Classifying topoi I

Let us recall the idea of a classifying topos from the first lecture of this seminar. We have still not precisely defined all the terms involved (nor will we resolve this in today's lecture) but I still think it is useful. Given a geometric theory $\mathbb{T}$ (e.g. Groups, Rings, ...) we call a topos $\mathcal{B}(\mathbb{T})$ a <u>classifying topos</u> if we have a family of equivalences parametrised by cocomplete topoi $\mathcal{E}$

$$\underline{\mathrm{Hom}}(\mathcal{E}, \mathcal{B}(\mathbb{T})) \cong \underline{\mathrm{Mod}}(\mathbb{T}, \mathcal{E}) \qquad (1)$$

geometric morphisms = models of $\mathbb{T}$ in $\mathcal{E}$

which is natural in $\mathcal{E}$. A <u>geometric morphism</u> $\mathcal{E} \longrightarrow \mathcal{B}(\mathbb{T})$ is an adjoint pair of functors

$$\begin{tikzcd} \mathcal{E} \arrow[r, shift right, "f_*"'] & \mathcal{B}(\mathbb{T}) \arrow[l, shift right, "f^*"'] \end{tikzcd} \qquad f^* \dashv f_*$$

in which f^* preserves finite limits.

The stated goal of these seminars was to study how geometric morphisms of classifying topoi can be used to organise mathematical knowledge. But we also want this organisation to be <u>effective</u>: for example, implementable in a computer logic calculus such as Isabelle. Let us begin by sketching why this a reasonable goal, then specify the <u>obstacle</u>, and then we will spend the rest of the lecture studying the example of <u>simplicial sets</u> which suggests this obstacle may be surmountable.

⌜Recall from Lecture 9 that the topos $\mathrm{T}(\mathcal{L})$ constructed from a type theory $\mathcal{L}$ has for its objects closed terms $\alpha : PA$ modulo $\alpha \sim \alpha'$ iff. $\vdash \alpha = \alpha'$.⌟

The example given in my first lecture was the following:

<u>Example</u> Let $\mathbb{A}b$, $\mathbb{R}ng$ denote the theories of abelian groups and rings respectively. Since any ring is an abelian group, the universal model

$$U_{\mathbb{R}ng} \in \underline{\mathrm{Mod}}(\mathbb{R}ng, \mathcal{B}(\mathbb{R}ng))$$

is also a model of abelian groups

$$U_{\mathbb{R}ng} \in \underline{\mathrm{Mod}}(\mathbb{A}b, \mathcal{B}(\mathbb{R}ng))$$

and this model corresponds uniquely to a geometric morphism

$$\mathcal{B}(\mathbb{R}ng) \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathcal{B}(\mathbb{A}b).$$

This is a prototypical example of geometric morphisms "organising" mathematical concepts.

<u>Question</u>: to what extent can the objects

$$\mathcal{B}(\mathbb{R}ng),\ \mathcal{B}(\mathbb{A}b),\ f^*,\ f_*$$

be described effectively?

this statement to be justified later

- $\mathcal{B}(\mathbb{R}ng)$, $\mathcal{B}(\mathbb{A}b)$ may be constructed as categories $\mathrm{Sh}_J(\mathcal{C})$ of sheaves on categories $\mathcal{C}$ which are <u>syntactic</u>, i.e. their objects and morphisms are equivalence classes of terms in the underlying language of the theory.

These topoi are subcanonical (i.e. $\mathcal{C} \hookrightarrow Sh_J(\mathcal{C})$ via Yoneda) so every object is a colimit of objects in $\mathcal{C}$, so $Sh_J(\mathcal{C})$ is "generated" by the objects and morphisms of $\mathcal{C}$ (which recall, have an effective syntactic character). This suggests that at least those sheaves which are constructed by <u>finite</u> colimits from $\mathcal{C}$ should be amenable to automated reasoning.

<u>Def</u> Let $\mathcal{B}^{fin}(\mathcal{A}b), \mathcal{B}^{fin}(\mathcal{R}ng)$ denote the smallest subcategories containing the representable sheaves (i.e. $\mathcal{C}$) and closed under finite colimits.

We believe it should be possible to reason in an effective way in a computational tool about $\mathcal{B}^{fin}(\mathcal{A}b), \mathcal{B}^{fin}(\mathcal{R}ng)$.

