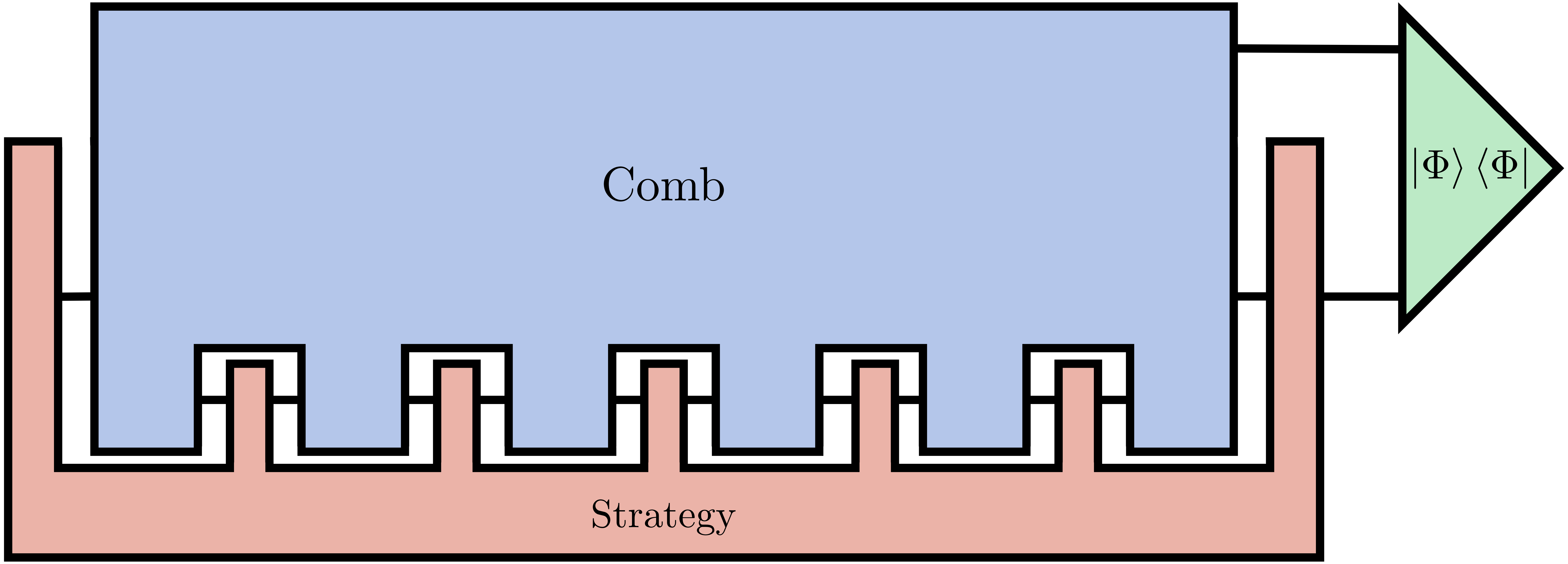
Quantum Combs, the min entropy, and their uses



Isaac D. Smith, *Metauni Festival 2023*, 12.01.2023

Broad Aim

Learn an unknown property of a quantum system by interacting with it.

Representing Quantum States and Dynamics

States: $\rho \in \mathcal{L}(\mathcal{H})$ $\rho \geq 0,\ \mathrm{Tr}[\rho] = 1$

Quantum Operations: $\mathcal{E} \in \mathcal{L}(\mathcal{L}(\mathcal{H}), \mathcal{L}(\mathcal{H}'))$ $\mathrm{Tr}[\rho] \geq \mathrm{Tr}[\mathcal{E}(\rho)],\ (\mathcal{I} \otimes \mathcal{E})(\rho) \geq 0$

$$\equiv \quad \mathcal{E}(\rho) = \sum_k E_k \rho E_k^\dagger \qquad \sum_k E_k^\dagger E_k \leq I$$

$$\equiv \quad \mathcal{E}(\rho) = \mathrm{Tr}_{\mathrm{env}}[U(\rho \otimes \rho_{\mathrm{env}})U^\dagger]$$

