Lecture 15 : Abstraction and adjunction

27/7/18

In my last lecture I explained the idea of "non-additive tensor products" and how to view geometric realisation as an example. This was motivated by the desire to understand more generally the inverse image part $f^*: \mathcal{B}(\mathbb{T}) \to \mathcal{E}$ of a geometric morphism $\mathcal{E} \longrightarrow \mathcal{B}(\mathbb{T})$ corresponding to a model of a geometric theory $\mathbb{T}$ in a cocomplete topos $\mathcal{E}$. This in turn was motivated by a desire to understand in some effective computational way how we might use classifying topoi to organise mathematical knowledge.

This lecture directly addresses the question of organising mathematical knowledge, with a focus on formalising the idea of abstraction (or hiding). We begin with an informal example. A mathematician $\mathcal{M}$ is working in some underlying logical system (say ZF set theory or a type theory in the sense of Lambek & Scott, as treated in Lecture 9) associated to which is a topos $\mathcal{E}$ (e.g. Sets or the topos $\mathcal{T}(\mathcal{L})$ associated to a type theory). Consider a piece of knowledge of the following kind:

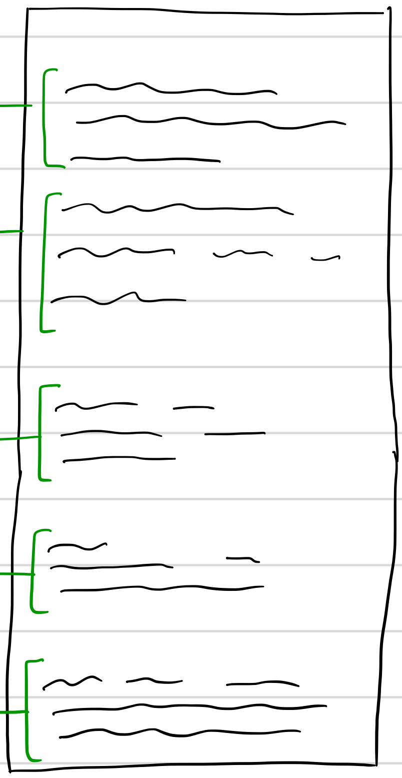
① $\mathcal{M}$ defines a set A

② $\mathcal{M}$ defines a function $f: A^2 \to A$, a function $- : A \to A$ and $0 \in A$

③ $\mathcal{M}$ proves $(A, f, -, 0)$ is an abelian group

④ $\mathcal{M}$ uses some general theorem about groups

⑤ $\mathcal{M}$ continues to prove something specifically about the group A



We imagine that the above is a completely <u>formal proof</u>, so that the "usage" in step 4 actually consists of an explicit specialisation of the proof of the general theorem about groups, with the generic group G replaced everywhere by the particular group A. In actual practice, of course, we do not do this (or at least we often do not do this). Instead the formal proof has the following shape:

(-1) $\mathcal{M}'$ proves a theorem about an abstract group G

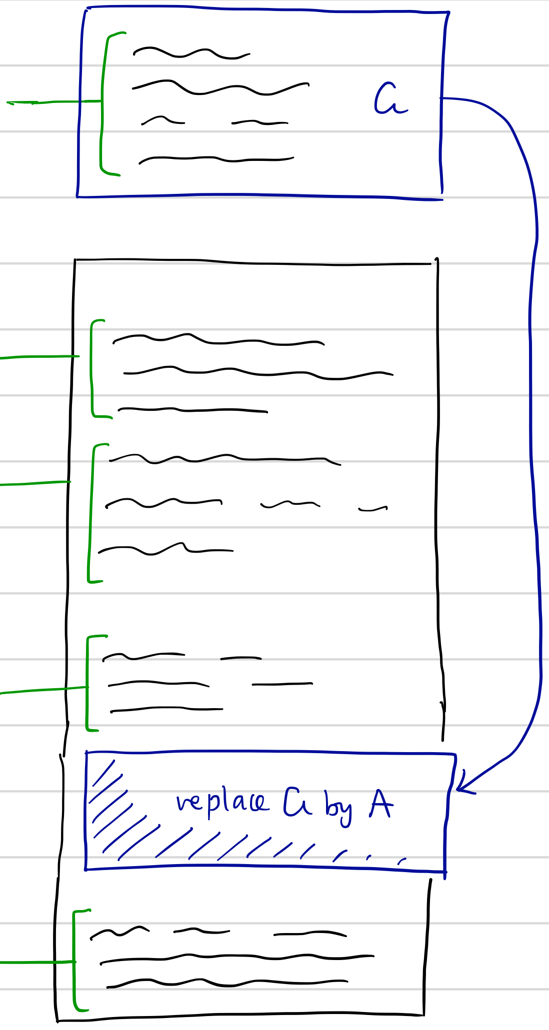
(1) $\mathcal{M}$ defines a set A

(2) $\mathcal{M}$ defines a function $f : A^2 \to A$, a function $- : A \to A$ and $0 \in A$

