Introduction

1. Homological algebra is rather young. Its subject descends from two areas of mathematics studied at the end of the previous century; these areas later became combinatorial topology and "modern algebra" (in the sense of van der Waerden) respectively. As the examples of main notions inherited from this early period, we can mention Betti numbers of a topological space and D. Hilbert's "syzygy theorem" (1890).

At present we easily recognize a general construction which underlies these notions. A topological space X is glued from cells (or simplices) of various dimensions i; the boundary of a cell is a linear combination of other cells. The i-th Betti number is the number of linearly independent chains with zero boundary modulo chains that are boundaries themselves; in other words, the i-th Betti number is the rank of the group $\operatorname{Ker}\partial_i/\operatorname{Im}\partial_{i-1}$, where $\partial_i : C_i \to C_{i-1}$ is the boundary operator and C_i is the group of i-dimensional chains. "Syzygies" occur in a different problem. Let M be a graded module with a finite number of generators over the ring $A = k[x_1, \ldots, x_n]$ of polynomials with coefficients in a fixed field k. Hilbert considered the case when M is an ideal in A generated by several forms (homogeneous polynomials). In general, generators of M can not be chosen to be independent. Fixing a set of r_0 generators we obtain a submodule in A^{r_0} consisting of coefficients of all relations among these generators. This submodule has a natural grading and is called "the first syzygy module" $Z_0(M)$ of the module M. For $i > 1$ let $Z_i(M) = Z_0(Z_{i-1}(M))$ (on each step we have a freedom in choosing the generators of $Z_{i-1}(M)$). The Hilbert theorem asserts that $Z_{n-1}(M)$ is a free module so that we can always assume $Z_n(M) = 0$.

The algebraic framework of both constructions is the notion of a complex; a complex is a sequence of modules and homomorphisms $\ldots \to K_i \xrightarrow{\partial_i} K_{i-1} \to \ldots$ with the condition $\partial_{i-1}\partial_i = 0$. The complex of chains of a topological space determines its homology $H_i(X) = \operatorname{Ker}\partial_i/\operatorname{Im}\partial_{i-1}$. The Hilbert complex consists of free modules. It is acyclic everywhere but at the end: $Z_i(M)$ is both the group of cycles and the group of boundaries in a free resolution of the module M:

$$0 \to A^{r_n} \to A^{r_{n-1}} \to \ldots \to A^{r_1} \xrightarrow{\partial_1} A^{r_0} \xrightarrow{\partial} 0$$
$$M \simeq \operatorname{Ker}\partial_0 = A^{r_0}/\operatorname{Im}\partial_1.$$

Both the complex of chains of a space X and the resolution of a module M, are defined non-uniquely: they depend on the decomposition of X into cells or on the choice of generators of subsequent syzygy modules. The essence of the first theorems in homological algebra is that there is something that does not depend on this ambiguity in the choice of a complex, namely the Betti numbers (or the homology groups themselves) in the first case, and the maximal length of a complex (the last non-zero place) in the second case.

The first stage of homological algebra was marked by the acquisition of data. Combinatorial and, later, homotopic topology supplied plentiful examples of

– types of complexes;

– operations over complexes that reflect some geometrical constructions: the product of spaces led to the tensor product of complexes, the multiplication in cohomology led to the notion of a differential graded algebra, homotopy resulted in the algebraic notion of a homotopy between morphisms of complexes, the algebraic framework of the geometrical study of fiber spaces is the notion of a spectral sequence associated to a filtered complex, and so on and so forth;

– algebraic constructions imitating topological ones; examples are cohomology of groups, of Lie algebras, of associative algebras, etc.

2. The famous "Homological algebra" by H. Cartan and S. Eilenberg, published in 1956 (and written some time between 1950 and 1953) summarized the achievements of this first period, and introduced some very important new ideas which determined the development of this branch of algebra for many years ahead. It seems that the very name "homological algebra" became generally accepted only after the publication of this book.

First of all, this book contains a detailed study of the main algebraic formalism of (co)homology groups and of working instructions that do not depend on the origin of the complex. Second, this book gave a conceptially important answer to the question about the nature of homological invariants (as opposed to complexes themselves, which cannot be considered as invariants). This answer can be formulated as follows. The application of some basic operations over modules, such as tensor products, the formation of the module of homomorphisms, etc., to short exact sequences violates the exactness; for example, if the sequence $0 \to M' \to M \to M'' \to 0$ is exact, the sequence $0 \to N \otimes M' \to N \otimes M \to N \otimes M'' \to 0$ can have non-trivial cohomology at the left term. One can define the "torsion product" $\mathrm{Tor}_1(N, M'')$ in such a way that the complex

$$\mathrm{Tor}_1(N,M') \to \mathrm{Tor}_1(N,M'') \to \mathrm{Tor}_1(N,M'') \to \\ \to N \otimes M' \to N \otimes M \to N \otimes M'' \to 0$$

is acyclic. However, to extend this complex further to the left one must introduce $\mathrm{Tor}_2(N, M'')$, etc.

