
\alpha_{cd} \longmapsto (-1)^{|\alpha_{cd}|} \alpha_{cd} \circ d_Y &= (-1)^{|\alpha_{cd}|} e_c^X \circ (e_d^Y)^* \circ d_Y \\
&= \sum_a (-1)^{|\alpha_{cd}|} e_c^X \circ (e^Y)_a^* (d_Y)_{da} = \sum_a (-1)^{|\alpha_{cd}|} (d_Y)_{da} \alpha_{ca}
\end{aligned}$$

Hence $\phi \mapsto (-1)^{|\phi|} \phi \circ d_Y$ has matrix $(-)_{rs,cd} = (-1)^{|\alpha_{cd}|} \delta_{rc} \cdot (d_Y)_{ds}$.

The upshot is that as an operator on $\mathrm{Hom}_k(\tilde{Y}, \tilde{X}) \otimes_k R$, d_{Hom} which is the operator $\phi \mapsto d_X \circ \phi - (-1)^{|\phi|} \phi \circ d_Y$ has the tensor expression

⌜NOTE: in all tensor representations we ignore gradings, i.e. (4.1) says that $d_{\mathrm{Hom}}(\alpha_{cd}) = \sum_{a,b} [\cdots] \alpha_{ab}$ as shown⌟

$$d_{\mathrm{Hom}} = \sum_{a,b,c,d} \left[\delta_{bd} (d_X)_{ac} - (-1)^{|\alpha_{cd}|} \delta_{ac} (d_Y)_{db} \right] \alpha^*_{cd} \otimes \alpha_{ab} \qquad (4.1)$$

$$\cap$$

$$\mathrm{Hom}_k(\tilde{Y}, \tilde{X})^* \otimes_k \mathrm{Hom}_k(\tilde{Y}, \tilde{X}) \otimes_k R$$

$$= \sum_{a,c,d} (d_X)_{ac} \alpha^*_{cd} \otimes \alpha_{ad} - \sum_{a,b,d} (-1)^{|\alpha_{ad}|} (d_Y)_{db} \alpha^*_{ad} \otimes \alpha_{ab}$$

Thus $\hat{d}_{\mathrm{Hom}}$ is written in terms of the operators $[d_{X,ac}]^{\#}$ and $[d_{Y,db}]^{\#}$. Let us write

$$d_{X,uv} = \sum_{k=1}^{\mu} \sum_{\delta \in \mathbb{N}^n} \underbrace{(d_{X,uv})_{k\delta}}_{\text{in } k} \, 6(z_k) t^\delta$$

$$d_{Y,uv} = \sum_{k=1}^{\mu} \sum_{\delta \in \mathbb{N}^n} (d_{Y,uv})_{k\delta} \, 3(z_k) t^\delta \qquad (4.2)$$

Then by (4.1) as a matrix w.r.t. the chosen basis of α's,

$$(d_{\mathrm{Hom}})_{ab,cd} = \delta_{bd} (d_X)_{ac} - (-1)^{|\alpha_{cd}|} \delta_{ac} (d_Y)_{db}$$

$$= \sum_k \sum_\delta \left[\delta_{bd} (d_{X,ac})_{k\delta} - (-1)^{|\alpha_{cd}|} \delta_{ac} (d_{Y,db})_{k\delta} \right] 3(z_k) t^\delta$$

$$\therefore \{ (d_{\mathrm{Hom}})_{ab,cd} \}_{k\delta} = \delta_{bd} (d_{X,ac})_{k\delta} - (-1)^{|\alpha_{cd}|} \delta_{ac} (d_{Y,db})_{k\delta} \qquad (4.3)$$

Then according to ainfmf28, as a tensor (5.1)

$$\hat{d}_{Hom} \in (R/(\underline{t}))^* \otimes_k \mathrm{Hom}_k(\tilde{Y},\tilde{X})^* \otimes_k (R/(\underline{t})) \otimes_k \mathrm{Hom}_k(\tilde{Y},\tilde{X}) \otimes_k k[[\underline{t}]]$$

we have

$$\hat{d}_{Hom} = \sum_{\substack{i,a,b,c,d\\ \ell,\delta}} (\hat{d}_{Hom})^{i,c,d}_{\ell,a,b,\delta} \; z_i^* \otimes \alpha_{cd}^* \otimes z_\ell \otimes \alpha_{ab} \otimes t^\delta$$

$$(\hat{d}_{Hom})^{i,c,d}_{\ell,a,b,\delta} = \sum_{\epsilon+\kappa=\delta} \sum_{k=1}^{\mu} ((d_{Hom})_{ab,cd})_{k\epsilon} \, \gamma^{ki}_{\ell\kappa} \qquad (5.2)$$

$$= \sum_{\epsilon+\kappa=\delta} \sum_{k=1}^{\mu} \Big[\delta_{bd} (d_{X,ac})_{k\epsilon} - (-1)^{|\alpha_{cd}|} \delta_{ac} (d_{Y,db})_{k\epsilon} \Big] \gamma^{ki}_{\ell\kappa}$$

where γ is a tensor defined on p.25 of ainfmf28. Thus, dividing it into two parts and using the Dirac deltas,

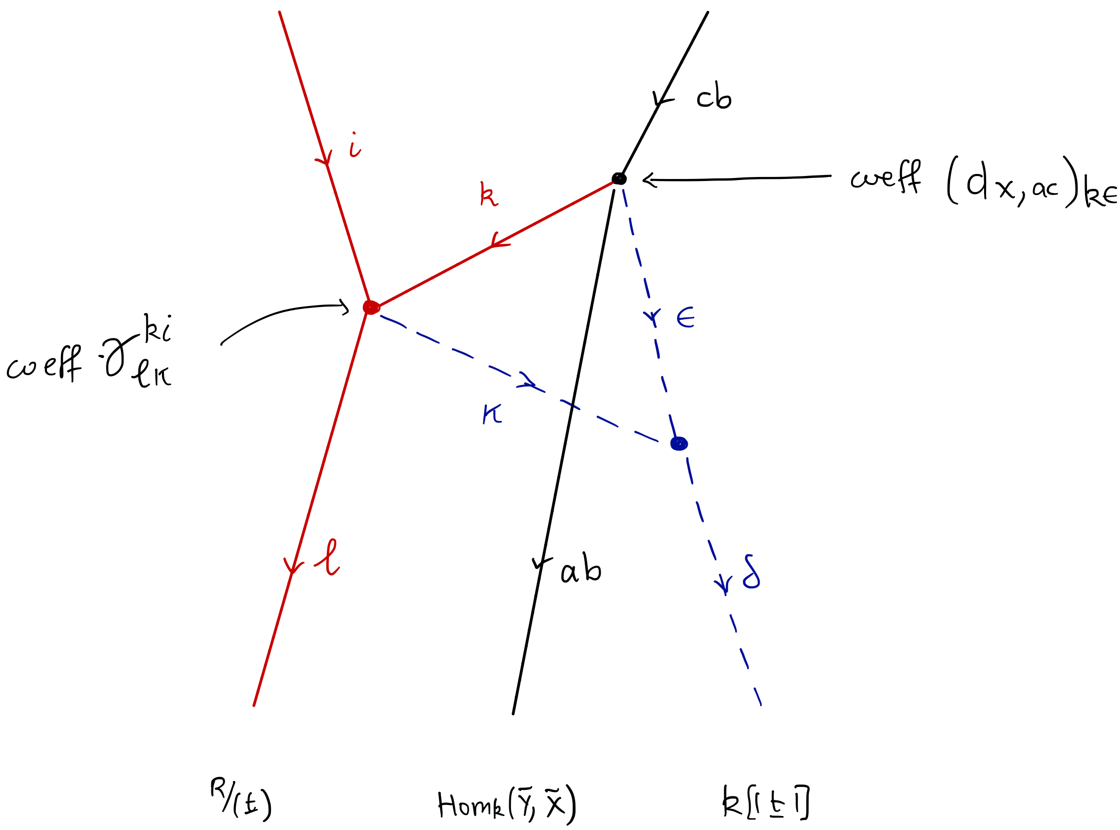
$$\hat{d}_{Hom} = \sum_{\substack{i,a,b,c,\ell,\delta\\ \epsilon+\kappa=\delta,k}} (d_{X,ac})_{k\epsilon} \, \gamma^{ki}_{\ell\kappa} \, z_i^* \otimes \alpha_{cb}^* \otimes z_\ell \otimes \alpha_{ab} \otimes t^\delta \qquad (5.3)$$

$$- \sum_{\substack{i,a,b,d,\ell,\delta\\ \epsilon+\kappa=\delta,k}} (-1)^{|\alpha_{ad}|} (d_{Y,db})_{k\epsilon} \, \gamma^{ki}_{\ell\kappa} \, z_i^* \otimes \alpha_{ad}^* \otimes z_\ell \otimes \alpha_{ab} \otimes t^\delta.$$

Let us record the string diagram corresponding to the first summand, with <u>external</u> indices i, a, b, c, ℓ, δ fixed and an implicit sum over <u>internal</u> indices k, ϵ, κ, i.e.

$$\sum_{\substack{\cancel{i},\cancel{a},\cancel{b},\cancel{c},\ell,\cancel{\delta} \\ \epsilon+\kappa=\cancel{\delta},\, k}} (d_{X,ac})_{k\epsilon}\, \gamma^{ki}_{\ell\kappa}\, z_i^* \otimes \alpha_{cb}^* \otimes z_\ell \otimes \alpha_{ab} \otimes t^\delta \qquad (6.1)$$

Below red lines ($\longrightarrow$) stand for $R/(\underline{t})$, black lines for $\mathrm{Hom}_k(\tilde{Y}, \tilde{X})$ and blue lines ($\dashrightarrow$) for $k[[\underline{t}]]$.