(3) $\mathcal{M}$ proves $(A, f, -, 0)$ is an abelian group

(4') $\mathcal{M}$ <u>invokes</u> the theorem proved by $\mathcal{M}'$ with $G \Longleftarrow A$

(5) $\mathcal{M}$ continues to prove something specifically about the group A



Where "invocation" involves, in part, rewriting the abstract G to the concrete A, the abstract $+$ on G to the concrete f on A, and so on.
We now proceed to formalise this picture in terms of <u>adjoint functors between topoi</u>, and then discuss the relationship to monads.

Categories of models in a topos

Let $\mathcal{T}$ be a type theory in the sense of Lambek & Scott, as discussed in Lecture 9 of this seminar series, and let $\mathcal{E}$ be the associated topos. We recall that the objects of $\mathcal{E}$ are "sets" i.e. equivalence classes of closed terms $\alpha : PA$ where A is a type and PA is to be understood as the "powerset" of A, and the morphisms from $\alpha : PA$ to $\beta : PB$ are equivalence classes of closed terms $F : P(A \times B)$ which provably "send" α into β and are functions. The equivalence relation says $F \sim F'$ iff. $\vdash F = F'$ where the entailment $\vdash$ is part of the data of the type theory.

[We take $\mathcal{E}$ as the underlying logical system of the mathematician $\mathcal{M}$.]

The language L_{Ab} has one "sort" X, no relation symbols, two function symbols $+ : XX \longrightarrow X$ (a function symbol is given together with an ordered nonempty list of sorts, to be interpreted as inputs, and a single sort to be interpreted as the output) and $- : X \longrightarrow X$ and one constant $0 : X$.

The terms : there is a countable list of variables $x, y, z, \ldots$ of sort X, which are terms of sort X, and 0 is a term of sort X. (Note that there is no construction of sorts, i.e. $X \times X$ is not a sort and $\langle x, y \rangle : X \times X$ is not a term). If t_1, t_2 are terms (all terms are of sort X!) then $+(t_1, t_2)$, written $t_1 + t_2$, is a term, and $-t_1$ is a term.

The *formulas*: if t_1, t_2 are terms then $t_1 = t_2$ is an atomic formula, and the symbols $\top$ (true) and $\bot$ (false) are atomic formulas. Atomic formulas are formulas and if p, q are formulas then so are

infinitary disjunction / conjunction

$$p \wedge q, \; p \vee q, \; p \Rightarrow q, \; \neg p, \; \forall x \in X\, p, \; \exists x \in X\, p, \; \bigvee_{i \in I} p_i, \; \bigwedge_{i \in I} p_i$$

where x is any variable. Occurrences of variables are declared free and bound in the usual way (see e.g. Lecture 9), and we identify formulas up to α-equivalence as defined there (we do not impose α-equivalence on terms, only on formulas, in any case all variables in $t_1 = t_2$ are free and so there are only trivial instances of $=_\alpha$ between terms).

So far what we have defined is a first-order language L_{Ab}. The *theory* of abelian groups Ab defined over L_{Ab} consists of a set of formulas (called axioms). For a general theory this set would be empty, or infinite. The axioms are:

$$\begin{aligned}
\phi_1 &: \quad (x+y)+z = x+(y+z). \\
\phi_2 &: \quad x+y = y+x. \\
\phi_3 &: \quad x+0 = x. \\
\phi_4 &: \quad x+(-x) = 0.
\end{aligned}$$

A *model* of Ab in the topos $\mathcal{E}$ is an object X^M of $\mathcal{E}$, say the equivalence class of a closed term $\alpha : PA$, a pair of morphisms in $\mathcal{E}$

$$+^M : X^M \times X^M \longrightarrow X^M \qquad (\text{repr. by } F_+ : P(A \times A \times A))$$

$$-^M : X^M \longrightarrow X^M \qquad (\text{repr. by } F_- : P(A \times A))$$

and a morphism $0^M : \mathbb{1} \longrightarrow X^M$ (represented by $F_0 : A$).