Quantum operations include:

State preparations $\rho \in \mathcal{L}(\mathbb{C}, \mathcal{L}(\mathcal{H}))$ Noise Channels $E_0 = \sqrt{p_0} I,\ E_1 = \sqrt{1-p_0} Z$

Unitaries $E_k = U$ Measurements $\mathcal{M} \in \mathcal{L}(\mathcal{L}(\mathcal{H}), \mathbb{C})$

§8.5 Limitations of the quantum operations formalism

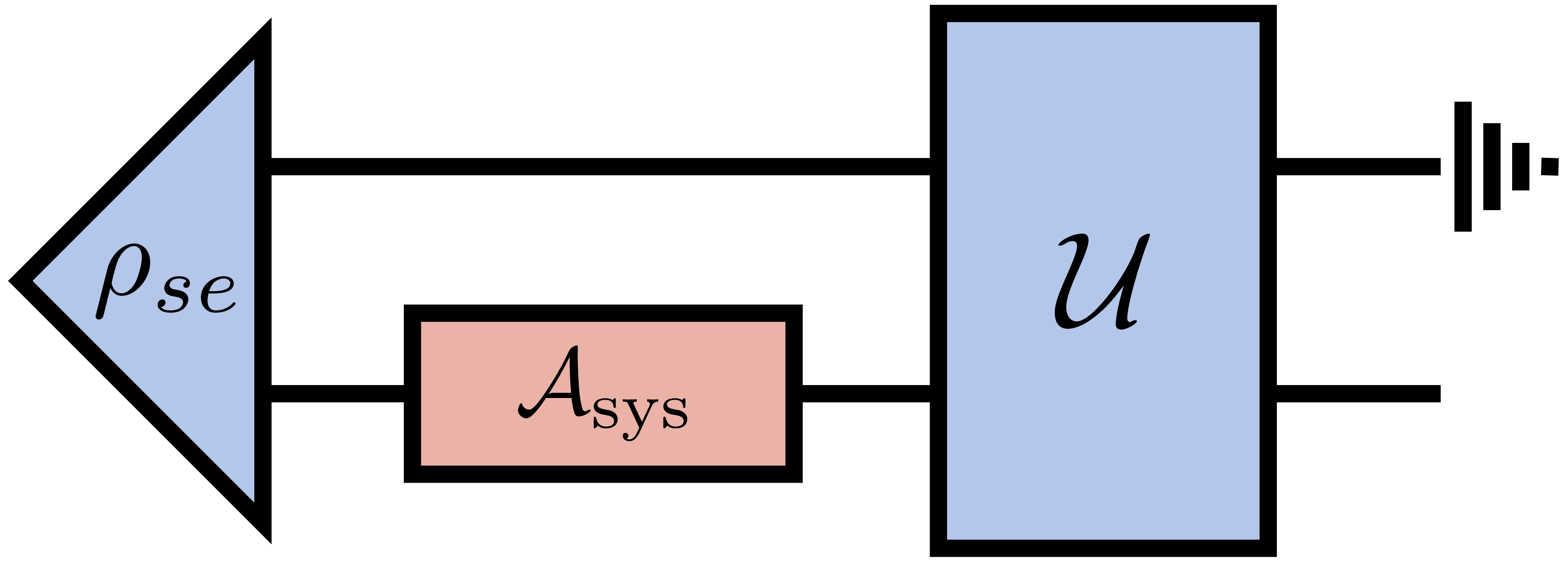
Nielsen and Chuang, 2002

Limitations and Resolutions

