An invitation to topos theory

The aim of the seminar series this semester is to understand how to organise mathematical knowledge using topoi (a kind of category) and adjoint functors. Let us begin with the big picture.

Concrete $\longleftrightarrow$ Abstract

Structured sets	$\overset{①}{\rightsquigarrow}$	Theories	$\overset{②}{\rightsquigarrow}$	Categories of theories
$(\mathbb{Z}, +), (\mathbb{Z}[x], +), \ldots$		Abelian groups (operations, laws)		
$(\mathbb{Z}, +, \cdot), (\mathbb{R}, +, \cdot), \ldots$		Ring		

$$\begin{tikzcd} \mathcal{A}b \arrow[d, shift right] \\ \mathcal{R}ng \arrow[u, shift right] \end{tikzcd}$$

there is some relation between these theories, since every ring is also an abelian group

We are used to the process ① above of abstraction, which allows us to deal simultaneously with many examples using concepts like "abelian group". These concepts are how we organise mathematical knowledge. As we all know, the development of mathematical logic in the 20th century allowed us to view the theorems about abelian groups and their proofs, as <u>mathematical objects</u> in their own right. This is important, because we know how to reason reliably about mathematical objects and how to <u>automate</u> (to some degree) this reasoning. This seminar series is about the next step, marked ② above.

QUESTION

Our theories, theorems and proofs are organised in lecture notes, books and brains, but is this hierarchical system of knowledge itself a mathematical object? And if so, what kind of object is it?

(why care? To make human reasoning more reliable, and eventually to automate it)

Our aim is to approach this question from the point of view of Lawvere, who emphasised the deep role of adjoint functors in foundations, many aspects of which are realised concretely in the theory of <u>topoi</u> (plural of <u>topos</u>). We illustrate the general idea with an example:

- Every ring is an abelian group, so there should be a morphism

$$\mathcal{A}b \underset{?}{\overset{?}{\rightleftarrows}} \mathcal{R}ng \qquad \text{(but in which direction?)}$$

  But what is the "correct" notion of a <u>morphism of theories</u>? Clearly if we are to make mathematical knowledge a mathematical object, we must have a good answer to this question!

- Topos theory gives a beautiful answer, at least for so-called "geometric" theories (includes $\mathcal{A}b$, $\mathcal{R}ng$). First, we construct from a theory a category

$$\mathcal{A}b \rightsquigarrow \mathcal{B}(\mathcal{A}b)$$

  called the <u>classifying topos</u>. Then we define

$$\mathrm{Hom}(\mathcal{R}ng, \mathcal{A}b) := \underbrace{\underline{\mathrm{Hom}}(\mathcal{B}(\mathcal{R}ng), \mathcal{B}(\mathcal{A}b))}_{\text{(geometric) functors and natural transformations}}$$

- We will give a brief introduction to category theory in talks #2, #3, but for the moment: a category has <u>objects</u> $X, Y, Z, \ldots$ <u>morphisms</u> $f: X \to Y, \ldots$ and <u>composition</u> $(g: Y \to Z, f: X \to Y) \longmapsto g \circ f: X \longrightarrow Z$. A <u>functor</u> $F: \mathcal{C} \longrightarrow \mathcal{D}$ between categories sends objects X to objects $F(X)$, morphisms to morphisms and preserves composition.

- Why does topos theory give the "right" answer for the notion of morphisms of theories, and what is it good for? In my view the main reason to buy the topos theoretic answer is:

e.g. **Sets**

<u>Theorem</u> (Universal property of classifying topoi) For any cocomplete topos $\mathcal{E}$ and geometric theory $\mathbb{T}$ there is an equivalence of categories

$$\underline{\mathrm{Hom}}(\mathcal{E}, \mathcal{B}(\mathbb{T})) \simeq \{\text{models of } \mathbb{T} \text{ in } \mathcal{E}\}$$

<u>What is a topos?</u>

<u>Def^n</u> (Vague) A topos is a "universe of mathematical discourse".

<u>Def^n</u> A topos is a category with

- finite limits and colimits
- exponentials $\qquad \mathrm{Hom}(X \times Y, Z) \cong \mathrm{Hom}(X, Z^Y)$
- a subobject classifier $\qquad \mathrm{Sub}(X) \cong \mathrm{Hom}(X, \Omega)$

<u>Examples</u> (1) <u>Sets</u>
(2) $Sh(X)$ $\quad X$ topological space
(3) simplicial sets $\quad$ (or $\underline{\text{Sets}}^{\mathcal{C}^{op}}$ any small $\mathcal{C}$)
(4) category of types / terms in a HOL

<u>What is a theory?</u> It is a set of axioms (i.e. formulas) in a first-order language (i.e. there are sorts, constants, function and relation symbols, variables and formulas built from relations $R(t)$, equalities $t = t'$ and the connectives $\wedge, \vee, \Rightarrow, \neg, \forall, \exists$). I will not explain "geometric" today.

(e.g. Ab has $\times : A \times A \to A$, $- : A \to A$, $0 : A$)

Example (1) $\underline{\mathrm{Hom}}(\underline{\mathrm{Sets}}, \mathcal{B}(\mathcal{A}b)) \simeq \{\text{abelian groups}\}$

(2) $\underline{\mathrm{Hom}}(Sh(X), \mathcal{B}(\mathcal{A}b)) \simeq \{\text{sheaves of ab. grps. on } X\}$

(3) Corresponding to $1_{\mathcal{B}(T)} : \mathcal{B}(T) \longrightarrow \mathcal{B}(T)$ is a special model of T in $\mathcal{B}(T)$, the <u>universal model</u> U_T, e.g.

