Higher-order logic and topoi

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1 Recall

Definition 1 (Subobject Classifier)**.** *Let $\mathscr{C}$ be a Category with a terminal object* 1*. A* ***subobject classifier*** *is a monic* $true : 1 \to \Omega$*, such that for any monic* $m : S \rightarrowtail M$*, there exists a unique* $\phi : M \to \Omega$ *such that*

$$\begin{tikzcd} S \arrow[r] \arrow[d, tail, "m"'] & 1 \arrow[d, tail, "true"] \\ M \arrow[r, dashed, "\phi"'] & \Omega \end{tikzcd}$$

is a pullback diagram.

Theorem 1. *A Category $\mathscr{C}$ with finite limits has a subobject classifier if and only if there is an object Ω, and for each object $X \in \mathscr{C}$, a natural bijection,*

$$Sub(X) \cong Hom(X, \Omega)$$

Definition 2 (Exponential)**.** *An* ***exponential*** *for an object $E \in \mathscr{E}$ is a collection of objects $\{Z^E\}_{Z \in \mathscr{E}}$ and a collection of isomorphisms*

$$Hom(E \times X, Z) \cong Hom(X, Z^E)$$

natural in X and Z.

ie, an exponential for $E \in \mathscr{E}$ is a right adjoint to the functor $E \times \_$.

Definition 3 (Topos)**.** *A topos is a Cartesian Closed Category $\mathscr{E}$ which admits all finite limits, and a subobject classifier.*

2 Introduction

In the text [1], which we are following in this seminar, the notion of an *elementary topos* is used in the sections relevent to Higher Order Logic. In particular, the notion of a power set operator is used, which is coming from how functions are spoken about in the topos Set. The goal of this talk is to give this definition, and prove that it is equivalent to the definition of a topos. One key family of objects in the proof will be morphisms into the subobject classifier of an arbitrary topos, which can be viewed as predicates. This will motivate a small section on predicates in arbitrary topoi before the proof is presented.

First, the definition of an *elementary topos*,

Definition 4 (Elementary Topos)**.** *An* ***Elementary Topos*** *is a Category $\mathscr{E}$ which admits*

- *all pullbacks*
- *a terminal object*
- *a subobject classifier* Ω
- *for every object $E \in \mathscr{E}$, a collection of objects $\{\Omega^E\}_{E \in \mathscr{E}}$ and a collection of bijections*

  $$Hom(E \times Y, \Omega) \cong Hom(Y, \Omega^E)$$

  which are all natural in Y.

There are a few immediate notes to make.

This is **not** to say that $\mathscr{E}$ admits an exponential for Ω. This definition is not asking for Ω varying, so this is not an adjunction, and so in particular is not an exponential. Also, the following notation and terminology will be used,

Notation 1. *The image of a map $\varphi \in Hom(X \times Y, \Omega)$ will be notated by $\ulcorner\varphi\urcorner$, and will be refered to as the* transpose *of φ. The image of a map $\psi \in Hom(Y, \Omega^X)$ will be refered to as the inverse transpose, and in fact there is an explicit map for this direction,*

$$\psi \mapsto ev_X(id_X \times \varphi)$$

where $ev_X : X \times \Omega^X \to \Omega$ is inverse transpose of 1_{Ω^X} (for which there is no explicit map for).

3 Predicates

3.1 True$_B$

The predicate

$$B \longrightarrow 1 \xrightarrow{\text{true}} \Omega$$

can be thought of as being "true" for any value of B. This composition will be refered to as true_B.

3.2 Evaluation as membership

In the topos Set, a map from the terminal object to a set $1 \to B$ picks out an element $b \in B$. This provides intuition for maps out of the terminal object in an arbitrary topos. With this intuition, the evaluation map of the form $\text{ev}_B : B \times \Omega^B \to \Omega$, for some $B \in \mathscr{E}$, can be seen as the membership predicate. Indeed, given $b : 1 \to B$, and a subobject $B' \rightarrowtail B \in \text{Sub}(B)$, the following diagram commutes

$$\begin{tikzcd}
1 \times 1 \arrow[r, "b \times 1_1"] \arrow[d, "\cong"'] & B \times 1 \arrow[r, "1_B \times s"] \arrow[d, "\cong"'] & B \times \Omega^B \arrow[d, "\text{ev}_B"] \\
1 \arrow[r, "b"'] & B \arrow[r, "\chi_{B'}"'] & \Omega
\end{tikzcd}$$