These modules $\mathrm{Tor}_i(N, M)$ are the derived functors (in one of the arguments) of the functor $\otimes$. They are uniquely determined by the requirement that the exact triples are mapped to acyclic complexes. To compute these functors one can use, say, free resolutions of the module M and define $\mathrm{Tor}_i(M, N)$ as homology groups of the tensor product of such a resolution with the module N.

Hence, a homological invariant of the module N is the value on N of some higher derived functor which can be uniquely characterized by a list of properties and can be computed using resolutions.

This idea, which first originated in the algebraic context, immediately returned to topology in the extremely important paper by A. Grothendieck "Sur quelques questions d'algèbre homologique", published in 1957. In order to pursue the point of view of Cartan and Eilenberg, Grothendieck had to revise completely the system of basic notions of combinatorial topology. Before his paper it was clear that the (co)homology depends, first of all, on the space X, and the axioms of homology described the behavior of $H(X)$ in passing to an open subspace (the excision axiom), under homotopy, etc. However, spaces X look quite unlike modules over a ring, and in this context the groups $H(X)$ do not behave like the derived functors. Grothendieck stressed the role of a second "hidden" parameter of the cohomology theory, the group of coefficients. It occurs that if we consider the cohomology $H^i(X, \mathcal{F})$ of X with coefficients in an arbitrary sheaf of abelian groups $\mathcal{F}$ on X (at the beginning of the fifties this notion was introduced and studied in detail due to the needs of the theory of functions in several complex variables), we can almost completely "ignore" the space X! Namely, $H^i(X, \mathcal{F})$ becomes in this context the i-th derived functor of the functor $\mathcal{F} \to \Gamma(X, \mathcal{F})$ (the global sections functor) in the spirit of Cartan–Eilenberg.

This idea turned out to be extremely fruitful for topology (understood in a wide sense). Being widely developed and generalized by Grothendieck himself and by his students and collabarators, it led to algebraic topology of algebraic varieties over an arbitrary field (the "Weil program"). The jewel of this theory is P.Deligne's proof of Riemann–Weil conjectures. We must mention also the cohomological version of class field theory (Chevalley and Tate among others), the modern version of Hodge theory (Griffiths, Deligne,...), theory of perverse sheaves, and the general penetration of the homological language into various areas of mathematics.

3. In the sixties homological algebra was enriched by yet another important construction. We mean here the notions of derived and triangulated categories.

While earlier the main concern of a mathematician working with homology were homological invariants, in the last twenty years the role of complexes themselves was emphasized; the complexes are viewed as objects of a rather complicated and not very explicit category. The idea is that, say, a resolution of a module is not only a tool to compute various Ext's and Tor's, but, in a sense, a rightful representative of this module. What we only need is a method that enables us to identify all resolutions of a given module. In the same way the chain complex of a space together with a sufficient set of auxiliary structures, is an adequate substitute of this space.

Although the axioms and the initial constructions of the theory of derived and triangulated categories are rather cumbersome, the approach itself is rather flexible and in the last few years this approach turned out to be indispensable in topology, representation theory, theory of analytical spaces, not to mention, of course, algebraic geometry which initiated all this (the Grothendieck seminars, the Verdier thesis, the Hartshorne notes).

One of the paradoxes of homological algebra, which now slowly becomes to be understood, is that in some cases an appropriately chosen triangulated category is simpler than the abelian category studied before. For example, the derived category of coherent sheaves on a projective space is understood better than the category of sheaves themselves. Next, one triangulated category can have several abelian "cores". Such a phenomenon leads to various meaningful versions of classical duality theories.

4. This volume of the Encyclopaedia is not intended to be a complete survey of all known results in homological algebra. This task could not presumably be solved both because of authors' limitations and the huge amount of data involved.

The volume can be roughly divided into three parts. The introductory Chap. 1–3 contains the most classical aspects of the theory; even now the main technical methods of homological algebra are based on these ideas (complemented from time to time by new constructions). For example, a comparatively new subject is cyclic (co)homology.

Chapters 4 and 5 describe derived and triangulated categories.

Finally, Chap. 6–8 contain geometrical applications of the modern homological algebra to mixed Hodge structures, perverse sheaves and $\mathcal{D}$-modules. In other topological and algebraic geometry volumes of the Encyclopaedia the reader can find several parallel expositions and of additional material; in this volume we mostly emphasize the categorical and homological aspects of the theory.

The bibliography, inevitable quite incomplete, can help the interested reader to learn more about topics involved.