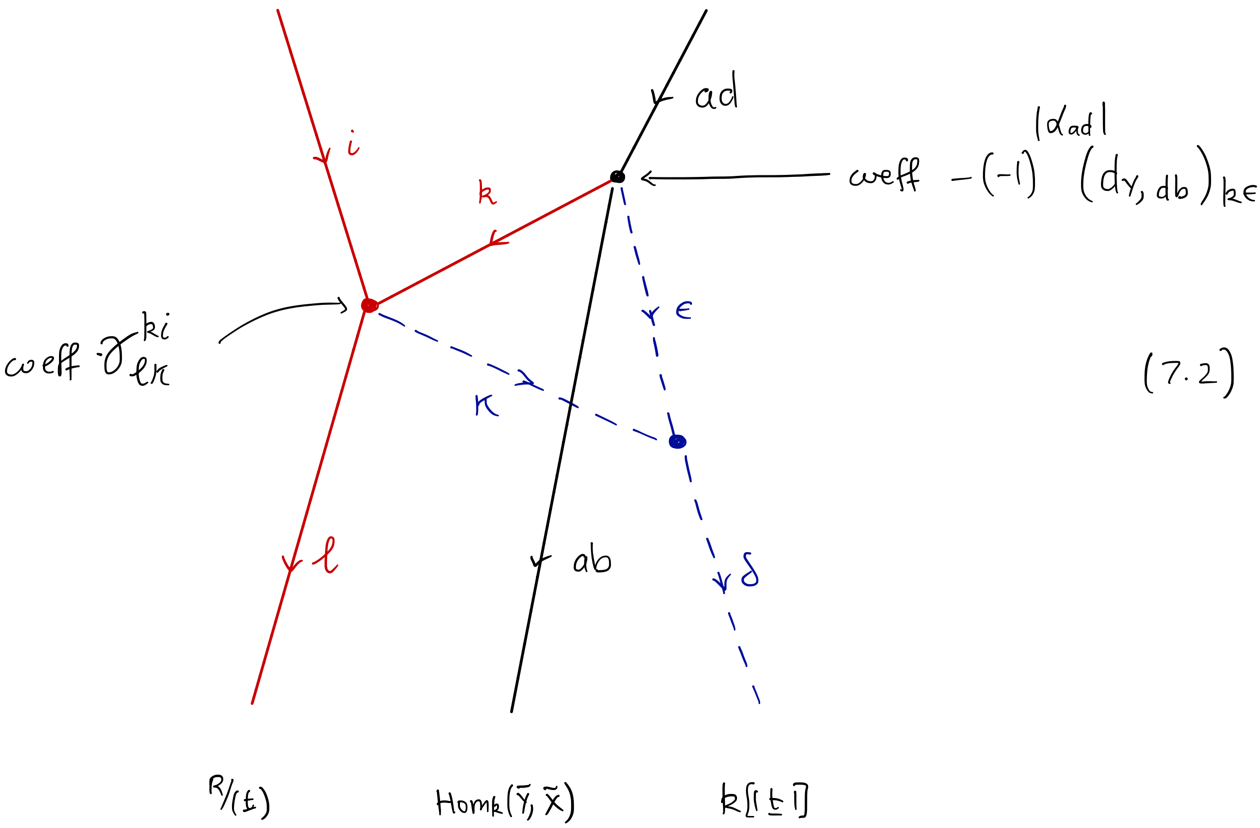


(6.2)

And for the other summand in (5.3), namely

$$- \sum_{\substack{\cancel{i},\cancel{a},\cancel{b},\cancel{d},\cancel{\ell},\cancel{\delta}\\ \epsilon+\kappa=\cancel{\delta},\, k}} (-1)^{|\alpha_{ad}|} (d_{Y,db})_{k\epsilon}\, \gamma^{ki}_{\ell\kappa}\, z_i^* \otimes \alpha_{ad}^* \otimes z_\ell \otimes \alpha_{ab} \otimes t^{\delta}. \qquad (7.1)$$

we have similarly



(7.2)

We want to return now to the calculation of $\hat{\mathfrak{z}}_\infty$. But first we need a compact way of recording the t-dependence of $\hat{d}_{Hom}$. From (5.1) and (5.2) we derive for $\delta \in \mathbb{N}^n$ a tensor (as usual we omit the Y, X dependence)

$$\hat{d}_{Hom}^{(\delta)} \in \left(R/(\underline{t})\right)^* \otimes_k \mathrm{Hom}_k(\tilde{Y}, \tilde{X})^* \otimes_k \left(R/(\underline{t})\right) \otimes_k \mathrm{Hom}_k(\tilde{Y}, \tilde{X}) \qquad (8.1)$$

$$\hat{d}_{Hom}^{(\delta)} = \sum_{i,a,b,c,d,\ell} \left(\hat{d}_{Hom}\right)_{\ell,a,b,\delta}^{i,c,d} \; z_i^* \otimes \alpha_{cd}^* \otimes z_\ell \otimes \alpha_{ab}$$

such that

$$\hat{d}_{Hom} = \sum_\delta \hat{d}_{Hom}^{(\delta)} \otimes t^\delta \qquad (8.2)$$

To compute $\hat{\mathfrak{z}}_\infty$ we need to calculate the following for $m \geq 0$ (writing $\zeta = [\hat{d}_K, \nabla_t]^{-1}$)

$$\begin{aligned}
(\hat{H}\hat{d}_{Hom})^m \hat{\mathfrak{z}} &= (\hat{H}\hat{d}_{Hom}) \cdots (\hat{H}\hat{d}_{Hom})\hat{\mathfrak{z}} \qquad (8.3)\\
&= \sum_{\delta_1,\dots,\delta_m} \left(\hat{H} \circ (\hat{d}_{Hom}^{(\delta_m)} \otimes t^{\delta_m})\right) \cdots \left(\hat{H} \circ (\hat{d}_{Hom}^{(\delta_1)} \otimes t^{\delta_1})\right) \hat{\mathfrak{z}}\\
&= \sum_{\delta_1,\dots,\delta_m} \left(\zeta \nabla_t \circ (\hat{d}_{Hom}^{(\delta_m)} \otimes t^{\delta_m})\right) \cdots \left(\zeta \nabla_t \circ (\hat{d}_{Hom}^{(\delta_1)} \otimes t^{\delta_1})\right) \hat{\mathfrak{z}}
\end{aligned}$$

Now we know that ζ only cares about the Θ-degree and the t-degree, both of which are zero after applying $\hat{\mathfrak{z}}$. After an application of $\hat{d}_{Hom}^{(\delta)} \otimes t^\delta$ the Θ-degree is unchanged and the t-degree increases by $|\delta|$. After an application of ∇_t the Θ-degree increases by 1 and the t-degree decreases by 1, so the sum is unchanged. Hence the ζ's in (8.3) contribute the following scalar factor

$$\frac{1}{|\delta_1|+\cdots+|\delta_m|} \cdots \frac{1}{|\delta_1|+|\delta_2|}\frac{1}{|\delta_1|} \tag{9.1}$$

That is, $(\hat{H}\hat{d}_{Hom})^m\hat{z}$ equals (9.2)

$$\sum_{\delta_1,\dots,\delta_m}\prod_{r=1}^{m}\frac{1}{|\delta_1|+\cdots+|\delta_r|}\left(\nabla_t\circ\left(\hat{d}_{Hom}^{(\delta_m)}\otimes t^{\delta_m}\right)\right)\cdots\left(\nabla_t\circ\left(\hat{d}_{Hom}^{(\delta_1)}\otimes t^{\delta_1}\right)\right)\hat{z}$$

Now, since $\nabla_t^2=0$ and $\nabla_t\hat{z}=0$ this may be rewritten in terms of commutators

$$[\nabla_t,\ \hat{d}_{Hom}^{(\delta)}\otimes t^{\delta}]$$

which act on a tensor as follows

$$\begin{aligned}
&[\nabla_t,\hat{d}_{Hom}^{(\delta)}\otimes t^{\delta}](\omega\otimes z_i\otimes\alpha\otimes t^{J}) \\
&= \nabla_t\left((-1)^{|\omega|}\omega\otimes\hat{d}_{Hom}^{(\delta)}(z_i\otimes\alpha)\otimes t^{J+\delta}\right) \\
&\quad + (\hat{d}_{Hom}^{(\delta)}\otimes t^{\delta})\nabla_t(\omega\otimes z_i\otimes\alpha\otimes t^{J}) \\
&= \sum_i(-1)^{|\omega|}\theta_i\omega\otimes\hat{d}_{Hom}^{(\delta)}(z_i\otimes\alpha)\otimes\frac{\partial}{\partial t_i}(t^{J}t^{\delta}) \\
&\quad + \sum_i(-1)^{|\omega|+1}\theta_i\omega\otimes\hat{d}_{Hom}^{(\delta)}(z_i\otimes\alpha)\otimes t^{\delta}\frac{\partial}{\partial t_i}(t^{J}) \\
&= \sum_i(-1)^{|\omega|}\theta_i\omega\otimes\hat{d}_{Hom}^{(\delta)}(z_i\otimes\alpha)\otimes t^{J}\frac{\partial}{\partial t_i}(t^{\delta}) \\
&= \left[\sum_i\theta_i\circ\left(\hat{d}_{Hom}^{(\delta)}\otimes\frac{\partial}{\partial t_i}(t^{\delta})\right)\right](\omega\otimes z_i\otimes\alpha\otimes t^{J})
\end{aligned} \tag{9.3}$$

That is,

$$[\nabla_t, \hat{d}^{(\delta)}_{Hom} \otimes t^\delta] = \sum_i \mathcal{O}_i \circ \left(\hat{d}^{(\delta)}_{Hom} \otimes \frac{\partial}{\partial t_i}(t^\delta) \right) \qquad (10.1)$$