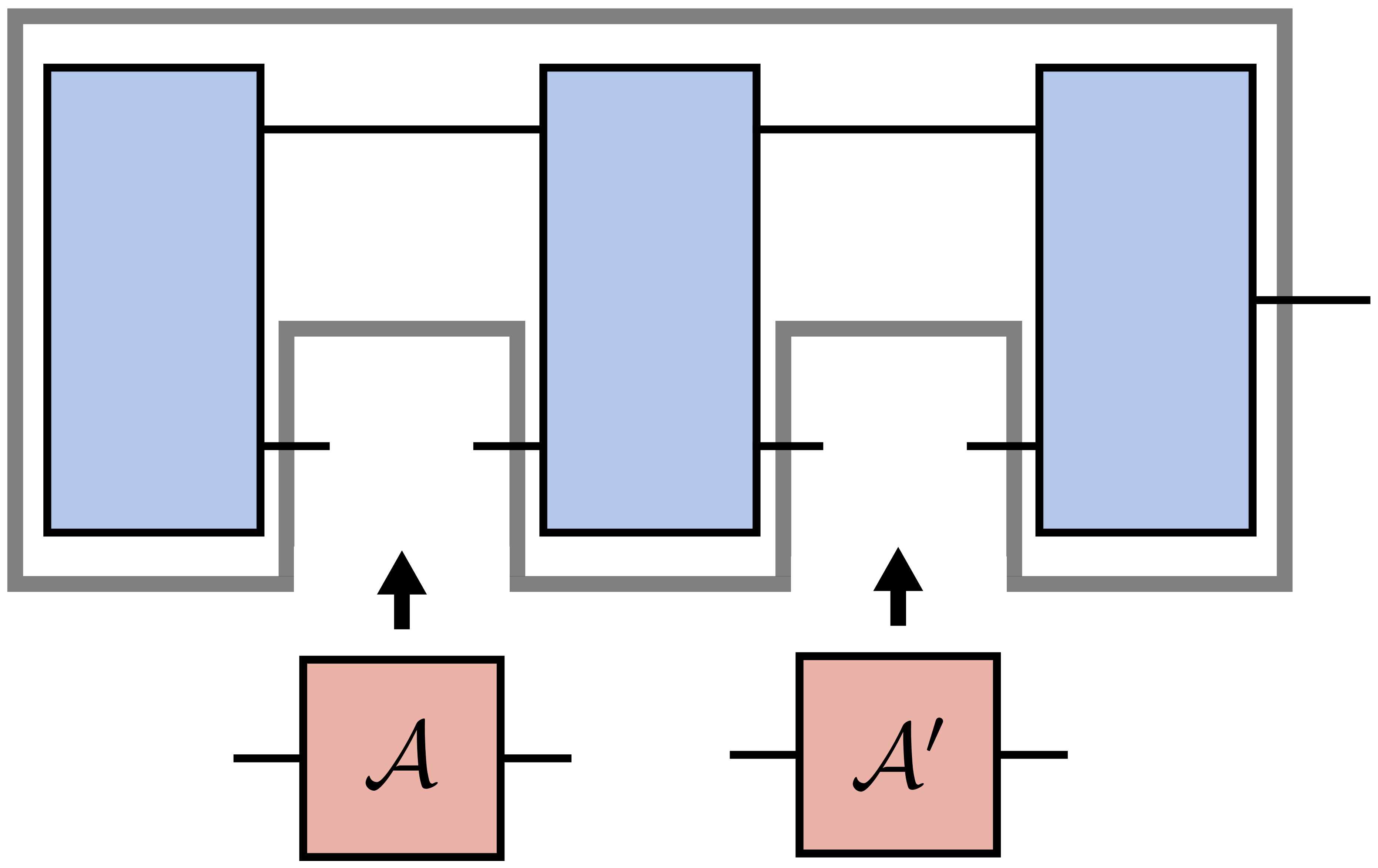
The problem: $\mathcal{E}(\rho) = \mathrm{Tr}_{\mathrm{env}}[U(\underline{\rho \otimes \rho_{\mathrm{env}}})U^\dagger]$

First Resolutions: Give up Complete Positivity or Linearity (or …)

Better Resolution: $\mathcal{C}(\mathcal{A}_{\mathrm{sys}}) := \mathrm{Tr}_{\mathrm{env}}[U(\mathcal{A}_{\mathrm{sys}} \otimes \mathcal{I}_{\mathrm{env}}(\rho_{se}))U^\dagger]$