Let us remark also that the references in the text give section and subsection numbers, e.g. Chap. 2, 2.1, or Chap. 1, 1.5.1. In references inside the current chapter the chapter number is omitted.

Chapter 1
Complexes and Cohomology

§ 1. Complexes and the Exact Sequence

1.1. Complexes. A *chain complex* is a sequence of abelian groups and homomorphisms

$$C_{\cdot}: \dots \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \dots$$

with the property $d_n \circ d_{n+1} = 0$ for all n. Homomorphisms d_n are called *boundary operators* or *differentials.* A *cochain complex* is a sequence of abelian groups and homomorphisms

$$C^{\cdot}: \dots \xrightarrow{d^{n-1}} C^n \xrightarrow{d^n} C_{n+1} \xrightarrow{d_{n+1}} \dots$$

with the property $d^n \circ d_{n-1} = 0$. A chain complex can be considered as a cochain complex by reversing the enumeration: $C^n = C_{-n}$, $d^n = d_{-n}$. This is why we will usually consider only cochain complexes. A complex of A-modules is a complex for which C_n (respectively C^n) are modules over a ring A and d_n (resp. d^n) are homomorphisms of modules.

1.2. Homology and Cohomology. Since $d_n \circ d_{n+1} = 0$, we have $\operatorname{im} d_{n+1} \subset \ker d_n$. A *homology* of a chain complex is the group $H_n(C_{\cdot}) = \ker d_n / \operatorname{im} d_{n+1}$. A *cohomology* of a cochain complex is the group $H^n(C^{\cdot}) = \ker d^n / \operatorname{im} d^{n-1}$. The standard terminology is as follows: elements of C_n are called *n-dimensional chains,* elements of C^n are *n-dimensional cochains,* elements of $\ker d_n = Z_n$ are *n-dimensional cycles,* elements of $\ker d^n = Z^n$ are *n-dimensional cocycles,* those of $\operatorname{im} d_{n+1} = B_n$ are *boundaries,* those of $\operatorname{im} d^{n-1} = B^n$ are *coboundaries.* If $C^{\cdot}$ is a complex of A-modules, its cohomology is an A-module. A complex is said to be *acyclic* (or an *exact sequence*) if $H^n(C^{\cdot}) = 0$ for all n.

1.3. Morphisms of Complexes. A morphism $f\colon C^{\cdot} \to D^{\cdot}$ is a family of group (module) homomorphisms $f^n\colon C^n \to D^n$ commuting with differentials: $f^{n+1} \circ d^n_C = d^n_D \circ f^n$. A morphism f induces a morphism of cohomology $H^{\cdot}(f) = \{H^n(f)\colon H^n(C^{\cdot}) \to H^n(D^{\cdot})\}$ by the formula $\{$*the class of a cocycle* $c\} \mapsto \{$*the class of a cocycle* $f(c)\}$.

A *homotopy* between morphisms of complexes $f, g\colon C^{\cdot} \to D^{\cdot}$ is a family of group homomorphisms $h^n\colon C^n \to D^{n+1}$ such that $f^n - g^n = h^{n+1} \circ d^n + d^{n-1} \circ h^n$. The class of morphisms homotopic to zero form "an ideal," i.e. it is stable under addition and the composition with an arbitrary morphism.

1.3.1. Lemma. *If f and g are homotopic then $H^n(f) = H^n(g)$ for each n.*

Indeed, if c is a cocycle then $f(c) = g(c) + d(h(c))$, so that the classes of $f(c)$ and $g(c)$ coincide.

1.4. Exact Triple of Complexes. A sequence of complexes and morphisms $O \to K^{\cdot} \to L^{\cdot} \to M^{\cdot} \to 0$ is said to be *exact* (or an *exact triple*) if for each n the sequence of groups (modules) $O \to K^n \to L^n \to M^n \to 0$ is exact.

1.5. Connecting Homomorphism. Let $O \to K^{\cdot} \to L^{\cdot} \to M^{\cdot} \to 0$ be an exact triple of complexes. For any n define a homomorphism $\delta^n = \delta^n(f,g) : H^n(M^{\cdot}) \to H^{n+1}(K^{\cdot})$ as follows. Let $m \in M^n$ be a cycle. Choose $l \in L^n$ such that $g^n(l) = m$. Then $g^{n+1}(d^n(l)) = 0$, so that $d^n(l)) = f^{n+1}(k)$ for some $k \in K^{n+1}$. It is clear that $d^{n+1}k = 0$. Set δ(the class of m) = (the class of k). Direct computations show that δ^n does not depend on the choices.

1.5.1. Theorem. *The cohomology sequence*

$$\ldots \longrightarrow H^n(K^{\cdot}) \xrightarrow{H^n(f)} H^n(L^{\cdot}) \xrightarrow{H^n(g)}$$
$$\xrightarrow{H^n(g)} H^n(M^{\cdot}) \xrightarrow{\delta^n(f,g)} H^{n+1}(K^{\cdot}) \longrightarrow \ldots$$

is exact.

