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[ALGEBRAIC GEOMETRY]

CONVENTIONS

All rings are commutative with identity element 1. All homomorphisms take 1 to 1. In an integral domain or a field $0 \neq 1$. A prime ideal (resp. maximal) is an ideal $\mathfrak{p}$ in a ring A s.t. $A/\mathfrak{p}$ is a domain (resp. field) Thus the ring itself is not prime or maximal.

A multiplicative system in a ring A is a subset S, containing 1, and closed under multiplication. The localisation $S^{-1}A$ is defined to be the ring formed by the equivalence classes of fractions a/s, $a \in A$, $s \in S$ where $a/s = a'/s'$ iff $\exists s'' \in S$ s.t. $s''(s'a - sa') = 0$. Two special cases which are used constantly are the following. If $\mathfrak{p}$ is a prime ideal in A, then $S = A - \mathfrak{p}$ is multiplicatively closed, and the corresponding localisation is denoted by $A_{\mathfrak{p}}$. If f is an element of A, $S = \{1\} \cup \{f^n \mid n \geq 1\}$ is multiplicative, and the corresponding localisation is denoted by A_f. (Note if f is nilpotent, A_f is the zero ring).

1. AFFINE VARIETIES

Let k be an algebraically closed field. We define affine n-space over k, denoted $\mathbb{A}^n_k$ or simply $\mathbb{A}^n$, to be the set of n-tuples of elements of k. An element $P \in \mathbb{A}^n$ will be called a point, and if $P = (a_1, \dots, a_n)$ with $a_i \in k$, then the a_i will be called the coordinates of P.

Let $A = k[x_1, \dots, x_n]$ be the polynomial ring in n variables over k. We will interpret the elements of A as functions from the affine n-space to k, by defining $f(P) = f(a_1, \dots, a_n)$ where $f \in A$ and $P \in \mathbb{A}^n$. Thus if $f \in A$ is a polynomial, we can talk about the set of zeros of f, namely

$$Z(f) = \{P \in \mathbb{A}^n \mid f(P) = 0\}$$

More generally, if T is any subset of A, we define the zero set of T to be the common zeros of all the elements of T, namely

$$Z(T) = \{P \in \mathbb{A}^n \mid f(P) = 0 \text{ for all } f \in T\}$$

Clearly if $\mathfrak{a}$ is the ideal of A generated by T, then $Z(T) = Z(\mathfrak{a})$. Furthermore, since A is a noetherian ring, any ideal has a finite set of generators $f_1, \dots, f_r$. Thus $Z(T)$ can be expressed as the common zeros of the finite set of polynomials $f_1, \dots, f_r$.

DEFINITION Subset $Y \subseteq \mathbb{A}^n$ is an algebraic set if there exists a subset $T \subseteq A$ such that $Y = Z(T)$.

PROPOSITION 1.1 The union of two algebraic sets is an algebraic set. The intersection of any family of algebraic sets is an algebraic set. The empty set and the whole space are algebraic sets.

PROOF If $Y_1 = Z(T_1)$ and $Y_2 = Z(T_2)$ then $Y_1 \cup Y_2 = Z(T_1T_2)$ since if $x \in Y_1 \cup Y_2$ then either $x \in Y_1$ or $x \in Y_2$ so x is a zero of every element of T_1T_2. Conversely, if $P \in Z(T_1T_2)$ and $P \notin Y_1$, then for some $f \in T_1$, $f(P) \neq 0$. Hence for $g \in T_2$, $(fg)(P) \neq 0$ unless $g(P) = 0$ for each $g \in T_2$. Hence $P \in Y_2$.
If $Y_\alpha = Z(T_\alpha)$ is a family of algebraic sets then $\bigcap Y_\alpha = Z(\bigcup T_\alpha)$, so $\bigcap Y_\alpha$ is an algebraic set.
Finally, $\emptyset = Z(1)$ and $\mathbb{A}^n = Z(0)$. □

DEFINITION We define the Zariski topology on $\mathbb{A}^n$ by taking the open subsets to be the complements of the algebraic sets. This is a topology, because according to the proposition, the intersection of two open sets is open, and the union of any family of open sets is open, etc.

EXAMPLE 1.1.1 Let us consider the Zariski topology on the affine line $\mathbb{A}^1$. Every ideal in $A = k[x]$ is principal, so every algebraic set is the set of zeros of some polynomial. Since k is algebraically closed, every nonzero polynomial $f(x)$ can be written $f(x) = c(x-a_1)\cdots(x-a_n)$ with $c, a_1, \dots, a_n \in k$. Then $Z(f) = \{a_1, \dots, a_n\}$. Thus the algebraic sets in $\mathbb{A}^1$ are just the finite subsets (including $\emptyset$) and the whole space ($f=0$). Thus the open sets are the empty set and the complements of finite subsets. Notice in particular that this topology is not Hausdorff.
(since the complement of an open set is finite and so can't contain another open set)(take k infinite).

DEFINITION A nonempty subset Y of a topological space X is irreducible if it cannot be expressed as the union $Y = Y_1 \cup Y_2$ of two proper subsets, each one of which is closed in Y. The empty set is not considered to be irreducible.

EXAMPLE 1.1.2 $\mathbb{A}^1$ is irreducible, because its only proper closed subsets are finite, yet it is infinite.
(because k is algebraically closed, hence infinite)

EXAMPLE 1.1.3 Any nonempty open subset of an irreducible space is irreducible and dense.

EXAMPLE 1.1.4 If Y is an irreducible subset of X, then its closure $\overline{Y}$ in X is also irreducible.

PROOF If Y is irreducible and Z is any subset of X containing Y, the only way for Z to be reducible is if one of the closed sets avoids Y completely — meaning the other contains Z. If $Z = \overline{Y}$ such a set cannot be proper.

DEFINITION An affine algebraic variety (or simply affine variety) is an irreducible closed subset of $\mathbb{A}^n$ (with the induced topology). An open subset of an affine variety is called a quasi-affine variety.

Note the closed sets are the algebraic sets. These affine and quasi-affine varieties are our first objects of study. But before we can go further, in fact before we can even give any interesting examples, we need to explore the relationship between subsets of $\mathbb{A}^n$ and ideals in A more deeply. So for any subset $Y \subseteq \mathbb{A}^n$, let us define the ideal of Y in A by

$$I(Y) = \{f \in A \mid f(P) = 0 \text{ for all } P \in Y\}$$

Now we have a function Z which maps subsets of A to algebraic sets, and a function I which maps subsets of $\mathbb{A}^n$ to ideals. Their properties are summarised in the following proposition.

PROPOSITION 1.2 (a) If $T_1 \subseteq T_2$ are subsets of A, $Z(T_2) \subseteq Z(T_1)$

(b) If $Y_1 \subseteq Y_2$ are subsets of $\mathbb{A}^n$, $I(Y_1) \supseteq I(Y_2)$

(c) For any two subsets Y_1, Y_2 of $\mathbb{A}^n$, we have $I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$

(d) For any ideal $\mathfrak{a} \subseteq A$, $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$, the radical of $\mathfrak{a}$.

(e) For any subset $Y \subseteq \mathbb{A}^n$, $Z(I(Y)) = \overline{Y}$, the closure of Y.

PROOF (a), (b), and (c) are obvious. (d) is a direct consequence of Hilbert's Nullstellensatz.
To prove (e), we note that $Y \subseteq Z(I(Y))$, which is a closed set, so clearly $\overline{Y} \subseteq Z(I(Y))$.
On the other hand, let W be any closed set containing Y. Then $W = Z(\mathfrak{a})$ for some ideal $\mathfrak{a}$.
So $Y \subseteq Z(\mathfrak{a})$ and by (b) $IZ(\mathfrak{a}) \subseteq I(Y)$. But certainly $\mathfrak{a} \subseteq IZ(\mathfrak{a})$ so by (a) we have $W = Z(\mathfrak{a}) \supseteq ZI(Y)$
□

THEOREM 1.3A (Hilbert's Nullstellensatz) Let k be an algebraically closed field, let $\mathfrak{a}$ be an ideal in $A = k[x_1,\dots,x_n]$, and let $f \in A$ be a polynomial which vanishes at all points of $Z(\mathfrak{a})$. Then $f^r \in \mathfrak{a}$ for some integer $r > 0$.