exercise: why not $P(\mathbb{1} \times A)$? use (c3-4), def^n of morphism such that the axioms are <u>valid</u>, which means that certain subobjects in $\mathcal{E}$ associated canonically to $\phi_1, \phi_2, \phi_3, \phi_4$ (by the model) are improper. The def^n is recursive, with for example the subobject

$$\{(x,y,z) \mid \phi_1\}^M \subseteq X^M \times X^M \times X^M$$

being defined to be the equaliser of the two ways around the usual associativity square

$$\begin{array}{ccc} X^M \times X^M \times X^M & \xrightarrow{+^M \times 1} & X^M \times X^M \\ {\scriptstyle 1 \times +^M} \downarrow & & \downarrow {\scriptstyle +^M} \\ X^M \times X^M & \xrightarrow[+^M]{} & X^M \end{array}$$

Note that in $\mathcal{E}$ this equaliser is the subobject

(informally) $\{ u \in X^M \times X^M \times X^M \mid (+^M \circ (+^M \times 1))(u)$
$= (+^M \circ (1 \times +^M))(u) \}$.

More formally, $X^M \times X^M \times X^M$ is the equivalence class of the term $\alpha^3 : P(A \times A \times A)$,

$$\alpha^3 := \{ u \in A \times A \times A \mid (\exists x_1, x_2, x_3 \in A)(u = \langle x_1, \langle x_2, x_3 \rangle \rangle \wedge x_1 \in \alpha \wedge x_2 \in \alpha \wedge x_3 \in \alpha) \}$$

and the equaliser of the above diagram is the term $e : P(A \times A \times A)$ defined as follows (recall the composition from Lecture 9 p.8, where we use $|f|$ as there to refer to a closed term representing f, so for example $|+^M| = F_+$).

$$e = \{ u \in A \times A \times A \mid u \in \alpha^3 \wedge \Psi_1(u) \}$$

$$\begin{aligned} \Psi_1(u) \text{ is } \forall (x,y,z \in A)\{ & \langle x, \langle y, z\rangle\rangle = u \\ & \Longleftrightarrow (\forall t \in A)\big[\langle\langle x, \langle y, z\rangle\rangle, t\rangle \in | +^M \circ (+^M \times 1) | \\ & \qquad \Longleftrightarrow \langle\langle x, \langle y, z\rangle\rangle, t\rangle \in | +^M \circ (1 \times +^M) | \big] \} \end{aligned}$$

The inclusion $\vdash e \subseteq \alpha$ induces a morphism $[e : P(A^3)] \longrightarrow [\alpha : P(A^3)] = (X^M)^3$ which is the equaliser in $\mathcal{E}$.

So to say the subobject $\{(x,y,z) \mid \phi_1\}^M$ is improper is therefore equivalent to saying e is equivalent to $\{u \in A \times A \times A \mid \top\}$ which means provability $\vdash e = \{u \in A \times A \times A \mid \top\}$, or equivalently

$$\text{Axiom 1 holds} \Longleftrightarrow \vdash \forall u (u \in \alpha^3 \Rightarrow \Psi_1(u)) \text{ in the type theory } \mathcal{T} \tag{6.1}$$

which is what we expect. Similarly for the other axioms. So we have a notion of a <u>model</u> M of Ab in $\mathcal{E}$, and a natural notion of a <u>morphism</u> of models in $\mathcal{E}$ (as in Lecture 14) and hence a category <u>Mod</u>$(Ab, \mathcal{E})$. These definitions generalise in a straightforward way to general theories, and to arbitrary topoi $\mathcal{E}$.

The type theories in Lambek–Scott are sufficiently rich that they give rise immediately to topoi (e.g. they contain powerset types). Next we consider a similar but different construction of syntactic categories from <u>first-order logics</u> such as L_{Ab}, which only become topoi once we pass to sheaves.

The syntactic site

Next we describe the classifying topos $\mathcal{B}(Ab)$, and the geometric morphism $\mathcal{E} \longrightarrow \mathcal{B}(Ab)$ induced by a model M. Unfortunately MacLane & Moerdijk's treatment in §X.5 is hopeless:

- The definition of the topology on the syntactic site is wrong (arguably a typo, but it is in a crucial definition: θ_i does not appear!)
- All their definitions involve quantifiers $\forall \mathcal{E}$ over all topoi $\mathcal{E}$! In the philosophical context of constructive mathematics and foundations, this is an embarrassment. Nor is the fig leaf provided on p. 558 sufficient.