§ 2. Standard Complexes in Algebra and in Geometry

2.1. Simplicial Sets. Complexes in homological algebra are mostly either of topological nature or somehow appeal to topological intuition. A classical method to study a topological space is to consider its triangulation, i.e. to decompose it into simplexes: points, segments, triangles, tertrahedra, etc. The corresponding algebraic technique is the technique of simplicial sets.

2.1.1. Definition. A *geometrical n-dimensional simplex* is the topological space

$$\Delta_n = \left\{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} x_i = 1,\ x_i \geq 0\right\}.$$

The point e_i such that $x_i = 1$ is called its *i-th vertex*. To any nondecreasing mapping $f : [m] \to [n]$, where $[m] = 0, 1, \ldots, m$, we associate the mapping Δ_f, called "the f-th face," as follows: Δ_f is a unique linear mapping that sends the vertex $e_i \in \Delta_m$ to the vertex $e_{f(i)} \in \Delta_n$ for $i = 0, 1, \ldots, m$.

2.1.2. Definition. A *simplicial set* is a family of sets $X = (X_n)$, $n = 0, 1, \ldots,$ and mappings $X(f) : X_n \to X_m$, one for each nondecreasing map $f : [m] \to [n]$, such that

$$X(\mathrm{id}) = \mathrm{id}, \quad X(g \circ f) = X(f) \circ X(g).$$

One can consider X_n as the set of indices enumerating a family of n-dimensional geometrical simplexes. Mappings $X(f)$ describe how to glue all these simplexes together in order to obtain one topological space.

A *simplicial mapping* $\varphi : X \to Y$ is a family $\varphi_n : X_n \to Y_n$ such that $Y(f)\varphi_n = \varphi_m X(f)$ for each nondecreasing $f : [m] \to [n]$.

2.1.3. Definition. *Geometric realization* of a simplicial set X is the topological space

$$|X| = \coprod_{n=0}^{\infty} (\Delta_n \times X_n)/R,$$

where the equivalence relation R is defined as follows: $(s, x) \in \Delta_n \times X_n$ is identified with $(t, y) \in \Delta_m \times X_m$ if there exists a nondecreasing mapping $f : [m] \to [n]$ with $Y = X(f)x$, $s = \Delta_f t$. The topology on $|X|$ is the weakest one for which the factorization by R is continuous.

2.2. Homology and Cohomology of Simplicial Sets. Let X be a simplicial set. Denote by $C_n(X, \mathbb{Z})$, $n > 0$, the free abelian group generated by the set X_n, and set $C_n = 0$ for $n < 0$. For any abelian group F set $C_n(X, F) = C_n(X, \mathbb{Z}) \otimes_{\mathbb{Z}} F$. Hence, elements of $C_n(X, F)$, called chains of X with coefficients in F, are formal linear combinations of the form $\sum_{x \in X_n} a(x)x$, $a(x) \in F$. The boundary operator is defined as follows. Let $\partial_n^i : [n-1] \to [n]$ be a unique decreasing mapping whose image does not contain i. We set $d_0 = 0$, and then

$$d_n\Big(\sum_{x \in X_n} a(x)x\Big) = \sum_{x \in X_n} a(x) \sum_{i=0}^{n} (-1)^i X(\partial_n^i)(x), \quad n \geq 1,$$

Cochains $C^n(X, F)$ are defined dually: $C^n(X, F)$ consists of functions on X_n with values in F, and

$$(d^n f)(x) = \sum_{i=0}^{n+1} (-1)^i f(X(\partial_{n+1}^i)(x).$$

Set

$$H_n(X, F) = H_n(C_.(X, F)), \quad H^n(X, F) = H^n(C^{\cdot}(X, F)).$$

2.3. The Singular Complex. Let Y be a topological space. By a *singular n-simplex* of Y we mean a continuous mapping $\varphi : \Delta_n \to Y$. Define

$$X_n \text{ is the set of singular } n\text{-simplexes of } Y;$$
$$X(f)\varphi = \varphi \circ \Delta_f, \text{ where } f : [m] \to [n] \text{ does not decrease, } \Delta_f : \Delta_n \to \Delta_m.$$

(Co)homology Y with coefficients in an abelian group F is defined as $H_n(X, F)$ and $H^n(X, F)$ and denoted $H_n^{\mathrm{sing}}(X, F)$ and $H^n_{\mathrm{sing}}(X, F)$.

2.4. Coefficient Systems. In the definition of an n-chain of a simplicial set coefficients we can assume that coefficients at different simplexes are taken from different group. However, to define the boundary operator in this case one has to impose to these coefficient groups the following compatibility conditions.