COROLLARY 1.4 There is a one-to-one inclusion-reversing correspondence between algebraic sets in $\mathbb{A}^n$ and radical ideals (i.e. ideals which are equal to their own radical) in A, given by $Y \mapsto I(Y)$ and $\mathfrak{a} \mapsto Z(\mathfrak{a})$. Furthermore, an algebraic set is irreducible if and only if its ideal is a prime ideal.

PROOF Only the last part is new. If Y is irreducible, we show that $I(Y)$ is prime. Suppose $fg \in I(Y)$. Then $Y \subseteq Z(fg) = Z(f) \cup Z(g)$. Hence $Y = (Y \cap Z(f)) \cup (Y \cap Z(g))$ where both sets are closed in Y. Thus wlog $Y = Y \cap Z(f)$, so $Y \subseteq Z(f)$ and hence $f \in I(Y)$.

Conversely, let $\mathfrak{p}$ be a prime ideal, and suppose that $Z(\mathfrak{p}) = Y_1 \cup Y_2$. Then $\mathfrak{p} = I(Y_1) \cap I(Y_2)$. Since ideals are f.g. say $I(Y_1) = (f_1,\dots,f_k)$, $I(Y_2) = (g_1,\dots,g_s)$. Then $f_1 g_i \in \mathfrak{p}$ $i = 1,\dots,s$ so either $\mathfrak{p} = I(Y_2)$ or $f_1 \in \mathfrak{p}$. Repeat to find $\mathfrak{p} = I(Y_1)$ or $\mathfrak{p} = I(Y_2)$. Thus $Z(\mathfrak{p}) = Y_1$ or Y_2 and is hence irreducible.

EXAMPLE 1.4.1 $\mathbb{A}^n$ is irreducible, since it corresponds to the zero ideal in A, which is prime.

EXAMPLE 1.4.2 Let f be an irreducible polynomial in $A = k[x,y]$. Then f generates a prime ideal in A, since A is a UFD. Hence $Y = Z(f)$ is irreducible. We call it the affine curve defined by the equation $f(x,y) = 0$. If f has degree d, we say that Y is a curve of degree d.

EXAMPLE 1.4.3 More generally, if f is an irreducible polynomial in $A = k[x_1,\dots,x_n]$, we obtain an affine variety $Y = Z(f)$, which is called a surface if $n = 3$ or a hypersurface if $n > 3$.

EXAMPLE 1.4.4 A maximal ideal $\mathfrak{m}$ of $A = k[x_1,\dots,x_n]$ corresponds to a minimal irreducible closed set of $\mathbb{A}^n$, which must be a point, say $P = (a_1,\dots,a_n)$. This shows that every maximal ideal of A is of the form $\mathfrak{m} = (x_1 - a_1,\dots,x_n - a_n)$ for some $a_1,\dots,a_n \in k$.

EXAMPLE 1.4.5 If k is not algebraically closed, these results do not hold. For example, if $k = \mathbb{R}$, the curve $x^2 + y^2 + 1 = 0$ in $\mathbb{A}^2_{\mathbb{R}}$ has no points. So (1.2d) is false. See also Ex 1.12.

DEFINITION If $Y \subseteq \mathbb{A}^n$ is an affine algebraic set, we define the affine coordinate ring $A(Y)$ of Y to be the ring $A/I(Y)$.

REMARK 1.4.6 If Y is an affine variety, then $A(Y)$ is an integral domain. Furthermore, $A(Y)$ is a finitely generated k-algebra. Conversely, any f.g. k-algebra B which is a domain is the affine coordinate ring of some affine variety. Indeed, write B as the quotient of a polynomial ring $A = k[x_1,\dots,x_n]$ by a prime ideal $\mathfrak{a}$, and let $Y = Z(\mathfrak{a})$.

Next we will study the topology of our varieties. To do so we introduce an important class of topological spaces which includes all varieties.

DEFINITION A topological space X is called noetherian if it satisfies the descending chain condition for closed subsets: for any sequence $Y_1 \supseteq Y_2 \supseteq \cdots$ of closed subsets, there is an integer r such that $Y_r = Y_{r+1} = \cdots$

EXAMPLE 1.4.7 $\mathbb{A}^n$ is a noetherian topological space. Indeed, if $Y_1 \supseteq Y_2 \supseteq \cdots$ is a descending chain of closed subsets, then $I(Y_1) \subseteq I(Y_2) \subseteq \cdots$ is an ascending chain of ideals in $A = k[x_1, \ldots, x_n]$. Since A is a noetherian ring, this chain of ideals is eventually stationary. But for each i, $Y_i = Z(I(Y_i))$, so the chain Y_i is also stationary.

PROPOSITION 1.5 In a noetherian topological space X, every nonempty closed subset Y can be expressed as a finite union $Y = Y_1 \cup \cdots \cup Y_r$ of irreducible closed subsets Y_i. If we require that $Y_i \not\supseteq Y_j$ for $i \neq j$ then the Y_i are uniquely determined. They are called the irreducible components of Y.

PROOF First we show the existence of such a representation of Y. Let $\mathfrak{S}$ be the set of nonempty closed subsets of X which cannot be written as a finite union of irreducible closed subsets. If $\mathfrak{S}$ is nonempty, then since X is noetherian, it must contain a minimal element, say Y. Then Y is not irreducible by construction of $\mathfrak{S}$. Thus we can write $Y = Y' \cup Y''$, where Y' and Y'' are proper closed subsets of Y. By minimality of Y, each of Y', Y'' can be expressed as a finite union of closed irreducible subsets, hence Y also, which is a contradiction.

Now let $Y = Y_1 \cup \cdots \cup Y_r$ and $Y = Y_1' \cup \cdots \cup Y_s'$ be two such representations. Then

$$Y_1' \subseteq Y = Y_1 \cup \cdots \cup Y_r$$

and hence $Y_1' = \bigcup (Y_1' \cap Y_j)$. But Y_1' is irreducible and hence $Y_1' \subseteq Y_j$ for some j, say $j=1$. Similarly $Y_1 \subseteq Y_i'$ some i. Then $Y_1' \subseteq Y_i'$ and hence $i=1$ and so $Y_1 = Y_1'$. Now let

$$Z = \mathrm{Cl}(Y - Y_1)$$

Clearly $Y_2 \cup \cdots \cup Y_r$ and $Y_2' \cup \cdots \cup Y_s'$ are closed sets containing $Y - Y_1$, and for $2 \le i \le r$ note that $Y_1 \cap Y_i$ is closed, and $\mathrm{Cl}(Y - Y_1) \cap Y_i$ is closed, and their union is Y_i. Since $Y_i \not\subseteq Y_1$, we see that $\mathrm{Cl}(Y - Y_1) \cap Y_i = Y_i$ and so

$$Z = Y_2 \cup \cdots \cup Y_r = Y_2' \cup \cdots \cup Y_s'.$$

So proceeding by induction we find $r=s$ (else some $Y_j^{(1)}$ is $\emptyset$) and $Y_1 = Y_1', \ldots, Y_r = Y_r'$. □

COROLLARY 1.6 Every algebraic set in $\mathbb{A}^n$ can be expressed uniquely as a union of varieties, no one containing the other. (except for $\emptyset$).

DEFINITION If X is a topological space, we define the dimension of X (denoted $\dim X$) to be the supremum of all integers n such that there exists a chain

$$Z_0 \subset Z_1 \subset \cdots \subset Z_n$$

of distinct irreducible closed subsets of X. We define the dimension of an affine or quasi-affine variety to be its dimension as a topological space.

EXAMPLE 1.6.1 The dimension of $\mathbb{A}^1$ is 1. Indeed, the only irreducible closed subsets of $\mathbb{A}^1$ are the whole space and single points.

