Minimal models for MFs 33 (rough)

In this note we explore the geometric ideas behind the strong deformation retract appearing in our other notes in this series, for instance ainfmf28.
We employ the following references

[CQ] Cuntz and Quillen, "Algebra extensions and nonsingularity" 1995.

[B] Burszytn et al "Splitting theorems for Poisson and related structures" 2017

[L] Lipman "Residues and traces of differential forms via Hochschild homology" 1987.

[H] Hirsch "Differential topology".

We begin with by recalling some aspects of the theory of tubular neighborhoods, following mainly [B]. Let $Y \hookrightarrow X$ be a submanifold and denote by N the normal bundle. A <u>tubular neighborhood of Y in X</u> [H, Ch 4, Section 5] is an embedding $f: N \longrightarrow X$ such that $f(N) \subseteq X$ is open and

$$\begin{tikzcd} N \arrow[d] \arrow[dr, "f"] & \\ Y \arrow[r, hook] & X \end{tikzcd}$$

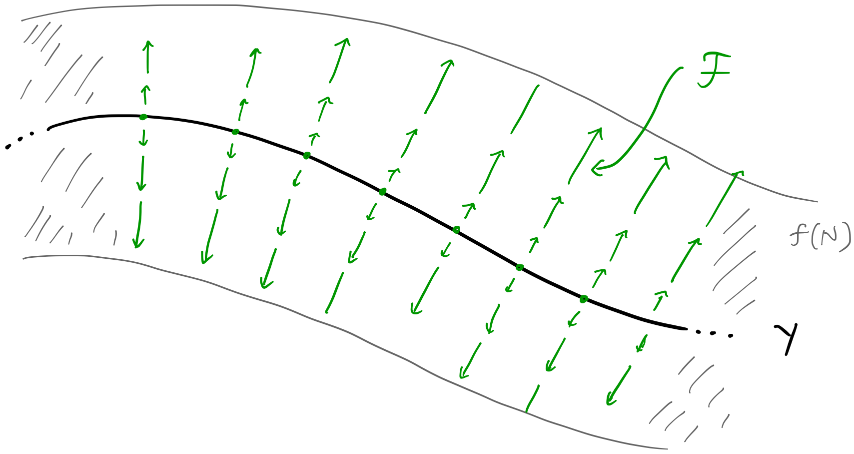
commutes. The action $\mathbb{R} \times N \longrightarrow N$ which scales the fibers has as its infinitesimal generator a vector field on N, the <u>Euler field</u> $\mathcal{E}$. In local bundle coordinates on N, with t_i being the coordinates in the fiber direction and y_i the coordinates in the base,

$$\mathcal{E} = \sum_i t_i \frac{\partial}{\partial t_i}.$$

Thus from our tubuluar neighborhood f we acquire a vector field on $f(N)$:

$$\mathcal{F} = f_*(\mathcal{E}) \in T_\infty(f(N), TX).$$

(2.1)



Conversely, any vector field $\mathcal{F}$ on an open neighborhood of Y with $\mathcal{F}|_Y = 0$, under some conditions on the 1-jet of $\mathcal{F}$, induces a tubular neighborhood f with $f_*\mathcal{E} = \mathcal{F}$, see [B, §2.3]. As explained in [B], the vector field $\mathcal{F}$ induces a splitting of the exact sequence of vector bundles on Y

$$0 \longrightarrow TY \longrightarrow TX|_Y \longrightarrow N \longrightarrow 0, \qquad (2.2)$$

see in particular [B, Remark 2.2 (iii)].

<u>Algebraic case</u>

Let k be a commutative ring, R a commutative k-algebra and $t_1, \dots, t_n \in R$ a quasi-regular sequence such that, with $I = (t_1, \dots, t_n)$, the k-module R/I is f.g. and projective. Let $\nabla_{\mathfrak{z}}$ be the usual connection induced by a section $\mathfrak{z}$,

$$\nabla_{\mathfrak{z}} : \hat{R} \longrightarrow \hat{R} \otimes_{k[\underline{z}]} \Omega^1_{k[\underline{z}]/k}$$

and let $d_K = \sum_i t_i \theta_i^*$ be the Koszul differential

$$(K, d_K) = \left(\Lambda(k\theta_1 \oplus \cdots \oplus k\theta_n) \otimes_k \hat{R},\ \sum_i t_i \theta_i^* \right).$$

As usual we interpret $\Lambda(k\theta_1 \oplus \cdots \oplus k\theta_n) \otimes_k \hat{R}$ as $\hat{R} \otimes_{k[\underline{z}]} \Omega^*_{k[\underline{z}]/k}$ and so $\nabla_{\mathfrak{z}}$ extends to a k-linear operator on K. Then

$$d_K \nabla^\circ_{\mathfrak{z}} = \sum_i t_i \frac{\partial}{\partial z_i} : \hat{R} \longrightarrow \hat{R}$$

is a k-linear operator. All of this is of course induced by the $k[[\underline{z}]]$-linear isomorphism

$$\mathfrak{z}^* : R/I[[z_1, \dots, z_n]] \longrightarrow \hat{R}.$$

This far the analogies with the manifold situation are as follows:

Differential	Algebraic
X	R
Y	R/I
N	$\mathrm{Sym}_{R/I}(I/I^2) \cong R/I[\underline{z}]$
$f(N)$	$\hat{R}$
$Y \subset U \subset N$	$R/I[[\underline{z}]]$
f	$\mathfrak{z}^*$
$\mathcal{F}$	$d_K \nabla^\circ_{\mathfrak{z}}$

The analogy is not perfect because $d_K \nabla^\circ_{\mathfrak{z}}$ is <u>not</u> a differential operator on $\hat{R}$, but this is tied to the fact that even if R is smooth, R/I is not assumed smooth, as compared to Y which is of course smooth.