Combining (9.2), (9.3) we arrive at

$$(\hat{H}\hat{d}_{Hom})^m \hat{z}$$

$$= \sum_{\delta_1,\dots,\delta_m} \prod_{r=1}^m \frac{1}{|\delta_1|+\cdots+|\delta_r|} \left(\nabla_t \circ (\hat{d}^{(\delta_m)}_{Hom} \otimes t^{\delta_m}) \right) \cdots \left(\nabla_t \circ (\hat{d}^{(\delta_1)}_{Hom} \otimes t^{\delta_1}) \right) \hat{z}$$

$$= \sum_{\delta_1,\dots,\delta_m} \prod_{r=1}^m \frac{1}{|\delta_1|+\cdots+|\delta_r|} \left\{ \sum_{i_m} \mathcal{O}_{i_m} \left(\hat{d}^{(\delta_m)}_{Hom} \otimes \frac{\partial}{\partial t_{i_m}} t^{\delta_m} \right) \right\} \cdots$$

$$\cdots \left\{ \sum_{i_1} \mathcal{O}_{i_1} \left(\hat{d}^{(\delta_1)}_{Hom} \otimes \frac{\partial}{\partial t_{i_1}} t^{\delta_1} \right) \right\} \hat{z} \qquad (10.2)$$

$$= \sum_{\substack{\delta_1,\dots,\delta_m \\ i_1,\dots,i_m}} \prod_{r=1}^m \frac{1}{|\delta_1|+\cdots+|\delta_r|} \left\{ \mathcal{O}_{i_m} \left(\hat{d}^{(\delta_m)}_{Hom} \otimes \frac{\partial}{\partial t_{i_m}} t^{\delta_m} \right) \right\} \cdots$$

$$\cdots \left\{ \mathcal{O}_{i_1} \left(\hat{d}^{(\delta_1)}_{Hom} \otimes \frac{\partial}{\partial t_{i_1}} t^{\delta_1} \right) \right\} \hat{z}$$

and hence finally,

⌜Note if $\delta_1 = 0$ the summand in (10.3) is zero⌟

<u>Lemma</u> We have the following expansion, with $\delta_1, \dots, \delta_m$ ranging over $\mathbb{N}^n$, and $i_1, \dots, i_m$ ranging over $\{1, \dots, n\}$, and $m = 0$ giving just $\hat{z}$

$$\hat{z}_\infty = \sum_{m \geq 0} (-1)^m \sum_{\substack{\delta_1,\dots,\delta_m \\ i_1,\dots,i_m}} \prod_{r=1}^m \left(\frac{1}{|\delta_1|+\cdots+|\delta_r|} \right) \left\{ \mathcal{O}_{i_m} \left(\hat{d}^{(\delta_m)}_{Hom} \otimes \frac{\partial}{\partial t_{i_m}} t^{\delta_m} \right) \right\} \cdots$$

$$\cdots \left\{ \mathcal{O}_{i_1} \left(\hat{d}^{(\delta_1)}_{Hom} \otimes \frac{\partial}{\partial t_{i_1}} t^{\delta_1} \right) \right\} \hat{z} \qquad (10.3)$$

$\boxed{\hat{\phi}_\infty}$ is the $\mathbb{Q}$-bilinear map $\hat{\mathcal{H}} \longrightarrow \hat{\mathcal{H}}$ given by the formula (p. 21 ainfmf28)

$$\hat{\phi}_\infty = \sum_{m \geq 0} (-1)^m \big(\hat{H} \hat{d}_{Hom} \big)^m \hat{H}. \tag{11.1}$$

Here $\hat{H}$ is as in (3.2) and $\hat{d}_{Hom}$ is given by (8.1), (8.2), so we may proceed just as we did for $\hat{z}_\infty$ on p. 8, 9, 10. The difference is, of course, that there is a $\hat{H}$ at the end rather than a $\hat{z}$, and we apply $\hat{\phi}_\infty$ to tensors which already involve θ's and t's, whereas $\hat{z}_\infty$ applies to elements of $\vec{\mathcal{K}}$.

Let us begin with the analogue of (8.3), (writing $\zeta = [d_K, \nabla_t]^{-1}$ as before)

$$\begin{aligned} \big(\hat{H}\hat{d}_{Hom}\big)^m \hat{H} &= \big(\hat{H}\hat{d}_{Hom}\big) \cdots \big(\hat{H}\hat{d}_{Hom}\big)\hat{H} \qquad (11.2)\\ &= \sum_{\delta_1,\ldots,\delta_m} \big(\hat{H}\circ(\hat{d}_{Hom}^{(\delta_m)} \otimes t^{\delta_m})\big) \cdots \big(\hat{H}\circ(\hat{d}_{Hom}^{(\delta_1)} \otimes t^{\delta_1})\big)\hat{H} \\ &= \sum_{\delta_1,\ldots,\delta_m} \big(\zeta \nabla_t \circ (\hat{d}_{Hom}^{(\delta_m)} \otimes t^{\delta_m})\big) \cdots \big(\zeta \nabla_t \circ (\hat{d}_{Hom}^{(\delta_1)} \otimes t^{\delta_1})\big)\zeta \nabla_t \end{aligned}$$

Now suppose we apply this to a tensor $\omega \otimes z_i \otimes \alpha \otimes t^J$ with $\omega \in \Lambda F_\theta$, $J \in \mathbb{N}^n$ and suppose ω, α are homogeneous. Now by the arguments given earlier the ζ's will contribute an overall factor of (writing $a = |\omega| + |J|$)

from the rightmost ζ

$$\frac{1}{a + |\delta_1| + \cdots + |\delta_m|} \cdots \frac{1}{a + |\delta_1| + |\delta_2|} \cdot \frac{1}{a + |\delta_1|} \cdot \frac{1}{a} \tag{11.3}$$

This results in the equality

$$(\hat{H}\hat{d}_{Hom})^m \hat{H}$$

$$= \sum_{\delta_1,\dots,\delta_m} \prod_{r=0}^{m} \frac{1}{a+|\delta_1|+\cdots+|\delta_r|} \left(\nabla_t \circ (\hat{d}_{Hom}^{(\delta_m)} \otimes t^{\delta_m})\right) \cdots \left(\nabla_t \circ (\hat{d}_{Hom}^{(\delta_1)} \otimes t^{\delta_1})\right) \nabla_t$$

$r=0$ contributes $\frac{1}{a}$

which, by converting to commutators and using (10.1), may be written as

$$= \sum_{\delta_1,\dots,\delta_m} \prod_{r=0}^{m} \frac{1}{a+|\delta_1|+\cdots+|\delta_r|} \left[\nabla_t , (\hat{d}_{Hom}^{(\delta_m)} \otimes t^{\delta_m})\right] \cdots \left[\nabla_t , (\hat{d}_{Hom}^{(\delta_1)} \otimes t^{\delta_1})\right] \nabla_t$$

$$= \sum_{\substack{\delta_1,\dots,\delta_m \\ i_0, i_1,\dots,i_m}} \prod_{r=0}^{m} \frac{1}{a+|\delta_1|+\cdots+|\delta_r|} \left\{ \theta_{i_m} \left(\hat{d}_{Hom}^{(\delta_m)} \otimes \frac{\partial}{\partial t_{i_m}} t^{\delta_m}\right)\right\} \cdots \cdots \left\{ \theta_{i_1} \left(\hat{d}_{Hom}^{(\delta_1)} \otimes \frac{\partial}{\partial t_{i_1}} t^{\delta_1}\right)\right\} \theta_{i_0} \frac{\partial}{\partial t_{i_0}} \tag{12.1}$$

Again, we summarise the calculation as follows: ⌜again if $\delta_1 = 0$ the summand in (12.2) is zero⌟

<u>Lemma</u> We have the following expansion, with $\delta_1, \dots, \delta_m$ ranging over $\mathbb{N}^n$, and $i_0, i_1, \dots, i_m$ ranging over $\{1, \dots, n\}$, of $\hat{\phi}_\infty$ applied to a tensor

$$\omega \otimes z_i \otimes \alpha \otimes t^{J} \in \hat{\mathcal{H}},$$

where $|\omega| + |J| = a$. On the space of such tensors, $\hat{\phi}_\infty$ is equal to the operator

$$\sum_{m \geq 0} (-1)^m \sum_{\substack{\delta_1,\dots,\delta_m \\ i_0, i_1,\dots,i_m}} \prod_{r=0}^{m} \left(\frac{1}{a+|\delta_1|+\cdots+|\delta_r|}\right) \left\{ \theta_{i_m} \left(\hat{d}_{Hom}^{(\delta_m)} \otimes \frac{\partial}{\partial t_{i_m}} t^{\delta_m}\right)\right\} \cdots \cdots \left\{ \theta_{i_1} \left(\hat{d}_{Hom}^{(\delta_1)} \otimes \frac{\partial}{\partial t_{i_1}} t^{\delta_1}\right)\right\} \theta_{i_0} \frac{\partial}{\partial t_{i_0}} \tag{12.2}$$

$\boxed{\hat{\Xi}'}$ is the Q-bilinear operator on $\hat{\mathcal{H}} \tilde{\otimes}_Q \hat{\mathcal{H}}$ given by (p. 33 ainfmf28)

$$\hat{\Xi}' = \sum_{i=1}^{n} [\lambda_i, -]^{\wedge} \otimes \mathcal{O}_i^* \tag{13.1}$$

