(ainfmf30)

[Minimal models of Koszul factorisations]

Minimal models for MFs 30 (checked)

In this note we take the setting of ainfmf29 and specialise to a set of Koszul-type factorisations of the potential W. We also assume the chosen homotopies are "Koszul type". However, everything else remains generic.

<u>Def</u>

Def^n Given sequences $f_1, \dots, f_e$ and $g_1, \dots, g_e$ in R with $\sum_{i=1}^e f_i g_i = W$ the associated <u>Koszul factorisation</u> is

$$\{\underline{f}, \underline{g}\} := \Big(\bigwedge(k\xi_1 \oplus \cdots \oplus k\xi_e) \otimes_k R\,,\ \sum_{i=1}^e f_i \xi_i^* + \sum_{i=1}^e g_i \xi_i \Big) \qquad (1.1)$$

where $|\xi_i| = 1$. As our homogeneous basis we choose the product $\xi_{i_1} \wedge \cdots \wedge \xi_{i_p}$ for indices $i_1 < \cdots < i_p$, and thus with $\{\underline{f}, \underline{g}\}^\sim = \bigwedge(k\xi_1 \oplus \cdots \oplus k\xi_e)$ we have as $\mathbb{Z}_2$-graded R-modules $\{\underline{f}, \underline{g}\} \cong \{\underline{f}, \underline{g}\}^\sim \otimes_k R$.

<u>Lemma</u> There is an isomorphism of matrix factorisations of $-W$

$$\{\underline{g}, -\underline{f}\} \xrightarrow{\cong} \{\underline{f}, \underline{g}\}^\vee \qquad (1.2)$$

$$\xi_{i_1} \cdots \xi_{i_p} \longmapsto (-1)^{\binom{p}{2}} (\xi_{i_1} \cdots \xi_{i_p})^* \qquad (i_1 < \cdots < i_p)$$

<u>Proof</u> See p. 3 kdmf2. $\square$

Another way to put this is the following:

<u>Lemma</u> There is an isomorphism of matrix factorisations of $-W$

$$\eta : \Big(\bigwedge(k\xi_1^* \oplus \cdots \oplus k\xi_e^*),\ \sum_i g_i \xi_i \lrcorner(-) - \sum_i f_i \xi_i^* \wedge (-) \Big) \xrightarrow{\cong} \{\underline{f}, \underline{g}\}^\vee$$

$$\eta(\xi_{i_1}^* \wedge \cdots \wedge \xi_{i_p}^*) = (-1)^{\binom{p}{2}} (\xi_{i_1} \wedge \cdots \wedge \xi_{i_p})^* \qquad (1.3)$$

This gives an isomorphism of complexes of R-modules, for another sequence $\underline{f}', \underline{g}'$ of length e' with $\sum_i f_i' g_i' = W$

$$\mathrm{Hom}_R(\{\underline{f},\underline{g}\}, \{\underline{f}',\underline{g}'\}) \cong \{\underline{f},\underline{g}\}^\vee \otimes_R \{\underline{f}',\underline{g}'\}$$ (§2.6 defects & adjoints paper)

$$\cong \{\underline{f}',\underline{g}'\} \otimes_R \{\underline{f},\underline{g}\}^\vee \qquad (1.5.1)$$

$$\cong \Big(\textstyle\bigwedge(k\xi_1' \oplus \cdots \oplus k\xi_{e'}') \otimes_k R, \sum_i f_i'\xi_i'^* + \sum_i g_i'\xi_i' \Big) \qquad (\eta^{-1})$$
$$\otimes_k \Big(\textstyle\bigwedge(k\xi_1^* \oplus \cdots \oplus k\xi_e^*) \otimes_k R, \sum_i g_i\xi_i - \sum_i f_i \xi_i^* \Big)$$

$$\cong \Big(\textstyle\bigwedge(\oplus_{i=1}^{e'} k\xi_i' \oplus \oplus_{i=1}^{e} k\xi_i^*) \otimes_k R, \sum_i f_i'\xi_i'^* + \sum_i g_i'\xi_i' + \sum_i g_i\xi_i - \sum_i f_i\xi_i^* \Big)$$

<u>Def^n</u> Let ν denote the isomorphism of $\mathbb{Z}_2$-graded k-modules making

$$\begin{array}{ccc} \mathrm{Hom}_k(\{\underline{f},\underline{g}\}^\sim, \{\underline{f}',\underline{g}'\}^\sim) \otimes_k R & \cong & \mathrm{Hom}_R(\{\underline{f},\underline{g}\}, \{\underline{f}',\underline{g}'\}) \\ & {}_{\nu\otimes 1}\searrow \quad \swarrow_{(1.5.1)} & \\ & \bigwedge(\oplus_i k\xi_i' \oplus \oplus_i k\xi_i^*) \otimes_k R & \end{array} \qquad (1.5.2)$$

<u>Lemma</u> For a homogeneous linear map ϕ,

$$\nu(\phi) = \sum_{i_1<\cdots<i_p} (-1)^{\binom{p}{2}} \phi(\xi_{i_1} \wedge \cdots \wedge \xi_{i_p}) \wedge \xi_{i_1}^* \wedge \cdots \wedge \xi_{i_p}^* \qquad (2.1)$$

$$\nu^{-1}(\xi_{b_1}' \wedge \cdots \wedge \xi_{b_q}' \wedge \xi_{a_1}^* \wedge \cdots \wedge \xi_{a_p}^*) = (-1)^{\binom{p}{2}} \underbrace{(\xi_{b_1}' \wedge \cdots \wedge \xi_{b_q}') \circ (\xi_{a_1} \wedge \cdots \wedge \xi_{a_p})^*}_{\text{no signs, just projection}}$$

Proof Under the isomorphism

$$\Phi : \widetilde{\{f,g\}}^* \otimes_k \widetilde{\{f',g'\}} \longrightarrow \operatorname{Hom}_k\big(\widetilde{\{f,g\}}, \widetilde{\{f',g'\}}\big)$$
$$\Phi(z \otimes y)(x) = (-1)^{|z||y|} z(x) \cdot y$$

we have

$$\Phi\Big(\sum_{i_1 < \cdots < i_p} (-1)^{p|\phi| + p} (\xi_{i_1} \cdots \xi_{i_p})^* \otimes \phi(\xi_{i_1} \cdots \xi_{i_p})\Big)(\xi_{a_1} \cdots \xi_{a_{p'}})$$
$$= (-1)^{p'(|\phi| + p') + p'(|\phi| + p')} \phi(\xi_{a_1} \cdots \xi_{a_{p'}}) = \phi(\xi_{a_1} \cdots \xi_{a_{p'}})$$

the rest is clear. $\square$

Lemma Under the isomorphism $\nu \otimes 1$ of (1.5.2) the operator $\alpha \longmapsto d_{\{f',g'\}} \circ \alpha$ on the left corresponds to

$$\sum_{i=1}^{e'} f_i' \xi_i'^* + \sum_{i=1}^{e'} g_i' \xi_i' \qquad (2.3)$$

while the operator $\alpha \longmapsto -(-1)^{|\alpha|} \alpha \circ d_{\{f,g\}}$ corresponds to

$$-\sum_{i=1}^{e} f_i \xi_i^* + \sum_{i=1}^{e} g_i \xi_i .$$

with ξ_i^* multiplying on the left and ξ_i as $(\xi_i^*)^* \lrcorner (-)$

Proof This is a restatement of (1.5.1), but we feel it is useful to check directly as well:

$$\nu(d_{\{f',g'\}} \circ -)\nu^{-1}(\xi_{b_1}' \cdots \xi_{b_q}' \xi_{a_1}^* \cdots \xi_{a_p}^*)$$
$$= \nu(d_{\{f',g'\}} \circ -)\Big((\xi_{b_1}' \cdots \xi_{b_q}') \circ (\xi_{a_1} \cdots \xi_{a_p})^*\Big) \cdot (-1)^{\binom{p}{2}}$$

