Minimal models for MFs 34 (checked)

For some time we erroneously believed that we needed the base ring k to be noetherian for some of our perturbation arguments, especially for the relation between matrix factorisations over R, $\hat{R}$, where $R = k[\underline{x}]$ and the completion is I-adic for $I = (\partial_{x_1} W, \ldots, \partial_{x_n} W)$. But this is not necessary, as we explain in this note (offending files such as ainfmf28, ainfmf29 will have been corrected, but our older written notes may still have un-necessary noetherianness hypotheses). The references are:

[CM] Carqueville, Murfet "Computing Khovanov-Rozansky homology and defect fusion"

[DM] Dyckerhoff, Murfet "Pushing forward matrix factorisations"

[AM] Atiyah, Macdonald "Commutative algebra"

[M] Matsumura "Commutative algebra".

[M-Cut] Murfet, "The cut operation on matrix factorisations"

[L] Lipman "Residues and traces of differential forms via HH".

Throughout all rings are commutative, k is a ring and R a k-algebra, and $t_1, \ldots, t_n$ is a quasi-regular sequence in R. We write $\hat{R}$ for the I-adic completion, where $I = (t_1, \ldots, t_n) \subseteq R$.

<u>Lemma</u> $t_1, \ldots, t_n$ is quasi-regular in $\hat{R}$.

<u>Proof</u> As recalled in (tdq), by [AM, Prop 10.13] if M is a f.g. R-module, the canonical map

$$\hat{R} \otimes_R M \xrightarrow{\delta} \hat{M}$$

is surjective, and so in particular for a f.g. ideal $J \subseteq R$,

$$J\hat{R} = \mathrm{Im}(\hat{J} \longrightarrow \hat{R}). \qquad (2.1)$$

Applying $\hat{R} \otimes_R -$ to $0 \to J \to R \to R/J \to 0$ of course shows that

$$\hat{R} \otimes_R R/J \cong \hat{R}/J\hat{R}.$$

<u>First claim</u> The canonical map $R/I \longrightarrow \hat{R}/I\hat{R}$ is an isomorphism.

<u>Proof of claim</u> Writing elements $r \in \hat{R}$ as $r = (r_n)_{n \geq 1}$, $r_n \in R/I^n$, define an R-bilinear map

$$\begin{aligned} \hat{R} \times R/I &\longrightarrow R/I \\ (r, s) &\longmapsto r_1 s \end{aligned}$$

This induces an R-linear map $\phi : \hat{R} \otimes_R R/I \longrightarrow R/I$, and thus $\hat{R}/I\hat{R} \longrightarrow R/I$ by simply sending $\bar{r}$ to $\bar{r_1}$. Now clearly $R/I \to \hat{R}/I\hat{R} \to R/I$ is the identity, and $\hat{R}/I\hat{R} \to R/I \to \hat{R}/I\hat{R}$ sends $\bar{r}$ to the residue class of $(\ldots, r_1, r_1)$. But in $\hat{R}$,

$$r - (\ldots, r_1, r_1) = (\ldots, r_3 - r_1, r_2 - r_1, 0) \in \hat{I}.$$

Since $\mathrm{Im}(\hat{I} \to \hat{R}) = I\hat{R}$, we're done. $\square$

Consider the diagram

$$\begin{tikzcd} R/I[x_1,\dots,x_n] \arrow[r, "\phi"', "\cong"] \arrow[d, "\alpha"', "\cong"] & R/I \oplus I/I^2 \oplus \cdots \arrow[d, "\beta"] \\ \hat{R}/I\hat{R}[x_1,\dots,x_n] \arrow[r, "\phi'"'] & \hat{R}/I\hat{R} \oplus I\hat{R}/I^2\hat{R} \oplus \cdots \end{tikzcd}$$

where $\phi(x_i)=t_i$, $\phi'(x_i)=t_i$, α,β are canonical. By hypothesis ϕ is an iso and we have just shown α is an iso. We need to show that the canonical map

$$\theta : I^k/I^{k+1} \longrightarrow I^k\hat{R}/I^{k+1}\hat{R}$$

is an isomorphism also. An element $r \in I^k\hat{R}$ has $r_n \in I^k(R/I^n)$ for $n \geq 1$, and thus has the form