Here we perform the same analysis of $[\lambda_i, -]^{\wedge}$ as we did on $\hat{d}_{Hom}$, starting on p. 3. Recall that by definition ((8.2) of ainfmf28) given $\alpha \in \mathrm{Hom}_k(\tilde{Y}, \tilde{X})$,

$$[\lambda_i, -](\alpha) = \lambda_i^X \circ \alpha - (-1)^{|\alpha|} \alpha \circ \lambda_i^Y \tag{13.2}$$

which is similar to the formula for d_{Hom}. Hence the earlier derivation may be repeated verbatim to show that, as a tensor (we switch to j to avoid clashes)

$$[\lambda_j, -]^{\wedge} \in (R/(\underline{t}))^* \otimes_k \mathrm{Hom}_k(\tilde{Y}, \tilde{X})^* \otimes_k (R/(\underline{t})) \otimes_k \mathrm{Hom}_k(\tilde{Y}, \tilde{X}) \otimes_k k[[\underline{t}]]$$

we have

$$[\lambda_j, -]^{\wedge} = \sum_{\substack{i,a,b,c,d \\ \ell, \delta}} \left([\lambda_j, -]^{\wedge}\right)^{i,c,d}_{\ell,a,b,\delta} \; z_i^* \otimes \alpha_{cd}^* \otimes z_\ell \otimes \alpha_{ab} \otimes t^{\delta} \tag{13.3}$$

where

$$\left([\lambda_j, -]^{\wedge}\right)^{i,c,d}_{\ell,a,b,\delta} = \sum_{\epsilon+\kappa=\delta} \sum_{k=1}^{\mu} \left[\delta_{bd} (\lambda_j^X{}_{,ac})_{k\epsilon} - (-1)^{|\alpha_{cd}|} \delta_{ac} (\lambda_j^Y{}_{,db})_{k\epsilon} \right] \gamma^{ki}_{\ell\kappa} \tag{13.4}$$

As we did on p. 8 for $\hat{d}_{Hom}$ we also introduce the tensors

$$[\lambda_j,-]^{\wedge(\delta)} \in \left(R/(\underline{t})\right)^* \otimes_k \mathrm{Hom}_k(\tilde{Y},\tilde{X})^* \otimes_k \left(R/(\underline{t})\right) \otimes_k \mathrm{Hom}_k(\tilde{Y},\tilde{X})$$

$$[\lambda_j,-]^{\wedge(\delta)} = \sum_{i,a,b,c,d,\ell} \left([\lambda_j,-]^{\wedge}\right)^{i,c,d}_{\ell,a,b,\delta}\, z_i^* \otimes \alpha_{cd}^* \otimes z_\ell \otimes \alpha_{ab} \qquad (14.1)$$

such that

$$[\lambda_j,-]^{\wedge} = \sum_\delta [\lambda_j,-]^{\wedge(\delta)} \otimes t^\delta \qquad (14.2)$$

Let us now turn to an expression of $\hat{\mu}_2 \exp(-\hat{\Xi}')$. We have

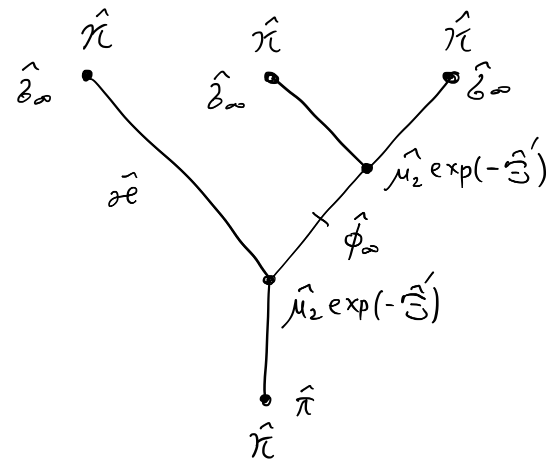
$$\begin{aligned}
\hat{\mu}_2 \exp(-\hat{\Xi}') &= \sum_{m\geq 0} (-1)^m \frac{1}{m!} \hat{\mu}_2 (\hat{\Xi}')^m \qquad (14.3)\\
&= \sum_{m\geq 0} (-1)^m \frac{1}{m!} \hat{\mu}_2 \Big(\sum_{i=1}^n [\lambda_i,-]^{\wedge} \otimes \theta_i^* \Big)^m \\
&= \sum_{m\geq 0} (-1)^m \frac{1}{m!} \sum_{i_1,\dots,i_m} \hat{\mu}_2 \prod_{r=1}^m \left([\lambda_{i_r},-]^{\wedge} \otimes \theta_{i_r}^* \right) \\
&= \sum_{m\geq 0} (-1)^m \frac{1}{m!} \sum_{\substack{\delta_1,\dots,\delta_m\\ i_1,\dots,i_m}} \hat{\mu}_2 \prod_{r=1}^m \left([\lambda_{i_r},-]^{\wedge(\delta_r)} \otimes t^{\delta_r} \right) \otimes \theta_{i_r}^* \\
&= \sum_{m\geq 0} (-1)^m \frac{1}{m!} \sum_{\substack{\delta_1,\dots,\delta_m\\ i_1,\dots,i_m}} t^{\delta_1+\dots+\delta_m} \hat{\mu}_2 \prod_{r=1}^m \left([\lambda_{i_r},-]^{\wedge(\delta_r)} \otimes \theta_{i_r}^* \right)
\end{aligned}$$

With these ingredients in place we can now turn to a proper study of eval_T from p. 33 ainfmf28.

Evaluating the tree

Let $T \in \mathcal{T}_p$ be given, $p \geq 2$, and consider the decoration of $A(T)$ from p. 32 ainfmf28

- $\hat{\mathcal{H}}$ to each internal edge of T and $\hat{\mathcal{K}}$ to each leaf of T,



- $\hat{\delta}_\infty : \hat{\mathcal{K}} \longrightarrow \hat{\mathcal{H}}$ to each input,
- $\hat{\pi} : \hat{\mathcal{H}} \longrightarrow \hat{\mathcal{K}}$ to the root,
- $\hat{\phi}_\infty : \hat{\mathcal{H}} \longrightarrow \hat{\mathcal{H}}$ to each internal edge of T,
- $\hat{\mu}_2 \exp(-\hat{\Xi}')$ to each interval vertex.

By definition

$$\mathrm{eval}_T : \hat{\mathcal{K}}^{\tilde{\otimes}_Q p} \longrightarrow \hat{\mathcal{K}}$$

is the branch denotation of this decoration in the category of <u>ungraded</u> modules.

We have described $\hat{\delta}_\infty$ in (10.3), $\hat{\phi}_\infty$ in (12.2) and $\hat{\mu}_2 \exp(-\hat{\Xi}')$ in (14.3), and we now assemble these results. Each of these operators involves a sum over an integer $m \geq 0$. Let $\mathrm{Loc}(T)$ denote the set of <u>locations</u> in T, by which we mean non-root vertices of $A(T)$. Then we can write

$$\mathrm{eval}_T = \sum_{f : \mathrm{Loc}(T) \to \mathbb{N}} (-1)^{|f|} \, \mathrm{eval}_{T,f}$$

where $|f| = \sum_{x \in Loc(T)} f(x)$ and $\mathrm{eval}_{T,f}$ denotes the denotation of the decoration with $\hat{\mathfrak{z}}_{\infty, f(x)}$, $\hat{\phi}_{\infty, f(x)}$, resp. $\hat{\mu}_2 \exp(-\hat{\Xi}')_{f(x)}$ assigned to locations in place of $\hat{\mathfrak{z}}_\infty$, $\hat{\phi}_\infty$, $\hat{\mu}_2 \exp(\hat{\Xi}')$, where

$$\hat{\mathfrak{z}}_{\infty,m} = \sum_{\substack{\delta_1,\dots,\delta_m \\ i_1,\dots,i_m}} \prod_{r=1}^{m} \left(\frac{1}{|\delta_1| + \cdots + |\delta_r|} \right) \prod_{r=1}^{m} \left(\theta_{i_r} \left(\hat{d}^{(\delta_r)}_{\mathrm{Hom}} \otimes \frac{\partial}{\partial t_{i_r}} t^{\delta_r} \right) \right) \hat{\mathfrak{z}} \qquad (16.1)$$

on the space of tensors of weight $|\omega| + |J| = a$

$$\hat{\phi}_{\infty,m} = \sum_{\substack{\delta_1,\dots,\delta_m \\ i_0, i_1,\dots,i_m}} \prod_{r=0}^{m} \left(\frac{1}{a + |\delta_1| + \cdots + |\delta_r|} \right) \prod_{r=1}^{m} \left(\theta_{i_r} \left(\hat{d}^{(\delta_r)}_{\mathrm{Hom}} \otimes \frac{\partial}{\partial t_{i_r}} t^{\delta_r} \right) \right) \theta_{i_0} \frac{\partial}{\partial t_{i_0}}, \qquad (16.2)$$

and

$$\hat{\mu}_2 \exp(-\hat{\Xi}')_m = \frac{1}{m!} \sum_{\substack{\delta_1,\dots,\delta_m \\ i_1,\dots,i_m}} t^{\delta_1 + \cdots + \delta_m} \hat{\mu}_2 \prod_{r=1}^{m} \left([\lambda_{i_r}, -]^{\wedge(\delta_r)} \otimes \theta^*_{i_r} \right) \qquad (16.3)$$

In all cases the products of operators are expanded from left $(r=1)$ to right $(r=m)$. Let us write

$$\mathrm{Loc}(T) = \mathrm{In}(T) \amalg \mathrm{Edge}(T) \amalg \mathrm{Vert}(T)$$

where $\mathrm{In}(T)$ is the set of input vertices, $\mathrm{Edge}(T)$ the internal edges of T (or more precisely the corresponding vertices of $A(T)$) and $\mathrm{Vert}(T)$ the internal vertices.

<u>Def^n</u> The <u>weight</u> of a tensor $\omega \otimes z_i \otimes \alpha \otimes t^J$ in $\hat{\mathcal{H}}$ is $|\omega| + |J| \in \mathbb{N}$.