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$$= \nu\Big(\sum_{i=1}^{e'} f_i' \xi_i'^{*}(\xi_{b_1}'\cdots\xi_{b_q}') \circ (\xi_{a_1}\cdots\xi_{a_p})^* + \sum_{i=1}^{e'} g_i'\xi_i'(\xi_{b_1}'\cdots\xi_{b_q}')\circ(\xi_{a_1}\cdots\xi_{a_p})^* \Big)\cdot(-1)^{\binom{p}{2}}$$

$$= \nu\Big\{ \sum_i f_i' \big[\xi_i'^{*} \lrcorner (\xi_{b_1}'\cdots\xi_{b_q}')\big]\circ(\xi_{a_1}\cdots\xi_{a_p})^* + \sum_i g_i'\xi_i'\xi_{b_1}'\cdots\xi_{b_q}'\circ(\xi_{a_1}\cdots\xi_{a_p})^*\Big\}(-1)^{\binom{p}{2}}$$

$$= \sum_i f_i'\big[\xi_i'^{*}\lrcorner(\xi_{b_1}'\cdots\xi_{b_q}')\big]\wedge\xi_{a_1}^*\wedge\cdots\wedge\xi_{a_p}^*$$

$$+ \sum g_i'\xi_i'\xi_{b_1}'\wedge\cdots\wedge\xi_{b_q}'\wedge\xi_{a_1}^*\wedge\cdots\wedge\xi_{a_p}^*$$

$$= \Big(\sum_i f_i'\xi_i'^{*} + \sum_i g_i'\xi_i'\Big)\Big(\xi_{b_1}'\cdots\xi_{b_q}'\xi_{a_1}^*\cdots\xi_{a_p}^*\Big)$$

which proves (2.3). For (3.1) we similarly compute

$$\nu(\underbrace{-\circ d_{\{f,g\}}}_{\alpha\mapsto(-1)^{|\alpha|}\alpha\circ d_{\{f,g\}}})\nu^{-1}(\xi_{b_1}'\cdots\xi_{b_q}'\xi_{a_1}^*\cdots\xi_{a_p}^*)$$

$$= \nu(-\circ d_{\{f,g\}})\Big((\xi_{b_1}'\cdots\xi_{b_q}')\circ(\xi_{a_1}\cdots\xi_{a_p})^*\Big)(-1)^{\binom{p}{2}}$$

$$= \nu\Big((-1)^{p+q}(\xi_{b_1}'\cdots\xi_{b_q}')\circ(\xi_{a_1}\cdots\xi_{a_p})^*\circ d_{\{f,g\}}\Big)(-1)^{\binom{p}{2}}$$

$$(\xi_{a_1}\cdots\xi_{a_p})^*\xi_i^*(\xi_i\xi_{a_1}\cdots\xi_{a_p}) = 1$$

$$= (-1)^{p+q+\binom{p}{2}}\,\nu\Big(\sum_{i=1}^{e} f_i(\xi_{b_1}'\cdots\xi_{b_q}')\circ(\xi_{a_1}\cdots\xi_{a_p})^*\circ\xi_i^* + \sum_{i=1}^{e} g_i(\xi_{b_1}'\cdots\xi_{b_q}')\circ(\xi_{a_1}\cdots\xi_{a_p})^*\circ\xi_i\Big)$$

$$= (-1)^{p+q+\binom{p}{2}}\,\nu\Big(\sum_i f_i(\xi_{b_1}'\cdots\xi_{b_q}')\circ(\xi_i\xi_{a_1}\cdots\xi_{a_p})^* + \sum_i g_i(\xi_{b_1}'\cdots\xi_{b_q}')\circ(\xi_i^*(\xi_{a_1}\cdots\xi_{a_p}))^*\Big)$$

$\binom{p}{2} = \frac{1}{2}p(p-1) \quad \binom{p+1}{2} = \frac{1}{2}(p+1)p$

$\therefore \binom{p}{2} + \binom{p+1}{2} = \frac{1}{2}(p^2 - p + p^2 + p) = p^2 \equiv p$

$$= (-1)^{p+q+\binom{p}{2}} \Big((-1)^{\binom{p+1}{2}} \textstyle\sum_i f_i \, \xi'_{b_1} \cdots \xi'_{b_q} \, \xi_i^* \xi_{a_1}^* \cdots \xi_{a_p}^*$$
$$+ (-1)^{\binom{p-1}{2}} \textstyle\sum_i g_i \, \xi'_{b_1} \cdots \xi'_{b_q} \wedge \xi_i \lrcorner (\xi_{a_1}^* \wedge \cdots \wedge \xi_{a_p}^*) \Big)$$

$$= (-1)^q \textstyle\sum_i f_i (\cdots) + (-1)^{q+1} \sum_i g_i (\cdots)$$

$$= \Big(\textstyle\sum_i f_i \xi_i^* - \sum_i g_i \xi_i \Big) \Big(\xi'_{b_1} \cdots \xi'_{b_q} \, \xi_{a_1}^* \cdots \xi_{a_p}^* \Big)$$

as claimed. $\square$

<u>Defn</u> The differential on $\bigwedge(\oplus_i k\xi_i' \oplus \oplus_i k\xi_i^*) \otimes_k R$ corresponding under $\nu \otimes 1$ of (1.5.2) to the differential d_{Hom} is denoted

$$\tilde{d}_{Hom} = \sum_{i=1}^{e'} f_i' \xi_i'^* + \sum_{i=1}^{e'} g_i' \xi_i' - \sum_{i=1}^{e} f_i \xi_i^* + \sum_{i=1}^{e} g_i \xi_i . \qquad (4.2)$$

Setup

Let $\mathcal{C}$ be a full sub-DG-category of $mf(R,W)$ whose objects are all Koszul-type matrix factorisations. Given $X \in ob(\mathcal{C})$ we write

$$X = \{\underline{f}^X, \underline{g}^X\} = \Big(\bigwedge(k\xi_1^X \oplus \cdots \oplus k\xi_{e(X)}^X)\otimes_k R\, ,\, \sum_{i=1}^{e(X)} f_i^X \xi_i^{X*} + \sum_{i=1}^{e(X)} g_i^X \xi_i^X\Big) \quad (5.1)$$

where $e(X) \geq 0$ depends on X. As in ainfmf29 we have our quasi-regular sequence $t_1, \dots, t_n$ and we have to choose a homotopy λ_i^X for $t_i \cdot 1_X \simeq 0$. We assume that these homotopies have the following form,

$$\lambda_i^X = \sum_{j=1}^{e(X)} F_{ij}^X \xi_j^{X*} + \sum_{j=1}^{e(X)} G_{ij}^X \xi_j^X \quad (5.2)$$

where the $F_{ij}^X, G_{ij}^X \in R$ satisfy $\sum_j \{ F_{ij}^X g_j^X + G_{ij}^X f_j^X \} = t_i$ for $1 \leq i \leq n$.

Building off the isomorphism ν of p. (1.5) we introduce $\mathbb{Z}_2$-graded Q-bimodules

$$\begin{aligned}
\widetilde{\mathcal{H}} &= \bigoplus_{Y,X\in ob(\mathcal{C})} \bigwedge F_\theta \otimes_k R/(\underline{t}) \otimes_k \bigwedge\Big(\bigoplus_{i=1}^{e(X)} k\xi_i^X \oplus \bigoplus_{i=1}^{e(Y)} k\xi_i^{Y*}\Big)\otimes_k k[[\underline{t}]] \\
\widetilde{\mathcal{K}} &= \bigoplus_{Y,X\in ob(\mathcal{C})} R/(\underline{t}) \otimes_k \bigwedge\Big(\bigoplus_{i=1}^{e(X)} k\xi_i^X \oplus \bigoplus_{i=1}^{e(Y)} k\xi_i^{Y*}\Big) \quad (5.1)
\end{aligned}$$

There are $\mathbb{Z}_2$-graded Q-bimodule isos induced from ν, and for which we use the same letter

$$\nu : \hat{\mathcal{H}} \xrightarrow{\cong} \widetilde{\mathcal{H}}, \qquad \nu : \hat{\mathcal{K}} \xrightarrow{\cong} \widetilde{\mathcal{K}}$$

We define $\{\tilde{\rho}_k\}_{k\geq 1}$ to be the higher products on $\widetilde{\mathcal{K}}$ induced by $\hat{\rho}_k$ on $\hat{\mathcal{K}}$ via ν in the obvious way, similarly for $\tilde{\rho}_T$.

Let $k \geq 2$ and $T \in \mathcal{T}_k$ have only internal vertices of valency 3. In the following we give $\widetilde{\mathcal{H}}$ and $\widetilde{\mathcal{K}}$ the modified $\mathbb{Q}$-bimodule structure of ainfmf28 (30.2), when interpreting diagrams.

Def^n Consider the following decoration of $A(T)$ by $\mathbb{Q}$-bimodules

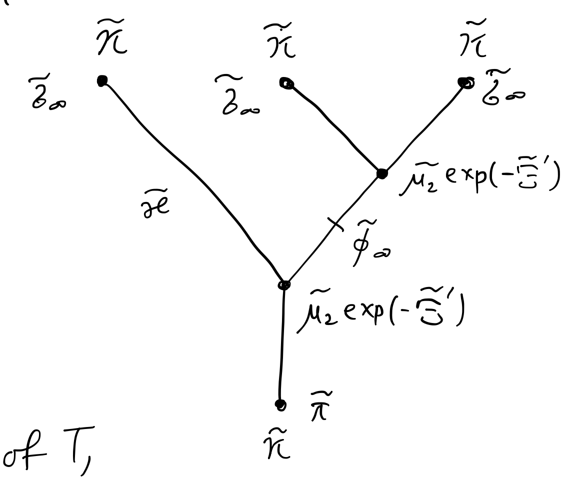
- $\widetilde{\mathcal{H}}$ to each internal edge of T and $\widetilde{\mathcal{K}}$ to each leaf of T,

- $\widetilde{\zeta}_\infty = \nu \hat{\zeta}_\infty \nu^{-1} : \widetilde{\mathcal{K}} \longrightarrow \widetilde{\mathcal{H}}$ to each input,