We therefore follow Johnstone "Sketches of an elephant" §D1.4. We retain the setting of the language L_{Ab} (with $\times, +, -, 0$) and theory Ab, with its model M in the topos $\mathcal{E} = \mathcal{T}(\mathbb{J})$, where $\mathbb{J}$ is a type theory. To the pair (L_{Ab}, Ab) we associate a site $(\mathcal{C}, J)$, the syntactic site, whose category of sheaves (see Lectures 6, 7, 10, 11) denoted

$$\mathcal{B}(Ab) := Sh_J(\mathcal{C}) \subseteq \underline{Sets}^{\mathcal{C}^{op}}$$

has the right universal property to be the classifying topos of Ab. In particular the geometric morphism $\mathcal{E} \longrightarrow \mathcal{B}(Ab)$ is completely determined by a functor

$$\mathcal{C} \hookrightarrow Sh_J(\mathcal{C}) \xrightarrow{f^*} \mathcal{E}$$

which we will describe in syntactic terms.

The <u>syntactic site</u> $(\mathcal{C}, J)$ of the theory Ab has for its objects equivalence classes of geometric formulas. A formula of L (which recall had one sort X, function symbols $+, -$ and a constant $0 : X$) is called <u>geometric</u> according to the following recursive definition:

- The atomic formulas $t = t'$, $\top$, $\bot$ are geometric
- If ϕ, ψ are geometric so are $\phi \wedge \psi$, $\bigvee_{i \in I} \phi_i$
- If ϕ is geometric then so is $\exists x \in X\, \phi$ is geometric.

A <u>context</u> is a finite list $\underline{x} = (x_1, \dots, x_n)$ of distinct variables. The empty context is allowed and is denoted $[\,]$. A <u>formula-in-context</u> is a pair consisting of a formula ϕ and a context $\underline{x}$ such that $FV(\phi) \subseteq \{x_1, \dots, x_n\}$. We denote the pair by $\phi(\underline{x})$ or $\{\underline{x}.\phi\}$. We say two formulas in context $\{x_1, \dots, x_n . \phi\}$ and $\{y_1, \dots, y_n . \psi\}$ are <u>A-equivalent</u> if ψ can be obtained from ϕ by (possibly renaming bound variables in ϕ in the usual way, and) replacing every free occurrence of x_i in ϕ by y_i, for $1 \le i \le n$ (read "A" as a variant of "α").

<u>Def^n</u> The objects of $\mathcal{C}$ are A-equivalence classes of geometric formulas-in-context denoted $[\underline{x}.\phi]$. A morphism $[\theta] : [\underline{x}.\phi] \longrightarrow [\underline{y}.\psi]$ is an equivalence class of geometric formulas θ with $FV(\theta) \subseteq \{x_1, \dots, x_n, y_1, \dots, y_m\}$ such that

⌜we may and do assume $\{x_1, \dots, x_n\} \cap \{y_1, \dots, y_m\} = \phi$⌟

- $\theta \vdash_{\underline{x}, \underline{y}} \phi \wedge \psi$ ⌜says $\{(x,y) \mid \theta\} \subseteq \{x \mid \phi\} \times \{y \mid \psi\}$⌟
- $\theta \wedge \theta[\underline{z}/\underline{y}] \vdash_{\underline{x}, \underline{y}, \underline{z}} \underline{y} = \underline{z}$ ⌜another way of expressing the $\exists!$ part of Lecture 9 p. 8. Note $\underline{y} = \underline{z}$ means $\bigwedge_{i=1}^{m} y_i = z_i$⌟
- $\phi \vdash_{\underline{x}} \exists \underline{y}\, \theta$ ⌜$\exists \underline{y} = \exists y_1 \exists y_2 \cdots \exists y_m$⌟

The equivalence relation is $\theta \sim \theta'$ iff. $\theta \vdash_{\underline{x}, \underline{y}} \theta'$ and $\theta' \vdash_{\underline{x}, \underline{y}} \theta$, and composition is defined as in Lecture 9 for the topos of a type theory.

<u>Remark</u> Here $\vdash$ means provability or entailment of a sequent in an associated logic, as defined in Johnstone §D1.3 (see also Caramello "Theories, sites, toposes" §1.2) which includes the axioms of the theory Ab (e.g. $\vdash x+y=y+x$ may apper at a leaf). This is similar to entailment in Lambek-Scott with some key differences: it is much simpler owing to the fact first-order logics are simpler than type theories, and in Johnstone $\vdash$ involves rules for infinite disjunctions (e.g. $\phi_i \vdash \bigvee_i \phi_i$).