$\mathcal{C}(\mathcal{A}_{\mathrm{sys}})$



$\mathcal{C}(\mathcal{A}, \mathcal{A}') \in \mathcal{L}(\mathcal{H})$

S. Milz et al., 2017; Pechukas 1994, 1995; Alicki 1995;

Detour: Choi Operator

Consider the linear map

$$\mathcal{E} : \mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_B)$$

Define

$$E := (\mathcal{E} \otimes \mathcal{I}_{A'})(|\Phi^+\rangle \langle \Phi^+|)$$

where

$$\mathcal{H}_{A'} \simeq \mathcal{H}_A \qquad \{|i\rangle\}_{i=0}^{d_A-1}$$

$$|\Phi^+\rangle := \sum_{i=0}^{d_A-1} |ii\rangle_{AA'}$$

Lemma:

$$E \geq 0 \text{ iff } \mathcal{E} \text{ is CP.}$$

$$\mathrm{Tr}_B[E] \leq I_A \; (= I_A) \text{ iff } \mathcal{E} \text{ is trace non-increasing (TP).}$$

M-D. Choi 1975; A. Jamiolkowski 1972; G. Chiribella et al. 2009

Quantum Combs (a.k.a Process Tensors)

$C \in \mathcal{L}(\bigotimes_{j=1}^{n} \mathcal{H}_{A_j^{\text{in}}} \otimes \mathcal{H}_{A_j^{\text{in}}})$ is a **quantum comb** if there exists a sequence of operators $C_k \in \mathcal{L}(\bigotimes_{j=1}^{k} \mathcal{H}_{A_j^{\text{in}}} \otimes \mathcal{H}_{A_j^{\text{in}}})$ such that

- $C_k \geq 0;$ (CP)
- $\mathrm{Tr}_{A_k^{\text{out}}}[C_k] = I_{A_k^{\text{in}}} \otimes C_{k-1};$ (TP)
- $C_n = C,\ C_0 = 1\ (C_0 > 0).$

Denote the set of combs by $\mathrm{Comb}(A_1^{\text{in}} \to A_1^{\text{out}}, ..., A_n^{\text{in}} \to A_n^{\text{out}})$

G. Chiribella et al., 2008; G. Chiribella et al. 2009; || O. Oreshkov et al., 2012; F.A. Pollock et al., 2018;

An n**-comb** between two families of objects $X_0, \ldots, X_n$ and $Y_0, \ldots, Y_n$ is an element of

$$Comb_n(X, Y) := \int^{M_0, \ldots, M_{n-1}} \prod_{i=0}^{n} \mathcal{D}(M_{i-1} \otimes X_i, M_i \otimes Y_i),\ \text{with}\ M_{-1}, M_n := I$$

(Coend)

Cat Theory Def

M. Roman 2003.06214

Quantum Combs (and related) - Some Applications

Abstract Nonsense:

- CT ML - “optics”, “lenses”, their relation to Backprop

D. Shiebler et al., 2103.01931; Categories for AI - e.g., Week 3

- Logic - “*-autonomous categories”, their relation to Linear Logic

A. Kissinger and S. Uijlen, 1701.04732

Less Abstract Not-Nonsense:

- Characterising Non-Markovian noise in quantum computers

F.A. Pollock et al., 2018; G.A.L. White et al., 2020;