- $\widetilde{\pi} = \nu \hat{\pi} \nu^{-1} : \widetilde{\mathcal{H}} \longrightarrow \widetilde{\mathcal{K}}$ to the root,

- $\widetilde{\phi}_\infty = \nu \hat{\phi}_\infty \nu^{-1} : \widetilde{\mathcal{H}} \longrightarrow \widetilde{\mathcal{H}}$ to each internal edge of T,

- $\widetilde{\mu}_2 \exp(-\widetilde{\Xi}')$ to each internal vertex, where we define $\widetilde{\mu}_2$ to be



$$\widetilde{\mathcal{H}} \widetilde{\otimes}_{\mathbb{Q}} \widetilde{\mathcal{H}} \xrightarrow{\nu^{-1} \otimes \nu^{-1}} \hat{\mathcal{H}} \widetilde{\otimes}_{\mathbb{Q}} \hat{\mathcal{H}} \xrightarrow{\hat{\mu}_2} \hat{\mathcal{H}} \xrightarrow{\nu} \widetilde{\mathcal{H}},$$

and $\widetilde{\Xi}'$ to be $[\lambda_i, -]^{\sim} \otimes O_i^*$ where $[\lambda_i, -]^{\sim}$ is

$$\widetilde{\mathcal{H}} \xrightarrow{\nu^{-1}} \hat{\mathcal{H}} \xrightarrow{[\lambda_i, -]^{\wedge}} \hat{\mathcal{H}} \xrightarrow{\nu} \widetilde{\mathcal{H}}.$$

Def^n Let eval_T denote the $\mathbb{Q}$-bilinear map $\widetilde{\mathcal{K}}^{\widetilde{\otimes}_{\mathbb{Q}} k} \longrightarrow \widetilde{\mathcal{K}}$ obtained by evaluating the above decorated tree, without Koszul signs. That is, we simply feed in inputs at the top of the tree and evaluate each operator in succession. More precisely, we take the branch denotation in the category of <u>ungraded</u> $\mathbb{Q}$-bimodules.

Then by p.33 ainfmf28

<u>Proposition</u> Given $x_1,\dots,x_k \in \tilde{\mathcal{K}}$ and a tree T as above,

$$\tilde{\rho}_T(x_1,\dots,x_k) = (-1)^{e_i(T)+\sum_{i<j}\tilde{x}_i\tilde{x}_j+\sum_i \tilde{x}_i P_i + k+1}\, \mathrm{eval}_{\hat{T}}(x_k,\dots,x_1)$$

where P_i is the parity of the ith leaf (see p.15 ainfmf2).

So it remains to describe $\tilde{\delta}_\infty$, $\tilde{\pi}$, $\tilde{\phi}_\infty$, $\tilde{\mu}_2 \exp(-\tilde{\Xi}')$.

$\boxed{\tilde{\delta}_\infty}$ From (3.1) of ainfmf29 we have

$$\tilde{\delta}_\infty = \nu\hat{\delta}_\infty\nu^{-1} = \sum_{m\geq 0}(-1)^m \nu(\hat{H}\hat{d}_{\mathrm{Hom}})^m\hat{\delta}\nu^{-1}$$

$$= \sum_{m\geq 0}(-1)^m(\tilde{H}\tilde{d}_{\mathrm{Hom}})^m\tilde{\delta}$$

where $\tilde{H} = \nu\hat{H}\nu^{-1}$ is given by the same formula as $\hat{H}$, $\tilde{\delta}: \mathcal{K} \to \tilde{\mathcal{H}}$ is the obvious inclusion, and $\tilde{d}_{\mathrm{Hom}} = \nu\hat{d}_{\mathrm{Hom}}\nu^{-1}$ is derived from the operator of (4.2), i.e.

$$\sum_{i=1}^{e(X)} f_i^X \xi_i^{X*} + \sum_{i=1}^{e(X)} g_i^X \xi_i^X - \sum_{i=1}^{e(Y)} f_i^Y \xi_i^{Y*} + \sum_{i=1}^{e(Y)} g_i^Y \xi_i^Y$$

$$\circlearrowright$$

$$\Lambda\left(\bigoplus_{i=1}^{e(Y)} k\xi_i^{Y*} \oplus \bigoplus_{i=1}^{e(X)} k\xi_i^X\right)\otimes_k R$$

by applying the same logic as in p.26 ainfmf28. That is, as an operator on $\tilde{\mathcal{H}}$

$$\tilde{d}_{\mathrm{Hom}} = \sum_{i=1}^{e(X)}(f_i^X)^\# \xi_i^{X*} + \sum_{i=1}^{e(X)}(g_i^X)^\#\xi_i^X - \sum_{i=1}^{e(Y)}(f_i^Y)^\#\xi_i^{Y*} + \sum_{i=1}^{e(Y)}(g_i^Y)^\#\xi_i^Y.$$

Using the notation of ainfmf28 (26.3) we may write, parroting (8.2) of ainfmf29,

$$\tilde{d}_{Hom} = \sum_{\delta} \tilde{d}_{Hom}^{(\delta)} \otimes t^{\delta} \qquad (8.1)$$

where (8.2)

$$\tilde{d}_{Hom}^{(\delta)} = \sum_{i=1}^{e(X)} (f_i^{X\#})_{\delta}\, \xi_i^{X*} + \sum_{i=1}^{e(X)} (g_i^{X\#})_{\delta}\, \xi_i^{X} - \sum_{i=1}^{e(Y)} (f_i^{Y\#})_{\delta}\, \xi_i^{Y*} + \sum_{i=1}^{e(Y)} (g_i^{Y\#})_{\delta}\, \xi_i^{Y}.$$

recalling that for $r \in R$,

$$r_{\delta}^{\#}(z_j) = \sum_{\alpha+\beta=\delta} \sum_{\ell,k=1}^{\mu} r_{k\alpha} \gamma_{\ell\beta}^{kj} \cdot z_{\ell}$$

The explicit description of $\hat{\mathfrak{z}}_{\infty}$ given in (10.3) ainfmf29 applies mutatis mutandis to $\tilde{\mathfrak{z}}_{\infty}$, just replacing $\hat{d}_{Hom}$ everywhere by $\tilde{d}_{Hom}$.

$\boxed{\tilde{\pi}}$ is just the obvious projection $\tilde{\mathcal{H}} \longrightarrow \tilde{\pi}$.

$\boxed{\tilde{\phi}_{\infty}}$ is, again following p. 11 ainfmf29 and the above,

$$\tilde{\phi}_{\infty} = \sum_{m \geq 0} (-1)^m (\tilde{H} \tilde{d}_{Hom})^m \tilde{H},$$

and the explicit description of (12.2), ainfmf29 again holds with $\hat{d}_{Hom}^{(\delta)}$ replaced by $\tilde{d}_{Hom}^{(\delta)}$.

$\boxed{\tilde{\mu}_2}$ See ainfmf28 (32.2) for $\hat{\mu}_2$.

<u>Lemma</u> Given $X, Y, Z \in \mathrm{ob}(\mathcal{C})$ we have commutativity of

$$
\begin{tikzcd}
 & \mathrm{Hom}_k(\{f^Y, g^Y\}^\sim, \{f^X, g^X\}^\sim) \otimes_k \mathrm{Hom}_k(\{f^Z, g^Z\}^\sim, \{f^Y, g^Y\}^\sim) \arrow[dl, "\nu\otimes\nu"'] \arrow[dr, "- \circ -"] & \\
\begin{array}{l}\Lambda(\oplus_i k\xi_i^X) \otimes_k \Lambda(\oplus_i k\xi_i^{Y*}) \\ \quad \otimes_k \Lambda(\oplus_i k\xi_i^Y) \otimes_k \Lambda(\oplus_i k\xi_i^{Z*})\end{array} \arrow[dr, "1\otimes c\otimes 1"'] & & \mathrm{Hom}_k(\{f^Z, g^Z\}^\sim, \{f^X, g^X\}^\sim) \arrow[dl, "\nu"] \\
 & \Lambda(\oplus_i k\xi_i^X) \otimes_k \Lambda(\oplus_i k\xi_i^{Z*}) &
\end{tikzcd}
\tag{9.1}
$$

where c is the contraction operator (not from extk)

$$\Lambda(\oplus_i k\xi_i^{Y*}) \otimes_k \Lambda(\oplus_i k\xi_i^Y) \cong \Lambda(\oplus_i k\xi_i^Y)^* \otimes_k \Lambda(\oplus_i k\xi_i^Y) \xrightarrow{\mathrm{ev}} k$$

$$(\xi_{i_1}^{Y*} \wedge \cdots \wedge \xi_{i_p}^{Y*}) \otimes (\xi_{i_1}^Y \wedge \cdots \wedge \xi_{i_q}^Y) \longmapsto \mathrm{ev}\Big((-1)^{\binom{p}{2}} [\xi_{i_1}^Y \wedge \cdots \wedge \xi_{i_p}^Y]^*, \ \xi_{i_1}^Y \wedge \cdots \wedge \xi_{i_q}^Y\Big)$$

