Higher-order logic & topoi II

In this talk we make the connection between higher-order logic (HOL) and topoi, by constructing a topos out of any such logic. The references are:

[1] Mac Lane & Moerdijk, "Sheaves in geometry and logic"
[2] Lambek & Scott "Introduction to higher-order categorical logic"

To begin with we follow [2, §II.1] except that our type theories and topoi do not necessarily contain a natural numbers object. As we will see, what we actually produce out of a type theory (which is synonymous for us with HOL) is an <u>elementary</u> topos in the sense of Will's previous talk. By the main theorem of his talk, every elementary topos is a topos.

<u>Def^N</u> A <u>type theory</u> is given by the following data:

(a) a class of <u>types</u> including special types $\mathbb{1}, \Omega$ (think of Ω as the type of propositions)
(b) a class of <u>terms</u> of each type, including countably many variables of each type
(c) for each finite set X of variables a binary relation $\vdash_X$ of <u>entailment</u> between terms of type Ω all free variables of which are elements of X.

These data are subject to the following conditions:

(a) If A, B are types so are $A \times B$ and PA.

(b) As in λ-calculus, we first define <u>preterms</u> and then <u>terms</u> are α-equivalence classes of preterms (this is left implicit in [2]).

There is a prescribed set of <u>basic preterms</u> which include (but are not limited to) the variables, $*$ of type $\mathbb{1}$ and $\top$ and $\bot$ of type Ω. There is a set of <u>term formation operations</u> which include (but are not limited to) the following. We write

$$t : A \quad \text{for} \quad \text{“}t \text{ is a preterm of type } A\text{”}$$

(b1) If $a:A$ and $b:B$ then $\langle a,b\rangle : A\times B$
(b2) If $a:A$ and $\alpha : PA$ then $a\in\alpha : \Omega$
(b3) If $\varphi:\Omega$ and x is a variable of type A, then $\{x\in A \mid \varphi\} : PA$
(b4) If $p:\Omega$ and $q:\Omega$ then $p\wedge q:\Omega$, $p\vee q:\Omega$ and $p\Rightarrow q:\Omega$.
(b5) If $\varphi:\Omega$ and x is a variable of type A, then $\forall(x\in A)\varphi:\Omega$, $\exists(x\in A)\varphi:\Omega$.

Finally, the class of preterms is freely generated from the basic preterms by the term formation rules, or more precisely: let $\mathcal{T}_0$ be the basic preterms, and $\mathcal{T}_{i+1}$ the set of preterms which are either in $\mathcal{T}_i$ or can be formed from preterms in $\mathcal{T}_i$ by a term formation rule. Then the set $\mathcal{T}$ of preterms is $\bigcup_{i\geq 0}\mathcal{T}_i$.

To each preterm t we associate a finite set $FV(t)$ of free variables in the usual way, with $FV(x)=\{x\}$ if x is a variable, $FV(*)=FV(\bot)=FV(\top)=\emptyset$ and FV defined inductively by

[we assume any other basic terms have no free variables]

- $FV(\langle a,b\rangle) = FV(a)\cup FV(b)$
- $FV(a\in\alpha) = FV(a)\cup FV(\alpha)$
- $FV(\{x\in A\mid\varphi\}) = FV(\varphi)\setminus\{x\}$.

- $FV(p \wedge q) = FV(p \vee q) = FV(p \Rightarrow q) = FV(p) \cup FV(q)$

- $FV(\forall (x \in A) \varphi) = FV(\exists (x \in A) \varphi) = FV(\varphi) \setminus \{x\}$.

By an <u>occurence</u> of a variable x in a term t we mean an x which is not

$$\{x \in A \mid \varphi\}, \quad \forall (x \in A) \varphi, \quad \exists (x \in A) \varphi \qquad (*)$$

An occurence of x is <u>free</u> or <u>bound</u> according to the usual rule, where the three term formation rules in $(*)$ are those which "capture" variables.