<u>Example</u> The formula-in-context $\{(x,y).\ x=-y\}$ which determines an object E of $\mathcal{C}$, should be viewed as the syntactic antecedent of the object in $\mathcal{E}$ which is the equaliser of the two arrows

$$X^M \times X^M \xrightarrow{\pi_1} X^M, \quad X^M \times X^M \xrightarrow{\pi_2} X^M \xrightarrow{-^M} X^M$$

<u>Def^n</u> A basis for the topology J is given at an object $[\underline{y}.\psi]$, where $\underline{y}=(y_1,\dots,y_m)$, by those collections of arrows

$$\{[\theta^i] : [\underline{x}^i.\phi^i] \longrightarrow [\underline{y}.\psi]\}_{i=1}^{n} \qquad (|\underline{x}^i| = n_i)$$

for which we have the following entailment holds:

$$\vdash \forall y_1,\dots,y_m \Big(\psi(\underline{y}) \Rightarrow \bigvee_{i=1}^{n} \exists x^i_1,\dots,x^i_{n_i} \big(\phi^i(\underline{x}^i) \wedge \theta^i(\underline{x}^i,\underline{y})\big)\Big)$$

<u>Theorem</u> The pair $(\mathcal{C},J)$ is a site, and the associated sheaf topos

$$\mathcal{B}(Ab) := Sh_J(\mathcal{C})$$

is a classifying topos for Ab.

Moreover the same construction works for any geometric theory, for instance the theory of linear orders Lin, which arose in Lecture 12 and which we will revisit next lecture. The proof of the Theorem is quite involved and I have not fully understood it, so I won't try to explain it except to elaborate some of the details in the particular case of the model $(X^M, +^M, -^M, 0^M)$ of Ab in the topos $\mathcal{E}$.

<u>The universal model</u> For $\mathcal{B}(Ab)$ to be a classifying topos it must possess a universal model $(U, +, -, 0)$ of the theory. It is (all variables of type X)

$$U := [x.\top], \qquad \begin{array}{lll} + : [\{x_1,x_2\}.\top] \longrightarrow [y.\top] & \text{is} & [\{x_1,x_2,y\}. +(x_1,x_2,y)] \\ - : [x.\top] \longrightarrow [y.\top] & \text{is} & [\{x,y\}. -(x,y)] \\ 0 : [y.\, y=0] \longrightarrow [x.\top] & \text{is} & [\{x,y\}.\, y=0 \wedge x=y] \end{array}$$

<u>Geometric morphisms</u> The universal property of $\mathcal{B}(Ab)$ means the model $(X^M, +^M, 0^M)$ in $\mathcal{E}$ must induce a unique (up to isomorphism) geometric functor

$$\begin{tikzcd} \mathcal{E} \arrow[r, shift right, "f_*"'] & \mathcal{B}(Ab) \arrow[l, shift right, "f^*"'] \end{tikzcd}$$

with $f^*(U,+,-,0) \cong (X^M, +^M, 0^M)$. That is, as objects $f^*(U) \cong X^M$ in such a way that all relevant diagrams commute. Now consider the composite

$$\mathcal{C} \xhookrightarrow{\text{inc}} Sh_J(\mathcal{C}) = \mathcal{B}(Ab) \xrightarrow{f^*} \mathcal{E}$$

This functor sends formulas-in-context $[\underline{x}.\phi]$ to objects in $\mathcal{E}$, which are themselves equivalence classes of terms. We next consider these objects $f^*[\underline{x}.\phi]$ in a concrete example, which will also serve to return us to the context at the beginning of the talk (i.e. invocation of a theorem about abelian groups).

<u>Def^n</u> An element x in an abelian group A is <u>torsion</u> if $nx=0$ for some $n\geq 1$.

Define the following formula in $\mathbb{L}Ab$:

$$T(x) \quad : \quad \bigvee_{n=1}^{\infty} nx=0. \qquad \qquad (nx \equiv \overbrace{x+x+\cdots+x}^{n})$$

This is a geometric formula and in the syntactic category $\mathcal{C}$ there is a morphism

$$U_{tors} := [y.\,T(y)] \xrightarrow{\ \psi\ } [x.\,\top] = U$$

where ψ is the geometric formula $T(y)\wedge x=y$. This is a subobject (see Johnstone D1.4.4(iv)) in $\mathcal{C}$ and thus also in $\mathcal{B}(Ab)$, as $\mathcal{C}\hookrightarrow[\mathcal{C}^{op},\underline{Sets}]$ preserves limits, and $Sh_J(\mathcal{C})$ is closed under limits.