DEFINITION In a ring A, the height of a prime ideal $\mathfrak{p}$ is the supremum of all integers n such that there exists a chain $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n = \mathfrak{p}$ of distinct prime ideals. We define the dimension (or Krull dimension) of A to be the supremum of the heights of all prime ideals.

NOTE The Zariski topology is T_0 – that is, let $\mathfrak{p}, \mathfrak{q}$ be primes of a ring, $\mathfrak{p} \neq \mathfrak{q}$. Then wlog let $x \in \mathfrak{p}$ s.t. $x \notin \mathfrak{q}$. Then the ideal $I = (x)$ is s.t. $\mathfrak{q} \in D(I)$ but $\mathfrak{p} \notin D(I)$.

NOTE We know that the natural topology on the spectrum of a ring makes it a compact T_0-space, but also note that in this topology, each of the basic open sets X_f, $f \in R$, is also compact. First note the easy equivalence,

$$X_f = \bigcup_\alpha X_{f_\alpha} \iff \text{No prime ideal disjoint from } \{1, f, f^2, \dots\} \text{ contains all the } f_\alpha.$$

Then noting $X_f = \operatorname{Spec} R_f$ or using the above, $(f_\alpha) = R_f$ and so

$$1 = \frac{r_1}{f^{n_1}} \cdot \frac{f_1}{1} + \dots + \frac{r_k}{f^{n_k}} \cdot \frac{f_k}{1} \qquad \text{some } f_i \in f_\alpha.$$

Then picking N large, $f^N = r_1 f_1 + \dots + r_k f_k$ in R_f, so $f^{N+s} = r_1 f^s f_1 + \dots + r_k f^s f_k \in (f_\alpha)$ in R, or more accurately, is $(f_1, \dots, f_k)$ in R. But then if $f \notin \mathfrak{p}$, clearly some $f_i \notin \mathfrak{p}$ (else $f^{N+s} \in \mathfrak{p} \Rightarrow f \in \mathfrak{p}$) so that $\bigcup_{i=1}^k X_{f_i} = X_f$. Hence X_f is compact.

NOTE Given $n \geq 1$ there is no polynomial $f \in k[x_1, \dots, x_n]$ with $Z(f) = \mathbb{A}^n \setminus (0, \dots, 0)$. Because then

$$\mathbb{A}^n = (0, \dots, 0) \cup Z(f)$$

Since $\mathbb{A}^n$ is irreducible, one of these sets must be $\mathbb{A}^n$, a contradiction.

PROPOSITION 1.7 If Y is an affine algebraic set, then the dimension of Y is equal to the dimension of its affine coordinate ring $A(Y)$.

PROOF Let Y be a closed subset of $\mathbb{A}^n$. Then the closed irreducible subsets of Y correspond to prime ideals containing $I(Y)$, which are the prime ideals of $A(Y)$. Hence a chain of length n in Y exists iff. a chain of primes of length n exists in $A(Y)$. Hence $\dim A(Y) = \dim Y$ (both may be infinite). $\square$

If we agree that $\dim \emptyset = -1$ and the dimension of the zero ring is -1, then 1.7 holds for $Y = \emptyset$ also. By Ex 1.10 if Y is any closed subset of $\mathbb{A}^n$ then $\dim Y \leq n$ and $\dim Y = n$ iff. $Y = \mathbb{A}^n$.

This Proposition allows us to apply results from the dimension theory of noetherian rings to algebraic geometry.

THEOREM 1.8A Let k be a field, and let B be an integral domain which is a finitely-generated k-algebra. Then

(a) The dimension of B is equal to the transcendence degree of the quotient field $K(B)$ of B over k (which is $\leq r$ if B is generated over k by r elements)

(b) For any prime ideal $\mathfrak{p}$ in B, we have

$$\operatorname{ht}\mathfrak{p} + \dim B/\mathfrak{p} = \dim B$$

PROOF See our Atiyah & Macdonald Notes. $\square$

PROPOSITION 1.9 The dimension of $\mathbb{A}^n$ is n.

PROOF But by (1.7) $\dim \mathbb{A}^n = \dim k[x_1, \dots, x_n] = n$. $\square$

PROPOSITION 1.10 If Y is a quasi-affine variety, then $\dim Y = \dim \overline{Y}$. (closure in $\mathbb{A}^n$)

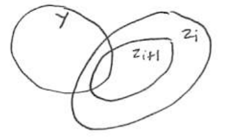
NOTE If Y is an open subset of an affine variety Z, then since Z is irreducible, $Z - Y$, $\overline{Y}$ closed we have $(Z-Y) \cup \overline{Y} = Z$ so provided $Y \neq \emptyset$ $\overline{Y} = Z$. But by assumption quasi-affine $\Rightarrow$ nonempty.

PROOF If $Z_0 \subset \dots \subset Z_n$ is a sequence of distinct closed irreducible subsets of Y, then $\overline{Z_0} \subset \overline{Z_1} \subset \dots \subset \overline{Z_n}$ is a sequence of distinct closed irreducible subsets of $\overline{Y}$. (closures in $Z = \overline{Y}$, distinct since if $Z_i = Y \cap Z_i'$, each Z_i' closed in Z, and $\overline{Z_i} = \overline{Z_{i+1}}$ then $Z_{i+1} \subseteq Y \cap \overline{Z_{i+1}} = Y \cap \overline{Z_i} \subseteq Y \cap Z_i' = Z_i$, a contradiction), so we have $\dim Y \leq \dim \overline{Y}$.

Since $\overline{Y}$ is an affine variety, $\dim \overline{Y} = n$ is finite. Suppose for a contradiction that $n > \dim Y$. Then pick any point $P \in Y$ and let $\mathfrak{m}$ be its associated maximal ideal. By (1.8A b) since $A(\overline{Y})/\mathfrak{m} \cong k$ (see Cor 5.24 A&M)

$$\operatorname{ht}\mathfrak{m} = \dim \overline{Y} = n > \dim Y \qquad \text{(height in } A(\overline{Y})\text{)}$$

Let $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \dots \subset \mathfrak{p}_n = \mathfrak{m}$ be a chain of primes, then there is a chain of distinct closed irreducible subsets of $\overline{Y}$, $P = Z(\mathfrak{m}) \subset Z_{n-1} \subset \dots \subset Z_0 = \overline{Y}$. Since $P \in Y$ each of these sets intersects Y. Since Y is open in $\overline{Y}$ and the Z_i are irreducible, $Z_n \cap Y, \dots, Z_0 \cap Y$ are irreducible closed subsets of Y (each $Z_i \cap Y$ is dense in Z_i). The sets are distinct since if $Z_i \cap Y = Z_{i+1} \cap Y$ $(0 \leq i \leq n-1)$



Since $Y \cap Z_{i+1} \subseteq Z_{i+1}$ taking closures in Z_i gives $\overline{Y \cap Z_{i+1}} \subseteq Z_{i+1}$. But $\overline{Y \cap Z_{i+1}} = \overline{Y \cap Z_i} = Z_i$ since $Z_i \cap Y$ is dense in Z_i. This contradicts $Z_{i+1} \subset Z_i$. Hence we have produced a chain of length $n > \dim Y$ in Y, which is a contradiction. Hence $\dim Y = \dim \overline{Y}$. $\square$

THEOREM 1.11A (Krull's Hauptidealsatz) Let A be a Noetherian ring, and let $f \in A$ be an element which is neither a zero divisor or a unit. Then every minimal prime ideal $\mathfrak{p}$ containing f has height 1.

PROOF see A&M. $\square$

PROPOSITION 1.12A A Noetherian integral domain A is a UFD iff. every prime ideal of height 1 is principal.