<u>Def^n</u> The relation $=_\alpha$ on the set $\mathcal{J}$ of preterms is the smallest equivalence relation with the property that it is closed under all term formation rules, so in particular

- if $s =_\alpha t$ then s, t have the same type
- if $a =_\alpha a'$ and $b =_\alpha b'$ then $\langle a, b \rangle =_\alpha \langle a', b' \rangle$
- if $a =_\alpha a'$ and $\beta =_\alpha \beta'$ then $a \in \beta =_\alpha a' \in \beta'$
- if $\varphi =_\alpha \varphi'$ then $\{x \in A \mid \varphi\} =_\alpha \{x \in A \mid \varphi'\}$.
- if $p =_\alpha p'$ and $q =_\alpha q'$ then $p \wedge q =_\alpha p' \wedge q'$, $p \vee q =_\alpha p' \vee q'$ and $p \Rightarrow q =_\alpha p' \Rightarrow q'$.
- if $\varphi =_\alpha \varphi'$ then $\forall_{x \in A} \varphi =_\alpha \forall_{x \in A} \varphi'$ and $\exists_{x \in A} \varphi =_\alpha \exists_{x \in A} \varphi'$.

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and such that

- for any pair x, y of variables of type A and $\varphi : \Omega$

replace all free occurrences of x by y

$$\{x \in A \mid \varphi\} =_\alpha \{y \in A \mid \varphi[y/x]\}$$

$$\forall (x \in A)\varphi =_\alpha \forall (y \in A)\varphi[y/x]$$

$$\exists (x \in A)\varphi =_\alpha \exists (y \in A)\varphi[y/x]$$

provided that no free occurrence of x appears in φ in such a way that a y in that position would be bound.

A term is an α-equivalence class of preterms. Terms of type Ω are called formulas. We introduce the following shorthand notation for terms

$\neg p$ means $p \Rightarrow \bot$ $\quad (p : \Omega)$

$p \Leftrightarrow q$ means $(p \Rightarrow q) \wedge (q \Rightarrow p)$ $\quad (p, q : \Omega)$

Leibniz equality

$a = a'$ means $\forall (u \in PA)(a \in u \Leftrightarrow a' \in u)$ $\quad (a, a' : A)$

$\{a\}$ means $\{x \in A \mid a = x\}$ $\quad (a : A)$

$\exists! (x \in A)\varphi$ means $\exists (x' \in A)(\{x \in A \mid \varphi\} = \{x'\})$

$\alpha \subseteq \beta$ means $\forall (x \in A)(x \in \alpha \Rightarrow x \in \beta)$ $\quad (\alpha, \beta : PA)$

(c) Entailment is a set $\mathcal{E}$ of tuples (p, q, X) where p, q are terms of type Ω and X is a finite set of variables (possibly empty) and $FV(p) \subseteq X$, $FV(q) \subseteq X$. We write $p \vdash_X q$ for $(p, q, X) \in \mathcal{E}$. These tuples satisfy the following rules:

(c1) Structural rules

(c1-1) • $p \vdash_X p$

(c1-2) • $\dfrac{p \vdash_X q \quad q \vdash_X r}{p \vdash_X r}$ ← this means if $(p, q, X), (q, r, X) \in \mathcal{E}$ then also $(p, r, X) \in \mathcal{E}$.

(c1-3) • $\dfrac{p \vdash_X q}{p \vdash_{X \cup \{y\}} q}$

(c1-4) • $\dfrac{\varphi \vdash_{X \cup \{y\}} \psi}{\varphi[b/y] \vdash_X \psi[b/y]}$

replacing only free occurrences of y

where y is a variable (it may be already in X), b is a term of the same type s.t. $FV(b) \subseteq X$ and we may assume (by α-equiv) that no free occurrence of a variable in b becomes bound in $\varphi[b/y], \psi[b/y]$.

(c2) Logical rules

(c2-1) • $p \vdash_X \top$, $\quad \bot \vdash_X p$

(c2-2) • $r \vdash_X p \wedge q$ iff. $r \vdash_X p$ and $r \vdash_X q$

(c2-3) • $p \vee q \vdash_X r$ iff. $p \vdash_X r$ and $q \vdash_X r$

(c2-4) • $p \vdash_X q \Rightarrow r$ iff. $p \wedge q \vdash_X r$ (and $\top \vdash_X q \Rightarrow r$ iff. $q \vdash_X r$)

(c2-5) • $p \vdash_X \forall(y \in B)\psi$ iff. $p \vdash_{X \cup \{y\}} \psi$

(c2-6) • $\exists(y \in B)\psi \vdash_X p$ iff. $\psi \vdash_{X \cup \{y\}} p$.

For example:

$$\dfrac{\dfrac{q \vdash_X \top \qquad \dfrac{p \vdash_X p}{\top \vdash_X p \Rightarrow p}}{q \vdash_X p \Rightarrow p}}{p \wedge q \vdash_X p}$$

We write $\vdash$ for $\vdash_\emptyset$ and $\vdash_X p$ for $\top \vdash_X p$.