where $s : 1 \to \Omega^B$ is the transpose of $\chi_{B'}$. The square on the right commutes by definition of the transpose of χ_B, given any subobject $\chi_{B'} : B \to \Omega$, which is a map $B \times 1 \to \Omega$ as $B \cong B \times 1$, there exists a unique $1 \to \Omega^B$ such that the square on the right commutes. The square on the left commutes because the isomorphism $B \times 1 \to B$ is the map π_B, and by definition, $\pi_B(b\varphi^{-1} \times 1_1) = b\varphi^{-1}$, where $\varphi : 1 \times 1 \to 1$ is an isomorphism.

The significance of the above diagram commuting, is that the map $\chi_{B'}b = \text{true}$ if and only if $\text{ev}_B(b \times s) = \text{true}$, and so interpretting this in the topos set, $\chi_{B'}b = \text{true}$ if and only if $b \in B'$, so $\text{ev}_B(b \times s) = \text{true}$ if and only if $b \in s$. So ev_B can be thought of as the predicate "$b \in s$".

3.3 The predicate "is a singleton"

For every object $E \in \mathscr{E}$, where $\mathscr{E}$ is a topos, there is a diagonal map $\Delta_E : E \to E \times E$. Indeed this map is monic, so there is a corresponding characteristic morphism $\chi_{\Delta_E} : E \times E \to \Omega$. In the topos Set, this map sends $(e_1, e_2) \mapsto 1$ if and only if $e_1 = e_2$. So this map can be seen as the *equality* predicate. The transpose of this map, $\ulcorner\chi_{\Delta_E}\urcorner : E \to \Omega^E$, sends an element $e \in E$ to the map $\chi_{\{e_1\}} : E \to \Omega$, this map will be notated δ_E, after the kronecker delta function. In the topos Set, this function sends an element $e \in E$ to the singleton subset $\{e\} \subseteq E$. For this reason, the notation $\{\cdot\}_B := \ulcorner\chi_{\Delta_E}\urcorner$ will be adopted.

In fact, as lemma 1 (below) states, this map too is monic, and so has a corresponding map $\chi_{\{\cdot\}_E}$. To avoid cumbersome notation, $\sigma_E : \Omega^E \to \Omega$ will be written in place of $\chi_{\{\cdot\}_E}$. So to summaries the notation,

$$\sigma_E = \chi_{\{\cdot\}_E} = \chi_{\ulcorner\chi_{\Delta_E}\urcorner} : \Omega^E \to \Omega$$

Lemma 1. *The map $\{\cdot\}_E$ is monic, for any object $E \in \mathscr{E}$.*

Let $b, b' : B \to E$ be two maps such that $\{\cdot\}_E b = \{\cdot\}_E b'$. The proof will use that the following square is a pullback diagram,

$$\begin{tikzcd} B \arrow[r, "b"] \arrow[d, "\langle 1_B, b\rangle"'] & E \arrow[d, "\Delta_E"] \\ B \times E \arrow[r, "b \times 1_E"'] & E \times E \end{tikzcd}$$

Commutativity comes from the fact that $\langle \mathrm{id}_B, b\rangle$ is the map such that diagrams

$$\begin{tikzcd} B \arrow[r, "\langle 1_B, b\rangle"] \arrow[dr, "b"'] & B \times E \arrow[d, "\pi_2"] \\ & E \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} B \arrow[r, "\langle 1_B, b\rangle"] \arrow[dr, "b"'] & B \times E \arrow[d, "b\pi_1"] \\ & E \end{tikzcd}$$

commute. Thus

$$\Delta_E b = \langle 1_E, 1_E\rangle b = \langle b, b\rangle \stackrel{*}{=} \langle b\pi_1, \pi_2\rangle\langle 1_B, b\rangle = (b \times 1_E)\langle 1_B, b\rangle$$

where the equality labeled by $*$ is the where commutativity of the diagrams has been used.