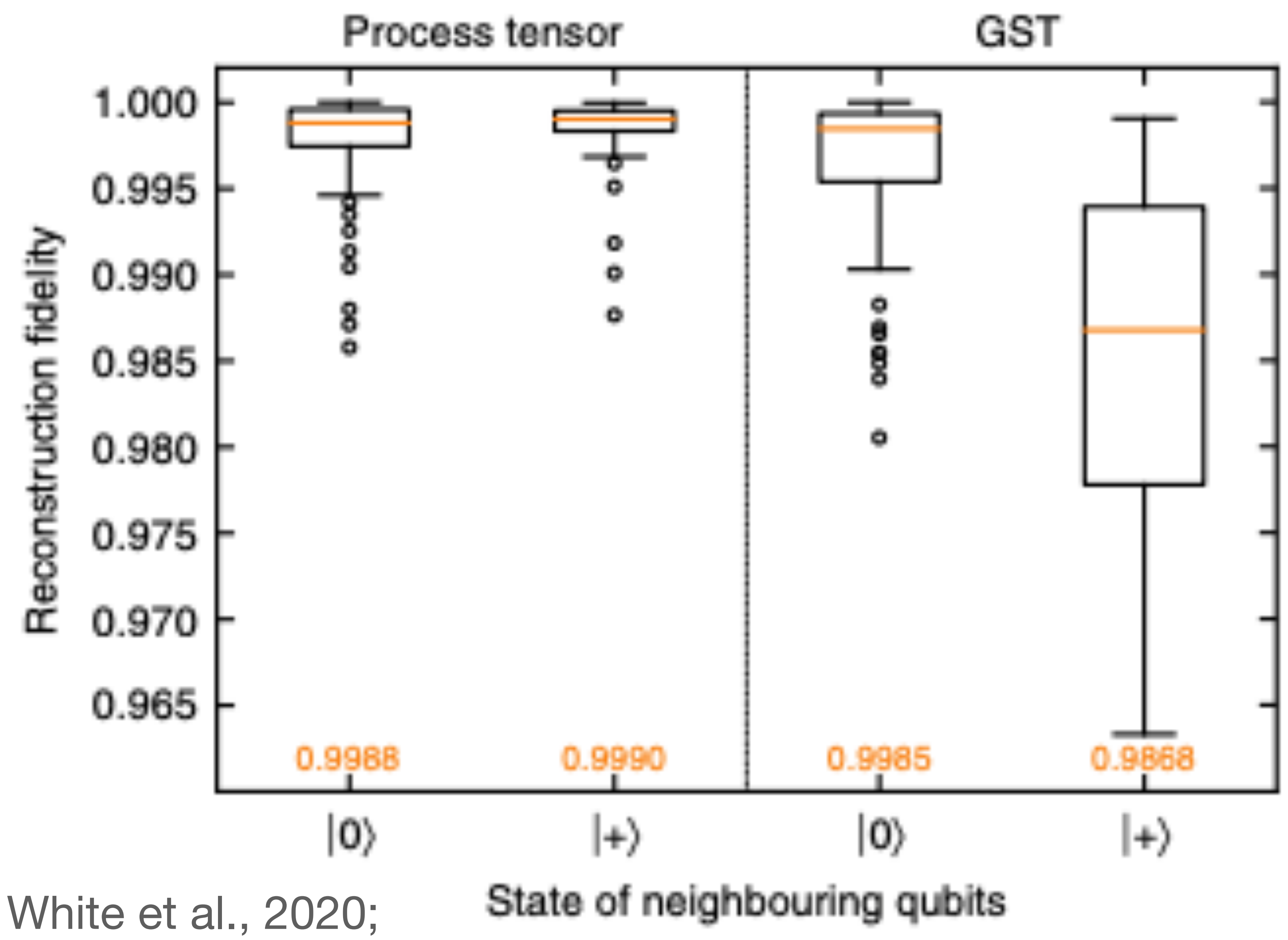


Fig 4., G.A.L. White et al., 2020;

Min Entropy and Guessing Probability

Definition: $\rho \in \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$

SDP!

$$H_{\min}(A|B)_\rho := -\log[\min_{\gamma_B \in \mathcal{L}(\mathcal{H}_B)} \min\{\lambda \in \mathbb{R} | \lambda(I_A \otimes \gamma_B) \geq \rho\}]$$

Operational Meaning:

Maximising overlap with maximally entangled state

$$2^{-H_{\min}(A|B)_\rho} = d_A \max_{\mathcal{E}} F((I_A \otimes \mathcal{E})(\rho), |\Phi\rangle\langle\Phi|_{AA'})^2$$

Also, does this kind of look familiar?