2.4.1. Definition. a. A *homological coefficient system* $\mathcal{A}$ on a simplicial set X is a family of abelian group $\mathcal{A}_x$, one for each simplex $x \in X$, and a family of group homomorphisms $\mathcal{A}(f,x) : \mathcal{A}_x \to \mathcal{A}_{X(f)x}$, one for each pair $(x \in X_n,\ f : [m] \to [n])$, such that the following conditions are satisfied:

$$\mathcal{A}(\mathrm{id}, x) = \mathrm{id}; \quad \mathcal{A}(fg, x) = \mathcal{A}(g, X(f)x)\mathcal{A}(f, x).$$

b. A *cohomological coefficient system* $\mathcal{B}$ on a X is a similar family of abelian group $\{\mathcal{B}_x\}$, and a similar family of group homomorphisms $\mathcal{B}(f,x) : \mathcal{B}_{X(f)x} \to \mathcal{B}_x$ such that

$$\mathcal{B}(\mathrm{id}, x) = \mathrm{id}; \quad \mathcal{B}(fg, x) = \mathcal{B}(f, x)\mathcal{B}(g, X(f)x).$$

2.5. Homology and Cohomology with Coefficients. In the notation of 2.4, set

$$C_n(X, \mathcal{A}) = \Big\{ \sum_{x \in X_n} a(x)x \mid a(x) \in \mathcal{A}_x \Big\},$$

$$d_n\Big(\sum_{x \in X_n} a(x)x \Big) = \sum_{x \in X_n} \sum_{i=0}^{n} \mathcal{A}(\partial_n^i, x)(a(x))(-1)^i X(\partial_n^i x), \quad n \geq 1,$$

and similarly

$$C^n(X, \mathcal{B}) = \Big\{ \text{functions } f : X_n \to \coprod_{x \in X} \mathcal{B}_x, \quad f(x) \in \mathcal{B}_x \Big\},$$

$$(d^n f)(x) = \sum_{i=0}^{n+1} (-1)^i \mathcal{B}(\partial_n^i, x)(f(X(\partial_{n+1}^i))), \quad x \in X_{n+1}.$$

If the groups $\mathcal{A}_x$ (resp. $\mathcal{B}_x$) do not depend on x, and all mappings $\mathcal{A}(f,x)$ (resp. $\mathcal{B}(f,x)$) are the identity homomorphisms, we recover the definition from 2.2. (Co)homology of $C_.(X, \mathcal{A})$ and $C^{\cdot}(X, \mathcal{B})$ are called the (co)homology of the simplicial set with the coefficient system.

2.6. Čech Cohomology with Coefficients in a Sheaf. Let Y be a topological space, $U = (U_\alpha)$, $\alpha \in A$, be its open or closed covering. The *nerve* of the covering U is the following simplicial set X:

$$X_n = \{(\alpha_0, \ldots, \alpha_n) \mid U_{\alpha_0} \cap \cdots \cap U_{\alpha_n} \neq \emptyset\} \subset A^{n+1};$$
$$X(f)(\alpha_0, \ldots, \alpha_n) = (\alpha_{f(0)}, \ldots, \alpha_{f(n)}) = \quad \text{for } f : [m] \to [n].$$

Let $\mathcal{F}$ be a sheaf of abelian groups on Y. It determines a cohomological coefficient system on the nerve of Y as follows:

$$\mathcal{F}_{\alpha_0, \ldots, \alpha_n} = \Gamma(U_{\alpha_0} \cap \cdots \cap U_{\alpha_n}, \mathcal{F}),$$

$$\mathcal{F}(f,(\alpha_0,\dots,\alpha_n)) \quad \text{maps the section } \varphi \in \Gamma(U_{\alpha_{f(0)}} \cap \dots \cap U_{\alpha_{f(n)}}, \mathcal{F}) \\ \text{to its restriction to } U_{\alpha_0} \cap \dots \cap U_{\alpha_n}.$$

Cohomology of X with this coefficient system is called the Čech cohomology of the covering U with coefficients in the sheaf $\mathcal{F}$.

2.7. Group Cohomology. Let G be a group. Define a simplicial set BG as follows:

$$(BG)_n = G^n;$$
$$\text{for } f:[m]\to[n], \quad BG(f)(g_1,\dots,g_n) = (h_1,\dots,h_m),$$
$$\text{where } h_i = \prod_{j=f(i-1)+1}^{f(i)} g_j \quad (= e \text{ if } f(i-1)=f(i)).$$

The geometric realization $|BG|$ is called the *classifying space* of the group G.