PROOF Suppose A is a UFD, and let $\mathfrak{p}$ be a prime ideal of height 1. Say $0 \neq a \in \mathfrak{p}$ (ht $0 = 0$) and let $a = up_1^{r_1} \dots p_n^{r_n}$ be a prime factorisation. Then some $p_i \in \mathfrak{p}$ whence

$$(0) \subset (a) \subseteq (p_i) \subseteq \mathfrak{p}$$

Since ht $\mathfrak{p} = 1$, $\mathfrak{p} = (p_i)$ and we are done. For the converse, it suffices to show that every nonzero (by Th5 Kapslansky) prime contains a prime element (i.e. $0 \neq p$, p nonunit $p|ab \Rightarrow p|a \vee p|b$). Here is where we use the Noetherian hypothesis and (1.11A). Let $\mathfrak{p} \neq 0$ and $0 \neq f \in \mathfrak{p}$. Since A is a Noetherian domain, f is a non zero-divisor. Applying Th10 Kapslansky let $\mathfrak{q} \subseteq \mathfrak{p}$ be a minimal prime containing (f). By (1.11A) $\mathfrak{q}$ has height 1. By assumption $\mathfrak{q} = (q)$ is principal, whence q is a prime contained in $\mathfrak{p}$, completing the proof. $\square$

PROPOSITION 1.13 An affine variety $Y \subseteq \mathbb{A}^n$ has dimension $n-1$ if and only if it is the zero set $Z(f)$ of a single nonconstant irreducible polynomial in $A = k[x_1, \dots, x_n]$

PROOF If f is irreducible then by def^n f is not constant, and (f) is prime so $Z(f)$ is an affine variety. Its ideal is $\mathfrak{p} = (f)$ and by (1.11A) $\mathfrak{p}$ has height 1, so by (1.8A) $Z(f)$ has dimension $n-1$. Conversely, a variety of dimension $n-1$ corresponds to a prime ideal of height 1. Now a polynomial ring A is a UFD, so by (1.12A) $\mathfrak{p}$ is principal, necessarily generated by an irreducible polynomial f. Hence $Y = Z(f)$. $\square$

NOTE If Y is a closed subset of a Noetherian space X in which points are closed (i.e. T_0) then $\dim Y = 0$ iff. Y is a finite collection of points.

PROOF Since $Y \neq \emptyset$ ($\dim \emptyset = -1$) by (1.5) $Y = Y_1 \cup \dots \cup Y_n$ with each Y_i closed and irreducible. (hence $\neq \emptyset$). Let $x_i \in Y_i$. Then by assumption $\{x_i\} \subseteq Y_i$ is closed and irreducible, implying $Y_i = \{x_i\}$ since $\dim Y = 0$. Conversely if $Y = \{x_1, \dots, x_n\}$ then any subset of Y is closed, so clearly $\dim Y = 0$. $\square$

One may expect that for an affine variety $Y \subseteq \mathbb{A}^n$ $(n \geqslant 2)$ $\dim Y = n-2$ iff. $I(Y)$ can be generated by two elements. Ex 1.11 shows this is not so.

NOTE (Primary components) Let Y be a nonempty closed subset of $\mathbb{A}^n$, $\mathfrak{a} = I(Y)$. Since $k[x_1, \dots, x_n]$ is Noetherian and $\mathfrak{a}$ is radical, a primary decomposition $\mathfrak{a} = \mathfrak{q}_1 \cap \dots \cap \mathfrak{q}_n$ becomes $\mathfrak{a} = r(\mathfrak{a}) = \bigcap r(\mathfrak{q}_i) = \mathfrak{p}_{i_1} \cap \dots \cap \mathfrak{p}_{i_m}$ where $\mathfrak{p}_{i_1}, \dots, \mathfrak{p}_{i_m}$ are the minimal prime ideals containing $\mathfrak{a}$. Then

$$Y = Z(\mathfrak{a}) = Z(\mathfrak{p}_{i_1} \cap \dots \cap \mathfrak{p}_{i_m})$$
$$= Z(\mathfrak{p}_{i_1}) \cup \dots \cup Z(\mathfrak{p}_{i_m})$$

And this decomposition is s.t. $Z(\mathfrak{p}_{i_j}) \nsubseteq Z(\mathfrak{p}_{i_k})$ $j \neq k$, so this the unique decomposition of Y into its irreducible components. So the irred. components of Y are Z of the minimal primes of $I(Y)$.

NOTE As a consequence of the Note on dimension, if Y is an affine variety of dimension 0 is a point. Since the converse is obvious affine variety of dimension $0 \Leftrightarrow$ point.

Conics in $\mathbb{A}^2$ and $\mathbb{P}^2$

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Let k be a field with characteristic $\neq 2$ (we will make comments on the char$k = 2$ case at the end). A *conic* in $\mathbb{A}^2$ is the locus of an irreducible quadratic polynomial $f(x, y) \in k[x, y]$. Any such polynomial has the form

$$f(x, y) = ax^2 + 2bxy + cy^2 + dx + ey + g \tag{1}$$

where not all of a, b, c are zero. The *discriminant* $D(f)$ of f is $b^2 - ac$.

We claim that every conic in $\mathbb{A}^2$ is isomorphic as a variety to either $y - x^2$ or $xy - 1$. Since affine varieties are isomorphic if and only if their coordinate rings are isomorphic, it suffices to show for any irreducible quadratic $f(x, y)$ that $k[x, y]/(f)$ is isomorphic as a k-algebra to one of the following:

$$k[x, y]/(y - x^2) \qquad k[x, y]/(xy - 1)$$

The technique is to study automorphisms ϕ of the k-algebra $k[x, y]$, since a polynomial g is irreducible iff. $\phi(g)$ is irreducible and $k[x, y]/(g)$ is isomorphic as a k-algebra to $k[x, y]/(\phi(g))$. So to prove our claim for some irreducible quadratic f, it would suffice to produce an automorphism of $k[x, y]$ mapping f to either $y - x^2$ or $xy - 1$.

We now introduce the three classes of maps we will need:

1. For $0 \neq u \in k$, the map $k[x, y] \longrightarrow k[x, y]$ defined by $h(x, y) \mapsto uh(x, y)$ is not an automorphism, but clearly $k[x, y]/(g) = k[x, y]/(ug)$.

2. For $l, m \in k$, the automorphism defined by $x \mapsto x + l, y \mapsto y + m$;

3. For an invertible matrix

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in M_2(k)$$

   the automorphism defined by

$$\begin{aligned} x &\mapsto a_{11}x + a_{12}y \\ y &\mapsto a_{21}x + a_{22}y \end{aligned}$$

Note that $D(uf) = u^2 D(f)$, $D(f(x + l, y + m)) = D(f)$ and if ϕ is an automorphism of the third type defined by a matrix A, then $D(\phi(f)) = D(f)(detA)^2$. All of these are easily checked by expanding the determinants by hand. Hence the property of the discriminant being zero is invariant under these maps.

1 Case $D(f) = 0$

Let $f(x, y)$ be an irreducible quadratic polynomial. First we deal with the case $D(f) = b^2 - ac = 0$. By assumption k is algebraically closed, so there are solutions in k to the equations $x^2 = a, x^2 = c$ and $x^2 = -1$. Pick solutions and call them $\sqrt{a}, \sqrt{c}, i$. Since b is a solution to the equation $x^2 = (\sqrt{a}\sqrt{c})^2$ we must have either $b = \sqrt{a}\sqrt{c}$ or $b = -\sqrt{a}\sqrt{c}$. In the latter case we may replace $\sqrt{c}$ by $-\sqrt{c}$, so we assume henceforth that $b = \sqrt{a}\sqrt{c}$. Then we can write

$$f(x, y) = (\sqrt{a}x + \sqrt{c}y)^2 + dx + ey + g$$

Let us consider what restrictions are placed on the coefficients by requiring that f be irreducible.