⌜where $\mathrm{ev}(\eta \otimes x) = \eta(x)$⌟

<u>Proof</u> By (2.1) we have

$$c(\nu \otimes \nu)(\phi_1 \otimes \phi_2) = c(\nu(\phi_1) \otimes \nu(\phi_2)) \tag{9.2}$$

$$= \sum_{i_1 < \cdots < i_p} \sum_{j_1 < \cdots < j_q} c\Big\{ \phi_1(\xi_{i_1} \wedge \cdots \wedge \xi_{i_p}) \otimes \xi_{i_1}^* \wedge \cdots \wedge \xi_{i_p}^* (-1)^{\binom{p}{2}} \otimes \phi_2(\xi_{j_1} \wedge \cdots \wedge \xi_{j_q}) \otimes \xi_{j_1}^* \wedge \cdots \wedge \xi_{j_q}^* (-1)^{\binom{q}{2}} \Big\}$$

$$= \sum_{\substack{i_1 < \cdots < i_p \\ j_1 < \cdots < j_q}} \phi_2(\xi_{j_1} \wedge \cdots \wedge \xi_{j_q})_{i_1 \cdots i_p} \, \phi_1(\xi_{i_1} \wedge \cdots \wedge \xi_{i_p}) \otimes \xi_{j_1}^* \wedge \cdots \wedge \xi_{j_q}^* \, (-1)^{\binom{q}{2}}.$$

Going the other way,

$$\nu(\phi_1 \circ \phi_2) = \sum_{i_1<\cdots<i_p} \phi_1\big(\phi_2(\xi_{i_1}\cdots\xi_{i_p})\big) \otimes \xi_{i_1}^* \cdots \xi_{i_p}^* \; (-1)^{\binom{p}{2}}$$

$$= \sum_{\substack{i_1<\cdots<i_p \\ j_1<\cdots<j_q}} \phi_1\big(\phi_2(\xi_{i_1}\cdots\xi_{i_p})_{j_1\cdots j_q}\, \xi_{j_1}\cdots\xi_{j_q}\big) \otimes \xi_{i_1}^* \cdots \xi_{i_p}^* (-1)^{\binom{p}{2}} \tag{10.1}$$

$$= \sum_{\substack{i_1<\cdots<i_p \\ j_1<\cdots<j_q}} \phi_2(\xi_{i_1}\cdots\xi_{i_p})_{j_1\cdots j_q}\, \phi_1(\xi_{j_1}\cdots\xi_{j_s}) \otimes \xi_{i_1}^* \cdots \xi_{i_p}^* (-1)^{\binom{p}{2}}$$

as claimed. □

Now, $\tilde{\mu}_2$ is defined on p. 6 to be

$$\tilde{\mathcal{H}} \tilde{\otimes}_{\mathbb{Q}} \tilde{\mathcal{H}} \xrightarrow{\nu^{-1}\otimes\nu^{-1}} \hat{\mathcal{H}} \tilde{\otimes}_{\mathbb{Q}} \hat{\mathcal{H}} \xrightarrow{\hat{\mu}_2} \hat{\mathcal{H}} \xrightarrow{\nu} \tilde{\mathcal{H}}, \tag{10.2}$$

and using (32.2) of ainfmf28 we have ($\alpha \in \Lambda(\oplus k\xi^x)$, $\alpha' \in \Lambda(\oplus k\xi^{y*})$, $\beta \in \Lambda(\oplus k\xi^y)$, $\beta' \in \Lambda(\oplus k\xi^{z*})$)

$$\tilde{\mu}_2\big([\omega \otimes z_i \otimes \alpha \otimes \alpha'] \otimes [\omega' \otimes z_j \otimes \beta \otimes \beta']\big) \tag{10.3}$$

$$\parallel$$

$$(-1)^{|\omega'||\alpha| + |\omega'||\alpha'|} \sum_{k=1}^{\mu} \sum_{\delta} \gamma^{ij}_{k\delta}\, c(\alpha', \beta) \cdot \omega \wedge \omega' \otimes z_k \otimes \alpha \otimes \beta' \otimes t^{\delta}$$

$\boxed{\tilde{\Xi}'}$ is defined to be $[\lambda_i,-]^{\sim} \otimes O_i^*$ where $[\lambda_i,-]^{\sim}$ is

$$\tilde{\mathcal{H}} \xrightarrow{v^{-1}} \hat{\mathcal{H}} \xrightarrow{[\lambda_i,-]^{\wedge}} \hat{\mathcal{H}} \xrightarrow{v} \tilde{\mathcal{H}}, \tag{11.1}$$

Here is where we use the hypothesis that λ_i^X is as in (5.2), that is,

$$\lambda_i^X = \sum_{j=1}^{e(X)} F_{ij}^X \xi_j^{X*} + \sum_{j=1}^{e(X)} G_{ij}^X \xi_j^X. \tag{11.2}$$

Note $[\lambda_i,-]$ is given by (13.2) ainfmf29 and is the same in form as d_{Hom}. So by the analysis given above in (8.1), (8.2), we have

$$[\lambda_i,-]^{\sim} = \sum_{\delta} [\lambda_i,-]^{\sim(\delta)} \otimes t^{\delta} \tag{11.3}$$

where

$$\begin{aligned} [\lambda_i,-]^{\sim(\delta)} = & \sum_{j=1}^{e(X)} (F_{ij}^X)_{\delta}^{\#} \xi_j^{X*} + \sum_{j=1}^{e(X)} (G_{ij}^X)_{\delta}^{\#} \xi_j^X \\ & - \sum_{j=1}^{e(Y)} (F_{ij}^Y)_{\delta}^{\#} \xi_j^{Y*} + \sum_{j=1}^{e(Y)} (G_{ij}^Y)_{\delta}^{\#} \xi_j^Y . \end{aligned} \tag{11.4}$$

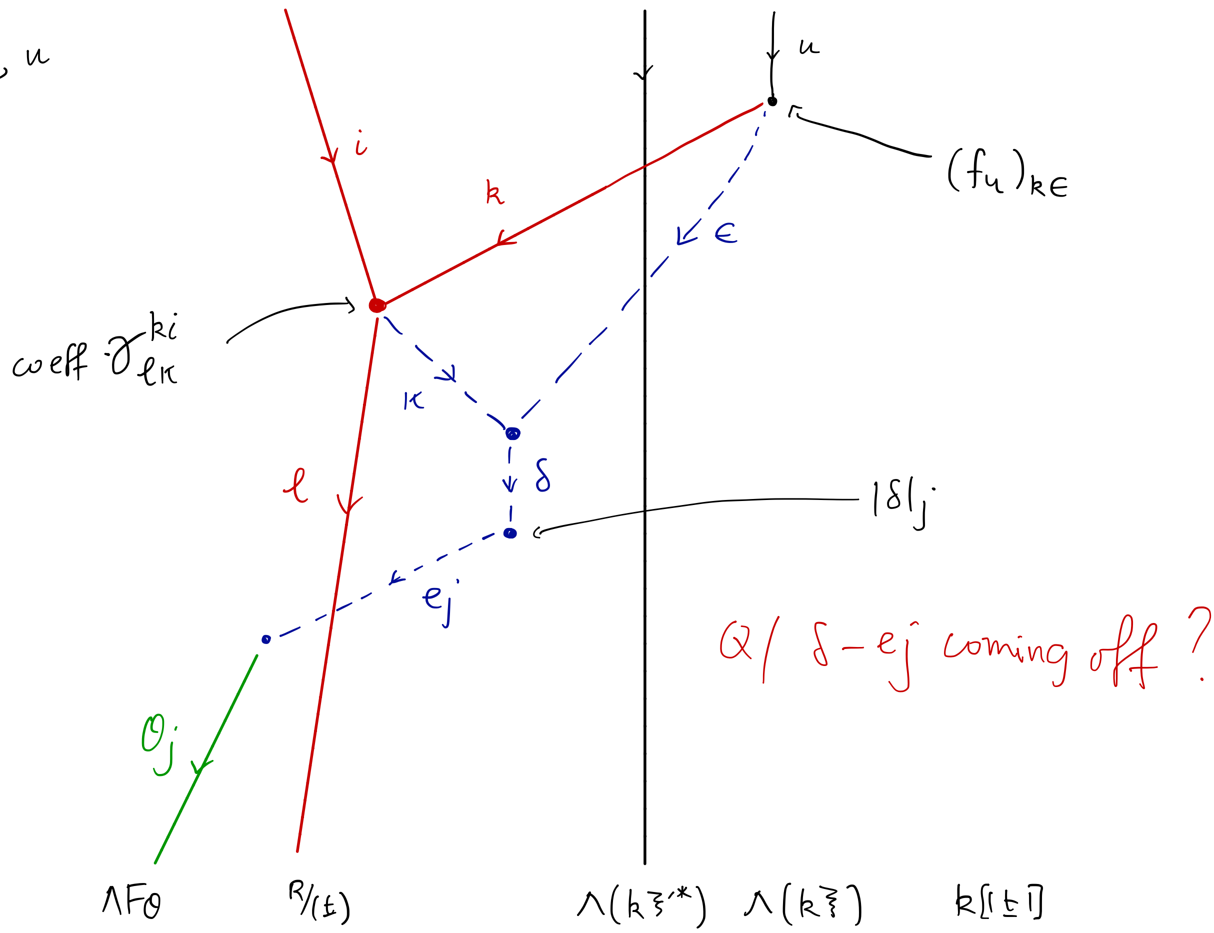
note the sign!