- f^*, f_* : here the situation is less clear, <u>a priori</u>. The basic question is the following. Consider the restrictions

$$f^* : \mathcal{B}^{fin}(\mathcal{A}b) \longrightarrow \mathcal{B}(\mathcal{R}ng),$$

$$f_* : \mathcal{B}^{fin}(\mathcal{R}ng) \longrightarrow \mathcal{B}(\mathcal{A}b).$$

We imagine the input objects and morphisms to these functors are represented by terms in some formal language (i.e. they are explicitly constructed from some basic data by specified rules). Can we describe the <u>outputs</u> on these inputs in a similar way?

Or, more succinctly: can we give an <u>algorithmic</u> description of the action of f^* and f_* on objects and morphisms? In the remainder of this lecture we study this question.

Tensor products

We begin with what may seem like a detour. Recall that the tensor product $M \otimes_R N$ of a right R-module M and a left R-module N is an abelian group, and there is a function $M \times N \longrightarrow M \otimes_R N$ which is the <u>universal bilinear map</u>. There is an alternative way of characterising tensor products which we will now explain; for details see Theorem 5.2 of Chapter IV of B. Mitchell, "Theory of categories". Throughout rings are associative and unital but not necessarily commutative, <u>Mod</u>R means the category of right R-modules and R<u>Mod</u> the category of left R-modules. A category $\mathcal{C}$ is <u>additive</u> if each $\mathcal{C}(a,b)$ is an abelian group and composition is bilinear, a functor between additive categories is <u>additive</u> if it preserves addition of morphisms.

<u>Remark</u> Let $\mathcal{P} \subseteq \underline{\text{Mod}}R$ be the full subcategory containing just the object R (as an R-module in the usual way). An additive functor $F : \mathcal{P} \longrightarrow \underline{\text{Mod}}S$ is the data of

- a right S-module $B := F(R)$
- a morphism of rings

$$F_{RR} : R = \mathcal{P}(R,R) \longrightarrow \text{Hom}_S(B,B)$$

(left multiplication)

If we define $R \times B \longrightarrow B$ by $(r,b) \longmapsto F_{RR}(r)(b)$ we make B into an R-S-bimodule, and in the fact there is a bijection between additive functors F and such bimodules.

<u>Theorem</u> Any additive functor $F : \mathcal{P} \longrightarrow \underline{\mathrm{Mod}}\, S$ can be extended uniquely (up to natural isomorphism) to a colimit preserving functor

$$\overline{F} : \underline{\mathrm{Mod}}\, R \longrightarrow \underline{\mathrm{Mod}}\, S .$$

<u>Sketch of proof</u> For each R-module M we choose a presentation

$$\bigoplus_{i\in I} R \xrightarrow{\ \alpha\ } \bigoplus_{j\in J} R \xrightarrow{\ \beta\ } M \longrightarrow 0$$

with I, J allowed to be infinite. Observe that

$$\alpha_i := R \xhookrightarrow{\ u_i\ } \bigoplus_{i\in I} R \xrightarrow{\ \alpha\ } \bigoplus_{j\in J} R$$

factors as

$$R \xrightarrow{(\alpha_{ij})_{j\in J_i}} \bigoplus_{j\in J_i} R \hookrightarrow \bigoplus_{j\in J} R$$

for some finite $J_i \subseteq J$, where $\alpha_{ij} \in R$. We define $\overline{F}(M)$ to be the cokernel

$$\underbrace{\bigoplus_{i\in I} F(R) \xrightarrow{(F(\alpha_{ij}))_{i,j}} \bigoplus_{j\in J} F(R)}_{S\text{-linear map}} \longrightarrow \overline{F}(M) \longrightarrow 0$$

An R-linear map $f : M \longrightarrow M'$ can be lifted to the presentations and in this way induces $\overline{F}(f) : \overline{F}(M) \longrightarrow \overline{F}(M')$. One checks $\overline{F}$ preserves colimits and $\overline{F} \circ \mathrm{inc} \cong F$ by construction. $\square$

<u>Exercise</u> What does uniqueness mean? How is this related to the notion of a Kan extension?