$$\mathcal{E} : \mathcal{L}(\mathcal{H}_B) \to \mathcal{L}(\mathcal{H}_{A'}),\ \mathcal{H}_{A'} \simeq \mathcal{H}_A \qquad |\Phi\rangle_{AA'} := \frac{1}{\sqrt{d_A}} \sum_i |ii\rangle_{AA'}$$

Guessing Probability: $\rho_{XB} = \sum_x P(x) |x\rangle\langle x| \otimes \rho_B^x$

$$p_{\text{guess}}(X|B) = 2^{-H_{\min}(X|B)} = \max_{\{E_x\}} \sum_x P(x) \mathrm{Tr}[E_x \rho_B^x]$$

N. Datta and R. Renner, 2009; R. König et al., 2009; M. Tomamichel 2016; G. Chiribella and D. Ebler, 2016;

The Comb Min Entropy and Classical-Quantum Combs

The Comb min entropy:

$$C \in \mathrm{Comb}(A_1^{\mathrm{in}} \to A_1^{\mathrm{out}}, \dots, A_n^{\mathrm{in}} \to A_n^{\mathrm{out}})$$

$$H_{\min}(t_n|t_1, \dots, t_{n-1})_C := -\log[\min_\Gamma \min\{\lambda \in \mathbb{R} | \lambda(I_{A_n^{\mathrm{in}} A_n^{\mathrm{out}}} \otimes \Gamma) \geq C\}]$$

Operational Meaning:

Maximal correlations with a distinct system

$$2^{-H_{\min}(t_n|t_1,\dots,t_{n-1})_C} = \max_E \mathrm{Tr}[CE^T] = \max_{\widehat{E},\mathcal{E}} d_{A_n^{\mathrm{out}}} F((I_{A_n^{\mathrm{out}}} \otimes \mathcal{E})(C\widehat{E}^T), |\Phi\rangle\langle\Phi|)^2$$

Learning strategy

Classical-Quantum Combs:

$$C := \sum_{x \in X} P(x) |x\rangle\langle x| \otimes \sigma_x$$

$$\sigma_x \in \mathrm{Comb}(A_1^{\mathrm{in}} \to A_1^{\mathrm{out}}, \dots, A_n^{\mathrm{in}} \to A_n^{\mathrm{out}})$$

$$C \in \mathrm{Comb}(A_1^{\mathrm{in}} \to A_1^{\mathrm{out}}, \dots, A_n^{\mathrm{in}} \to A_n^{\mathrm{out}}, \mathbb{C} \to X)$$

Unknown property

Quantum system undergoing dynamics

G. Chiribella and D. Ebler, 2016; IDS et al., 2212.00553

Blind Quantum Computing Application

BQC: Cryptographic protocol between client (restricted computational power) and a server (unrestricted computational power).

Mantri et al. Protocol: Classical client and entirely classical communication - server implements a Measurement-Based Quantum Computation (MBQC)

$$C = \sum_{\text{comp}} P(\text{comp}) \left|\text{comp}\right\rangle \left\langle\text{comp}\right| \otimes \sigma^{\text{comp}}_{\text{protocol}}$$

Classical-classical comb

Client's known choice of computation

Client's side of the protocol

$H_{\min}(\text{comp}|\text{protocol})$ Quantifies how much the server has yet to find out about the choice of computation after a single round of the protocol.

Results: $H_{\min}(\text{comp}|\text{protocol}) > 0$, $H_{\min}(\text{comp}|\text{protocol}^m) > 0$, $H_{\min}(\text{comp}|\text{protocol}) > H_{\min}(\text{comp}|\text{protocol}^2)$

Analytical — Analytical — Example, Numerical

A. Broadbent et al., 2009; JF Fitzsimmons 2017; Mantri et al., 2017; R. Raussendorf and H.J. Briegel 2001; IDS et al., 2212.00553

Quantum Causal Models

(Classical) Causal Models:

- $X_1, ..., X_n$ RVs;
- DAG G, X_i as vertices;
- for each X_i, $P(X_i|\mathrm{Pa}(X_i))$.