Moreover (14.3) ainfmf29 applies to describe $\tilde{\mu}_2 \exp(-\tilde{\Xi}')$, replacing $\wedge$'s by $\sim$'s.

ainfmf30 (12)

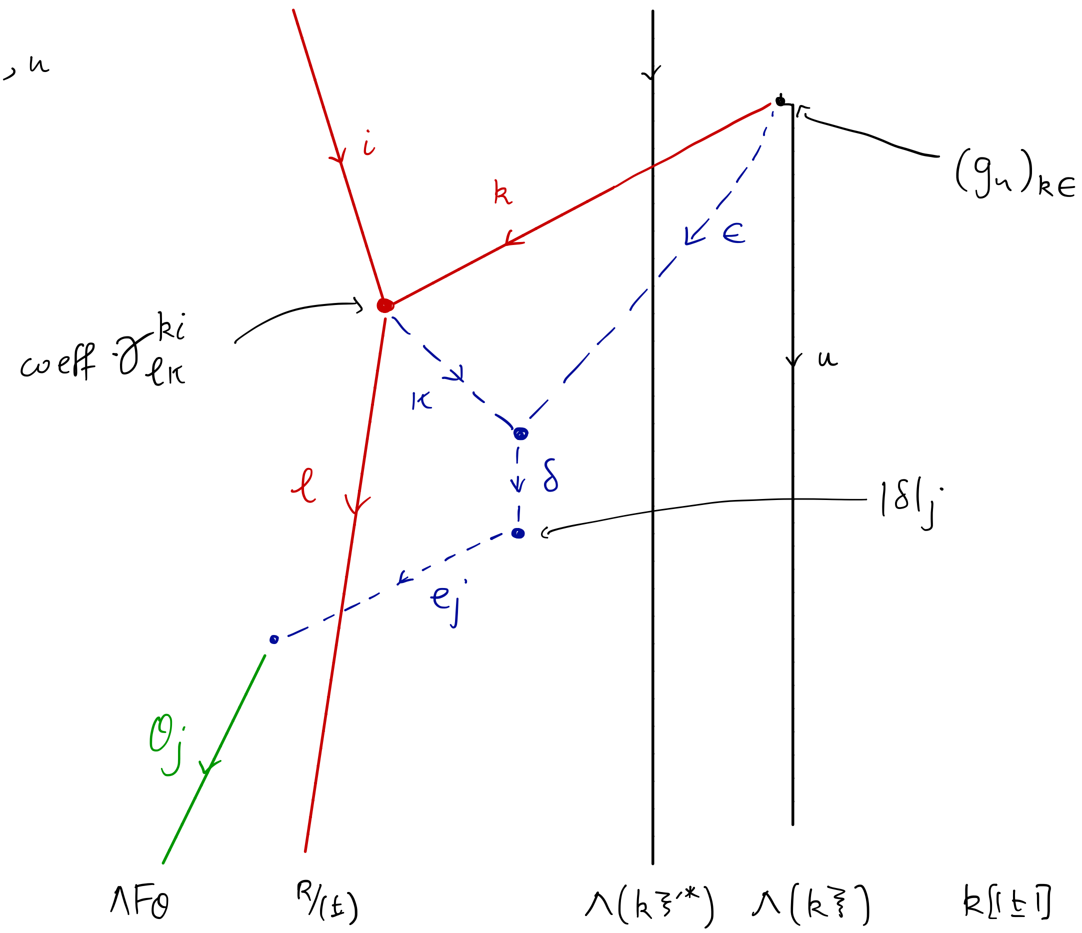
$$d_X = \underbrace{\sum_{u=1}^{e} f_u \xi_u^*}_{(A)} + \underbrace{\sum_{u=1}^{e} g_u \xi_u}_{(B)}$$

(A), u



Q/ $\delta - e_j$ coming off?

(B), u



Remark We have

$$\left[\lambda^x_a, \lambda^x_b\right] = \Big[\sum_{j=1}^{e(x)} F^x_{aj}\xi^{x*}_j + \sum_{j=1}^{e(x)} G^x_{aj}\xi^x_j ,$$

$$\sum_{j=1}^{e(x)} F^x_{bj}\xi^{x*}_j + \sum_{j=1}^{e(x)} G^x_{bj}\xi^x_j \Big]$$

$$= \sum_{j=1}^{e(x)} F^x_{aj} G^x_{bj} + \sum_{j=1}^{e(x)} G^x_{aj} F^x_{bj}$$

The default choice $F^x_{ij} = \frac{\partial}{\partial x_i}(f^x_j)$, $G^x_{ij} = \frac{\partial}{\partial x_i}(g^x_j)$ makes this

$$= \sum_{j=1}^{e(x)} \Big(\frac{\partial}{\partial x_a}(f^x_j) \frac{\partial}{\partial x_b}(g^x_j)$$

$$+ \frac{\partial}{\partial x_a}(g^x_j) \frac{\partial}{\partial x_b}(f^x_j)\Big)$$

$$= \sum_{j=1}^{e(x)} \frac{\partial}{\partial x_a}\frac{\partial}{\partial x_b}\left(f^x_j g^x_j \right)$$

$$= \frac{\partial^2}{\partial x_a \partial x_b}(W) - \sum_{j=1}^{e(x)} \Big\{ \frac{\partial^2}{\partial x_a \partial x_b}(f^x_j) g^x_j$$

$$+ f^x_j \frac{\partial^2}{\partial x_a \partial x_b}(g^x_j)\Big\}.$$

There is no reason to think this is zero, in general.

Alternative presentation for endos

We recheck a claim from ainfmf6b p.4 which asserts that for $e \geq 1$ with $T = \bigoplus_{i=1}^{e} k\psi_i$ for $|\psi_i| = 1$ we have an isomorphism of $\mathbb{Z}_2$-graded k-modules

$$\Lambda(k\psi_1 \oplus \cdots \oplus k\psi_e) \otimes_k \Lambda(k\xi_1 \oplus \cdots \oplus k\xi_e) \xrightarrow[\rho]{\cong} \mathrm{End}_k(\Lambda T).$$

$$\rho(\psi_{i_1} \wedge \cdots \wedge \psi_{i_r} \otimes \xi_{j_1} \wedge \cdots \wedge \xi_{j_s}) = \psi_{i_1} \circ \cdots \circ \psi_{i_r} \circ \psi_{j_1}^* \circ \cdots \circ \psi_{j_s}^* \qquad (14.1)$$

where ψ_i^* means contraction and on the RHS ψ_i means $\psi_i \wedge (-)$ as usual. Note that this formula gives a well-defined degree zero map if we view it as defined on the basis where $i_1 < \cdots < i_r$, $j_1 < \cdots < j_s$. But clearly the formula also correctly computes the image on a sequence out of order or with repeats, so in fact ρ is given by the indicated formula for any sequences $\underline{i}, \underline{j}$. The isomorphism is stated in §2.4 of the cut systems paper.

By induction on the length any product of creation and annihilation operators can be put into the form on the RHS of (14.1) and since these span, (14.1) is surjective. To see that (14.1) is injective notice that we can include the LHS into the Clifford algebra C of cliff4 (via $\psi_i \leftrightarrow \theta_i$, $\xi_i \leftrightarrow \theta_i^*$), so it suffices to prove the $\theta_{i_1} \cdots \theta_{i_r} \theta_{j_1}^* \cdots \theta_{j_s}^*$ in C form a k-basis of that algebra. But this is implicit in the proof of the lemma on p.2 of cliff4. So actually ρ is the isomorphism of $\mathbb{Z}_2$-graded k-modules underlying an isomorphism of the Clifford algebra C with $\mathrm{End}_k(\Lambda T)$, as $\mathbb{Z}_2$-graded algebras.