<u>Theorem</u> Let A be an abelian group, $A_{tors}\subseteq A$ the set of torsion elements. Then A_{tors} is a subgroup.

The internalised version of this theorem is the statement that in $\mathcal{C}$ (hence in $\mathcal{B}(Ab)$) there exist morphisms f_1, f_2 making the diagrams below commute

$$\begin{tikzcd} U_{tors}\times U_{tors} \arrow[r, "\psi\times\psi"] \arrow[d, dashed, "f_1"'] & U\times U \arrow[d, "+"] \\ U_{tors} \arrow[r, "\psi"'] & U \end{tikzcd} \qquad \begin{tikzcd} \mathbb{1} \arrow[r, "0"] \arrow[dr, dashed, "f_2"'] & U \\ & U_{tors} \arrow[u, "\psi"'] \end{tikzcd} \qquad (11.1)$$

<u>Remark</u> $[y.\,y=0]\cong\mathbb{1}$ in $\mathcal{B}(Ab)$ since if $[\theta]\in\mathcal{C}([\underline{x}.\,\phi],[y.\,y=0])$ then θ is provably equivalent to $y=0$.

Unravelling the definitions, the existence of f_1 is equivalent to the provability (from the axioms of Ab) of the sequent

$$J(y_1)\wedge J(y_2) \vdash_{y_1,y_2} J(y_1+y_2). \qquad (12.1)$$

Such a proof establishes that $[\theta] : [\{y_1,y_2\}.\, J(y_1)\wedge J(y_2)] \longrightarrow [y.\, J(y)]$ given by taking θ to be the formula $y = y_1+y_2 \wedge J(y_1)\wedge J(y_2)$ satisfies the first axiom of a morphism (i.e. $\theta \vdash \phi\wedge\psi$), and is therefore an arrow $U_{tors}\times U_{tors} \longrightarrow U_{tors}$, and it is then immediate that the diagram commutes. Note that the proof of (12.1) involves the rules for infinite disjunctions. Similar observations apply to the other diagram. The upshot is that

- The provability of the theorem "torsion elements form a subgroup" is encoded in the existence of factorisations f_1, f_2 in the category $\mathcal{B}(Ab)$.

The model of Ab in $\mathcal{E}$ induces f^* which sends these commuting diagrams to commuting diagrams in $\mathcal{E}$ (note f^* sends U to X^M and preserves finite limits) and picks out a subobject $X^M_{tors} := f(U_{tors}) \subseteq f(U) \cong X^M$, as in

$$\mathcal{C} \xhookrightarrow{inc} Sh_J(\mathcal{C}) = \mathcal{B}(Ab) \xrightarrow{f^*} \mathcal{E}$$

$$\begin{tikzcd} U_{tors}\times U_{tors} \arrow[r, "\psi\times\psi"] \arrow[d, dashed, "f_1"'] & U\times U \arrow[d, "+"] \\ U_{tors} \arrow[r, "\psi"'] & U \end{tikzcd} \qquad \begin{tikzcd} \mathbb{1} \arrow[r, "0"] \arrow[dr, dashed, "f_2"'] & U \\ & U_{tors} \arrow[u, "\psi"'] \end{tikzcd}$$

$$\xrightarrow{\quad f^* \quad}$$

$$\begin{tikzcd} X^M_{tors}\times X^M_{tors} \arrow[r] \arrow[d, dashed] & X^M\times X^M \arrow[d, "+^M"] \\ X^M_{tors} \arrow[r] & X^M \end{tikzcd} \qquad \begin{tikzcd} \mathbb{1} \arrow[r, "0^M"] \arrow[dr, dashed] & X^M \\ & X^M_{tors} \arrow[u] \end{tikzcd}$$

Moreover when we unravel the categorical structure of $U_{tors} \subseteq U$ in $\mathcal{B}(Ab)$ we see that as subobjects it is a union (a colimit) of

$$U_{tors} = \bigcup_{n\geq 1} [x . nx = 0]$$

and hence (since f^* preserves finite limits and all colimits)

$$X^M_{tors} = f^*(U_{tors}) = \bigcup_{n\geq 1} f^*[x . nx = 0]$$

$$= \bigcup_{n\geq 1} \text{Equaliser}\left(X^M \longrightarrow \mathbb{1} \xrightarrow{0} X^M,\ X^M \xrightarrow{\Delta} (X^M)^n \xrightarrow{(+^M)^{n-1}} X^M \right)$$

which if we look inside the type theory underlying $\mathcal{E}$ says what we expect (as on p. 5). So the functorial approach is constructing the "right thing" from a logical point of view, and the existence of the factorisations $f^*(f_1)$, $f^*(f_2)$ say precisely that in the particular model of Ab within the type theory (think of X^M as a concrete set) the torsion elements form a subgroup.