Firstly we claim that e, d cannot both be zero. Suppose otherwise, so that $f(x, y) = (\sqrt{a}x + \sqrt{c}y)^2 + g$. Since not all of a, b, c are zero, we cannot have both $a = 0$ and $c = 0$. Assume for the moment that $a \neq 0$ (the case $c \neq 0$ is similar) and consider the automorphism determined by $x \mapsto \sqrt{a}x + \sqrt{c}y, y \mapsto y$, which is of the third type above since the associated matrix has determinant $\sqrt{a} \neq 0$. Applying the inverse of this automorphism to f we would find that the polynomial $x^2 + g$ is irreducible in $k[x, y]$. This is clearly impossible since k is algebraically closed. Hence one of e, d must be nonzero.

We assume $e \neq 0$ (the case $d \neq 0$ is similar) with no other assumptions about the coefficients. Consider the automorphism determined by $x \mapsto x, y \mapsto -\frac{d}{e}x + \frac{1}{e}y$. Applying the inverse of this automorphism to f we discover that the following polynomial is irreducible

$$\left(\frac{e\sqrt{a} - d\sqrt{c}}{e} x + \frac{\sqrt{c}}{e} y \right)^2 + y + g$$

Suppose $e\sqrt{a} - d\sqrt{c} = 0$. Then since a quadratic in y cannot be irreducible in $k[x, y]$, we would have $c = 0$. But then $e\sqrt{a} = d\sqrt{c} = 0$ would imply $a = 0$ (since $e \neq 0$). But of course a, c cannot both be zero, so this is a contradiction. Hence for any irreducible quadratic $f(x, y)$ with zero determinant, we must have $e\sqrt{a} - d\sqrt{c} \neq 0$.

This is fortunate, since it is implies that the morphism of k-algebras determined by

$$\begin{aligned} x &\mapsto \frac{\sqrt{a}}{i}x + \frac{\sqrt{c}}{i}y \\ y &\mapsto dx + ey \end{aligned}$$

is an automorphism. Applying the inverse of this automorphism to f and using another automorphism to exchange y and $y + g$ we end up with $y - x^2$, as required. Hence any conic in $\mathbb{A}^2$ defined by a polynomial with zero discriminant is isomorphic as a variety to $y - x^2$.

2 Case $D(f) \neq 0$

Let $f(x, y)$ be an irreducible quadratic polynomial, $D(f) = b^2 - ac \neq 0$. We claim that there is an automorphism of $k[x, y]$ mapping f to a polynomial of the following form

$$xy + d'x + e'y + g' \qquad d', e', g' \in k$$

If $a = c = 0$ then this is trivial, since then $b \neq 0$ and therefore we can simply apply the map $f \mapsto \frac{1}{2b} f$. So we assume that $a \neq 0$ (the case $c \neq 0$ is similar). One then checks the authenticity of the following algebraic manipulation:

$$\begin{aligned}
f(x,y) &= ax^2 + 2bxy + cy^2 + dx + ey + g \\
&= a\left(\left[x + \frac{b}{a}y^2\right]^2 + \frac{c}{a}y^2 - \frac{b^2}{a^2}y^2\right) + dx + ey + g \\
&= a\left(x + \frac{b}{a}y\right)^2 + \frac{ac - b^2}{a}y^2 + dx + ey + g \\
&= \left(\sqrt{a}x + \frac{b}{\sqrt{a}}y\right)^2 + \left(\sqrt{\frac{ac-b^2}{a}}y\right)^2 + dx + ey + g \\
&= \left(\sqrt{a}x + \left(\frac{b}{\sqrt{a}} + \sqrt{\frac{ac-b^2}{a}}i\right)y\right)\left(\sqrt{a}x + \left(\frac{b}{\sqrt{a}} - \sqrt{\frac{ac-b^2}{a}}i\right)y\right) \\
&\quad + dx + ey + g
\end{aligned}$$

Since $b^2 - ac \neq 0$ the following defines an automorphism of the third type:

$$\begin{aligned}
x &\mapsto \sqrt{a}x + \left(\frac{b}{\sqrt{a}} + \sqrt{\frac{ac-b^2}{a}}i\right)y \\
y &\mapsto \sqrt{a}x + \left(\frac{b}{\sqrt{a}} - \sqrt{\frac{ac-b^2}{a}}i\right)y
\end{aligned}$$

Applying the inverse of this automorphism to the expression for $f(x, y)$ obtained above, we see that f is mapped to a polynomial of the form $xy + d'x + e'y + g'$ for coefficients $d', e', g' \in k$, as claimed.

So to complete the proof, we need to show how to map any polynomial of the form $xy + d'x + e'y + g'$ to $xy - 1$ under an automorphism of $k[x, y]$. Firstly, write

$$xy + d'x + e'y + g' = (x + e')(y + d') + g - e'd'$$

And replace $x + e'$ by x and $y + d'$ by y to reduce to the case $xy + g - e'd'$. Since this polynomial is irreducible, $g - e'd' \neq 0$. Let $u \in k$ be such that $u(g - e'd') = -1$. Then we apply the map $h \mapsto uh$ and consider the polynomial $(ux)y - 1$. Using the automorphism of the third type $x \mapsto \frac{1}{u}x, y \mapsto y$ we complete the chain of automorphisms and arrive at $xy - 1$. Hence any conic in $\mathbb{A}^2$ defined by a polynomial with nonzero discriminant is isomorphic as a variety to $xy - 1$.

3 Conics in $\mathbb{P}^2$

Let k be any field (the following works fine in characteristic 2). A conic C in $\mathbb{P}^2$ is the locus of an irreducible quadratic homogenous polynomial $f(x, y, z) \in k[x, y, z]$ which has the form

$$f(x, y, z) = ax^2 + by^2 + cz^2 + dxy + eyz + gxz \tag{2}$$

where not all of a, b, c are zero. We claim that every such conic C is isomorphic as a variety to $\mathbb{P}^1$.

By Exercise 2.8 the locus $Y = Z(f)$ has dimension 1 and is therefore an infinite set. Pick any two distinct points $P, Q \in Y$ and let L be the unique line passing through these two points (see our Section 2 solutions for more detail). Pick a point $T \in Y$ not on L (the polynomial defining L is linear so its locus cannot contain Y). Recall that points P, Q, T are *collinear* in $\mathbb{P}^2$ if there is a nonzero linear polynomial $mx + ny + pz$ which admits P, Q, T as solutions. Writing

$$P = (p_1, p_2, p_3),\ Q = (q_1, q_2, q_3),\ T = (t_1, t_2, t_3)$$

the points P, Q, T are collinear if and only if the matrix

$$A = \begin{pmatrix} p_1 & q_1 & t_1 \\ p_2 & q_2 & t_2 \\ p_3 & q_3 & t_3 \end{pmatrix}$$

has row rank < 3 which is iff. it has column rank < 3 which is iff. the tuples P, Q, T are linearly dependent in $\mathbb{A}^3$. Since L is the *unique* line passing through P, Q and $T \notin L$, we conclude that the tuples P, Q, T are linearly independent in $\mathbb{A}^3$ and so the matrix A is invertible.

The automorphism $\varphi : \mathbb{P}^2 \longrightarrow \mathbb{P}^2$ determined by A^{-1} has the following property:

$$\varphi(P) = (1, 0, 0), \quad \varphi(Q) = (0, 1, 0), \quad \varphi(T) = (0, 0, 1)$$

The morphism φ maps conics to conics, so C is isomorphic to a conic in $\mathbb{P}^2$ containing $(1, 0, 0), (0, 1, 0), (0, 0, 1)$. Considering Equation 2 we see that any such conic is defined by a polynomial of the form

$$d'xy + e'yz + g'xz$$

We can use another automorphism to absorb the coefficients d', e', g' into x, y, z respectively, and so every conic in $\mathbb{P}^2$ is isomorphic to the variety $xy + yz + xz$. To complete the proof it suffices to show that one particular conic is isomorphic to $\mathbb{P}^1$.

The 2-uple embedding $\rho : \mathbb{P}^1 \longrightarrow \mathbb{P}^2$ gives an isomorphism of $\mathbb{P}^1$ with the variety $Im\rho$. We claim that $Im\rho$ is equal to the conic $xz - y^2$. Since $\rho(a, b) = (a^2, ab, b^2)$ one inclusion is clear. In the other direction, suppose $(e, f, g) \in Z(xz - y^2)$. Then $eg = f^2$ and if $f = 0$ either $e = 0$ or $g = 0$. In either case $\rho(\sqrt{e}, \sqrt{g}) = (e, f, g)$.