In particular, with $F_\psi = \bigoplus_{i=1}^{e} k\psi_i$ and $F_\xi = \bigoplus_{i=1}^{e} k\xi_i$ the diagram

$$\begin{tikzcd}
(\wedge F_\psi \otimes \wedge F_\xi) \otimes (\wedge F_\psi \otimes \wedge F_\xi) \arrow[r, "\rho\otimes\rho"'] \arrow[d, "m"'] & \mathrm{End}_k(\wedge T) \otimes \mathrm{End}_k(\wedge T) \arrow[d, "-\circ-"] \\
\wedge F_\psi \otimes \wedge F_\xi \arrow[r, "\rho"'] & \mathrm{End}_k(\wedge T)
\end{tikzcd}$$

commutes where m is multiplication in the Clifford algebra C, identifying this with $\wedge F_\psi \otimes \wedge F_\xi$ in the natural way (i.e. $\psi_{i_1} \cdots \psi_{i_r} \otimes \xi_{j_1} \cdots \xi_{j_s}$ is identified with $\psi_{i_1} \cdots \psi_{i_r} \psi_{j_1}^* \cdots \psi_{j_s}^*$). For example

$$\begin{aligned} m(\psi_1 \xi_1 \otimes \psi_2 \xi_2) &= \psi_1 \xi_1 \psi_2 \xi_2 \qquad \text{(Clifford algebra product)} \\ &= -\psi_1 \psi_2 \xi_1 \xi_2 \end{aligned}$$

where in this calculation ξ_i stands for a generator of the Clifford algebra, and

$$\begin{aligned} m(\psi_1 \xi_1 \otimes \psi_1 \xi_2) &= \psi_1 \xi_1 \psi_1 \xi_2 \\ &= \psi_1 (1 - \psi_1 \xi_1) \xi_2 \\ &= \psi_1 \xi_2 - \psi_1^2 \xi_1 \xi_2 \\ &= \psi_1 \xi_2 \end{aligned}$$

In ainfmf6b we state that ρ identifies

$$\begin{aligned} 1 \otimes \xi_i^* \lrcorner (-) &\quad \text{with} \quad [\psi_i, -] \\ \psi_i^* \lrcorner (-) \otimes 1 &\quad \text{with} \quad [\psi_i^*, -]. \end{aligned} \tag{15.2}$$

where on the RHS we have graded commutators with $\psi_i \wedge (-)$, $\psi_i^* \lrcorner (-)$, that is, the graded commutator in $\mathrm{End}_k(\wedge T)$ as a $\mathbb{Z}_2$-graded algebra.

But since ρ is an isomorphism of algebras, this means that on

$$C := \Lambda F_\psi \otimes \Lambda F_\xi$$

we have

$$(16.1) \qquad \begin{aligned} 1 \otimes \xi_i^* \lrcorner (-) &= [\psi_i, -] \qquad \text{(graded commutator of } C) \\ \psi_i^* \lrcorner (-) \otimes 1 &= [\xi_i, -] \qquad \text{(graded commutator of } C) \end{aligned}$$

and in particular these operators are both graded derivations (see Lemma 4.18 of the cut-systems paper) with respect to the Clifford multiplication on C. We present Feynman diagrams by commuting annihilation operators upwards in trees, and so the fact that the operators in (16.1) are derivations means we simply have two summands: one where a ξ_i^* (resp. ψ_i^*) annihilates with creation operators on the left branch and one where it annihilates on the right.

The effect of μ_2 in the (9.1) picture is similar, but there is a boundary condition (no ξ_i^y's after the vertex) and an effective interaction

$$\exp\Big(\textstyle\sum_i \underbrace{(\xi_i^{y*})^* \otimes (\xi_i^y)^*}_{\text{both contraction operators}} \Big) \qquad (16.2)$$

With respect to the ξ_i^x, ξ_i^z contractions this μ_2 is a derivation, so the Feynman diagrams have the same interpretation. We make this precise in the following way.

We write $F_{\xi,x} = \bigoplus_i k\xi_i^x$ and similarly for other spaces, e.g. $F_{\xi,z}$.
In the context of p. 9 let P denote the operator which projects onto the subspace $\bigwedge F_{\xi,x} \otimes \bigwedge F_{\xi,z}^* \subseteq \bigwedge F_{\xi,x} \otimes \bigwedge F_{\xi,y}^* \otimes \bigwedge F_{\xi,y} \otimes \bigwedge F_{\xi,z}^*$. Then if we write $\psi_i := \xi_i^{y*}$ for the generator in $F_{\xi,y}^*$ (i.e. <u>not</u> contraction with $\xi_i^y \in F_{\xi,y}$)

<u>Lemma</u> The following diagram commutes

from (9.1), $1 \otimes c \otimes 1$ there

$$
\begin{tikzcd}
\bigwedge\left(F_{\xi,x} \oplus F_{\xi,y}^* \oplus F_{\xi,y} \oplus F_{\xi,z}^*\right) \arrow[r, "c"] \arrow[d, "\exp(-\sum_i \psi_i^* \xi_i^{y*})"'] & \bigwedge\left(F_{\xi,x} \oplus F_{\xi,z}^*\right) \arrow[d, equal] \\
\bigwedge\left(F_{\xi,x} \oplus F_{\xi,y}^* \oplus F_{\xi,y} \oplus F_{\xi,z}^*\right) \arrow[r, "P"'] & \bigwedge\left(F_{\xi,x} \oplus F_{\xi,z}^*\right)
\end{tikzcd}
\qquad (17.1)
$$

updated 14/3/19 to fix this sign

where ψ_i^* means $\psi_i^* \lrcorner (-)$ and ξ_i^{y*} means $\xi_i^{y*} \lrcorner (-)$, as in

$$\bigwedge\left(F_{\xi,x} \oplus F_{\xi,y}^* \oplus F_{\xi,y} \oplus F_{\xi,z}^*\right)$$

ψ_i^* means contraction with respect to $(\xi_i^{y*})^*$

ξ_i^{y*} means contraction with respect to the dual of a generator here

<u>Proof</u> We just have to check both ways around the diagram agree on the k-basis $\xi_{\underline{i}}^x \psi_{\underline{j}} \xi_{\underline{k}}^y \xi_{\underline{\ell}}^{z*}$ but we have already calculated that under c this maps to $(-1)^{\binom{|\underline{j}|}{2}} \delta_{\underline{j}=\underline{k}} \xi_{\underline{i}}^x \xi_{\underline{\ell}}^{z*}$. The other way, it maps to

$$P\left(\exp\left(-\sum_i \psi_i^* \xi_i^{y*}\right)\left(\xi_{\underline{i}}^x \psi_{\underline{j}} \xi_{\underline{k}}^y \xi_{\underline{\ell}}^{z*} \right)\right)$$

$$= \xi_{\underline{i}}^x \cdot P\left(\exp\left(-\sum_i \psi_i^* \xi_i^{y*}\right)\left(\psi_{\underline{j}} \xi_{\underline{k}}^y\right)\right) \cdot \xi_{\underline{\ell}}^{z*}$$

$$= \xi_{\underline{i}}^{x} \cdot P(\exp(-\textstyle\sum_i \psi_i^* \xi_i^{y*})(\psi_{\underline{j}} \xi_{\underline{k}}^{y})) \cdot \xi_{\underline{\ell}}^{z*}$$

and, supposing $\underline{j} = j_1 < \cdots < j_r$

$$P(\exp(-\textstyle\sum_i \psi_i^* \xi_i^{y*})(\psi_{\underline{j}} \xi_{\underline{k}}^{y}))$$

$$= \delta_{\underline{j}=\underline{k}} \frac{1}{r!} \sum_{\sigma \in S_r} (\psi_{j_{\sigma(1)}}^* \xi_{j_{\sigma(1)}}^{y*}) \cdots (\psi_{j_{\sigma(r)}}^* \xi_{j_{\sigma(r)}}^{y*})(\psi_{\underline{j}} \xi_{\underline{j}}^{y}) \cdot (-1)^r$$

$$= \delta_{\underline{j}=\underline{k}} \frac{1}{r!} \sum_{\sigma \in S_r} (\psi_{j_r}^* \xi_{j_r}^{y*}) \cdots (\psi_{j_1}^* \xi_{j_1}^{y*})(\psi_{\underline{j}} \xi_{\underline{j}}^{y}) \cdot (-1)^r$$

$$= \delta_{\underline{j}=\underline{k}} (-1)^{\binom{r+1}{2}} \xi_{j_r}^{y*} \cdots \xi_{j_1}^{y*} \psi_{j_r}^* \cdots \psi_{j_1}^* (\psi_{\underline{j}} \xi_{\underline{j}}^{y}) \cdot (-1)^r$$

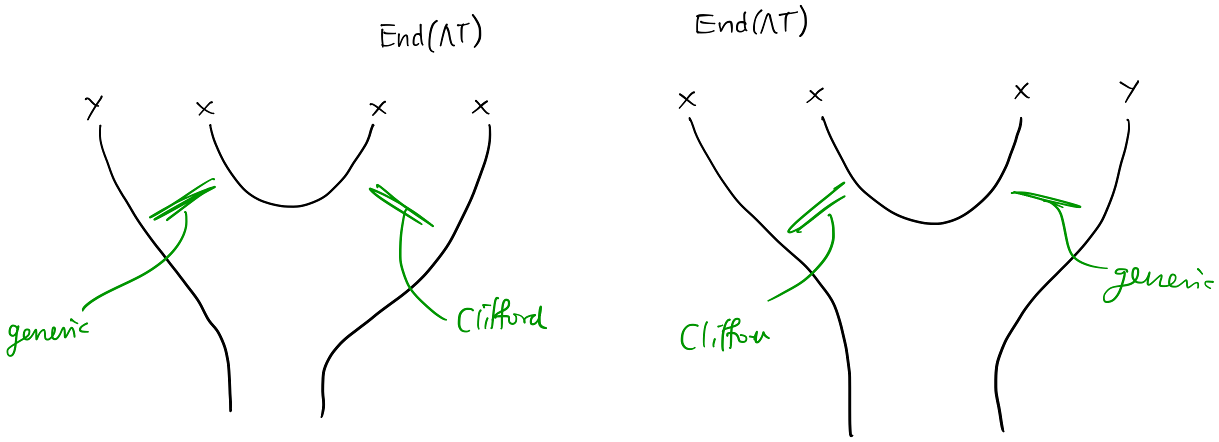
$$= (-1)^{\binom{r}{2}} \delta_{\underline{j}=\underline{k}}$$

as claimed. □

The upshot is that when $\{f, g\} \neq \{f', g'\}$ we stick with the isomorphism ν of (1.5.2) as a means of presenting the state space as a tensor product of exterior algebras and operators in terms of creation and annihilation operators. This means we have to include special interaction vertices (16.2) at internal vertices. When $\{f, g\} = \{f', g'\}$ we may choose instead to present the state spaces using ρ of (14.1). This has the advantage that d_ζ involves $[\psi_i, -]$, $[\psi_i^*, -]$ which by (15.2) are expressed directly as annihilation operators. In this case (16.1) tells us how to understand internal vertices.