For universality, say $\alpha : C \to E$ and $\langle \beta_1, \beta_2\rangle : C \to B \times E$ are such that the outside square of the diagram

$$\begin{tikzcd} C \arrow[dr, dashed, "\gamma"] \arrow[drr, bend left, "\alpha"] \arrow[ddr, bend right, "\langle \beta_1, \beta_2\rangle"'] & & \\ & B \arrow[r, "b"] \arrow[d, "\langle 1_B, b\rangle"'] & E \arrow[d, "\Delta_E"] \\ & B \times E \arrow[r, "b \times 1_E"'] & E \times E \end{tikzcd}$$

commutes. Commutativity will hold once the three equations

$$\gamma = \beta_1, \qquad \beta_2 = b\gamma, \qquad \alpha = b\gamma$$

have been shown to hold. The first equation shows that if γ exists, then it must be β_1, thus the proof is reduced to showing the second two equations with β_1 substituted in for γ. Since the outer square commutes,

$$\Delta_E \alpha = (b \times 1_E)\langle \beta_1, \beta_2, \rangle$$

which implies

$$\pi_1 \Delta_E \alpha = \pi_1 (b \times 1_E)\langle \beta_1, \beta_2, \rangle$$

from which it follows that $\alpha = b\beta_1$, which gives the third equation. As for the second equation,

$$\pi_2 \Delta_E \alpha = \pi_2 (b \times 1_E)\langle \beta_1, \beta_2, \rangle$$

from which it follows that $\alpha = \beta_2$, and as already seen, $\alpha = b\beta_1$, thus $\beta_2 = b\beta_1$.

Now onto the proof of the lemma. Since this square is a pullback diagram, it then follows that the square

$$\begin{tikzcd} B \arrow[r, "b"] \arrow[d, "\langle 1_B, b\rangle"'] & E \arrow[r] \arrow[d, "\Delta_E"] & 1 \arrow[d, "\mathrm{true}"] \\ B \times E \arrow[r, "b \times 1_E"'] & E \times E \arrow[r, "\chi_{\Delta_E}"'] & \Omega \end{tikzcd}$$

is a pullback diagram. Running the same argument for b', it follows that $\langle 1_B, b\rangle$ and $\langle 1_B, b'\rangle$ represent the same subobject of $B \times E$, so by the definition of the equivalence relation on subobjects, there exists an isomorphism $h : B \to B$ such that $\langle 1_B, b\rangle h = \langle 1_B, b'\rangle$. Projecting onto the first component shows that $h = 1_B$, and projecting onto the second shows that $b = b'$. $\square$

4 The Theorem

Theorem 2. *A Category $\mathscr{E}$ is a topos if and only if it is an elementary topos.*

Before proving the theorem, consider the case when $\mathscr{E} = \underline{\text{Set}}$ for intuition. Let B, C be sets, then

$$C^B = \{\text{functions } f : B \to C\}$$

which can be seen as the subset $C^B \subseteq B \times C$ consisting of graphs, ie, C^B is the subset

$$C^B = \{\Gamma \subseteq B \times C \mid \Gamma \text{ is a graph}\}$$

Since $\text{Sub}(\Omega^{B\times C}) \cong \text{Hom}(\Omega^{B\times C}, \Omega)$, this can be realised as a function,

$$\begin{aligned} C^B : \Omega^{B\times C} &\to \Omega \\ \Gamma &\mapsto \begin{cases} 1, & \Gamma \text{ is a graph} \\ 0, & \text{else} \end{cases} \end{aligned}$$

ie,

$$\begin{aligned} C^B : \Omega^{B\times C} &\to \Omega \\ \Gamma &\mapsto \begin{cases} 1, & \forall b \in B, \exists! c \in C, (b, c) \in \Gamma \\ 0, & \text{else} \end{cases} \end{aligned}$$

There is a clever trick to describe the $\forall$ in terms of a preimage of a function, consider the function

$$\begin{aligned} u : \Omega^{B\times C} &\to \Omega^B \\ \Gamma &\mapsto \left(b \mapsto \begin{cases} 1, & \exists! c \in C, (b, c) \in \Gamma \\ 0, & \text{else} \end{cases} \right) \end{aligned}$$

then $\Gamma \in \Omega^{B\times C}$ is a graph if and only if $u(\Gamma) = \mathbb{1}_B$. Asking for the collection of objects $\Gamma \in \Omega^{B\times C}$ for which u to maps Γ into $\mathbb{1}_B$ is the same as asking for the pullback of the diagram

$$\begin{tikzcd} C^B \arrow[r, dashed] \arrow[d, dashed, "m"'] & 1 \arrow[d, "\ulcorner \text{true}_B \urcorner"] \\ \Omega^{B\times C} \arrow[r, "u"'] & \Omega^B \end{tikzcd}$$

Notice that m is the pullback of a monic, and so is itself monic. So, to generalise this construction to an arbitrary topos, it remains only to come up with a general way of describing the morphism u.