(c3) Extralogical axioms

(c3-1) • Comprehension: $\vdash_X \forall(x \in A)(x \in \{y \in A \mid \varphi\} \Leftrightarrow \varphi[x/y])$
where as usual we assume that in forming $\varphi[x/y]$ no free y appears in a position where an x would be bound.

• Extensionality $\vdash \forall(u \in PA)\forall(v \in PA)(\forall(x \in A)(x \in u \Leftrightarrow x \in v) \Rightarrow u = v)$ (c3-2)

$\vdash \forall(s \in \Omega)\forall(t \in \Omega)((s \Leftrightarrow t) \Rightarrow s = t)$ (c3-3)

• Products $\vdash \forall(z \in \mathbb{1})(z = *)$ (c3-4)

$\vdash \forall(z \in A \times B)\exists(x \in A)\exists(y \in B)(z = \langle x, y \rangle)$ (c3-5)

$\vdash \forall(x \in A)\forall(x' \in A)\forall(y \in B)\forall(y' \in B)$
$(\langle x, y \rangle = \langle x', y' \rangle \Rightarrow (x = x' \wedge y = y'))$ (c3-6)

What we have defined above is an <u>intuitionistic</u> type theory. It is called <u>classical</u> if in addition we add the axiom $\vdash \forall(t \in \Omega)(t \vee \neg t)$. <u>Pure</u> type theory is the type theory in which there are no types or terms other than those defined inductively by the above closure rules, given a set of atomic types, there are no non-trivial identifications between types, and $\vdash_X$ is the smallest binary relation between terms satisfying the stated axioms and deduction rules.

<u>Remark</u> The reason for the subscript X on the entailment symbol is to allow us to distinguish e.g. $\forall(x \in A)\varphi \vdash \exists(x \in A)\varphi$ (which we do not want) from $\forall(x \in A)\varphi \vdash_x \exists(x \in A)\varphi$ (which we do). The latter may be derived thus:

$$\dfrac{\dfrac{\forall(x\in A)\varphi \vdash \forall(x\in A)\varphi}{\forall(x\in A)\varphi \vdash_x \varphi} \qquad \dfrac{\exists(x\in A)\varphi \vdash \exists(x\in A)\varphi}{\varphi \vdash_x \exists(x\in A)\varphi}}{\forall(x\in A)\varphi \vdash_x \exists(x\in A)\varphi}$$

<u>If</u> there exists a closed term a of type A, we may further deduce by the last of the structural rules that

$$\dfrac{\begin{array}{c}\vdots \\ \forall(x\in A)\varphi \vdash_x \exists(x\in A)\varphi\end{array}}{\forall(x\in A)\varphi \vdash \exists(x\in A)\varphi}$$

However in the case where there is no closed term of type A (e.g. A is the <u>type of unicorns</u>) we do <u>not</u> want to be able to deduce that there exists a unicorn with horns (φ = has horns) from the fact that all unicorns have horns.

Henceforth we fix an arbitrary type theory $\mathcal{L}$ (8)

<u>Def^N</u> The category $T(\mathcal{L})$ has meaning t s.t. $FV(t)=\phi$

- <u>objects</u> are "sets" i.e. closed terms α of type PA for any type A, modulo the equivalence relation $\alpha \sim \alpha'$ if $\alpha, \alpha' : PA$ and $\vdash \alpha = \alpha'$.

- <u>morphisms</u> are "functions", i.e. a morphism from $\alpha : PA$ to $\beta : PB$ is a closed term $F : P(A \times B)$ such that

$$\vdash F \subseteq \alpha \times \beta$$

where $\alpha \times \beta$ means $\{ z \in A \times B \mid \exists (x \in A) \exists (y \in B) (x \in \alpha \wedge y \in \beta \wedge z = \langle x, y \rangle)\}$.

$$\vdash \forall (x \in A)(x \in \alpha \Rightarrow \exists ! (y \in B)(\langle x, y \rangle \in F)).$$

modulo the equivalence relation which says $F, F' : P(A \times B)$ determine the same morphism if $\vdash F = F'$. We usually write $f : \alpha \longrightarrow \beta$ for a morphism and $|f|$ for a representing closed term F (called the <u>graph</u> of f).