Let us now recapitulate some material from [CQ, p. 277], and discuss what corresponds to the idempotent operator on $TX|_Y$ in (2.1) associated to the vector field $\mathcal{F}$. Consider the diagram

$$\begin{tikzcd} \hat{R} \arrow[r, shift left, "\nabla_0^{\delta}"] \arrow[d, shift left, "\pi"] & \hat{R}\otimes\Omega^1 \arrow[l, shift left, "d_K"] \\ R/I \arrow[u, shift left, "\delta"] & \end{tikzcd}$$

in which we have

$$\pi\delta = 1_{R/I}, \qquad \text{and} \qquad 1_{\hat{R}} - d_K\,[d_K, \nabla^{\delta}]^{-1}\,\nabla_0^{\delta} = \delta\pi.$$

That is to say, $e := 1_{\hat{R}} - d_K\,[d_K, \nabla^{\delta}]^{-1}\,\nabla_0^{\delta}$ is k-linear, sends I to zero, and induces on the quotient $\delta : R/I \longrightarrow R$. Post composing with $\pi' : R \longrightarrow R/I^2$ gives a k-linear

$$\pi'\circ\delta : R/I \longrightarrow R/I^2$$

which is a section of $R/I^2 \twoheadrightarrow R/I$, that is, a splitting of

$$\begin{tikzcd} 0 \arrow[r] & I/I^2 \arrow[r] & R/I^2 \arrow[r] & R/I \arrow[r] \arrow[l, bend right, dashed, "\pi'\delta"'] & 0. \end{tikzcd}$$

<u>Remark</u> Recall that if R/I were a smooth algebra then there would exist a <u>k-algebra</u> morphism $R/I \longrightarrow R/I^2$ splitting the above sequence. So it makes sense to ask how close $\pi'\delta$ is to being a ring morphism. Consider, for $r, s \in R$

$$\pi'\delta(\overline{rs}) = \overline{rs} - d_K\nabla_\delta(rs)$$

$$\pi'\delta(\bar{r})\cdot\pi'\delta(\bar{s}) = \{\bar{r} - d_K\nabla_\delta(r)\}\cdot\{\bar{s} - d_K\nabla_\delta(s)\}$$

$$= \overline{rs} - \bar{r}\,d_K\nabla_\delta(s) - \bar{s}\,d_K\nabla_\delta(r) \quad (\text{mod } I^2)$$

So in other words,

$$\pi'\delta(\overline{rs}) - \pi'\delta(\bar{r})\pi'\delta(\bar{s}) = \bar{r}\,d_K\nabla_\delta(s) + \bar{s}\,d_K\nabla_\delta(r) - d_K\nabla_\delta(rs).$$

This is a measure of $d_K\nabla_\delta$'s failure to actually be a derivation on $\hat{R}$. well, see p. 8

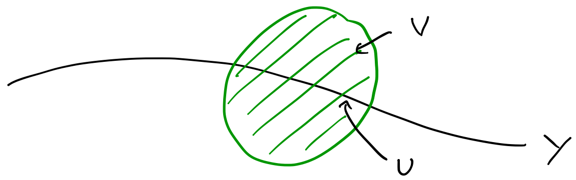
<u>Upshot</u> The k-linear idempotent $e = \delta\pi$ on $\hat{R}$, as the difference $1_{\hat{R}} - d_K\nabla_\delta^\circ$, is analogous to the vector field $1 - \mathcal{F}$ on $f(N)$, and it projects onto R/I, just as $1-\mathcal{F}$ gives rise to an idempotent on $TX|_Y$ which projects onto TY.

Thus the vector field $1-\mathcal{F}$ is analogous to the <u>strong deformation retract</u> of (K, d_K) onto $(R/I, 0)$, and ultimately of $(K, d_K)\otimes_{\hat{R}}\mathcal{C}$ onto $\mathcal{C}\otimes \mathrm{Jac}_W$. The idempotent arising on ΛF_θ is of a different nature, with no clear analogue in the differential picture, since it depends on cohomological support.

Moreover, as we have discussed above, in the case R/I is not smooth we cannot necessarily choose ∇_δ° to be such that $d_K\nabla_\delta^\circ$ is an actual derivation, so this is an <u>analogy</u> only.