Given our previous remark, we know F is "really" the data of a bimodule B. The functor

$$(-)_R \otimes_R {}_R B_S : \underline{\mathrm{Mod}}\, R \longrightarrow \underline{\mathrm{Mod}}\, S$$

is colimit preserving and $R \otimes_R B \cong B = F(R)$, so by uniqueness we must have $F \cong (-) \otimes_R B$. To see this more intuitively, recall how we construct $M \otimes_R B$ for a right R-module M:

- $M \otimes_R B$ is the quotient of the free abelian group on the set $M \times B$ by the relations

$$\begin{aligned} (m+m', b) &= (m,b) + (m', b) \\ (m, b+b') &= (m,b) + (m,b') \\ (mr, b) &= (m, rb) \qquad\qquad r \in R. \end{aligned}$$

The exact sequence

$$\begin{array}{ccccccc} \left(\bigoplus_{i\in I} R\right) \otimes_R B & \xrightarrow{\alpha \otimes 1_B} & \left(\bigoplus_{j\in J} R\right) \otimes_R B & \longrightarrow & M \otimes_R B & \longrightarrow & 0 \\ \cong & & \cong & & & & \\ \bigoplus_{i \in I} B & \longrightarrow & \bigoplus_{j\in J} B & & & & \end{array}$$

$$(i,b) \longmapsto \sum_j (j, \alpha_{ij} b) = \sum_j \alpha_{ij} (j,b)$$

presents the same abelian group as the quotient of the free R-module on symbols (j,b), $j \in J$, $b \in B$ by the relations

$$\begin{aligned} (j, b+b') &= (j,b) + (j,b') \\ r(j,b) &= (j, rb) \\ \sum_k r_k (k,b) &= \sum_\ell r_\ell (\ell, b) \qquad \text{whenever } \sum_k r_k e_k = \sum_\ell r_\ell e_\ell \text{ in } M. \end{aligned}$$

<u>Upshot</u> The bilinearity relations we think of as characterising the tensor product arise via <u>presentations</u> by <u>extending</u> a functor $F : \mathcal{P} \longrightarrow \underline{\mathrm{Mod}}\, S$ to a <u>colimit preserving functor</u> $F : \underline{\mathrm{Mod}}\, R \longrightarrow \underline{\mathrm{Mod}}\, S$.

A finite presentation of an R-module M (i.e. I, J above finite) is an <u>algorithm</u> for constructing M from copies of R, using finite colimits. The tensor product $- \otimes B$ sends this to an algorithm for constructing $M \otimes_R B$ from copies of B using finite colimits.

Next we turn to the non-additive analogue, and explain ultimately how geometric realisation of simplicial sets is analogous to a tensor product.

<u>Non-additive tensors</u>

The analogy is as follows:

Additive	Non-additive
Ring R, as additive cat. $\mathcal{P}$ (ringoid!)	Small category $\mathcal{C}$
left R-module, i.e. additive $\mathcal{P} \longrightarrow \underline{Ab}$	Functor $\mathcal{C} \longrightarrow \underline{\mathrm{Sets}}$
right R-module, i.e. additive $\mathcal{P}^{op} \longrightarrow \underline{Ab}$	Functor $\mathcal{C}^{op} \longrightarrow \underline{\mathrm{Sets}}$
$R\,\underline{\mathrm{Mod}}$	$\underline{\mathrm{Sets}}^{\mathcal{C}}$
$\underline{\mathrm{Mod}}\, R$	$\underline{\mathrm{Sets}}^{\mathcal{C}^{op}}$
R-S-bimodule, i.e. additive $\mathcal{P} \xrightarrow{B} \underline{\mathrm{Mod}}\, S$	Functor $\mathcal{C} \xrightarrow{A} \underline{\mathrm{Sets}}^{\mathcal{D}^{op}}$
Tensor product $(-) \otimes_R B : \underline{\mathrm{Mod}}\, R \to \underline{\mathrm{Mod}}\, S$	? $\underline{\mathrm{Sets}}^{\mathcal{C}^{op}} \longrightarrow \underline{\mathrm{Sets}}^{\mathcal{D}^{op}}$

(think $f^* : \mathcal{B}(\mathcal{A}b) \longrightarrow \mathcal{B}(\mathcal{R}ng)$)

Theorem Let $\mathcal{C}$ be a small category and $\mathcal{E}$ a cocomplete category, $A : \mathcal{C} \longrightarrow \mathcal{E}$ a functor. There is a unique (up to natural iso) extension of A to a colimit preserving functor

$$\bar{A} : \underline{Sets}^{\mathcal{C}^{op}} \longrightarrow \mathcal{E}.$$

Remark Here by "extension" we mean commutativity up to natural iso. of

$$\begin{tikzcd} \mathcal{C} \arrow[r, "A"] \arrow[d, "\text{Yoneda}"'] & \mathcal{E} \\ \underline{Sets}^{\mathcal{C}^{op}} \arrow[ur, "\bar{A}"'] & \end{tikzcd}$$