If $f \neq 0$ then we can assume $f = 1$, and $eg = 1$ implies both e and g are nonzero. Let $\sqrt{e}, \sqrt{g}$ be such that $\sqrt{e}\sqrt{g} = 1$. Then once again $\rho(\sqrt{e}, \sqrt{g}) = (e, f, g)$. Hence $Im\rho = Z(xz - y^2)$, which shows that $xz - y^2$ is isomorphic to $\mathbb{P}^1$. Hence any conic in $\mathbb{P}^2$ is isomorphic to $\mathbb{P}^1$.

4 Affine Conics in Characteristic 2

Let k be a field with $\mathrm{char} k = 2$ and let $f(x, y) \in k[x, y]$ be an irreducible quadratic polynomial

$$f(x, y) = ax^2 + bxy + cy^2 + dx + ey + g$$

If $b = 0$ then we can write $f(x, y) = (\sqrt{a}x + \sqrt{c}y)^2 + dx + ey + g$ and the proof in the section for affine conics with $D(f) = 0$ carries through unchanged to show that the conic defined by f is isomorphic to $y - x^2$.

Whereas if $b \neq 0$ then.... WHAT?

NOTE Intersection of Affine Conics and Lines

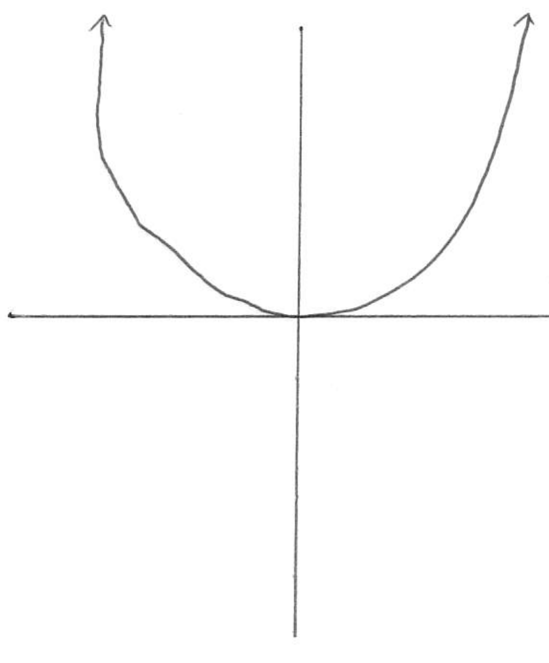
Let $C \subseteq \mathbb{A}^2$ be a conic and $L \subseteq \mathbb{A}^2$ a line

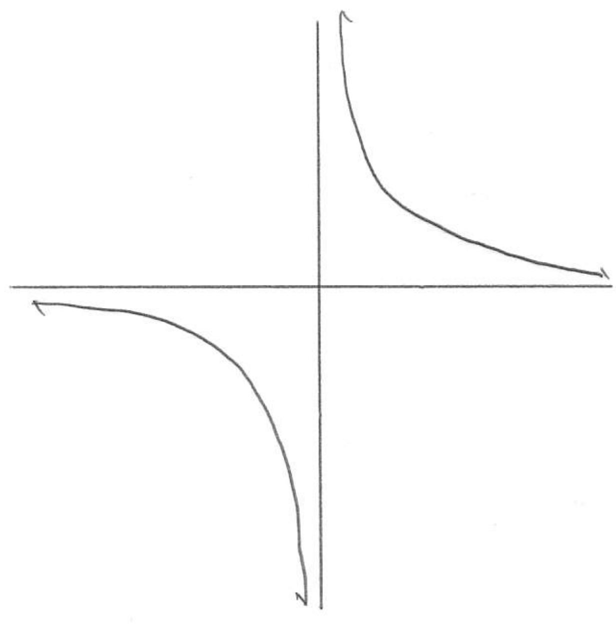
$$C: \ ax^2 + 2bxy + cy^2 + dy + ex + g$$
$$L: \ ax + by + c$$

Let $\varphi: \mathbb{A}^2 \longrightarrow \mathbb{A}^2$ be an automorphism carrying C to $y - x^2$ or $xy - 1$. By construction φ corresponds to a sequence of automorphisms whose associated maps $k[x,y] \longrightarrow k[x,y]$ are $h \longmapsto uh$, $x \longmapsto x+e$ $y \longmapsto y+m$, or $x \longmapsto a_{11}x + a_{12}y$ $y \longmapsto a_{21}x + a_{22}y$. All these operations preserve linear polynomials, so $\varphi(L)$ is a line. Moreover

$$C \cap L = \varphi^{-1}(\varphi(C)) \cap \varphi^{-1}(\varphi(L))$$
$$= \varphi^{-1}(\varphi(C) \cap \varphi(L))$$

So to count $C \cap L$ it suffices to look at $xy - 1$ and $y - x^2$.





The intersection of $y - x^2$ and a line $y = ax + b$ is $-x^2 + ax + b = 0$ which has precisely two solutions in k (alg. closed). Since we may have a double root, in the case where $D(c) = 0$,

$$|C \cap L| = \{1, 2\}$$

The intersection of $xy - 1$ and $y - ax - b$ is $x(ax+b) = 1 \iff ax^2 + bx = 1$ which again implies for $D(c) \neq 0$

$$|C \cap L) = \{1, 2\}$$

$\boxed{\text{Q1.1}}$ (a) Let $Y = V(y - x^2)$. We claim that $A(Y) \cong k[x]$. But since $A(Y) = k[x,y]/\sqrt{(y-x^2)}$ we map $x \mapsto x$, $y \mapsto x^2$, then this is well-defined and onto. Considering $k[x,y] = k[x][y]$ and the usual yoga, any $f(x,y) \in k[x,y]$ can be written $f(x,y) = g(x,y)(y-x^2) + f(x,x^2)$ so that the map is also injective.

(b) Let $Z = V(xy - 1)$. Then $A(Z)$ is $k[x,y]/(xy-1) \cong k[x,x^{-1}]$. If there were an isomorphism $A(Y) \cong k[z]$, it would send x, x^{-1} both to units, and hence into k. But then the image of the map would be in k.

(c) Let $f(x,y) \in k[x,y]$ be an irreducible quadratic polynomial. This is a standard result – this proof is largely from some notes on the web. Suppose

$$f(x,y) = ax^2 + 2bxy + cy^2 + dx + ey + g$$

with all coefficients in k, and not all of a, b, c zero. We define $D(f) = b^2 - ac \in k$. We first prove that $D(f) = 0$ or $D(f) \neq 0$ is preserved by the following operations on the polynomial f:

(i) $f \mapsto uf$ $\quad u \neq 0$ in k

(ii) $f(x,y) \longmapsto f(x+l, y+m)$ $\quad l, m \in k$

(iii) $f(x,y) \longmapsto f(A\binom{x}{y})$, that is under $\binom{x}{y} \mapsto A\binom{x}{y}$ for any invertible matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ over k.

And furthermore the affine coordinate ring $k[x,y]/(f)$ is preserved (up to isomorphism) by these operations. The proofs:

(i) $D(uf) = (bu)^2 - acu^2 = D(f)u^2$

(ii) When calculating $D(f)$, only the coefficients of x^2, xy and y^2 matter. But the substitution $x \mapsto x + l$, $y \mapsto y + m$ does not change these coefficients.