<u>Def^n</u> A <u>configuration</u> of T is the following data:

- a function $m : \mathrm{Loc}(T) \longrightarrow \mathbb{N}$,
- for each $x \in \mathrm{Loc}(T)$ sequences $\underline{i}(x) \in \{1,\ldots,n\}^{m(x)}$ and $\underline{\delta}(x) \in (\mathbb{N}^n)^{m(x)}$,
- for each $x \in \mathrm{Edge}(T)$ an index $i_0(x) \in \{1,\ldots,n\}$.

The set of configurations is denoted $\mathrm{Con}(T)$. This is an infinite set.

<u>Def^n</u> Given $m \geq 0$, $\underline{\delta} \in (\mathbb{N}^n)^m$ and $i_0, \underline{i} \in \{1,\ldots,n\}^m$ we define

$$\hat{6}_{\infty,m,\underline{\delta},\underline{i}} = \prod_{r=1}^{m}\Big(\frac{1}{|\delta_1|+\cdots+|\delta_r|}\Big)\prod_{r=1}^{m}\Big(\mathcal{O}_{i_r}\big(\hat{d}^{(\delta_r)}_{\mathrm{Hom}} \otimes \frac{\partial}{\partial t_{i_r}} t^{\delta_r}\big)\Big)\hat{6},$$

$$\hat{\phi}_{\infty,m,\underline{\delta},\underline{i},i_0} = \prod_{r=0}^{m}\Big(\frac{1}{a+|\delta_1|+\cdots+|\delta_r|}\Big)\prod_{r=1}^{m}\Big(\mathcal{O}_{i_r}\big(\hat{d}^{(\delta_r)}_{\mathrm{Hom}} \otimes \frac{\partial}{\partial t_{i_r}} t^{\delta_r}\big)\Big)\mathcal{O}_{i_0}\frac{\partial}{\partial t_{i_0}},$$

$$\hat{\mu}_2 \exp(-\hat{\Xi}')_{m,\underline{\delta},\underline{i}} = \frac{1}{m!}t^{\delta_1+\cdots+\delta_m}\hat{\mu}_2\prod_{r=1}^{m}\big([\lambda_{i_r},-]^{\wedge(\delta_r)} \otimes \mathcal{O}^*_{i_r}\big)$$

where the formula for $\hat{\phi}_{\infty,m,\underline{\delta},\underline{i},i_0}$ gives the action on tensors of weight a.

<u>Def^n</u> Given a configuration C of T, let $\mathrm{eval}_{T,C}$ denote the (ungraded, as usual) denotation of the decoration assigning $\hat{6}_{\infty,m(x),\underline{\delta}(x),\underline{i}(x)}$ to each input x, $\hat{\phi}_{\infty,m(x),\underline{\delta}(x),\underline{i}(x),i_0(x)}$ to each edge x, $\hat{\mu}_2\exp(-\hat{\Xi}')_{m(x),\underline{\delta}(x),\underline{i}(x)}$ to each vertex, and $\hat{\pi}$ to the root.

Lemma $\mathrm{eval}_T = \sum_{C \in \mathrm{Con}(T)} (-1)^{|m_C|} \mathrm{eval}_{T,C}$

Proof The only point to consider is that $\mathrm{Con}(T)$ is infinite, but for all but finitely many configurations C, $\mathrm{eval}_{T,C}$ is zero. In fact, $\mathrm{eval}_{T,C} = 0$ if for any $x \in \mathrm{Loc}(T)$, $m(x) > n$. $\square$

Given a configuration, let us consider the flow of $\mathcal{O}$'s and t's that it dictates through the tree. Every $\hat{z}_{\infty, m, \underline{\delta}, \underline{i}}$ adds m $\mathcal{O}$'s and increases the t-degree by $|\underline{\delta}| = \sum_j |\delta_j|$ minus m (for the derivatives). So overall the weight increases by $|\underline{\delta}|$. The same is true of $\hat{\phi}_{\infty, m, \underline{\delta}, \underline{i}, i_0}$, while $\hat{\mu}_2 \exp(\hat{\Xi}')_{m, \underline{\delta}, \underline{i}}$ increases the weight by $|\underline{\delta}| - m$.

Def^n Let C be a configuration as above. Then $\mathrm{wt} : \mathrm{Loc}(T) \longrightarrow \mathbb{Z}$ is the function defined by

$$\mathrm{wt}(x) = \begin{cases} |\underline{\delta}(x)| & x \in \mathrm{In}(T) \cup \mathrm{Edge}(T) \\ |\underline{\delta}(x)| - m(x) & x \in \mathrm{Vert}(T) \end{cases}$$

and the cumulative weight is the function $\mathrm{cwt} : \mathrm{Loc}(T) \longrightarrow \mathbb{Z}$ given by

$$\mathrm{cwt}(x) = \sum_{y < x} \mathrm{wt}(y)$$

where the sum is over locations y above x in the tree (i.e. for which the path from y to the root includes x).

Note Observe that a configuration could have negative weight at a location, if say $m(x) > 0$ but $|\underline{\delta}(x)| = 0$, but if $\mathrm{cwt}(x) < 0$ for any x then $\mathrm{eval}_{T,C} = 0$ since this must arise from more $\mathcal{O}_i^*$'s than $\mathcal{O}$'s at some vertex. Similarly if $\mathrm{cwt}(x) = 0$ at any edge x then $\mathrm{eval}_{T,C} = 0$ as the $\frac{\partial}{\partial t_{i_0}}$ annihilates $1 \in k[\![\underline{t}]\!]$.

<u>Note</u> In the evaluation of the tree computing $\mathrm{eval}_{T,C}$, $\mathrm{cwt}(x)$ is the weight of the tensor being fed to the operator at x.

The overall contribution of scalars from ζ to $\mathrm{eval}_{T,C}$ is

$$\prod_{x \in \mathrm{In}(T)} \prod_{r=1}^{m(x)} \left(\frac{1}{|\delta(x)_1| + \cdots + |\delta(x)_r|} \right) \cdot \qquad (19.1)$$

$$\prod_{x \in \mathrm{Edge}(T)} \frac{1}{\mathrm{cwt}(x)} \prod_{r=1}^{m(x)} \left(\frac{1}{\mathrm{cwt}(x) + |\delta(x)_1| + \cdots + |\delta(x)_r|} \right)$$

<u>Def^n</u> We denote (19.1) by $Z_C \in \mathbb{Q}$.

<u>Def^n</u> Given a configuration C of T, consider the following decoration of $A(T)$,

- $\hat{\mathcal{H}}$ to each leaf and $\hat{\mathcal{H}}$ to each internal edge
- $\prod_{r=1}^{m} \left(\mathcal{O}_{i_r} \left(\hat{d}^{(\delta_r)}_{\mathrm{Hom}} \otimes \frac{\partial}{\partial t_{i_r}} t^{\delta_r} \right) \right) \hat{\zeta}$ to an input labelled $m, \underline{\delta}, \underline{i}$ by C,
- $\prod_{r=1}^{m} \left(\mathcal{O}_{i_r} \left(\hat{d}^{(\delta_r)}_{\mathrm{Hom}} \otimes \frac{\partial}{\partial t_{i_r}} t^{\delta_r} \right) \right) \mathcal{O}_{i_0} \frac{\partial}{\partial t_{i_0}}$ to an edge labelled $m, \underline{\delta}, \underline{i}, i_0$ by C,
- $\frac{1}{m!} t^{\delta_1 + \cdots + \delta_m} \hat{\mu}_2 \prod_{r=1}^{m} \left([\lambda_{i_r}, -]^{\wedge(\delta_r)} \otimes \mathcal{O}^*_{i_r} \right)$ to a vertex labelled $m, \underline{\delta}, \underline{i}$ by C,
- $\hat{\pi}$ to the root

<u>Lemma</u> $\mathrm{eval}_{T,C}$ is Z_C times the denotation of the above decoration.

However we have not yet arrived at the Feynman rules, as $\hat{d}^{(\delta)}_{\mathrm{Hom}}$ and $[\lambda_i,-]^{\wedge(\delta)}$ consist of two parts (corresponding to pre- or post-composition), and these parts have their own indices which need to be fixed. Let us first name the parts.

By (5.3) we have

$$\hat{d}^{(\delta)}_{\mathrm{Hom}} = \hat{d}^{(\delta,\mathrm{post})}_{\mathrm{Hom}} + \hat{d}^{(\delta,\mathrm{pre})}_{\mathrm{Hom}} \tag{20.1}$$

where

$$\hat{d}^{(\delta,\mathrm{post})}_{\mathrm{Hom}} := \sum_{i,a,b,c,\ell,\epsilon+\kappa=\delta,k} (d_{X,ac})_{k\epsilon}\, \gamma^{ki}_{\ell\kappa}\, z_i^* \otimes \alpha_{cb}^* \otimes z_\ell \otimes \alpha_{ab}$$

$$\hat{d}^{(\delta,\mathrm{pre})}_{\mathrm{Hom}} := -\sum_{i,a,b,d,\ell,\epsilon+\kappa=\delta,k} (-1)^{|\alpha_{ad}|} (d_{Y,db})_{k\epsilon}\, \gamma^{ki}_{\ell\kappa}\, z_i^* \otimes \alpha_{ad}^* \otimes z_\ell \otimes \alpha_{ab} \tag{20.2}$$

and similarly

$$[\lambda_j,-]^{\wedge(\delta)} = [\lambda_j,-]^{\wedge(\delta,\mathrm{post})} + [\lambda_j,-]^{\wedge(\delta,\mathrm{pre})} \tag{20.3}$$

where

$$[\lambda_j,-]^{\wedge(\delta,\mathrm{post})} := \sum_{i,a,b,c,\ell,\epsilon+\kappa=\delta,k} (\lambda^X_{j,ac})_{k\epsilon}\, \gamma^{ki}_{\ell\kappa}\, z_i^* \otimes \alpha_{cb}^* \otimes z_\ell \otimes \alpha_{ab} \tag{20.4}$$

$$[\lambda_j,-]^{\wedge(\delta,\mathrm{pre})} := -\sum_{i,a,b,d,\ell,\epsilon+\kappa=\delta,k} (-1)^{|\alpha_{ad}|} (\lambda^Y_{j,db})_{k\epsilon}\, \gamma^{ki}_{\ell\kappa}\, z_i^* \otimes \alpha_{ad}^* \otimes z_\ell \otimes \alpha_{ab}$$

Def^n A full configuration of T is a configuration plus the extra data of

- for each $x \in Loc(T)$ a sequence $\underline{s}(x) \in \{pre, post\}^{m(x)}$.