In summary, we have realised the promised formalisation of abstraction via adjunction:

- We prove a theorem in the theory Ab of abelian groups
- We construct an abelian group in "sets", i.e. a model of Ab in $\mathcal{E}$.
- This induces a geometric morphism $f : \mathcal{E} \longrightarrow \mathcal{B}(Ab)$ consisting of an adjoint pair (f^*, f_*) whose inverse image part $f^* : \mathcal{B}(Ab) \longrightarrow \mathcal{E}$ sends that general theorem (expressed in categorical terms) to a theorem about our particular model.

Actually, if we define the theory TorsAb to have the same language as Ab but the additional axiom $T(x) : \bigvee_{n \in \mathbb{N}} nx = 0$, then TorsAb is geometric and the same construction produces $\mathcal{B}(\text{TorsAb})$, and $U_{\text{tors}} \in \mathcal{B}(\text{Ab})$ induces a geometric morphism $g : \mathcal{B}(\text{Ab}) \longrightarrow \mathcal{B}(\text{TorsAb})$ with $g^*(V) \cong U_{\text{tors}}$, where $V \in \mathcal{B}(\text{TorsAb})$ is the universal torsion abelian group. So we actually have constructed a chain of adjunctions

$$
\begin{tikzcd}
\mathcal{E} \arrow[r, shift right, "f_*"'] & \mathcal{B}(\text{Ab}) \arrow[l, shift right, "f^*"'] \arrow[r, shift right, "g_*"'] & \mathcal{B}(\text{TorsAb}) \arrow[l, shift right, "g^*"']
\end{tikzcd}
\tag{14.1}
$$

associated topos of a type theory, objects are equiv. classes of closed terms (i.e. "sets")

classifying topos objects are equiv. classes of geometric formulas

$f^*(U) \cong X^M$

$g^*(V) \cong U_{\text{tors}}$

where the functors f^*, g^* take objects (formulas) and "incarnate" them in the target language. In the case of f^* we view $\mathcal{E}$ and its type theory as being more expressive and low-level, and the first-order theory Ab and $\mathcal{B}(\text{Ab})$ as being a restricted and more abstract domain of mathematical knowledge. At least for <u>geometric</u> theories, classifying topoi and geometric morphisms (whose left adjoint parts have a strong algorithmic character) seem to be an effective means of formalising abstraction as a relation between logics. Ultimately this is a formalisation of abstraction in terms of <u>adjoint pairs</u>.

A related but distinct concept is that of a <u>monad</u>. However monads are a coarser notion than adjunctions, and at least from the topos-theoretic perspective <u>there is no reason to believe in a fundamental connection between monads and abstraction</u>.

To make this point concretely, consider the monads and comonads arising naturally from (14.1)

$$
\begin{tikzcd}
 & & M_f = f_* f^* & & \\
C_f = f^* f_* \circlearrowright \mathcal{E} \arrow[rr, shift right, "f_*"'] & & \mathcal{B}(Ab) \arrow[ll, shift right, "f^*"'] \arrow[rr, shift right, "g_*"'] & & \mathcal{B}(\mathrm{TorAb}) \arrow[ll, shift right, "g^*"'] \circlearrowleft M_g = g_* g^* \\
 & & C_g = g^* g_* & &
\end{tikzcd}
\tag{15.1}
$$

where C indicates a monad (e.g. $C_f = f^* f_* \xrightarrow{f^* \eta f_*} f^* f_* f^* f_* = C_f C_f$) and M a monad. Now the deep connections between <u>higher-order logic</u> and topoi (as developed in Lambek-Scott and exposited in Lecture 9 of this seminar) together with the deep theorems about the existence of <u>classifying topoi</u> and their syntactic construction (sketched at least in one example this lecture) means (it seems to me) that any credible general story formalising abstraction in terms of <u>monads</u> has to grapple with the simple example presented above:

$$
\mathbb{T} \qquad\qquad Ab \qquad\qquad \mathrm{TorAb} \tag{15.2}
$$

where $\mathbb{T}$ is a type theory (higher-order logic) rich enough to contain a model of Ab (which it does, as soon as it has a natural number object), and since $\mathcal{E}$, $\mathcal{B}(Ab)$, $\mathcal{B}(\mathrm{TorAb})$ are constructed out of the syntax, any monadic picture of (15.2) must interact with (14.1) in some way (e.g. the only obvious way to combine "monad" and "higher-order logic" is a (co) monad on $\mathbb{T}$, as in Moggi's original paper), and the <u>only obvious candidates</u> are the (co) monads in (15.1). But these obvious candidates contain less information than (14.1), as we will explain.