As already seen, the map $\text{ev}_{B\times C}$ can be seen as the membership predicate, the transpose of this map is

$$\begin{aligned} B \times \Omega^{B\times C} &\to \Omega^C \\ (b, \Gamma) &\mapsto \left(c \mapsto \begin{cases} 1, & (b, c) \in \Gamma \\ 0, & \text{else} \end{cases} \right) \end{aligned}$$

which is very close to u, but it needs to be checked that the function that (b, Γ) is mapped to maps exactly one $c \in C$ to 1 and everything else to 0. That is, it needs to be checked that the subobject (b, Γ) is mapped to is a singleton. So composing this with σ_C (the predicate "is a singleton") yields the function

$$\begin{aligned} B \times \Omega^{B\times C} \to \Omega^B \to \Omega \\ (b, \Gamma) \mapsto \left(f : c \mapsto \begin{cases} 1, & (b, c) \in \Gamma \\ 0, & \text{else} \end{cases} \right) \mapsto \begin{cases} 1, & f \text{ is a singleton} \\ 0, & \text{else} \end{cases} \end{aligned}$$

that is,

$$(b, \Gamma) \mapsto \begin{cases} 1, & \exists! c \in C, (b, c) \in \Gamma \\ 0, & \text{else} \end{cases}$$

the transpose of which is exactly the function u.

With this preamble out of the way, the proof will now be presented,

Proof of theorem 2: As mentioned by James Clift in his talk "the definition of a topos (part 1)", a Category $\mathscr{E}$ admits all finite limits if and only if it admits a terminal object, and all pullbacks. So it remains to show that an elementary topos has all exponentials.

To this end, let $B \in \mathscr{E}$. Then for any $C \in \mathscr{E}$, define C^B to be the pullback of the diagram

$$\begin{tikzcd} C^B \arrow[r, dashed] \arrow[d, dashed, "m"'] & 1 \arrow[d, "\ulcorner \mathrm{true}_B \urcorner"] \\ \Omega^{B\times C} \arrow[r, "u"'] & \Omega^B \end{tikzcd}$$

where $u = \ulcorner \sigma_C \ulcorner \mathrm{ev}_{B\times C} \urcorner\urcorner$. An exponential requires more than just a collection of objects, there also must exist a collection of morphisms $\mathrm{ev}_{B,C} : B \times C^B \to C$ such that for any morphism $f : B \times D \to C$, there exists a unique $g : D \to C^B$ such that the diagram

$$\begin{tikzcd} B\times D \arrow[d, "\mathrm{id}_B\times g"'] \arrow[dr, "f"] & \\ B\times C^B \arrow[r, "\mathrm{ev}_{B,C}"'] & C \end{tikzcd}$$

commutes.

What should $\mathrm{ev}_{B,C}$ be? Thinking again momentarily about the topos Set, the value of $\mathrm{ev}_{B,C}(b,\Gamma)$, for some function $\Gamma : B \to C$ should be the unique value c which Γ maps b to. Ie, want $\{\cdot\}_C \mathrm{ev}_{B,C} = \ulcorner \mathrm{ev}_{B\times C} \urcorner(\mathrm{id}_B \times m)$, where $m : \Omega^B \rightarrowtail \Omega^{C\times B}$ is the inclusion which sends a function to its graph. However nothing here uses anything specific about the topos Set, so thinking again about a general topos, the following diagram

$$\begin{tikzcd} B\times C^B \arrow[r, "1_B\times m"] & B\times \Omega^{C\times B} \arrow[r, "\ulcorner \mathrm{ev}_{C\times B}\urcorner"] & \Omega^C \arrow[d, "\sigma_C"] & C \arrow[l, "\{\cdot\}_C"'] \arrow[d] \\ & & \Omega & 1 \arrow[l, "\mathrm{true}"] \end{tikzcd}$$