<u>Splitting</u> with the notation of p.(2) we revisit the description from [B, CQ] of how the vector field gives rise to a splitting of (2.2). Recall that $TX|_Y$ is the bundle whose fiber over $y \in Y$ is the vector space T_yX, and a splitting of (2.2) means a bundle map $e : TX|_Y \longrightarrow TX|_Y$ with $e^2 = e$ and $e(TY) = 0$, with the induced map $\tilde{e} : N \longrightarrow TX|_Y$ satisfying $N \xrightarrow{\tilde{e}} TX|_Y \longrightarrow N = 1_N$.
If $U \subseteq Y$ is an open coordinate neighborhood, we need an $\mathcal{O}_Y(U)$-linear idempotent

$$e_U : T_\infty(U, TX|_Y) \longrightarrow T_\infty(U, TX|_Y).$$



Say $U = Y \cap V$, V a coordinate neighborhood in X. Let $y_1, \dots, y_k, x_1, \dots, x_n$ be the coordinates in V, with the y_i being the coordinates in U. Then any $\Psi \in T_\infty(U, TX|_Y)$ may be written as

$$\Psi = \sum_{i=1}^{k} g_i(\underline{y}) \frac{\partial}{\partial y_i} + \sum_{j=1}^{n} h_j(\underline{y}) \frac{\partial}{\partial x_j}$$

with g, h smooth. Viewing g, h as smooth on V by extension by zero, we may view Ψ as a section of TX. Then $[\mathcal{F}, \Psi] \in T_\infty(V, TX)$. In formulas, (see Boothby p. 152),

$$[\mathcal{F}, \Psi] = \sum_{j'} \left[x_{j'} \frac{\partial}{\partial x_{j'}}, \Psi\right]$$

$$= \sum_{i',j'} \left[x_{j'} \frac{\partial}{\partial x_{j'}}, g_{i'} \frac{\partial}{\partial y_{i'}}\right] + \sum_{j,j'} \left[x_{j'} \frac{\partial}{\partial x_{j'}}, h_j \frac{\partial}{\partial x_j}\right]$$

$$= \sum_{i',j'} \left(- g_{i'} \frac{\partial x_{j'}}{\partial y_{i'}}\right) \frac{\partial}{\partial x_{j'}} + \sum_{j,j'} \left(- h_j \frac{\partial x_{j'}}{\partial x_j}\right) \frac{\partial}{\partial x_{j'}}$$

So

$$[\mathcal{F}, \Psi] = -\sum_j h_j \frac{\partial}{\partial x_j}.$$

$$\therefore \; [\Psi, \mathcal{F}] = \sum_j h_j \frac{\partial}{\partial x_j}$$

So the Lie derivative with $\mathcal{F}$ picks out the normal part of Ψ. We can read $[\Psi, \mathcal{F}]$ as a section of $TX|_Y$, and so we have an idempotent operator

$$[-, \mathcal{F}] : TX|_Y \longrightarrow TX|_Y.$$

One way to make this more precise is given in [B].

<u>Remark</u> Let $i : Y \longrightarrow X$ be the inclusion, $i^{-1}(TX)$ the sheaf on Y induced in the usual way, via direct limits, and $\widetilde{TX}|_Y$ the étale space over Y defined on $\coprod_{y \in Y} i^{-1}(TX)_y$, with its obvious manifold structure. Since

$$T_y X = i^{-1}(TX)_y \otimes_{\mathcal{O}_{Y,y}} k(y)$$

we have a morphism of bundles

$$\begin{tikzcd} TX|_Y \arrow[dr] & & \widetilde{TX}|_Y \arrow[ll, "\kappa"'] \arrow[dl] \\ & Y & \end{tikzcd}$$

and it is an isomorphism. So in this way, "extending" Ψ to a section on U is not ill-defined.

We also note that $\mathcal{F}$ acting as a derivation on $T_\infty(U)$ is <u>not</u> idempotent:

$$\Big(\sum_j x_j \frac{\partial}{\partial x_j}\Big)\Big(\sum_{j'} x_{j'} \frac{\partial}{\partial x_{j'}}\Big)(f)$$

$$= \sum_{j,j'} x_j \frac{\partial}{\partial x_j}\Big(x_{j'} \frac{\partial f}{\partial x_{j'}}\Big)$$

$$= \sum_{j,j'} x_j\, \delta_{j=j'} \frac{\partial f}{\partial x_{j'}} + \sum_{j,j'} x_j x_{j'} \frac{\partial^2 f}{\partial x_j \partial x_{j'}}$$

$$= \sum_j x_j \frac{\partial f}{\partial x_j} + \sum_{j,j'} x_j x_{j'} \frac{\partial^2 f}{\partial x_j \partial x_{j'}}$$

So while $[-,\mathcal{F}]$ is an idempotent on vector fields, it needs correction to get an idempotent on smooth functions. This corresponds to the fact that

$$d_K \nabla^\delta\Big(\sum_M \delta(r_M) t^M\Big) = \sum_M |M|\, \delta(r_M) t^M$$

which is clearly not idempotent. But with $H = [d_K, \nabla^\delta]^{-1} \nabla^\delta$,

$$d_K H\Big(\sum_M \delta(r_M) t^M\Big) = \sum_{M \neq 0} \delta(r_M) t^M$$

which <u>is</u> idempotent