$$P(X_1, ..., X_n) = \prod_i P(X_i|\mathrm{Pa}(X_i))$$

What are they good for: Out of distribution learning! (They can handle interventions)

Quantum Causal Models:

- $A_1, ..., A_n$ pairs of Hilbert spaces $(\mathcal{H}_{A_i} \otimes \mathcal{H}^*_{A_i})$;
- DAG G, A_i as vertices;
- for each A_i, $\rho_{A_i|\mathrm{Pa}(A_i)} \in \mathcal{L}(\mathcal{H}_{A_i} \otimes \mathcal{H}^*_{\mathrm{Pa}(A_i)})$.

$$\sigma_{A_1,...,A_n} = \prod_i \rho_{A_i|\mathrm{Pa}(A_i)}$$

Achtung! Process matrix not tensor

Why do we need QCMs? Bell Inequality violating correlations can be a bit pesky…

Pearl 2009; B. Schölkopf et al., 2021; J. Barrett et al., 1906.10726; J.M.A. Allen et al., 2017; F. Costa and S. Shrapnel 2016; C.J. Wood and R. Speakers 2015

Quantum Causal Discovery Application

$$C = \sum_{\text{causal struc.}} P(\text{c.s.}) \left|\text{c.s.}\right\rangle \left\langle\text{c.s.}\right| \otimes \sigma_{\text{QCM}}^{\text{c.s.}}$$

$H_{\min}(\text{c.s.}|\text{QCM})$ Quantifies how much more information we need to know the causal structure with certainty.

From the SDP solver: E that gives $\max_E \mathrm{Tr}[CE^T]$

Optimal causal discovery strategy?

Our work:

- Demonstrated links between MBQC and QCMs;
- Calculated the min entropy for learning MBQC-related causal structure;
- Calculated the min entropy for learning an MBQC-related quantum reference frame.

IDS et al., 2212.00553;

Quantum Combs + Min Entropy = Much Future Work

Quantum Causality:

- Analysis of (dual) solutions of the SDP as optimal causal discovery strategies;

- Reanalysis of existing quantum causal inference literature.

Noise Analysis:

- Noise analysis for MBQC

K. Ried et al., 2015; G. Chiribella and D. Ebler 2019;

My Collaborators:

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To Help Translate…

TABLE I. "Process tensors" in different fields of quantum mechanics. Mathematical objects that are similar in spirit to the process tensor crop up frequently in quantum mechanics. The below table is an incomplete list of the respective fields and commonly used names. Note that, even within these fields, the respective names and concrete applications differ. Additionally, some of the objects that occur on this list might have slightly different properties than the process tensor (for example, process matrices do not have to display a global causal order), and might look very different than the process tensor (for example, it is a priori not obvious that the correlation kernels used in open quantum system dynamics are indeed variants of process tensors in disguise). These disparities notwithstanding, the objects in the table are close both in spirit, as well as the related mathematical framework.

	Name	Application
Quantum information	Quantum comb and causal box	Quantum circuit architecture
Open quantum system dynamics	Correlation kernel and process tensor	Study of temporal correlations
Quantum games	Strategy	Computation of winning probabilities
Quantum causality	Process matrix	Processes without definitive causal order
Quantum causal modeling	Process matrix	Causal relations in quantum processes
Quantum Shannon theory	Causal automaton and nonanticipatory channel	Quantum channels with memory

S. Milz and K. Modi 2021

Causal Models and AGI

- Causal models are a useful tool regarding out of distribution generalisation;
- Causal models have seen recent and strong representation in RL - RL is interventional;
- Causality plays a role in much of human explanation and consequently causal models support a variety of methods pertaining to explainability - XAI;
- There is strong evidence for causal modelling within human cognition and acquisition of concepts;
- Causality has a long history within the philosophy of science.

J. Peters et al. 2017; M. Waldmann 2017; J. Kaddour et al. 2206.15475; S. Carey 2009; P. Godfrey-Smith 2003