commutes, by definition of σ_C. Also, $u = \ulcorner \sigma_C \ulcorner \mathrm{ev}_{B\times C} \urcorner\urcorner$, so taking the transpose of this gives $\sigma_C \ulcorner \mathrm{ev}_{B\times C}\urcorner = \mathrm{ev}_B(\mathrm{id}_B \times u)$. So the diagram

$$\begin{tikzcd} B\times C^B \arrow[r, "1_B\times m"] & B\times \Omega^{C\times B} \arrow[r, "\ulcorner \mathrm{ev}_{C\times B}\urcorner"] \arrow[d, "(\mathrm{id}_B\times u)"'] & \Omega^C \arrow[d, "\sigma_C"] & C \arrow[l, "\{\cdot\}_C"'] \arrow[d] \\ & B\times\Omega^B \arrow[r, "\mathrm{ev}_B"'] & \Omega & 1 \arrow[l, "\mathrm{true}"] \end{tikzcd}$$

commutes. The final portion of the diagram can be extended to a commuting square by applying the functor $B \times \_$ to the definition of the exponential C^B, so the diagram

$$\begin{tikzcd} B\times C^B \arrow[r, "1_B\times m"] \arrow[d] & B\times \Omega^{C\times B} \arrow[r, "\ulcorner \mathrm{ev}_{C\times B}\urcorner"] \arrow[d, "(\mathrm{id}_B\times u)"'] & \Omega^C \arrow[d, "\sigma_C"] & C \arrow[l, "\{\cdot\}_C"'] \arrow[d] \\ B\times 1 \arrow[r, "1\times\ulcorner \mathrm{true}\urcorner"'] & B\times\Omega^B \arrow[r, "\mathrm{ev}_B"'] & \Omega & 1 \arrow[l, "\mathrm{true}"] \end{tikzcd}$$

commutes. In fact, there also exists a morphism from $B \times 1 \to 1$ such that the diagram

$$\begin{tikzcd} B\times C^B \arrow[r, "1_B\times m"] \arrow[d] & B\times \Omega^{C\times B} \arrow[r, "\ulcorner \mathrm{ev}_{C\times B}\urcorner"] \arrow[d, "(\mathrm{id}_B\times u)"'] & \Omega^C \arrow[d, "\sigma_C"] & C \arrow[l, "\{\cdot\}_C"'] \arrow[d] \\ B\times 1 \arrow[r, "1\times\ulcorner \mathrm{true}\urcorner"'] \arrow[rrr, bend right] & B\times\Omega^B \arrow[r, "\mathrm{ev}_B"'] & \Omega & 1 \arrow[l, "\mathrm{true}"] \end{tikzcd}$$

commutes. This is because the bottom square is a distorted version of the diagram

$$\begin{tikzcd}
B \times 1 \arrow[r, "1_B \times \ulcorner \mathrm{true}_B \urcorner"] \arrow[d] & B \times \Omega^B \arrow[d, "\mathrm{ev}_B"] \\
1 \arrow[r, "\mathrm{true}"'] & \Omega
\end{tikzcd}$$

which commutes as can be seen by the explicit formula for transpositions.

In fact, the square on the right is a pullback square, since Ω is a subobject classifier, so the fact that this diagram commutes shows there exists a unique map $e : B \times C^B \to C$ which will be taken to be the definition of $\mathrm{ev}_{B,C} : B \times C^B \to C$. The complete diagram is

$$\begin{tikzcd}
B \times C^B \arrow[r, "1_B \times m"] \arrow[d] \arrow[rrr, bend left, "e"] & B \times \Omega^{C \times B} \arrow[r, "\ulcorner \mathrm{ev}_{C \times B} \urcorner"] \arrow[d, "(\mathrm{id}_B \times u)"'] & \Omega^C \arrow[d, "\sigma_C"] & C \arrow[l, "\{\cdot\}_C"'] \arrow[d] \\
B \times 1 \arrow[r, "1 \times \ulcorner \mathrm{true} \urcorner"'] \arrow[rrr, bend right] & B \times \Omega^B \arrow[r, "\mathrm{ev}_B"'] & \Omega & 1 \arrow[l, "\mathrm{true}"]
\end{tikzcd} \tag{1}$$

The proof will then be complete once it has been shown that this map obeys the required universal property.