The set of full configurations is denoted $FCon(T)$. Given $C \in FCon(T)$ we denote by Z_C the constant associated to the underlying ordinary configuration.

Def^n Given a full configuration C of T, consider the following decoration of $A(T)$,

- $\hat{\mathcal{H}}$ to each leaf and $\mathcal{H}$ to each internal edge
- $\prod_{r=1}^{m} \Big(\mathcal{O}_{i_r} \big(\hat{d}_{Hom}^{\,(\delta_r, s_r)} \otimes \frac{\partial}{\partial t_{i_r}} t^{\delta_r} \big) \Big) \hat{z}$ to an input labelled $m, \underline{\delta}, \underline{i}, \underline{s}$ by C,
- $\prod_{r=1}^{m} \Big(\mathcal{O}_{i_r} \big(\hat{d}_{Hom}^{\,(\delta_r, s_r)} \otimes \frac{\partial}{\partial t_{i_r}} t^{\delta_r} \big) \Big) \mathcal{O}_{i_0} \frac{\partial}{\partial t_{i_0}}$ to an edge labelled $m, \underline{\delta}, \underline{i}, i_0, \underline{s}$ by C,
- $\frac{1}{m!} t^{\delta_1 + \cdots + \delta_m} \hat{\mu}_2 \prod_{r=1}^{m} \big([\lambda_{i_r}, -]^{\wedge(\delta_r, s_r)} \otimes \mathcal{O}_{i_r}^* \big)$ to a vertex labelled $m, \underline{\delta}, \underline{i}, \underline{s}$ by C,
- $\hat{\pi}$ to the root

Def^n Let $eval_{T,C}$ denote Z_C times the denotation of the above decoration.

Lemma $eval_T = \sum_{C \in FCon(T)} (-1)^{|m_C|} eval_{T,C}$

Now, each full configuration gives rise to many Feynman diagrams, since the tensors in (20.2), (20.4) contain internal indices ϵ, κ, R. But we prefer to elaborate this separately rather than continue to refine the notion of a configuration.

Feynman diagrams (A-type)

We draw diagrams downwards. We assume a full configuration has been fixed, so we are looking at some product

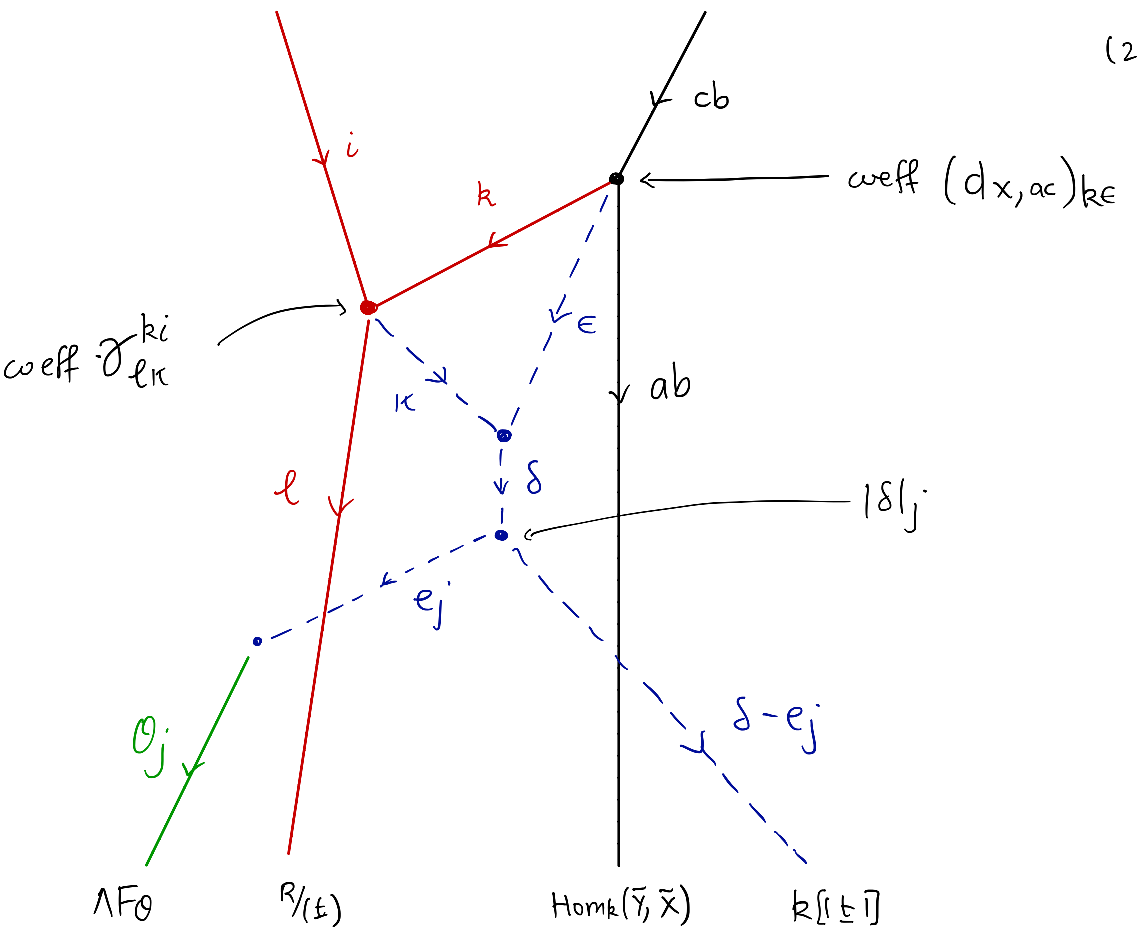
$$\prod_{r=1}^{m}\left(\mathcal{O}_{i_r}\left(\hat{d}^{(\delta_r, s_r)}_{\mathrm{Hom}} \otimes \frac{\partial}{\partial t_{i_r}} t^{\delta_r}\right)\right)\hat{\zeta}. \qquad (22.1)$$

Let us draw the diagram for

$$\mathcal{O}_j\left(\hat{d}^{(\delta, \mathrm{post})}_{\mathrm{Hom}} \otimes \frac{\partial}{\partial t_j} t^{\delta}\right)\hat{\zeta}, \qquad (22.2)$$

borrowing from (6.2)

(22.3)



The way to interpret this diagram is as a sum over internal indices (κ, ϵ, k) each multiplied by the indicated coefficients, i.e.

$$\mathcal{O}_j \left(\hat{d}^{(\delta, post)}_{Hom} \otimes \frac{\partial}{\partial t_j} t^\delta \right) \hat{\sigma} (z_i \otimes \alpha_{cb}) \qquad (23.1)$$

$$= \mathcal{O}_j \left(\hat{d}^{(\delta, post)}_{Hom} \otimes |\delta|_j t^{\delta - e_j} \right) (1 \otimes z_i \otimes \alpha_{cb} \otimes 1)$$

$$\overset{(5.2)}{=} \sum_{\substack{a, \ell, k \\ \epsilon + \kappa = \delta}} |\delta|_j (d_{x,ac})_{k\epsilon} \gamma^{ki}_{\ell\kappa} \mathcal{O}_j \otimes z_\ell \otimes \alpha_{ab} \otimes t^{\delta - e_j}$$

On a more general input,

$$\mathcal{O}_j \left(\hat{d}^{(\delta, post)}_{Hom} \otimes \frac{\partial}{\partial t_j} t^\delta \right) (\omega \otimes z_i \otimes \alpha_{cb} \otimes 1) \qquad (23.2)$$