$$\begin{array}{ccccccc} & k+1 & k & \cdots & & 2 & 1 \\ (\;\dots, & r_{k+1}, & 0, & \dots, & & 0, & 0\;) \end{array}$$

with $r_{k+1} \in I^k/I^{k+1}$. We define $\psi : I^k\hat{R}/I^{k+1}\hat{R} \longrightarrow I^k/I^{k+1}$ by $\psi(r) = r_{k+1}$. This is clearly well-defined. Moreover, $\psi\theta(\bar{r}) = \bar{r}$ and as above, (choosing a rep in I^k for r_{k+1})

$$r - \theta\psi(r) = (\dots, r_{k+2}, r_{k+1}, 0, \dots, 0) - (\dots, r_{k+1}, r_{k+1}, 0, \dots, 0)$$

$$= (\dots, \underbrace{r_{k+2} - r_{k+1}}_{\text{lies in } I^{k+1}/I^{k+2}}, 0, 0, \dots, 0) \in \widehat{I^{k+1}}$$

Since I^{k+1} is a f.g. ideal, we deduce this element lies in $I^{k+1}\hat{R}$, as required. Hence θ is an iso and the proof of quasi-regularity is complete. $\square$

<u>Lemma</u> Suppose that

(i) R, R/I are projective k-modules

(ii) the Koszul complex of $\underline{t}$ over R is exact in nonzero degrees.

Then, if $z : R/I \longrightarrow R$ is any k-linear section of the quotient $\pi : R \to R/I$, $(K, d_K) = (\Lambda(k\theta_1 \oplus \cdots \oplus k\theta_n) \otimes_k R,\ \Sigma_i t_i \theta_i^*)$ is the Koszul complex, π is viewed as a morphism of complexes $\pi : (K, d_K) \longrightarrow (R/I, 0)$, there is a degree -1 k-linear operator h as in the diagram

$$h \circlearrowright (K, d_K) \underset{z}{\overset{\pi}{\rightleftarrows}} (R/I, 0)$$

such that

- $\pi z = 1$,
- $z\pi = 1_K - [d_K, h]$ (graded commutator)
- $h^2 = 0$, $hz = 0$, $\pi h = 0$.

<u>Proof</u> Let $Z^i = \mathrm{Ker}(d_K^i : K^i \longrightarrow K^{i+1})$, and $Z^0 = I$ so that we have a diagram in which the top row is exact:

$$\begin{tikzcd}
\cdots & K^{-2} \arrow[rr, "d_K^{-2}"] \arrow[dr, shift right, "b^{-1}"'] & & K^{-1} \arrow[rr, "d_K^{-1}"] \arrow[dl, shift right, "c^{-1}"'] \arrow[dr, shift right, "b^{0}"'] & & K^{0} \arrow[r, shift left, "\pi"] \arrow[dl, shift right, "c^{0}"'] & R/I \arrow[l, shift left, "z"] \arrow[r] & 0 \\
& & Z^{-1} \arrow[ul, shift right, "d^{-1}"'] \arrow[ur, shift right, "a^{-1}"'] & & Z^{0} \arrow[ul, shift right, "d^{0}"'] \arrow[ur, shift right, "a^{0}"'] & & &
\end{tikzcd}$$

Define a^i, b^i as shown. Since K^i, R/I are all projective, there are k-linear c^i, d^i as indicated such that $a^i c^i = 1$, $d^i b^i = 1$, $c^i a^i + b^i d^i = 1$ for all i.

We set $h^i := d^i \circ c^i$. Then

- $\iota\pi = 1_K - [d_K, h]$

  This holds since $\iota\pi = 1_{K^0} - d_K^{-1}h^0$, and for $i>0$

$$\begin{aligned}[d_K, h]^i &= d_K^{i-1}h^i + h^{i-1}d_K^i \\ &= a^ib^id^ic^i + d^{i-1}c^{i-1}a^{i+1}b^{i+1} \\ &= a^ic^i + d^{i-1}b^{i+1} = 1_{K^i}.\end{aligned}$$

- $h^2 = 0$, $h\iota = 0$, $\pi h = 0$.