To this end, let $f : A \times B \to C$ be arbitrary. Then it needs to be shown that there exists a unique $g : A \to C^B$ such that $e(1_B \times g) = f$. Notice first that since $\{\cdot\}_C$ is monic,

$$\{\cdot\}_C f = \{\cdot\} e(1_B \times g) \Leftrightarrow f = e(1_B \times g)$$

this fact will be used when showing both existence and uniqueness. Also, the proof involves frequent switching between morphisms and their transpose or inverse transpose. This can be confusing, so to help with readability, the following commuting diagrams are mentioned and labeled,

$$\begin{tikzcd}
C \times C \arrow[d, "1_C \times \{\cdot\}_C"'] \arrow[dr, "\delta_C"] & \\
C \times \Omega^C \arrow[r, "\mathrm{ev}_C"'] & \Omega
\end{tikzcd} \tag{2}$$

$$\begin{tikzcd}
C \times B \times \Omega^{C \times B} \arrow[d, "1_C \times \ulcorner \mathrm{ev}_{C \times B} \urcorner"'] \arrow[dr, "ev_{C \times B}"] & \\
C \times \Omega^C \arrow[r, "\mathrm{ev}_C"'] & \Omega
\end{tikzcd} \tag{3}$$

Concerning uniqueness, say $\{\cdot\}_C f = \{\cdot\}_C e(1_B \times g)$. Then by (1),

$$\{\cdot\}_C f = \ulcorner \mathrm{ev}_{C \times B} \urcorner (1_B \times mg)$$

it then follows from (2) and (3) that

$$\delta_C(1 \times f) = \mathrm{ev}_{C \times B}(1_C \times 1_B \times mg)$$

this means that if there was a second map g' such that $e(1_B \times g) = f$, then

$$\mathrm{ev}_{C \times B}(1_C \times 1_B \times mg) = \mathrm{ev}_{C \times B}(1_C \times 1_B \times mg')$$

which by uniqueness of the transpose, this implies that $mg = mg'$, and then m is monic, so $g = g'$.

It remains to show existence. Say there is $f : A \times B \to C$. Then let $h : A \to \Omega^{B \times C}$ be the transpose of the map $\delta_C(1_C \times f)$. A map $g : A \to \Omega^B$ will be found by showing that the square

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "h"'] & 1 \arrow[d, "\mathrm{true}"] \\
\Omega^{B \times C} \arrow[r, "u"'] & \Omega
\end{tikzcd}$$

commutes. Then since C^B is defined to be a pullback, this will give a map $g : A \to C^B$. The functor $B \times \_$ can then be applied to the resulting commuting diagram, and appended onto the commuting diagram (1) which can then be seen by a diagram chase that

$$\{\cdot\}_C f = \{\cdot\} e(1_B \times g)$$

which will prove the result.

Indeed as already seen in the uniqueness argument,

$$\delta_C(1_C \times f) = \mathrm{ev}_{B\times C}(1_C \times 1_B \times h)$$

which by again using the same argument, follows from (2) and (3) that

$$\{\cdot\}_C f = \ulcorner \mathrm{ev}_{C\times B} \urcorner (1_B \times h)$$

composing with σ_C gives

$$\sigma_C \{\cdot\}_C f = \sigma_C \ulcorner \mathrm{ev}_{C\times B} \urcorner (1_B \times h)$$

which is

$$\mathrm{true}_C f = \sigma_C \ulcorner \mathrm{ev}_{C\times B} \urcorner (1_B \times h) \qquad (*)$$

there is now one more manipulation to be done, which is the observation that $\mathrm{true}_C f = \mathrm{true}_B \pi_B$ where $\pi_B : C \times B \to B$ is projection. It can then be shown that this has transpose $\ulcorner \mathrm{true}_B \urcorner !_A$, where $!_A : A \to 1$ is the terminal map. So, taking the transpose of $(*)$ gives

$$\ulcorner \mathrm{true}_B \urcorner !_A = uh$$

which is what was required to show. $\square$

References

[1] MacLane and Meordijk.