(iii) This is the substitution $x \mapsto a_{11}x + a_{12}y$, $y \mapsto a_{21}x + a_{22}y$. As in (ii), the substitution enacted on the lower degree terms $(x, y, 1)$ will have no effect on the value of D. Hence, assume $f(x,y) = ax^2 + 2bxy + cy^2$. Now note

$$ax^2 + 2bxy + cy^2 = (x\ y)\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

[One can check – and I have – that $D(f') = D(f)\det A^2$ by hand]

and $D(f) = -\det\begin{pmatrix} a & b \\ b & c \end{pmatrix}$. So under $(x\ y)^t \mapsto A(x\ y)^t$ we obtain

$$(x\ y)A^t\begin{pmatrix} a & b \\ b & c \end{pmatrix}A\begin{pmatrix} x \\ y \end{pmatrix}$$

and hence $D(f') = \det A^t D(f) \det A$. But A is invertible, so $D(f')$ is a nonzero constant multiple of $D(f)$ (f' being the transformed polynomial).

Now, $k[x,y]/(uf) = k[x,y]/(f)$, so the coordinate ring is clearly stable under (i). Under (ii), we have an isomorphism

$$k[x,y]/(f(x+l, y+m)) \longrightarrow k[x,y]/(f)$$

$$g(x,y) + (f(x+l, y+m)) \longmapsto g(x-l, y-m) + (f).$$

[Just note $x \mapsto x + l$, $y \mapsto y + h$ is an iso $k[x,y] \to k[x,y]$]

a linear transformation of the form (iii), $\binom{x}{y} \mapsto A\binom{x}{y}$ gives an isomorphism

$$k[x,y]/(f(A\tbinom{x}{y})) \longrightarrow k[x,y]/(f)$$

$$g(x,y) + (f(A\tbinom{x}{y})) \longmapsto g(A^{-1}\tbinom{x}{y}) + (f)$$

[Just note $x \mapsto a_{11}x + a_{12}y$, $y \mapsto a_{21}x + a_{22}y$ is inverse to using $B = A^{-1}$. Thus get automorphism of $k[x,y]$]

Now to finish the proof we show that any irreducible quadratic f can be transformed into one of the following cases using only operations (i), (ii) and (iii):

(a) $y - x^2$ $\quad$ if $D(f) = 0$

(b) $xy - 1$ $\quad$ if $D(f) \neq 0$

Suppose $D(f) = b^2 - ac = 0$. Then $f(x,y) = ax^2 + 2bxy + cy^2 + dx + ey + g = (\sqrt{a}x + \sqrt{c}y)^2 + dx + ey + g$, where the square roots exist since k is algebraically closed. Apply the transformations (using both (ii) and (iii))

$$\sqrt{a}/i\, x + \sqrt{c}/i\, y \longmapsto x$$
$$dx + ey + g \longmapsto y$$

[One means the inverse of $x \mapsto \sqrt{a}/i\,x + \sqrt{c}/i\,y$ etc. This needs to be checked to be invertible. See Additional ⊛]

Under this transformation, clearly $f \longmapsto y - x^2$. Now suppose $D(f) = b^2 - ac \neq 0$. Then

$$\begin{aligned}
f(x,y) &= ax^2 + 2bxy + cy^2 + dx + ey + g \\
&= a\left(\left[x + \tfrac{b}{a}y\right]^2 + \tfrac{c}{a}y^2 - \tfrac{b^2}{a^2}y^2 \right) + dx + ey + g \\
&= a\left(x + \tfrac{b}{a}y\right)^2 + \frac{(ac - b^2)}{a}y^2 + dx + ey + g \\
&= \left(\sqrt{a}x + \frac{b}{\sqrt{a}}y\right)^2 + \left(\sqrt{\frac{ac-b^2}{a}}\,y\right)^2 + dx + ey + g \\
&= \left\{ \sqrt{a}x + \left(\frac{b}{\sqrt{a}} + \sqrt{\frac{ac-b^2}{a}}\,i \right) y \right\}\left\{ \sqrt{a}x + \left(\frac{b}{\sqrt{a}} - \sqrt{\frac{ac-b^2}{a}}\,i \right) y \right\} + dx + ey + g
\end{aligned}$$

Take the transform (iii) s.t. (one checks these transformations are invertible, which follows since $b^2 - ac \neq 0$).

$$\sqrt{a}x + \left(\frac{b}{\sqrt{a}} + \sqrt{\frac{ac-b^2}{a}}\,i\right)y \longmapsto x$$
$$\sqrt{a}x + \left(\frac{b}{\sqrt{a}} - \sqrt{\frac{ac-b^2}{a}}\,i\right)y \longmapsto y$$

which takes f to $xy + d'x + e'y + g$ for some d', e' in k. But this can be written $(x+e')(y+d') + g - e'd'$ so take the transform $x + e' \mapsto x$, $y + d' \mapsto y$ which is (ii). Then f has become $xy + g - e'd'$. Since f is irreducible (and this is preserved by operations since $k[x,y]$ UFD irred $\Leftrightarrow$ prime and we have shown $k[x,y]/(f)$ is preserved up to iso) $g - e'd'$ is not zero, so there is u in k s.t. $u(g - e'd') = -1$. Then by (i) f becomes $(ux)y - 1$ and by (iii) with $ux \mapsto x$, $y \mapsto y$ we have $f = xy - 1$. So we have shown that any irreducible quadratic $f \in k[x,y]$ is s.t. $k[x,y]/(f)$ is isomorphic to $k[x] = A(Y)$ or $k[x,y]/(xy-1) = A(Z)$. □

Q1.2 The Twisted Cubic. Let $Y \subseteq \mathbb{A}^3$ be $Y = \{(t, t^2, t^3) \mid t \in k\}$. Clearly $Y \subseteq V(z - x^3, y - x^2)$, and the converse is clear as well. (We now show that $(z - x^3, y - x^2)$ is prime, by constructing an isomorphism

$$k[x,y,z]/(z - x^3, y - x^2) \longrightarrow k[u]$$

Let $\varphi : k[x,y,z] \longrightarrow k[u]$ be $x \mapsto u$, $y \mapsto u^2$, $z \mapsto u^3$. Clearly φ is onto and $(z - x^3, y - x^2) \subseteq \ker\varphi$. Suppose $P(x,y,z) \in \ker\varphi$. Divide $y - x^2$ into P as polynomials in y to find $P(x,y,z) = Q(y - x^2) + R(x,z)$. Then $\varphi(P) = 0$ implies $\varphi(R) = 0$, so let us divide $z - x^3$ into R as polynomials in z, so $R = Q'(z - x^3) + R'(x)$. Then $\varphi(R') = 0$ implies $R' = 0$, so that $P = Q(y - x^2) + Q'(z - x^3) \in (y - x^2, z - x^3)$. Hence $(y - x^2, z - x^3)$ is prime, so Y is an affine variety whose dimension is the Krull dimension of $A(Y) = k[u]$ which is 1.

Q1.3 Let $Y \subseteq \mathbb{A}^3$ be $V(x^2 - yz, xz - x)$. Then $(u,v,w) \in Y$ implies $u(w-1) = 0$, so

(i) $u = 0$, so that $vw = 0$, so $v = 0$ or $w = 0$, hence all triples $(0, v, 0), (0, 0, w)$

(ii) $w = 1$, so that $u^2 = v$, so all tuples $(u, u^2, 1)$.

Hence $Y = V(x,z) \cup V(x,y) \cup V(y^2 - x, z - 1)$. Here $k[x,y,z]/(x,z) = k[y]$, $k[x,y,z]/(x,y) = k[z]$ and $k[x,y,z]/(y^2 - x, z - 1) \cong k[t]$ via $\varphi : k[x,y,z] \longrightarrow k[t]$, $x \mapsto t$, $y \mapsto t^2$, $z \mapsto 1$. Clearly onto with $(y^2 - x, z - 1)$ in the kernel, and if $\varphi(P(x,y,z)) = 0$, let $P = Q(y - x^2) + R(x,z)$, then $\varphi(R) = 0$, say $R = Q'(z-1) + R'(x)$. Then $R' = 0$ so that $P = Q(y - x^2) + Q'(z-1) \in (y^2 - x, z - 1)$, and so we induce the desired isomorphism. Hence all of these sets are affine varieties of dimension 1.