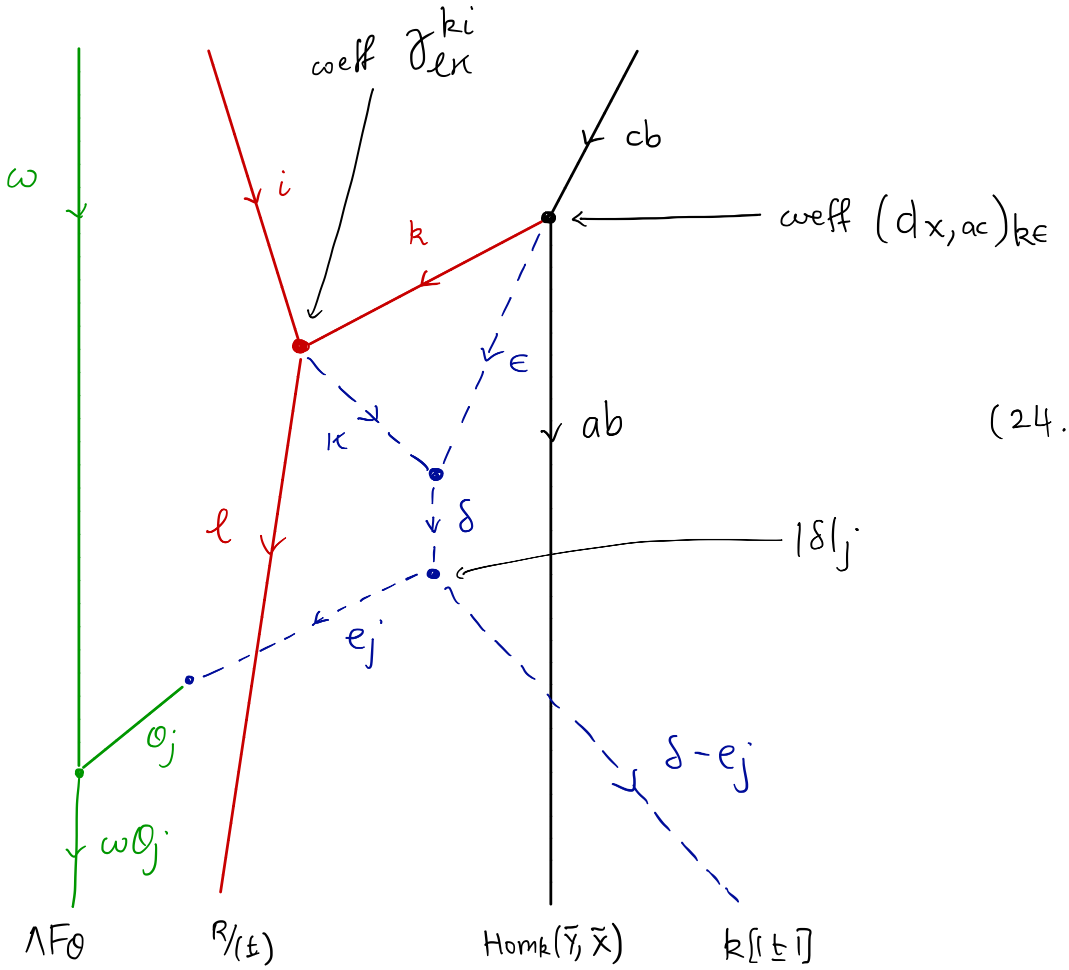
$$= (-1)^{|\omega|} \mathcal{O}_j \omega \left(\hat{d}^{(\delta, post)}_{Hom} \otimes \frac{\partial}{\partial t_j} t^\delta \right) (z_i \otimes \alpha_{cb} \otimes 1)$$

$$= \sum_{\substack{a, \ell, k \\ \epsilon + \kappa = \delta}} |\delta_j| (d_{x,ac})_{k\epsilon} \gamma^{ki}_{\ell\kappa} \omega \mathcal{O}_j \otimes z_\ell \otimes \alpha_{ab} \otimes t^{\delta - e_j}$$

which we represent by the diagram overleaf.

<u>Note</u> We tend to label $\mathcal{O}$-strands with products of $\mathcal{O}$'s, rather than some other indexing scheme (we could e.g. use ordered sequences $j_1 < \cdots < j_r$ in $\{1, \ldots, n\}$)

<u>Remark</u> For $\hat{d}^{(\delta, post)}_{Hom}$ the diagram is almost the same: just replace the incoming cb line with ad, and the $(d_{x,ac})_{k\epsilon}$ coefficient by $(-1)^{|\alpha_{ad}|+1} (d_{y,db})_{k\epsilon}$.



(24.1)

Now let us consider the diagram for a product

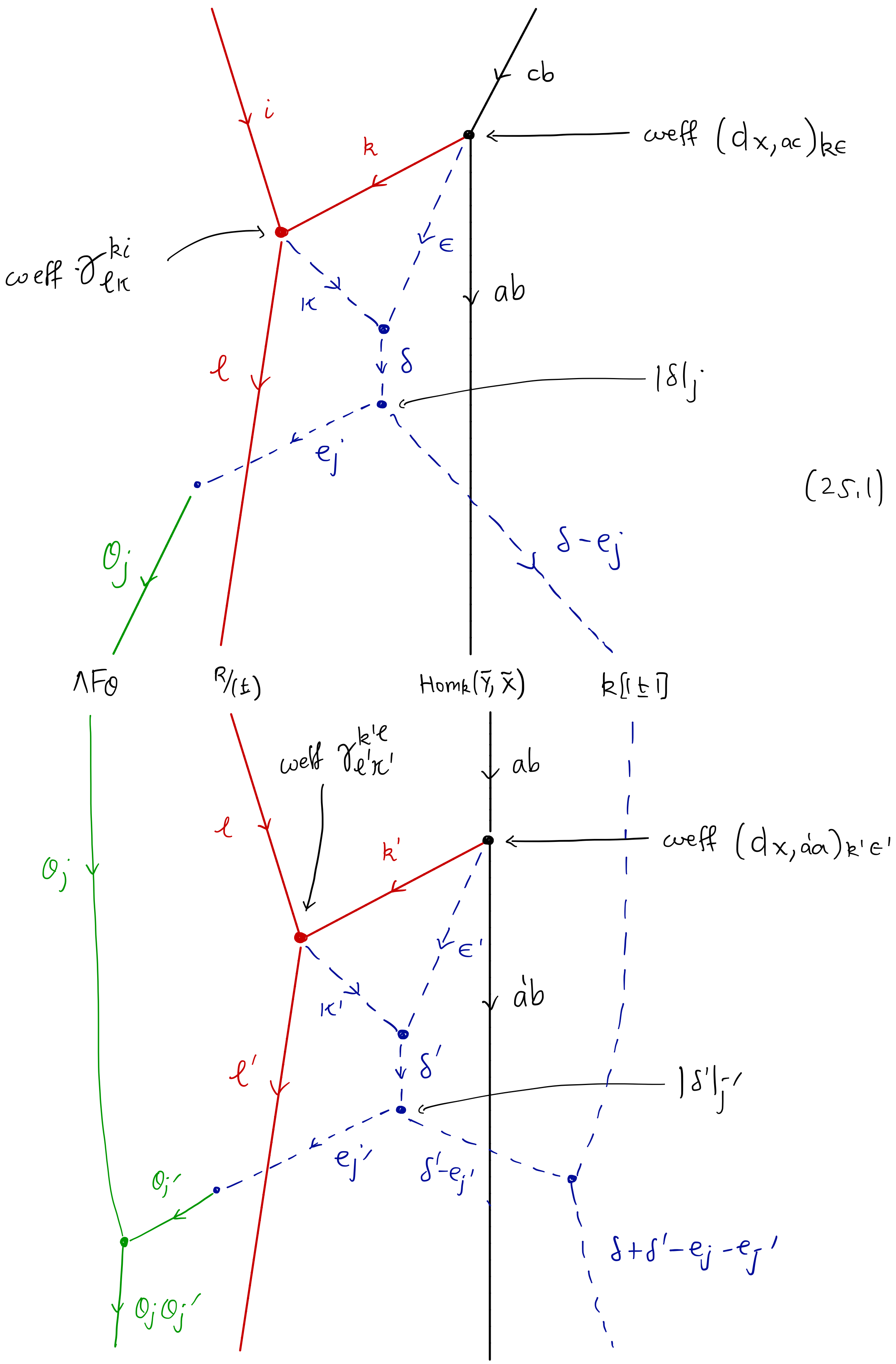
$$\theta_{j'}\Big(\hat{d}^{(\delta', post)}_{Hom} \otimes \frac{\partial}{\partial t_{j'}} t^{\delta'}\Big)\theta_j\Big(\hat{d}^{(\delta, post)}_{Hom} \otimes \frac{\partial}{\partial t_j} t^{\delta}\Big)\hat{\delta},$$

which when applied to $z_i \otimes \alpha_{cb}$ yields by (23.1),

$$\sum_{\substack{a,\ell,k\\ \epsilon+\kappa=\delta}} |\delta|_j (d_{x,ac})_{k\epsilon}\gamma^{ki}_{\ell\kappa}\,\theta_{j'}\Big(\hat{d}^{(\delta', post)}_{Hom} \otimes \frac{\partial}{\partial t_{j'}} t^{\delta'}\Big)\Big(\theta_j \otimes z_\ell \otimes \alpha_{ab} \otimes t^{\delta - e_j}\Big)$$

$$= \sum_{\substack{a,\ell,k\\ \epsilon+\kappa=\delta}} |\delta|_j (d_{x,ac})_{k\epsilon}\gamma^{ki}_{\ell\kappa} \sum_{\substack{a',\ell',k'\\ \epsilon'+\kappa'=\delta'}} |\delta'|_{j'} (d_{x,a'a})_{k'\epsilon'}\gamma^{k'\ell}_{\ell'\kappa'} \cdot \theta_j\theta_{j'} \otimes z_{\ell'} \otimes \alpha_{a'b} \otimes t^{\delta+\delta'-e_j-e_{j'}} \tag{24.2}$$

which is computed by the diagram



(25.1)

The "pre" version of $\hat{d}_{Hom}$ has a similar basic form (see (7.2)) so we do not present its diagrams separately. Since $\mathcal{O}_i \frac{\partial}{\partial t_i}$ has a familiar diagram we also omit it, and we focus now on the C-type.