- <u>composition</u> of $f : \alpha \longrightarrow \beta$ and $g : \beta \longrightarrow \gamma$ is the morphism $g \circ f : \alpha \longrightarrow \gamma$ determined by the closed term

$$|g \circ f| = \{ u \in A \times C \mid (\exists x \in A)(\exists z \in C)(u = \langle x, z \rangle \wedge \exists (y \in B)(\langle x, y \rangle \in |f| \wedge \langle y, z \rangle \in |g|))\}.$$

<u>Lemma</u> The composition is well-defined, i.e. the term $|g \circ f|$ satisfies the two conditions, and is independent (up to equivalence) of the choice of representatives.

Proof First we have to show $\vdash |g\circ f| \subseteq \alpha\times\gamma$, that is

$$\vdash \forall(t\in A\times C)(\,t\in|g\circ f| \Rightarrow t\in\alpha\times\gamma\,)$$

Intuitively, this is because everything "in" $|g\circ f|$ is of the form $\langle x,z\rangle$ where $\exists y$ s.t. $\langle x,y\rangle\in|f|\subseteq\alpha\times\beta$ and $\langle y,z\rangle\in|g|\subseteq\beta\times\gamma$ from which we deduce $x\in\alpha$ and $z\in\gamma$. But we have to package this as a proof tree. By (c2-5) it suffices to prove

$$\vdash_{\{t\}} t\in|g\circ f| \Rightarrow t\in\alpha\times\gamma$$

and by (c2-4) it is enough to show

$$t\in|g\circ f| \vdash_{\{t\}} t\in\alpha\times\gamma.$$

To manipulate the left hand side we use a comprehension. Write φ for the formula s.t. $|g\circ f| = \{u\in A\times C \mid \varphi\}$. Then

$$\vdash_{\{t\}} t\in\{u\in A\times C\mid\varphi\} \Leftrightarrow \varphi[t/u]$$

It is therefore enough to show

$$\varphi[t/u] \vdash_{\{t\}} t\in\alpha\times\gamma.$$

Applying another instance of comprehension to the definition of $\alpha\times\gamma$ it suffices to show that

$$\varphi[t/u] \vdash_{\{t\}} \exists(x\in A)\exists(z\in C)(x\in\alpha\wedge z\in\gamma\wedge t=\langle x,z\rangle)$$

But $\varphi[t/u]$ is

$$\exists(x\in A)\exists(z\in C)\Big(\overbrace{t=\langle x,z\rangle \wedge \exists(y\in B)\big(\langle x,y\rangle\in|f|\wedge\langle y,z\rangle\in|g|\big)}^{\text{call this } \psi}\Big)$$

so it suffices to prove

$$\psi \vdash_{\{t,x,z\}} x\in\alpha\wedge z\in\gamma\wedge t=\langle x,z\rangle$$

Aside —

Suppose for any formulas φ_1,φ_2 that $\varphi_1\vdash_{\{x\}}\varphi_2$. Then

$$\dfrac{\dfrac{\dfrac{\varphi_1\vdash_{\{x\}}\varphi_2}{\varphi_1\vdash\forall(x\in A)\varphi_2}}{\varphi_1\vdash_{\{x\}}\forall(x\in A)\varphi_2}\qquad \forall(x\in A)\varphi_2\vdash_{x}\exists(x\in A)\varphi_2}{\dfrac{\varphi_1\vdash_{\{x\}}\exists(x\in A)\varphi_2}{\exists(x\in A)\varphi_1\vdash\exists(x\in A)\varphi_2}}$$

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For this it suffices to prove

$$\exists(y\in B)\big(\langle x,y\rangle\in|f|\wedge\langle y,z\rangle\in|g|\big)\vdash_{\{t,x,z\}} x\in\alpha\wedge z\in\gamma$$

for which it suffices to prove

$$\langle x,y\rangle\in|f|\wedge\langle y,z\rangle\in|g|\vdash_{\{t,x,y,z\}} x\in\alpha\wedge z\in\gamma.$$

For this it suffices to prove separately

$$\langle x,y\rangle \in |f| \vdash_{\{f,x,y,z\}} x\in\alpha \qquad \langle y,z\rangle\in|g| \vdash_{\{f,x,y,z\}} z\in\gamma$$

But this is easy since $\vdash |f| \subseteq \alpha\times\beta$ and $\vdash |g| \subseteq \beta\times\gamma$. (apply the def^{N} of $\subseteq$ to reduce to proving e.g. $\langle x,y\rangle \in \alpha\times\beta \vdash_{\{x,y\}} x\in\alpha$ which we may do by comprehension applied to the def^{N} of $\alpha\times\beta$).

The rest of the lemma is proved in a similar way. □

Our aim in the next lecture is to prove:

<u>Theorem</u> $T(\mathcal{L})$ is a topos.