The mixed cases

There are also mixed cases, where we want to use (1.5.2) and ν to present one of the Homs coming into a composition, and (14.1) and ρ to present the other. Graphically, this looks like



(19.1)

The notation is a bit of a mess. We want $\Lambda T = \{\underline{f}^{x}, \underline{g}^{x}\}^{\sim}$ which means in our earlier notation $T = \bigoplus_{i=1}^{e} k\,\xi_i^{x}$. But this conflicts with p. 14 where T is spanned by ψ_i's, the duals of which are named ξ_i. For simplicity we restate ρ as an isomorphism of $\mathbb{Z}_2$-graded k-modules

$$\rho: \quad \Lambda(F_{\xi,x}) \otimes \Lambda(F^{*}_{\xi,x}) \longrightarrow \mathrm{End}_k(\{\underline{f}^{x}, \underline{g}^{x}\}^{\sim})$$

$$\rho(\xi_{i_1}^{x} \wedge \cdots \wedge \xi_{i_r}^{x} \wedge \xi_{j_1}^{x*} \wedge \cdots \wedge \xi_{j_s}^{x*}) = \xi_{i_1}^{x} \circ \cdots \circ \xi_{i_r}^{x} \circ \xi_{j_1}^{x*} \circ \cdots \circ \xi_{j_s}^{x*}.$$

We use ν as defined earlier. The F notation is from the top of p. 17.

<u>Lemma</u> The diagram (20.1)

$$
\begin{tikzcd}
\mathrm{Hom}_k(\{\underline{f}^X,\underline{g}^X\}^\sim,\{\underline{f}^Y,\underline{g}^Y\}^\sim)\otimes_k \mathrm{End}_k(\{\underline{f}^X,\underline{g}^X\}^\sim) \arrow[r, "-\circ-"] \arrow[d, "\nu\otimes\rho^{-1}"'] & \mathrm{Hom}_k(\{\underline{f}^X,\underline{g}^X\}^\sim,\{\underline{f}^Y,\underline{g}^Y\}^\sim) \arrow[d, "\nu"] \\
(*)\quad \bigwedge F_{\xi,Y}\otimes\bigwedge F_{\xi,X}^*\otimes\bigwedge F_{\xi,X}\otimes\bigwedge F_{\xi,X}^* \arrow[d, "\exp(-\Sigma_i \psi_i^*\xi_i^{X*})"'] & \bigwedge F_{\xi,Y}\otimes\bigwedge F_{\xi,X}^* \\
\bigwedge F_{\xi,Y}\otimes\bigwedge F_{\xi,X}^*\otimes\underline{\underline{\bigwedge F_{\xi,X}}}\otimes\bigwedge F_{\xi,X}^* \arrow[r, "P"'] & \bigwedge F_{\xi,Y}\otimes\bigwedge F_{\xi,X}^*\otimes\bigwedge F_{\xi,X}^* \arrow[u, "1\otimes m"']
\end{tikzcd}
$$

notice the sign

commutes, where P projects out the $\bigwedge F_{\xi,X}$ indicated, and m is multiplication in the exterior algebra, and ψ_i^* means $(\xi_i^{X*})^* \lrcorner (-)$ as in (17.1).

<u>Proof</u> Let us begin with $\xi_{\underline{i}}^Y \psi_{\underline{j}} \xi_{\underline{k}}^X \xi_{\underline{\ell}}^{X*}$ in the middle left $(*)$, which is by (2.1) given in the top left by (set $r = |\underline{j}|$)

$$
\begin{aligned}
&\nu^{-1}(\xi_{\underline{i}}^Y \psi_{\underline{j}}) \otimes \rho(\xi_{\underline{k}}^X \xi_{\underline{\ell}}^{X*}) \\
&= (-1)^{\binom{r}{2}} \xi_{\underline{i}}^Y \circ (\xi_{\underline{j}}^X)^* \otimes \xi_{k_1}^X \circ \cdots \xi_{\ell_1}^{X*} \circ \cdots
\end{aligned}
$$

meaning projection onto this basis vector

Then this is mapped by composition and ν to

$$
\sum_{t_1<\cdots<t_z} (-1)^{\binom{z}{2}} (-1)^{\binom{r}{2}} \Big[\xi_{\underline{i}}^Y \circ (\xi_{\underline{j}}^X)^* \circ \xi_{k_1}^X \circ \cdots \circ \xi_{\ell_1}^{X*} \circ \cdots \Big] (\xi_{\underline{t}}^X) \wedge \xi_{\underline{t}}^{X*} \qquad (20.2)
$$

For a summand indexed by $\underline{t}$ to be nonzero we must have $\underline{\ell} \subseteq \underline{t}$ (as sets) and $\underline{k} \subseteq \underline{j}$ with $\underline{j}\setminus\underline{k} = \underline{t}\setminus\underline{\ell}$. But then either $\underline{k} \not\subseteq \underline{j}$ in which case all summands are zero or $\underline{k} \subseteq \underline{j}$ and the only nonzero contribution is from $\underline{t} = \underline{\ell} \sqcup \underline{j}\setminus\underline{k}$ (if $\underline{\ell} \cap \underline{j}\setminus\underline{k} \neq \phi$ the sum is again zero). Thus

$$(20.2) = (-1)^{\binom{|\underline{\ell}|+|\underline{j}|-|\underline{k}|}{2} + \binom{|\underline{j}|}{2} + \binom{|\underline{\ell}|}{2} + c + d} \xi_{\underline{i}}^{Y} \xi_{\underline{t}}^{X*}$$

where $\xi_{\underline{t}}^{X} = (-1)^{c} \xi_{\underline{\ell}}^{X} \xi_{\underline{j}\setminus\underline{k}}^{X}$ and $\xi_{\underline{j}}^{X} = (-1)^{d} \xi_{\underline{k}}^{X} \xi_{\underline{j}\setminus\underline{k}}^{X}$.

Going the other way around the square gives (writing $s = |\underline{k}|$)

$$(1\otimes m) P\exp\Big(-\textstyle\sum_i \psi_i^{*} \xi_i^{X*}\Big)\Big(\xi_{\underline{i}}^{Y} \psi_{\underline{j}} \xi_{\underline{k}}^{X} \xi_{\underline{\ell}}^{X*}\Big)$$

$$= (1\otimes m)\Big[\xi_{\underline{i}}^{Y} P\exp\Big(-\textstyle\sum_i \psi_i^{*} \xi_i^{X*}\Big)(\psi_{\underline{j}} \xi_{\underline{k}}^{X}) \xi_{\underline{\ell}}^{X*}\Big]$$

$$= \xi_{\underline{i}}^{Y}\, m P\Big[\sum_{z \geq 0} \frac{(-1)^{z}}{z!}\Big[\textstyle\sum_i \psi_i^{*} \xi_i^{X*}\Big]^{z} (\psi_{\underline{j}} \xi_{\underline{k}}^{X}) \xi_{\underline{\ell}}^{X*}\Big]$$

$$= \xi_{\underline{i}}^{Y}\, m P\Big[\frac{(-1)^{|\underline{k}|}}{|\underline{k}|!} \sum_{\gamma \in S_{|\underline{k}|}} (\psi_{k_s}^{*} \xi_{k_s}^{X*}) \cdots (\psi_{k_1}^{*} \xi_{k_1}^{X*})(\psi_{\underline{j}} \xi_{\underline{k}}^{X}) \xi_{\underline{\ell}}^{X*}\Big]$$

$$= \xi_{\underline{i}}^{Y}\Big[(\psi_{k_s}^{*} \xi_{k_s}^{X*}) \cdots (\psi_{k_1}^{*} \xi_{k_1}^{X*})(\psi_{\underline{j}} \xi_{\underline{k}}^{X}) \xi_{\underline{\ell}}^{X*}\Big] \cdot (-1)^{|\underline{k}|}$$

$$= (-1)^{|\underline{k}||\underline{j}| + \binom{|\underline{k}|}{2} + e + |\underline{k}|} \xi_{\underline{i}}^{Y} \psi_{\underline{j}\setminus\underline{k}} \xi_{\underline{\ell}}^{X*} \cdot \delta_{\underline{k}\subseteq\underline{j}}$$

where $\psi_{k_s}^{*} \cdots \psi_{k_1}^{*}(\psi_{\underline{j}}) = (-1)^{e} \psi_{\underline{j}\setminus\underline{k}}$. Now note that $e = d$ so

$$= (-1)^{|\underline{k}||\underline{j}| + d + (|\underline{j}|-|\underline{k}|)|\underline{\ell}| + \binom{|\underline{k}|}{2} + |\underline{k}|} \xi_{\underline{i}}^{Y} \xi_{\underline{\ell}}^{X*} \xi_{\underline{j}\setminus\underline{k}}^{X*} \cdot \delta_{\underline{k}\subseteq\underline{j}}$$

$$= (-1)^{|\underline{k}||\underline{j}| + d + (|\underline{j}|-|\underline{k}|)|\underline{\ell}| + c + \binom{|\underline{k}|}{2} + |\underline{k}|} \xi_{\underline{i}}^{Y} \xi_{\underline{t}}^{X*} \cdot \delta_{\underline{k}\subseteq\underline{j}}$$