Let A be a left G-module, i.e. an additive group with the action of G by automorphisms. Such a module yields the following cohomological coefficient system $\mathcal{B}$ on BG:

$$\mathcal{B}_x = A \quad \text{for all } x;$$
$$\mathcal{B}(f,x)(a) = ha, \quad \text{where } h = \prod_{j=1}^{f(0)} g_j \quad (= e \text{ if } f(0)=0)$$
$$\text{for } f:[m]\to[n], \quad x=(g_1,\dots,g_n)\in(BG)_n, \quad a\in A.$$

Using the above definitions we can describe the complex $C^\cdot(BG,\mathcal{B})$ (denoted also by $C^\cdot(G,A)$) explicitly:

$$C^0(G,A) = A;$$
$$C^n(G,A) = \text{function on } G^n \text{ with values in } A.$$

Next, for an n-cochain f,

$$df(g_1,\dots,g_{n+1}) = g_1 f(g_2,\dots,g_{n+1}) \\ + \sum_{i=1}^{n}(-1)^i f(g_1,\dots,g_i g_{i+1},\dots,g_{n+1}) + (-1)^{n+1} g_{n+1} f(g_1,\dots,g_n).$$

Cohomology of this complex are denoted $H^n(C,A)$. Similarly, using A one can construct the following homological coefficient system $\mathcal{A}$ on BG:

$$\mathcal{A}_x = A \quad \text{for all } x;$$
$$\mathcal{A}(f,x)a = h^{-1}a, \quad \text{where } h = \prod_{j=1}^{f(0)} g_j.$$

It gives the homology $H_n(C,A)$.

2.8. The de Rham Complex. In the above examples the transition from geometry to algebra was performed using combinatorics and simplicial decomposition. In the case when the topological space X has the structure of a smooth manifold, the ring of smooth differential forms is a complex. More precisely, let $\Omega^i(X)$ be the space of i-forms. The exterior derivative is given in local coordinates $(x^1, \dots, x^n)$ by the formula

$$d\Big(\sum_{|I|=k} f_I dx^I\Big) = \sum_{|I|=k} \sum_i \frac{\partial f_I}{\partial x^i} dx^i \wedge dx^I,$$

where

$$I = (i_1, \dots, i_k), \quad |I| = i_1 + \dots + i_k, \quad dx^{i_1} \wedge \dots \wedge dx^{i_k}.$$

The cohomology of the complex $(\Omega^{\cdot}(X), d)$, denoted $H^{\cdot}_{\mathrm{DR}}(X)$, is called the *de Rham cohomology* of the manifold X.

The de Rham theorem established a canonical isomorphism

$$H^{\cdot}_{\mathrm{DR}}(X) = H^{\cdot}_{\mathrm{sing}}(X, \mathbb{R}).$$

On the level of chains this isomorphism associates to a differential i-form its integrals over smooth i-dimensional singular chains.

2.9. Lie Algebra Cohomology. Let us consider the de Rham complex of a connected Lie group G. The group G acts on this complex by the right shifts. Denote by $\Omega_{\mathrm{inv}}(G)$ the subcomplex consisting of G-invariant chains. It admits a purely algebraic description. Let $\mathfrak{g}$ be the Lie algebra of G considered as the Lie algebra of right-invariant vector fields on G. Then $\Omega^n_{\mathrm{inv}}(G) = L(\wedge^n \mathfrak{g}, \mathbb{R})$ is the space skew-symmetric n-linear real forms on $\mathfrak{g}$. The exterior derivative of an n-form considered as a polylinear function on vector fields (on an arbitrary smooth manifold) is given by the following Cartan formula:

$$\begin{aligned}(d\omega^n)(\xi_1, \dots, \xi_{n+1}) \\ = \sum_{1<j<l<n} (-1)^{j+l-1} \omega^n([\xi_j, \xi_l], \xi_1, \dots, \widehat{\xi_j}, \dots, \widehat{\xi_l}, \dots, \xi_{n+1}) \\ + \sum_{j=1}^{n+l} (-1)^j \xi_j [\omega^n(\xi_1, \dots, \widehat{\xi_j}, \dots, \xi_{n+1})]\end{aligned}$$

(here $\widehat{\ }$ means that the corresponding term is omitted). Applying this formula to $\Omega^n_{\mathrm{inv}}(G)$ we obtain the following formula for d on $C^{\cdot}(\mathfrak{g}) = L(\wedge^{\cdot} \mathfrak{g}, \mathbb{R})$:

$$(dc)(g_1, \dots, g_{n+1}) = \sum_{1<j<l<n} (-1)^{j+l-1} c([g_j, g_l], g_1, \dots, \widehat{g_j}, \dots, \widehat{g_l}, \dots, g_{n+1}).$$

Denote the cohomology of this complex by $H^{\cdot}(\mathfrak{g}, \mathbb{R})$. Merging the de Rham theorem with the averaging over a compact subgroup, we obtain the E. Cartan theorem: for a compact connected group G

$$H^{\cdot}_{\mathrm{sing}}(G, \mathbb{R}) = H^{\cdot}(\mathfrak{g}, \mathbb{R}).$$

The construction of $G^{\cdot}(\mathfrak{g})$ does not require the existence of the Lie group G associated to the Lie algebra $\mathfrak{g}$ and can be applied to an arbitrary Lie algebra over a field k.