Proof We sketch the proof from Maclane & Moerdijk Corollary 4 in §I.5 (p. 43).
One defines for a presheaf P

$$\bar{A}(P) := \operatorname{colim}\left(\int_{\mathcal{C}} P \longrightarrow \mathcal{C} \xrightarrow{A} \mathcal{E} \right)$$

objects are pairs (C, x), $x \in P(C)$ and morphisms $(C,x) \longrightarrow (C', y)$ are arrows $f : C \longrightarrow C'$ s.t. $P(f)(y) = x$.

sends $(C,x) \mapsto C$, $f \mapsto f$

⌜uses that every presheaf is a colimit of representable presheaves⌟

and checks all the desired properties. □

Example $\mathcal{E} = \underline{Sets}^{\mathcal{D}^{op}}$, so $A : \mathcal{C} \longrightarrow \underline{Sets}^{\mathcal{D}^{op}}$ is a "bimodule", and we think of $\bar{A}$ as a "tensor product" with A.

⌜**Aside** Recall that we construct the colimit of a diagram $F : \mathcal{I} \longrightarrow \mathcal{X}$ using coproducts and coequalisers as the coequaliser of

$$\coprod_{f : i \to i'} F(i)_{(f)} \underset{\text{post}}{\overset{\text{pre}}{\rightrightarrows}} \coprod_{i \in \mathcal{I}} F(i)$$

$$\begin{tikzcd} F(i)_{(f)} \arrow[r, "1"] \arrow[dr, "F(f)"'] & F(i) \\ & F(i') \end{tikzcd}$$
⌟

To strengthen this analogy let us recall that P itself is the colimit of

$$\int_{\mathscr{C}} P \longrightarrow \mathscr{C} \xrightarrow{h} \underline{\mathrm{Sets}}^{\mathscr{C}^{op}}$$

which means that P is the coequaliser

$$\coprod_{\substack{c, c' \in \mathscr{C} \\ x \in P(c),\, y \in P(c') \\ f : c \to c' \\ \text{s.t. } P(f)(y) = x}} h_c \underset{\text{post}}{\overset{\text{pre}}{\rightrightarrows}} \coprod_{\substack{c \in \mathscr{C} \\ x \in P(c)}} h_c \longrightarrow P$$

arrows in the diagram

vertices in the diagram

⌜Think of h_c as "R" in the additive case⌟

But observe that maps $h_c \longrightarrow h_{c'}$ are images of maps in $\mathscr{C}$, so we may apply A to them, and define $\bar{A}(P)$ as the coequaliser of

$$\coprod_{f, y} A(c) \rightrightarrows \coprod_{c, x} A(c) \longrightarrow \bar{A}(P)$$ ← in $\mathcal{E}$

which is precisely what the definition on the previous page says.

Observe that

$$\coprod_{C,x} A(C) = \{ (C,x,a) \mid C \in \mathcal{C},\ x \in P(C),\ a \in A(C) \}.$$

and the coequaliser, intuitively speaking, imposes the relation, for every $f : C \to C'$ in $\mathcal{C}$ and $y \in P(C')$

$$(C,\ P(f)(y),\ a) \sim (C',\ y,\ A(f)(a))$$

i.e. $(y \cdot f, a) \sim (y, f \cdot a)$

In categories like $\mathcal{E} = \underline{Sets}$ or $\mathcal{E} = \underline{Top}$, where the coequaliser is obtained by quotienting by an equivalence relation, this is a complete description of $\bar{A}(P)$ and not just an intuition.

<u>Example</u> Let $\mathcal{E} = \underline{Top}$, the category of topological spaces and continuous maps. This is a cocomplete category. Let $\mathcal{C} = \Delta$ be the simplex category, whose objects are $n \in \mathbb{N} = \{0, 1, \dots\}$ and where $\mathcal{C}(n,m)$ is the set of all morphisms of posets

$$[n] = \{0 \leq 1 \leq \dots \leq n\} \longrightarrow \{0 \leq 1 \leq \dots \leq m\} = [m]$$