- $U_{\mathcal{A}b} \in \mathcal{B}(T)$ is the "universal" abelian group
- $U_{Rng} \in \mathcal{B}(T)$ " " " ring.

But any ring is an abelian group! So

$$\underline{\mathrm{Hom}}(\mathcal{B}(Rng), \mathcal{B}(\mathcal{A}b)) \simeq \{\text{models of } \mathcal{A}b \text{ in } \mathcal{B}(Rng)\}$$

$$\begin{array}{ccc} \Psi & & \Psi \\ f & \longleftrightarrow & U_{Rng} \end{array}$$

There is some special geometric functor $f : \mathcal{B}(Rng) \longrightarrow \mathcal{B}(\mathcal{A}b)$, in fact it has a left adjoint f^* and $f^*(U_{\mathcal{A}b}) \cong U_{Rng}$.

Remark A good secondary reason is that these classifying topoi are naturally arising categories of interest in their own right. For example

- $\underbrace{\mathcal{B}(\mathcal{L}ocRng)}_{\text{local rings}}$ is the Zariski topos $\underbrace{Sh(J, (\text{fp-rings})^{op})}_{\text{includes schemes}/\mathbb{Z}}$

<u>Summary</u> The logical relation between theories $\mathbb{A}b$, $\mathbb{R}ng$ is promoted to a geometric morphism (adjoint pair of functors) between topoi.

$$\begin{array}{ccc} \mathbb{A}b & \qquad\qquad & \mathcal{B}(\mathbb{A}b) \\ ?\ \updownarrow & & \uparrow f \\ \mathbb{R}ng & & \mathcal{B}(\mathbb{R}ng) \end{array}$$

<u>Our aim</u>, more precisely stated, is to understand how to organise mathematical knowledge using the 2-category of classifying topoi and geometric morphisms (that is, this is the "arena" in which we propose to look for the mathematical objects which formalise our informal hierarchical system of mathematical knowledge). This has various aspects:

- logical (the $\mathcal{B}(T)$ are "syntactic")
- categorical
- computational (via Curry-Howard, as I will now explain)
- "geometric" ($\mathcal{B}(T)$ is a category of sheaves on a Grothendieck site)

Now, why should a computer scientist care about this?

Because programs are mathematical objects, which we want to reason about (perhaps to verify our software is safe) in an automated way (so that our ability to verify software scales with our capacity to create it, for example using machine learning methods).

$$\begin{tikzcd}
\text{Curry-Howard correspondence} \arrow[d] & & \text{This seminar} \arrow[d] \\
\{\text{intuitionistic logic}\} \arrow[rr, "\text{add Prop}"] \arrow[d, dash, "\cong"] & & \{\text{higher-order logic}\} \arrow[d, dashed, dash] \\
\{\text{simply-typed } \lambda\text{-calculus}\} \arrow[rr, "\text{add reasoning}"] \arrow[d, dash, "\cong"] & & \{\text{functional programs} + \text{equational reasoning}\} \arrow[d, dashed, dash] \\
\{\text{Cartesian-closed categories}\} \arrow[rr, "\text{add subobject class.}"] & & \{\text{Topoi}\}
\end{tikzcd}$$

(i.e. a type of Propositions)

(i.e. functional programs)

From a programming language (PL) point of view, the problem of finding mathematical objects to formalise "hierarchical systems of math. knowledge" is the problem of finding a principled PL approach to "hierarchical systems of libraries of programs and equational knowledge" since under Curry-Howard we can (perhaps) make the analogy

<u>logic</u>		<u>categories</u>		<u>computation</u>
theory	$\sim$	classifying topos	$\sim$	library

From the topos theoretic point of view

$$\begin{tikzcd} \mathcal{B}(Rng) \arrow[r, shift right, "f"'] & \mathcal{B}(Ab) \arrow[l, shift right, "f^*"'] \ni U_{Ab} \end{tikzcd}$$

universal abelian group

Our knowledge about abelian groups consists of theorems (zero is unique) their proofs, examples, additional definitions (cyclic groups) etc...

<u>theorems</u> formulas in the language of Ab, e.g.

$$\phi : \forall x \in U_{Ab} \left(\{ \forall y \in U_{Ab} (x + y = x) \} \Rightarrow x = 0 \right)$$

gives a morphism $\mathbb{1} \xrightarrow{\phi} \Omega$ in $\mathcal{B}(Ab)$, and its <u>proof</u> is an argument that $\phi = \text{true}$, where $\text{true} : \mathbb{1} \longrightarrow \Omega$ is some special morphism, i.e.

$$\begin{tikzcd} \mathbb{1} \arrow[r, bend left, "\text{true}"] \arrow[r, bend right, "\phi"'] \arrow[r, phantom, "\| \ \text{proof}"] & \Omega \end{tikzcd}$$

<u>geometric morphisms "act" on knowledge</u>

$$\begin{tikzcd} \mathbb{1} \cong f^*(\mathbb{1}) \arrow[r, bend left, "f^*(\text{true})"] \arrow[r, bend right, "f^*(\phi)"'] \arrow[r, phantom, "\|"] & f^*(\Omega) \arrow[r] & \Omega \end{tikzcd}$$

$$\| $$

using $f^* U_{Ab} \cong U_{Rng}$ and f^* commutes with $\forall, \Rightarrow$

$$\forall x \in U_{Rng} \left(\{ \forall y \in U_{Rng} (x + y = x) \} \Rightarrow x = 0 \right)$$

<u>Upshot</u> Every theorem about abelian groups gives a theorem about rings, via f^*.