Feynman diagrams (C-type)

Consider the operator

$$\frac{1}{m!} t^{\delta_1 + \cdots + \delta_m} \hat{\mu}_2 \prod_{r=1}^{m} \left([\lambda_{i_r}, -]^{\wedge(\delta_r, s_r)} \otimes \mathcal{O}_{i_r}^* \right) \qquad (26.1)$$

in the case $m=2$, so that (ignoring the $\frac{1}{m!}$) we have, with $\underline{i} = (j', j)$, $\underline{\delta} = (\delta', \delta)$

$$t^{\delta'} t^{\delta} \hat{\mu}_2 \left([\lambda_{j'}, -]^{\wedge(\delta', post)} \otimes \mathcal{O}_{j'}^* \right)\left([\lambda_j, -]^{\wedge(\delta, post)} \otimes \mathcal{O}_j^* \right) \qquad (26.2)$$

which applied to a tensor $(\omega \otimes z_i \otimes \alpha_{cb} \otimes 1) \otimes (\omega' \otimes z_{i'} \otimes \alpha_{c'b'} \otimes 1)$ yields

$$t^{\delta'} t^{\delta} \hat{\mu}_2 (-1)^{|\omega| + |\alpha_{cb}|} \left([\lambda_{j'}, -]^{\wedge(\delta', post)} \otimes \mathcal{O}_{j'}^* \right) \Big((-1)^{|\omega|} \big(\omega \otimes [\lambda_j, -]^{\wedge(\delta, post)} (z_i \otimes \alpha_{cb}) \otimes 1\big)$$
$$\otimes \left(\mathcal{O}_j^*(\omega') \otimes z_{i'} \otimes \alpha_{c'b'} \otimes 1 \right) \Big)$$

$$= t^{\delta'} t^{\delta} \hat{\mu}_2 (-1)^{|\omega| + |\alpha_{cb}| + |\omega| + |\omega| + |\alpha_{cb}| + 1 + |\omega|}$$
$$\left(\omega \otimes [\lambda_{j'}, -]^{\wedge(\delta', post)} [\lambda_j, -]^{\wedge(\delta, post)} (z_i \otimes \alpha_{cb}) \otimes 1 \right)$$
$$\otimes \left(\mathcal{O}_{j'}^* \mathcal{O}_j^* (\omega') \otimes z_{i'} \otimes \alpha_{c'b'} \otimes 1 \right)$$

$$= (-1)\, t^{\delta'} t^{\delta} \hat{\mu}_2 \Big(\left(\omega \otimes [\lambda_{j'}, -]^{\wedge(\delta', post)} [\lambda_{j}, -]^{\wedge(\delta, post)} (z_i \otimes \alpha_{cb}) \otimes 1 \right)$$
$$\otimes \left(\mathcal{O}_{j'}^* \mathcal{O}_j^* (\omega') \otimes z_{i'} \otimes \alpha_{c'b'} \otimes 1 \right) \Big)$$

(26.3)

By (20.4),

$$[\lambda_j, -]^{\wedge(\delta, post)} := \sum_{i,a,b,c,\ell,\epsilon+\kappa=\delta,k} (\lambda_{j,ac}^{\times})_{k\epsilon}\, \gamma_{\ell\kappa}^{ki}\, z_i^{*} \otimes \alpha_{cb}^{*} \otimes z_\ell \otimes \alpha_{ab} \tag{27.1}$$

Hence

$$[\lambda_{j'}, -]^{\wedge(\delta', post)} [\lambda_j, -]^{\wedge(\delta, post)} (z_i \otimes \alpha_{cb}) \tag{27.2}$$

$$= [\lambda_{j'}, -]^{\wedge(\delta', post)} \Big(\sum\nolimits_{a,\ell,\epsilon+\kappa=\delta,k} (\lambda_{j,ac}^{\times})_{k\epsilon}\, \gamma_{\ell\kappa}^{ki}\, z_\ell \otimes \alpha_{ab} \Big)$$

$$= \sum_{a,\ell,\epsilon+\kappa=\delta,k} (\lambda_{j,ac}^{\times})_{k\epsilon}\, \gamma_{\ell\kappa}^{ki}\, [\lambda_{j'}, -]^{\wedge(\delta', post)} (z_\ell \otimes \alpha_{ab})$$

$$= \sum_{a,\ell,\epsilon+\kappa=\delta,k} \; \sum_{a',\ell',\epsilon'+\kappa'=\delta',k'} (\lambda_{j,ac}^{\times})_{k\epsilon}\, \gamma_{\ell\kappa}^{ki}\, (\lambda_{j',a'c}^{\times})_{k'\epsilon'}\, \gamma_{\ell'\kappa}^{k'\ell}\, z_{\ell'} \otimes \alpha_{a'b}$$

Hence (26.3) equals

$$\sum_{\substack{a,\ell,\epsilon+\kappa=\delta,k \\ a',\ell',\epsilon'+\kappa'=\delta',k'}} (-1)\, t^{\delta'} t^{\delta} (\lambda_{j,ac}^{\times})_{k\epsilon}\, \gamma_{\ell\kappa}^{ki}\, (\lambda_{j',a'c}^{\times})_{k'\epsilon'}\, \gamma_{\ell'\kappa}^{k'\ell} \cdot \tag{27.3}$$
$$\hat{\mu}_2\big((\omega \otimes z_{\ell'} \otimes \alpha_{a'b} \otimes 1) \otimes (\theta_{j'}^{*} \theta_j^{*}(\omega') \otimes z_{i'} \otimes \alpha_{c'b'} \otimes 1) \big)$$

which using (32.2) of ainfmf28, i.e.

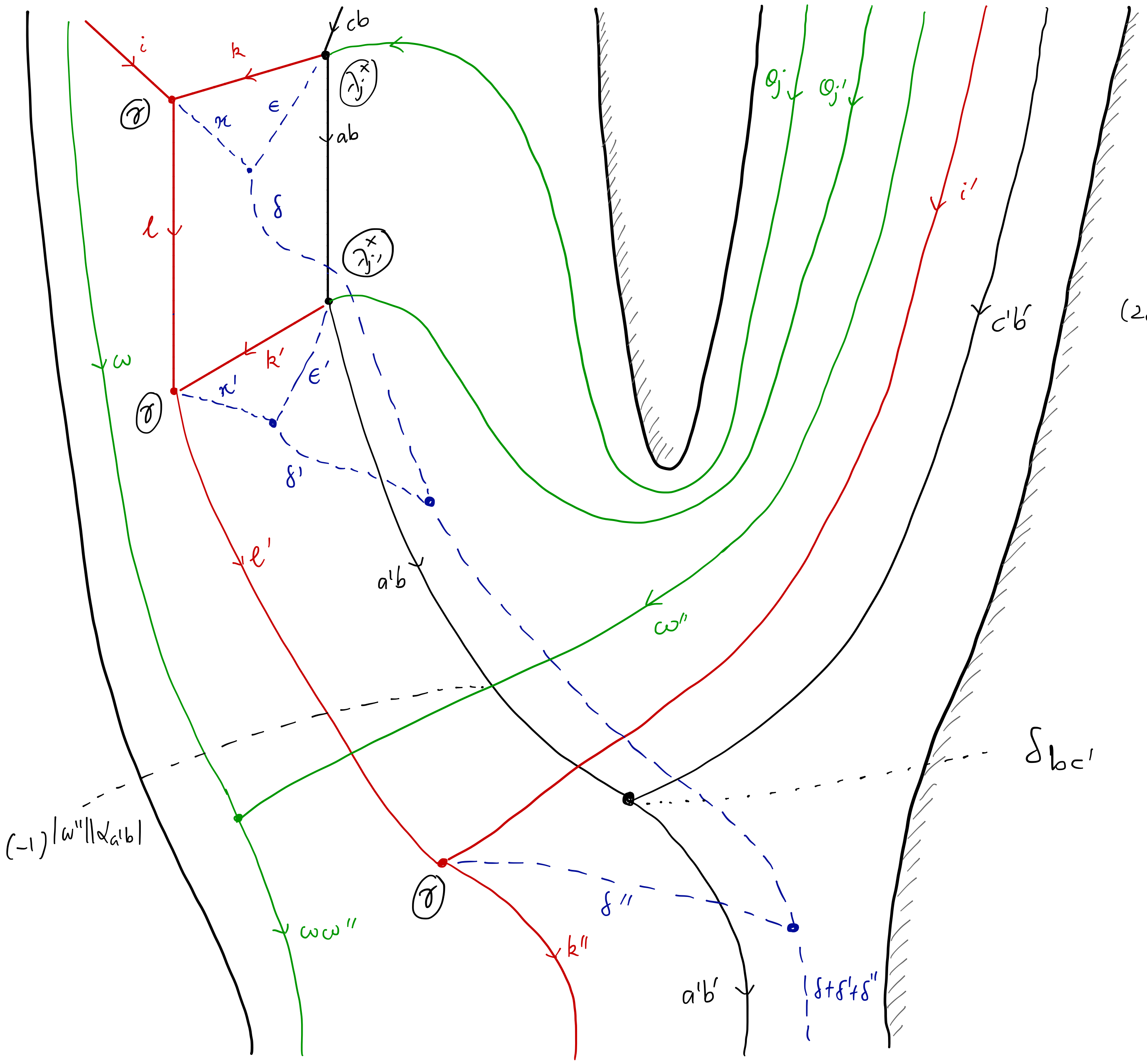
$$\hat{\mu}_2\big([\omega \otimes z_i \otimes \alpha] \otimes [\omega' \otimes z_j \otimes \beta] \big)$$
$$\|$$
$$(-1)^{|\alpha||\omega'|} \sum_{k=1}^{\mu} \sum_{\delta} \gamma_{k\delta}^{ij}\, \omega \wedge \omega' \otimes z_k \otimes \alpha\beta \otimes t^{\delta}$$

is equal to

we do not see this in the diagram

$$\sum_{\substack{a,\ell,\epsilon+\kappa=\delta,k\\ a',\ell',\epsilon'+\kappa'=\delta',k'\\ k'',\delta''}} (-1)^{|\alpha_{a'b}||w'|+1} (\lambda^{\times}_{j,ac})_{k\epsilon} (\lambda^{\times}_{j,j',a'c})_{k'\epsilon'} \gamma^{ki}_{\ell k} \gamma^{k'\ell}_{\ell' k} \gamma^{\ell' i'}_{k''\delta''} \cdot w\, O^{*}_{j;j'} O_j(w') \otimes z_{k''} \otimes \alpha_{a'b'} \otimes t^{\delta+\delta'+\delta''} \cdot \delta_{bc'} \quad (28.1)$$

Assuming $w' = O_j O_{j'} w''$ we have the corresponding diagram



(28.2)