We define

$$A : \Delta \longrightarrow \underline{Top} \quad \begin{cases} A(n) = \Delta^n = \{ (x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid x_i \geq 0,\ \sum_{i=0}^{n} x_i = 1 \} \\[1em] f : [n] \to [m] \quad A(f) : \Delta^n \longrightarrow \Delta^m \\ \tilde{A}(f)(e_i) \quad 0 \leq i \leq n \qquad \mathbb{R}^{n+1} \xrightarrow[\tilde{A}(f)]{} \mathbb{R}^{m+1} \\ = e_{f(i)} \end{cases}$$

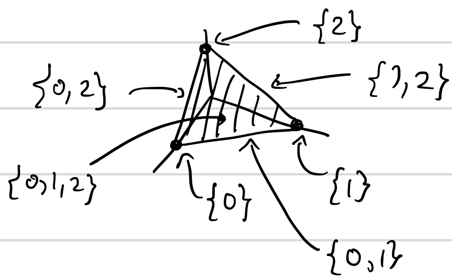
By the theorem A extends (essentially uniquely) to a colimit preserving functor

$$\bar{A} : \underline{Sets}^{\Delta^{op}} \longrightarrow \underline{Top}.$$

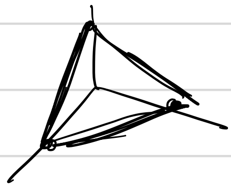
The category $\underline{sSets} := \underline{Sets}^{\Delta^{op}}$ is called the category of <u>simplicial sets</u>. What is this functor?

Recall that a <u>simplicial complex</u> is a set K of nonempty finite subsets of some set $\bar{K}$, s.t. if $X \in K$ and $Y \subseteq X$ is nonempty, then $Y \in K$. For example, take $\bar{K} = \{0,1,2\}$ and

$$K_{triangle} = \{\{0,1,2\}, \dots\}$$



$$K_{circle} = \{\{0,1\},\{1,2\},\{0,2\},\dots\}$$



The <u>geometric realisation</u> of K is $|K| = \bigcup_{\sigma \in K} co(\sigma)$, where $co(\sigma)$ denotes the convex hull in $\mathbb{R}\bar{K}$. We may generate from a simplicial complex K, assuming $\bar{K}$ partially ordered, a simplicial set

$$S_K : \Delta^{op} \longrightarrow \underline{Sets}$$

$$S_K(n) = \{(a_0,\dots,a_n) \in \bar{K}^{n+1} \mid a_0 \le \dots \le a_n \text{ and } \{a_0,\dots,a_n\} \in K\}$$

The morphisms in Δ are generated by

$$\varepsilon^i : [n-1] \longrightarrow [n] \qquad\qquad \eta^i : [n+1] \longrightarrow [n]$$

so to define $S_K : \Delta^{op} \longrightarrow \underline{Sets}$ we need only give

$$d_i := S_K(\varepsilon^i) : S_K([n]) \longrightarrow S_K([n-1]) \qquad (\underline{\text{face operator}})$$
$$(a_0, \dots, a_n) \longmapsto (a_0, \dots, \hat{a}_i, \dots, a_n)$$

$$s_i := S_K(\eta^i) : S_K([n]) \longrightarrow S_K([n+1]) \qquad (\underline{\text{degeneracy operator}})$$
$$(a_0, \dots, a_n) \longmapsto (a_0, \dots, a_i, a_i, a_{i+1}, \dots, a_n)$$

<u>Lemma</u> S_K is a simplicial set.

Let us now carefully analyse the value of the functor $\overline{A} : \underline{sSets} \longrightarrow \underline{Top}$ on this simplicial set $S_K : \Delta^{op} \to \underline{Sets}$. By what we said earlier

P

$$\overline{A}(S_K) = \{ (\overset{C}{n}, \overset{x}{x}, \overset{a}{a}) \mid n \geq 0, x \in S_K(n), a \in \Delta^n \} / \sim$$

where the relations were $(C, P(f)(y), a) \sim (C', y, A(f)(a))$ i.e.

$$(n-1, (x_0, \dots, \hat{x}_i, \dots, x_n), a \in \Delta^{n-1}) \sim (n, (x_0, \dots, x_n), A(\varepsilon^i)(a))$$

$$(n+1, (x_0, \dots, x_i, x_i, \dots, x_n), a \in \Delta^{n+1}) \sim (n, (x_0, \dots, x_n), A(\eta^i)(a))$$