More generally, let M be a $\mathfrak{g}$-module. Set $C^n(\mathfrak{g}, M) = L(\wedge^n \mathfrak{g}, M)$ and define the differential by the Cartan formula

$$\begin{aligned}(dc)(g_1, \dots, g_{n+1}) = &\sum_{1<j<l<n} (-1)^{j+l-1} c([g_j, g_l], g_1, \dots, \widehat{g_j}, \dots, \widehat{g_l}, \dots, g_{n+1}) \\ &+ \sum_{j=1}^{n+1} (-1)^j g_j c(g_1, \dots, \widehat{g_j}, \dots, g_{n+1}).\end{aligned}$$

Denote the cohomology of this complex by $H^{\cdot}(\mathfrak{g}, M)$.

To define the homology $H_{\cdot}(\mathfrak{g}, M)$ we must use the complex $H_{\cdot}(\mathfrak{g}, M) = M \otimes \wedge^{\cdot} \mathfrak{g}$ with the differential

$$\begin{aligned}&d(m \otimes (g_l \wedge \cdots \wedge g_n)) \\ &\quad = \sum_{1<j<l<n} (-1)^{j+l-1} m \otimes ([g_j, g_l], g_1 \wedge \cdots \wedge \widehat{g_j} \wedge \cdots \wedge \widehat{g_l} \wedge \cdots \wedge g_n) \\ &\quad + \sum_{j=1}^{n+l} (-1)^{j+1} g_j m \otimes (g_j \wedge \cdots \wedge \widehat{g_j} \wedge \cdots \wedge g_{n+1}).\end{aligned}$$

2.10. The Hochschild Complex. Let k be a commutative ring with unity, A be an associative k-algebra with unity. Consider the following chain complex $T_{\cdot}(A)$:

$$T_n(A) = \overbrace{A \otimes_k \cdots \otimes_k A}^{n+2 \text{ times}}, \quad n \geq -1,$$

$$d(a_0 \otimes \cdots \otimes a_{n+1}) = \sum_{i=0}^{n} (-1)^j (a_0 \otimes \cdots \otimes (a_i a_{i+1}) \otimes \cdots \otimes a_{n+1}).$$

It can be considered as a complex of k-modules, as a complex of A-bimodules, and, finally, as a complex of A^e -modules where $A^e = A \otimes_k A^t$, A^t is the opposite ring of A (i.e. the ring A with the oppposte multiplication). This complex is acyclic since its identity mapping is homotopic to the zero mapping: the homotopy is given by the formula

$$a_0 \otimes \cdots \otimes a_{n+1} \to 1 \otimes a_0 \otimes \cdots \otimes a_{n+1}.$$

Let M be an A-bimodule. Then we can consider the complexes of k-modules $M \otimes_{A^e} T_{\cdot}(A)$ and $\mathrm{Hom}_{A^e}(T_{\cdot}(A), M)$. The homology of these complexes are denoted by $H_n(A, M)$ and $H^n(A, M)$ respectively and called the Hochschild (co)homology of the algebra A with coefficient in the module M. We can get rid of the tensor product over A^e using the isomorphism

$$M \otimes_{A^e} T_n(A) = M \otimes_{A^e} A^e \otimes_k T_{n-2}(A) = M \otimes_{A^e} T_{n-2}(A).$$

In this setting $H_n(A, M)$ becomes the homology of the following complex $C_.(A, M)$:

$$\text{the } n\text{-th term is } M \otimes A^{\otimes n};$$
$$\begin{aligned} d(m \otimes a_1 \otimes \cdots \otimes a_n) &= ma_1 \otimes a_2 \otimes \cdots \otimes a_n \\ &+ \sum_{i=1}^{n-1} (-1)^i m \otimes a_1 \otimes \cdots \otimes (a_i a_{i+1}) \otimes \cdots \otimes a_n \\ &+ (-1)^n a_n m \otimes a_1 \otimes \cdots \otimes a_{n-1}. \end{aligned}$$

Similarly, is the cohomology of the complex $C^{\cdot}(A, M)$:

$$\text{the } n\text{-th term is } \mathrm{Hom}_k(A^{\otimes n}, M);$$
$$\begin{aligned} df(a_1 \otimes \cdots \otimes a_{n+1}) &= a_1 f(a_2 \otimes \cdots \otimes a_{n+1}) \\ &+ \sum_{i=0}^{n} (-1)^i f(a_1 \otimes \cdots \otimes (a_i a_{i+1}) \otimes \cdots \otimes a_{n+1}) \\ &+ (-1)^{n+1} f(a_1 \otimes \cdots \otimes a_n) a_{n+1}. \end{aligned}$$