For coalgebras over a comonad see MacLane & Moerdijk (henceforth [MM]) Ch. V. Section 8. Given a monad (comonad) T on a category $\mathcal{C}$ we write $\mathcal{C}^T$ for the category of algebras (coalgebras) over T and $\mathcal{C}_T$ for the category of free (cofree) algebras (coalgebras). Often $\mathcal{C}^T$ is called the Eilenberg-Moore category of $(\mathcal{C},T)$ and $\mathcal{C}_T$ the Kleisli category. By Prop. 4.2.1 of Borceux's "Handbook of categorical algebra II" any adjoint pair giving rise to T, which let us now fix to be a comonad, say

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift right, "R"'] & \mathcal{X} \arrow[l, shift right, "L"'] \end{tikzcd} \qquad\qquad T = L \circ R$$

induces a pair of J, K making the diagram

$$\begin{tikzcd} \mathcal{C}_T \arrow[r, "K"] \arrow[dr, bend right, "V"'] & \mathcal{X} \arrow[r, "J"] \arrow[d, "L"] & \mathcal{C}^T \arrow[dl, bend left, "U"] \\ & \mathcal{C} & \end{tikzcd}$$

commute up to natural isomorphism where U, V are forgetful functors, with $J \circ K$ the fully faithful inclusion $\mathcal{C}_T \subseteq \mathcal{C}^T$ and J is full while K is full and faithful. In the situation of (15.1) this means that we have a diagram (set $T = C_f$)

$$\begin{tikzcd} \mathcal{E}_T \arrow[r, "K"] \arrow[dr, bend right, "V"'] & \mathcal{B}(Ab) \arrow[r, "J"] \arrow[d, "f^*"] & \mathcal{E}^T \arrow[dl, bend left, "U"] \\ & \mathcal{E} & \end{tikzcd}$$

and the information of the geometric morphism f is recoverable from the pair $(\mathcal{E}, T)$ (i.e. "the monadic story about abstraction subsumes the adjunction story") if and only if either $\mathcal{E}_T = \mathcal{B}(Ab)$ or $\mathcal{B}(Ab) = \mathcal{E}^T$.
Now by [MM, Lemma VII. 3] we have

$$\mathcal{B}(Ab) = \mathcal{E}_T \text{ or } \mathcal{B}(Ab) = \mathcal{E}^T \implies f^* \text{ is faithful (as } U \text{ is faithful)}$$

$$\implies \text{for each } E \in \mathcal{B}(Ab) \text{ the map } \mathrm{Sub}(E) \longrightarrow \mathrm{Sub}(f^*E) \text{ is injective.}$$

Now in $\mathcal{B}(Ab)$ we have the proper subobject $U_{tors} \hookrightarrow U$ (since non-torsion groups exist!) but if the model X^M in $\mathcal{E}$ happens to have $X^M_{tors} = X^M$ (for example if the type theory has a natural numbers object and $X^M = \mathbb{Z}/n\mathbb{Z}$) then $f^* U_{tors} \cong X^M_{tors} = X^M \cong f^* U$ so the map

$$\mathrm{Sub}(U) \longrightarrow \mathrm{Sub}(X^M)$$

is not injective, hence f^* is not faithful and so $\mathcal{B}(Ab) \neq \mathcal{E}_T$, $\mathcal{B}(Ab) \neq \mathcal{E}^T$.
In this precise sense the diagram of adjunctions (14.1) cannot be recovered from (co)monads alone.

Since adjunctions have (via the theory of classifying topoi) a strong claim to the role of "organising mathematical knowledge" categorically, this counter-example raises doubt about any fundamental connection between monads and abstraction. Instead, the 2-category of Grothendieck topoi, as developed in Johnstone's "Sketches of an elephant" and Caramello's "Theories, sites, toposes" seems like the "correct" realisation of the underlying ideas (perhaps even in the programming context discussed by Moggi).