So the problem is to show that (mod 2)

$$\binom{|\underline{\ell}|+|\underline{j}|-|\underline{k}|}{2} + \binom{|\underline{j}|}{2} + \binom{|\underline{\ell}|}{2} + \cancel{c} + \cancel{d} \equiv |\underline{k}||\underline{j}| + \cancel{d} + (|\underline{j}|-|\underline{k}|)|\underline{\ell}| + |\underline{k}| + \cancel{c} + \binom{|\underline{k}|}{2}$$

Now the LHS is $(x = |\underline{\ell}|,\ y = |\underline{j}|,\ z = |\underline{k}|)$

$$\tfrac{1}{2}(x+y-z)(x+y-z-1) + \tfrac{1}{2}y(y-1) + \tfrac{1}{2}x(x-1)$$

$$= \tfrac{1}{2}\Big[x^2 + xy - xz - x + yx + y^2 - yz - y - zx - zy + z^2 + z + y^2 - y + x^2 - x\Big]$$

$$= \tfrac{1}{2}\Big[2(x^2-x) + 2(y^2-y) + 2xy - 2xz - 2yz + z^2 + z\Big]$$

$$= x^2 - x + y^2 - y + xy - xz - yz + \tfrac{1}{2}z(z+1)$$

$$\equiv xy + xz + yz + \tfrac{1}{2}z(z+1)$$

whereas the RHS is $yz + yx + zx + z + \frac{1}{2}z(z-1)$, which matches. □

Let $\bullet$ denote multiplication in the Clifford algebra $C = \bigwedge F_\psi \otimes_k \bigwedge F_\xi$.

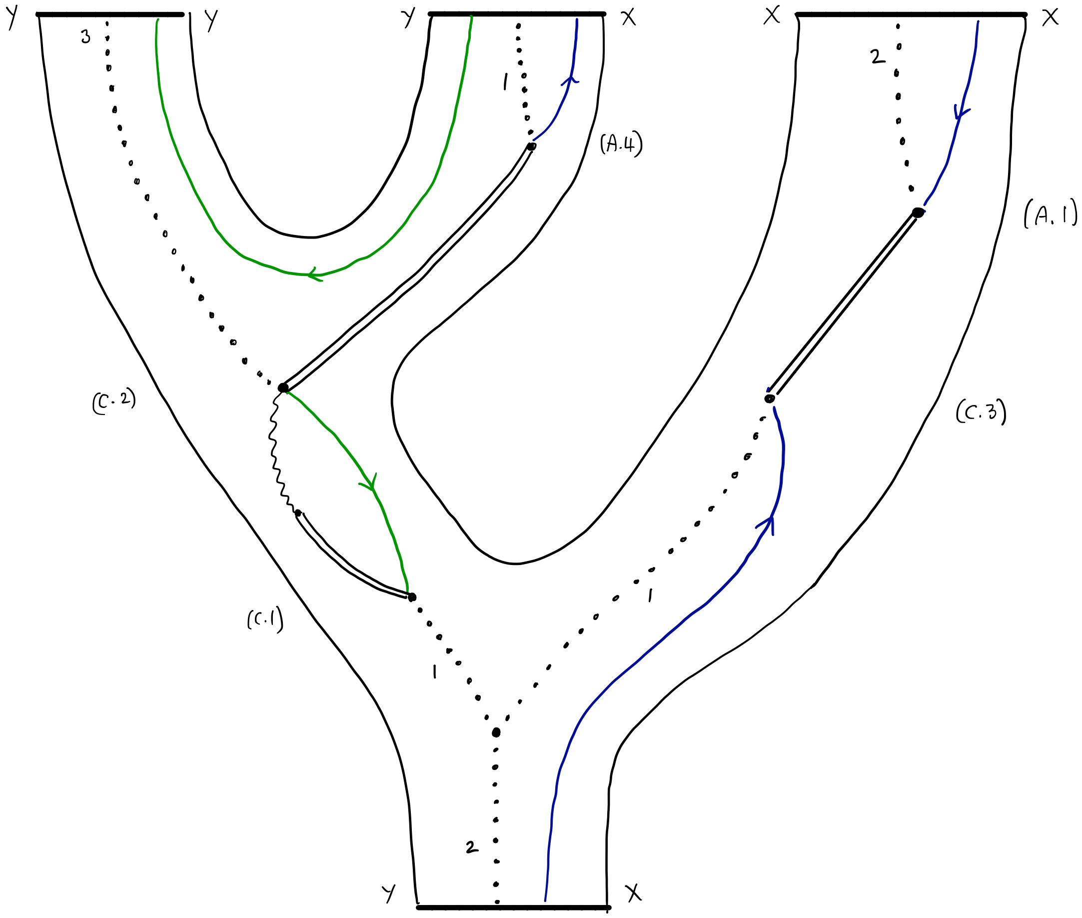
<u>Lemma</u> $\psi_i \bullet (-) = \psi_i \wedge (-) \otimes 1_{\wedge F_\xi}$ and $\xi_i \bullet (-) = \psi_i^* \lrcorner (-) \otimes 1 + 1 \otimes \xi_i \wedge (-)$.

<u>Proof</u> It suffices to compare the left and right on basis vectors. Then

$$\begin{aligned}
&\psi_i \bullet \left[\psi_{i_1} \wedge \cdots \wedge \psi_{i_r} \otimes \xi_{j_1} \wedge \cdots \wedge \xi_{j_s} \right] \\
&= \psi_i \bullet \psi_{i_1} \bullet \cdots \bullet \psi_{i_r} \bullet \xi_{j_1} \bullet \cdots \bullet \xi_{j_s} \\
&= \left[\psi_i \wedge \psi_{i_1} \wedge \cdots \wedge \psi_{i_r} \right] \otimes \left[\xi_{j_1} \wedge \cdots \wedge \xi_{j_s} \right],
\end{aligned}$$

and

$$\begin{aligned}
&\xi_i \bullet \left[\psi_{i_1} \wedge \cdots \wedge \psi_{i_r} \otimes \xi_{j_1} \wedge \cdots \wedge \xi_{j_s} \right] \\
&= \xi_i \bullet \psi_{i_1} \bullet \cdots \bullet \psi_{i_r} \bullet \xi_{j_1} \bullet \cdots \bullet \xi_{j_s} \\
&= \left[\xi_i, \psi_{i_1} \bullet \cdots \bullet \psi_{i_r} \right]_\bullet \bullet \xi_{j_1} \bullet \cdots \bullet \xi_{j_s} \\
&\qquad + (-1)^r \psi_{i_1} \bullet \cdots \bullet \psi_{i_r} \bullet \xi_i \bullet \xi_{j_1} \bullet \cdots \bullet \xi_{j_s} \\
&= \sum_{j=1}^r (-1)^{j-1} \psi_{i_1} \bullet \cdots \bullet \left[\xi_i, \psi_{i_j} \right]_\bullet \bullet \cdots \bullet \psi_{i_r} \bullet \xi_{j_1} \bullet \cdots \bullet \xi_{j_s} \\
&\qquad + (-1)^r \psi_{i_1} \bullet \cdots \bullet \psi_{i_r} \bullet \xi_i \bullet \xi_{j_1} \bullet \cdots \bullet \xi_{j_s} \\
&= \psi_i^* \lrcorner \left(\psi_{i_1} \wedge \cdots \wedge \psi_{i_r} \right) \otimes \xi_{j_1} \wedge \cdots \wedge \xi_{j_s} \\
&\qquad + (-1)^r \psi_{i_1} \wedge \cdots \wedge \psi_{i_r} \otimes \xi_i \wedge \xi_{j_1} \wedge \cdots \wedge \xi_{j_s}. \quad \square
\end{aligned}$$



Y
Y
Y
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(A.4)
(A.1)
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1
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X