__Example__ Let us take $K_{triangle}$, K_{circle} from above:

$\boxed{K = K_{circle}}$ $\quad S_K(0) = \{(0),(1),(2)\}$

degenerate simplices

$S_K(1) = \{(0,1),(1,2),(0,2),\ (0,0),(1,1),(2,2)\}$

$S_K(n)$ only contains degenerate simplices for $n > 1$.

$$\bar{A}(S_K) = \coprod_{c,x} A(c)/\sim$$

$\Delta^n \ni a$ denoted (n,x,a)

$$= \coprod_{n\geq 0,\, x\in S_K(n)} A(n)/\sim$$

degenerate stuff.

$$= \Big(A(0) \amalg A(0) \amalg A(0) \amalg \underset{(0)}{A(1)} \amalg \underset{(1)}{A(1)} \amalg \underset{(2)}{A(1)} \amalg \cdots\Big)/\sim$$

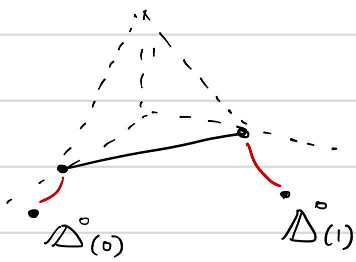
$$= \{\Delta^0 \amalg \Delta^0 \amalg \Delta^0 \qquad \text{3 points}$$

$$\amalg \underset{(0\,1)}{\Delta^1} \amalg \underset{(1\,2)}{\Delta^1} \amalg \underset{(0\,2)}{\Delta^1} \amalg \cdots\}/\sim \qquad \text{3 lines}$$

where the relations are

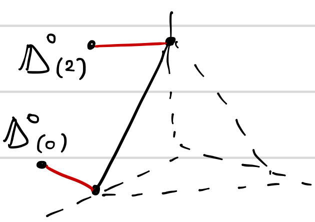
$$(0,(0),a\in\Delta^0) \sim (1,(0,1),A(\varepsilon^1)(a)\in\Delta^1)$$

$$(0,(1),a\in\Delta^0) \sim (1,(0,1),A(\varepsilon^0)(a)\in\Delta^1)$$



$$(0,(0),a\in\Delta^0) \sim (1,(0\,2),A(\varepsilon^1)(a)\in\Delta^1)$$

$$(0,(2),a\in\Delta^0) \sim (1,(0,2),A(\varepsilon^0)(a)\in\Delta^1)$$



etc...

__Exercise__: understand why the degenerate simplices can be ignored...

<u>Upshot</u> Once we choose a "standard model" $A : \Delta \longrightarrow \underline{Top}$ of all the n-simplexes, "tensoring" with A gives a colimit preserving functor

$$\bar{A} : \underline{Sets}^{\Delta^{op}} \longrightarrow \underline{Top}$$

which is nothing else than geometric realisation. Writing A as $\mathbb{\Delta}^{\bullet}$ and the geometric realisation as $|-|$ we may summarise this by

$$|S| \cong S \otimes_{\Delta} \mathbb{\Delta}^{\bullet} \qquad (\mathcal{C} = \Delta \text{ is like our ring } R)$$

Observe that a simplicial complex K on a finite set $\bar{K}$ is an <u>algorithm</u> for constructing a topological space $|K|$ (indeed a K s.t. $|K| \cong X$ is called a <u>triangulation</u> of X). The functor $\bar{A}(S_K)$ takes this algorithm and "executes" it using the data of $\mathbb{\Delta}^0, \mathbb{\Delta}^1, \mathbb{\Delta}^2, \ldots$ and how they fit together, as defined by $A = \mathbb{\Delta}^{\bullet}$.

This analogy will be made precise using the equivalence

$$\underline{sSets} \cong \mathcal{B}(\text{Lin}) \quad \text{theory of linear orders}$$

since objects on the right hand side are sheaves on a syntactic category, and the representable sheaves are closed terms in some formal language (the language of Lin). These terms formalise the idea of simplicial complexes as algorithms, as we will see. Finally, in the next lecture we will return to consider $f^* : \mathcal{B}(\mathcal{A}b) \longrightarrow \mathcal{B}(\mathcal{R}ng)$ as a tensor product, analogous to

$$(-) \otimes_{\Delta} \mathbb{\Delta}^{\bullet} : \mathcal{B}(\text{Lin}) = \underline{sSets} \longrightarrow \underline{Top}.$$

$$\text{i.e.} \quad f^* \text{ "=" } (-) \otimes_{Ab} U_{Rng}$$