2.11. Cyclic Homology of an Algebra. Let us keep the setup of the previous subsection and take in this setup $k \supset \mathbb{Q}$, $M = A$. The cyclic shift acts on the terms of the complex $C_.(A, A)$, whose homology is $H_n(A, A)$. Define

$$t(a_0 \otimes \cdots \otimes a_n) = (-1)^n a_n \otimes a_0 \otimes \cdots \otimes a_{n-1}.$$

The operator t does not commute with the differential. However, if we set

$$d'(a_0 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n,$$

then

$$d(1 - t) = (1 - t)d'.$$

Hence the image of $1 - t$ is a subcomplex of $C_n(A, A)$ so that we can define the quotient complex

$$C_n^{\lambda}(A) = C_n(A, A) / \mathrm{im}(1 - t),$$
$$d^{\lambda} = d \mod \mathrm{im}(1 - t).$$

Its homology is called the cyclic homology of the algebra A and are denoted $H_n^{\lambda}(A)$ or $HC_n(A)$.

2.12. Cyclic Cohomology of an Algebra. To define it we have to consider the subcomplex $C_{\lambda}^{\cdot}(A) \subset C^{\cdot}(A, A^*)$ consisting of t-invariant cochains, i.e. of k-linear functionals $f : A^{\otimes n} \to A^*$ with the property

$$f(a_0 \otimes \cdots \otimes a_n) = (-1)^n f(a_n \otimes a_0 \otimes \cdots \otimes a_{n-1}).$$

The coboundary operator is given by the last formula in 2.9.

Cohomology is denoted by $H^n_\lambda(A)$ or $HC^n(A)$.

2.13. (Co)Chain Complex of a Cell Decomposition. To use the singular (co)chain complex in the computation of (co)homology of a topological space X is non-economic because this complex is infinite-dimensional. Topologists often use (co)chains associated to a realization of X as a cell decomposition. Let us give the basic definitions.

A *cell decomposition* (or *CW*-complex) is a topological space X represented as a union $X = \bigcup_{n=0}^{\infty} \bigcup_{i \in I_n} e_i^n$ of disjoint sets e_i^n (cells) with mappings $f_i^n : B^n \to X$ of the closed unit ball into X such that the restriction of f_i^n to the interior $\operatorname{Int} B^n$ of B^n is a homeomorphism $f_i^n : \operatorname{Int} B^n \xrightarrow{\sim} e_i^n$, and the following conditions are satisfied:

a) The boundary $\dot{e}_i^n = \overline{e}_i^n \setminus e_i^n$ of any cell is contained in the union of a finite number of cell of smaller dimensions.

b) The set $Y \subset X$ is closed if and only if the preimage $(f_i^n)^{-1}(Y) \cap \overline{e}_i^n$ is closed in $\overline{e}_i^n$ for all n and all $i \in I_n$.

For a pair of cells e_i^n, e_j^{n-1} define the *incidence coefficient* $c(e_i^n, e_j^{n-1})$ as follows. Let X^r be the union of all cells of dimension $\leq r$. Then X^{n-1}/X^{n-2} (X^{n-2} is contracted to a point in X^{n-1}) is the wedge of $(n-1)$-dimensional spheres S^{n-1} in the number equal to the cardinality of I_{n-1}, and the cell e_j^{n-1} distinguishes one sphere in this wedge (denote it by S). Consider the composite mapping

$$S^{n-1} = \dot{B}^n \xrightarrow{f_i^n|S^{n-1}} X^{n-1} \longrightarrow X^{n-1}/X^{n-2} \xrightarrow{\pi} S,$$

where π is the projection of the wedge onto one of its components. The resulting mapping $S^{n-1} \to \dot{B}^n = S^{n-1}$ determines an element of the group $\pi_{n-1}(S^{n-1})$, i.e. an integer (the degree of the mapping), and we define the incidence coefficient $c(e_i^n, e_j^{n-1})$ to be equal to this integer.

Define now the group of integral n-dimensional chains as the free abelian group generated by e_i^n, $i \in I_n$, and define the differential by the formula

$$de_i^n = \sum_{j \in I_{n-1}} c(e_i^n, e_j^{n-1}) e_j^{n-1}.$$

By the condition a) above, this sum is finite.

Cochains, as well as chains and cochains with coefficients, can be defined similarly.

2.13.1. Theorem. (*Co*)*homology of a cell decomposition computed from cell* (*co*)*chains is canonically isomorphic to singular* (*co*)*homology.*