  Since $(h^2)^i = h^{i-1}h^i = d^{i-1}c^{i-1}d^ic^i = 0$, $h^2 = 0$.
  Clearly $h\iota = 0$ and $\pi h = 0$. $\square$

Remarks on [DM]

Firstly, in the context of [DM, §7] suppose X is not necessarily finite rank (but still free). Then all the arguments of that section go through, with the possible exception of Remark 7.5 (I did not check). Moreover Remark 7.7 there holds under weaker hypotheses: as in [DM, §7] let $\varphi: S \longrightarrow R$ be a ring morphism, $W \in S$ and (X, d) a matrix factorisation of W over R (not nec. finite rank), $t_1, \dots, t_n$ a quasi-regular sequence in R with $t_i \cdot 1_X \simeq 0$, with homotopies λ_i. We assume there is a deformation retract (of $\mathbb{Z}$-graded cpxs) over S

$$(R/\underline{t}R, 0) \underset{\delta}{\overset{\pi}{\rightleftarrows}} (K_R(\underline{t}), d_K), \quad h \qquad (6.1)$$

satisfying $h^2 = 0$, $h\delta = 0$, $\pi h = 0$, where π is the canonical map.

Let $\alpha: R \longrightarrow R'$ be a ring morphism such that

(i) $\alpha(t_1), \dots, \alpha(t_n)$ is quasi-regular in R', (we write t_i for $\alpha(t_i)$)

(ii) the induced map $R/\underline{t}R \longrightarrow R'/\underline{t}R'$ is an isomorphism,

(iii) there is a deformation retract (of $\mathbb{Z}$-graded cpxs over S)

$$(R'/\underline{t}R', 0) \underset{\delta'}{\overset{\pi'}{\rightleftarrows}} (K_{R'}(\underline{t}), d_K'), \quad h' \qquad (6.2)$$

satisfying $h'^2 = 0$, $h'\delta' = 0$, $\pi' h' = 0$.

Proposition The canonical morphism of linear factorisations of W over S, $\varphi_* X \longrightarrow (\alpha\varphi)_* \alpha^* X$ is an S-linear homotopy equivalence.

Proof As explained in [DM, Remark 7.7], the results there show that we have homotopy equivalences ($X' := X \otimes_R R'$) over S

$$\pi : X \otimes_R K_R(\underline{t}) \longrightarrow X/\underline{t}X$$

$$\pi' : X' \otimes_{R'} K_{R'}(\underline{t}) \longrightarrow X'/\underline{t}X'$$

and the canonical R-linear map $X/\underline{t}X \longrightarrow X'/\underline{t}X'$ is an isomorphism, by hypothesis (ii). It is clear that the canonical R-linear map $X \to X'$ induces a commutative diagram of linear factorisations

$$\begin{tikzcd}
X \otimes_R K_R(\underline{t}) \arrow[r, "\pi"] \arrow[d] & X/\underline{t}X \arrow[d, "\cong"] \\
X' \otimes_{R'} K_{R'}(\underline{t}) \arrow[r, "\pi'"'] & X'/\underline{t}X'
\end{tikzcd}$$

Hence a homotopy commutative diagram ($\psi := \varepsilon\pi^{-1}$, $\psi' := \varepsilon \circ (\pi')^{-1}$)

$$\begin{tikzcd}
X/\underline{t}X \arrow[r, "\pi^{-1}"] \arrow[d, "\cong"'] & X \otimes_R K_R(\underline{t}) \arrow[r, "\varepsilon"] \arrow[d] & X[n] \arrow[d] \\
X'/\underline{t}X' \arrow[r, "(\pi')^{-1}"'] & X \otimes_{R'} K_{R'}(\underline{t}) \arrow[r, "\varepsilon"'] & X'[n]
\end{tikzcd}$$

If we produce ϑ, ϑ' using λ_i and $\lambda_i' := \lambda_i \otimes_R R'$ then it is also clear that the following diagram commutes:

$$\begin{tikzcd}
X[n] \arrow[r, "\vartheta = (-1)^n \pi \lambda_1 \cdots \lambda_n"] \arrow[d] & X/\underline{t}X \arrow[d, "\cong"] \\
X'[n] \arrow[r, "\vartheta' = (-1)^n \pi' \lambda_1' \cdots \lambda_n'"'] & X'/\underline{t}X'
\end{tikzcd}$$

So in summary both squares in the following diagram commute in $HF(S,W)$

$$\begin{tikzcd}
X/\underline{t}X \arrow[r, "\psi"] \arrow[d, "\cong"'] & X[n] \arrow[r, "\vartheta"] \arrow[d, "can"'] & X/\underline{t}X \arrow[d, "\cong"] \\
X'/\underline{t}X \arrow[r, "\psi'"'] & X'[n] \arrow[r, "\vartheta'"'] & X'/\underline{t}X'
\end{tikzcd}$$

Now $\psi\vartheta \simeq 1$, $\psi'\vartheta' \simeq 1$, and the above shows (identifying $X/\underline{t}X, X'/\underline{t}X'$) that $\vartheta\psi \simeq \vartheta'\psi'$! Since they split the same idempotent, X, X' must be isomorphic in $HF(S,W)$, and the middle column is a homotopy equivalence. $\square$

Remarks on [M-Cut]

In the current version of the cut paper we assume the ground ring k is noetherian, purely so that $k[\underline{y}] \longrightarrow k[\underline{y}]^\wedge$ will be flat, which allows us to use [DM, Remark 7.7] to infer that the canonical map

$$\varepsilon : Y \otimes_{k[\underline{y}]} X \longrightarrow Y \otimes_{k[\underline{y}]} \widehat{k[\underline{y}]} \otimes_{k[\underline{y}]} X$$

is a homotopy equivalence, in Section 4.3. Using the above, we explain why everything in [M-Cut] holds for <u>any</u> commutative $\mathbb{Q}$-algebra.

- Set $t_i = \partial_{y_i} V$ for $1 \le i \le m$. By hypothesis this is a quasi-regular sequence in $R = k[\underline{y}]$, and $R/\underline{t}R$ is a f.g. projective k-module, and $K_R(\underline{t})$ is exact outside degree zero. Let $I = (\underline{t})$ and $\hat{R}$ the I-adic completion.

- By p.②, $\underline{t}$ is quasi-regular in $\hat{R}$, and $R/\underline{t}R \xrightarrow{\cong} \hat{R}/\underline{t}\hat{R}$. Then [DM, Appendix B] applies to the tuple $(k, \hat{R}, \underline{t})$ to give an isomorphism of $k[\underline{z}]$-modules $(P = \hat{R}/\underline{t}\hat{R})$

$$\zeta^* : P \otimes_k k[\![\underline{z}]\!] \longrightarrow \hat{R}.$$

Observe that $P \otimes_k K_{k[\![\underline{z}]\!]}(\underline{z}) \overset{\zeta^*}{\cong} K_{\hat{R}}(\underline{t})$ as complexes of $k[\underline{z}]$-modules, and in particular $K_{\hat{R}}(\underline{t})$ is exact outside degree zero. Moreover using connections (as recalled in [M-Cut] itself) we may produce a deformation retract of the form (6.2).

- By p.④ since R, R/I are projective k-modules and $K_R(\underline{t})$ is exact outside degree zero, we have a deformation retract (6.1).

Hence the Proposition on p. 6 applies to the data (notation of [M-Cut])

$$k[\underline{x},\underline{z}] \xrightarrow{\varphi} k[\underline{x},\underline{y},\underline{z}], \qquad (Y \otimes_{k[\underline{y}]} X,\ d_Y \otimes 1 + 1 \otimes d_X).$$

$$k[\underline{x},\underline{y},\underline{z}] \cong k[\underline{x},\underline{z}] \otimes_k k[\underline{y}] \xrightarrow{\alpha} k[\underline{x},\underline{z}] \otimes_k k[\underline{y}]^{\wedge}$$

and we conclude that in $HF(k[\underline{x},\underline{z}], U-W)$ the canonical map is an isomorphism

$$Y \otimes_{k[\underline{y}]} X \longrightarrow Y \otimes_{k[\underline{y}]} \widehat{k[\underline{y}]} \otimes_{k[\underline{y}]} X.$$

This completes the removal of the noetherian hypotheses from [M-Cut].