Q1.4 The set $V(x - y) = \{(t,t) \mid t \in k\}$ is closed in $\mathbb{A}^2$ with the Zariski topology, but it cannot be closed in the product topology because to include the diagonal a product of two closed sets would have to contain all of $\mathbb{A}^2$.

Q1.5 If B is the affine coordinate ring for some algebraic set in $\mathbb{A}^n$, say $B = A(Y)$, then $B = k[x_1, \ldots, x_n]/I(Y)$, where $I(Y)$ is radical, and hence B is a f.g k-algebra with no nilpotents. Conversely, $k[x]$ is a generator for k-alg, so any finitely generated object is a quotient of some $k[x]^{(n)} = \bigoplus_{i=1}^{n} k[x] = \bigotimes_{i=1}^{n} k[x] = k[x_1, \ldots, x_n]$, and if B is reduced then the kernel must also be radical, so that it corresponds to some algebraic set in $\mathbb{A}^n$, for which $B = k[x_1, \ldots, x_n]/I(Y)$ must be the coordinate ring.

Q1.6 Let X be an irreducible space, $Z \subseteq X$ nonempty and open. Then $X = \bar{Z} \cup (X \setminus Z)$ so since Z is nonempty, it is dense. Similarly if $Z = Y \cup Y'$, Y and Y' closed in Z (say $Y = Z \cap \hat{Y}$, $Y' = Z \cap \hat{Y}'$; $\hat{Y}$ and $\hat{Y}'$ closed in X) then we have $X = (X \setminus Z) \cup (\hat{Y} \cup \hat{Y}')$, so that $X = \hat{Y} \cup \hat{Y}'$ and hence wlog $X = \hat{Y}$. This implies $Z = Y$. Hence Z is irreducible and dense.

Suppose $Y \subseteq X$ is irreducible under the induced topology (X is now arbitrary). Let $\bar{Y}$ be the closure of Y, and suppose $\bar{Y} = Z_1 \cup Z_2$ for two closed subsets Z_1, Z_2 of $\bar{Y}$. Then as $Z_1 \cap Y$ and $Z_2 \cap Y$ are closed in Y and Y is irreducible, Y must be contained in (wlog) Z_1. But by def^N of closure, we have $Z_1 = \bar{Y}$.

Q1.7 (a) (i) $\Rightarrow$ (ii) Let $\mathcal{J}$ be a nonempty family of closed subsets – let $Q \in \mathcal{J}$. If Q is not minimal in $\mathcal{J}$, say $Q' \subset Q$. Then either Q' is minimal, etc. We produce a descending chain $Q \supset Q' \supset Q'' \supset \cdots$ which must terminate at a minimal element of $\mathcal{J}$. (i) $\Leftrightarrow$ (iii) Any decreasing chain of closed subsets produces the increasing chain of open complements, which halts iff. the original chain does. (ii) $\Leftrightarrow$ (iv) A closed family is equivalent to the open family of complements, and the former has a minimal element iff. the latter has a maximal. (i) $\Rightarrow$ (i). Let $Q_0 \supset Q_1 \supset \cdots$ be a descending chain of closed subsets. The family $\{Q_i\}$ must have a minimal element Q_n, and hence $Q_k = Q_n$ $\forall k \geq n$.

(b) Let U have an open cover $\{U_i\}_{i \in I}$. Since any strictly increasing chain of open subsets of X must terminate, if we pick $U_{j_0} \in \{U_i\}$, $U_{j_1} \in \{U_i\}$ etc and form $U_{j_0} \subseteq U_{j_0} \cup U_{j_1} \subseteq \cdots$ if $\{U_i\}$ contains no finite subcover, we can form a strictly increasing, infinite sequence of this form, which is a contradiction.

(c) Let $Y \subseteq X$ be a subspace and suppose $Q_0 \supset Q_1 \supset \cdots$ is a descending chain of closed subsets in Y, with say $Q_i = Y \cap P_i$, $P_i \subseteq X$ closed. Then if we assume P_i is the intesection of all those closed subsets of X which meet Y in Q_i, we have $P_0 \supset P_1 \supset \cdots$, since $(P_i \cap P_{i-1}) \cap Y = Q_{i-1} \cap Q_i = Q_i$, so that $P_i \cap P_{i-1} = P_i$; and hence $P_i \subseteq P_{i-1}$. If we were to have $P_i = P_{i-1}$, then $Q_i = Y \cap P_i = Y \cap P_{i-1} = Q_{i-1}$, which is false. But X is noethenan, so the chain $P_0 > P_1 \supset \cdots$ terminates, say $P_n = P_k$ $\forall k \geq n$. Then also $Q_n = Q_k$ $\forall k \geq n$, so Y is nuethenan.

(d) Let X be Noethenan and Hausdorff. If $V \subseteq X$ is irreducible closed with $x, y \in V$, then let $U_1, U_2 \subseteq X$ be open and $x \in U_1$, $y \in U_2$ $U_1 \cap U_2 = \emptyset$. Then $X = U_1^c \cup U_2^c$ and so $V = (U_1^c \cap V) \cup (U_2^c \cap V)$ as the union of proper closed subsets. Hence V is a singleton set. But by Prop 1.5 this implies that X is finite. Since a subset is irreducible closed iff. it is a singleton (equiv. singletons are open since if the points are $x_1, \ldots, x_n$ make open sets U_{i} each i seperating x_1 and x_i – then $\bigcap_i U_{i} = \{x_1\}$) Hence topology is discrete.

be a hypersurface in $\mathbb{A}^n$, f irreducible in $k[x_1, \ldots, x_n]$. Now $V((f) + \mathfrak{p}) = V(f) \cap V(\mathfrak{p}) = Y \cap H$ so that $I(Y \cap H) = \sqrt{(f) + \mathfrak{p}}$. An irreducible component of $Y \cap H$ is thus a minimal prime of $(f) + \mathfrak{p}$. Which is a prime $\mathfrak{q} \supseteq \mathfrak{p}$ minimal over (f) in $A/\mathfrak{p}$. But by 1.11A all such primes have height 1 in $A/\mathfrak{p}$, and since $\operatorname{ht}\mathfrak{q} + \dim A/\mathfrak{q} = \dim A/\mathfrak{p}$, $\dim A/\mathfrak{q} = r - 1$, as required.

Q1.9 Let $\mathfrak{a} \subseteq k[x_1, \ldots, x_n]$ be $\mathfrak{a} = (f_1, \ldots, f_r)$, a proper ideal. Then we have $(f_1) \subseteq (f_1, f_2) \subseteq \cdots \subseteq (f_1, \ldots, f_r) = \mathfrak{a}$ and hence

$$V(f_1) \supseteq V(f_1, f_2) \supseteq \cdots \supseteq V(\mathfrak{a})$$

and hence $\dim V(f_1, \ldots, f_{i+1}) \leq \dim V(f_1, \ldots, f_i)$.

f can be any polynomial, not necc. irreducible

Q1.8 Let Y be an affine variety of dimension r in $\mathbb{A}^n$ (so $0 \leq r \leq n$), and let $H = V(f)$ be a hypersurface in $\mathbb{A}^n$. We study the closed set $Y \cap H$ in $\mathbb{A}^n$. Let $\mathfrak{p} = I(Y)$.

$\boxed{\text{Case I}}$ $Y \subseteq H$: Clearly $Y \cap H = Y$ so $\dim Y \cap H = r$.

$\boxed{\text{Case II}}$ $Y \cap H = \emptyset$: Since $V((f) + \mathfrak{p}) = V(f) \cap V(\mathfrak{p}) = Y \cap H$ we have $Y \cap H = \emptyset$ iff. $(f) + \mathfrak{p} = A$. ($\sqrt{\mathfrak{a}} = A \Rightarrow \mathfrak{a} = A$)

$\boxed{\text{Case III}}$ $Y \cap H$ is nonempty and not Y: So $(f) + \mathfrak{p} \neq A$ and consequently f is neither a zerodivisor or a unit in $A/\mathfrak{p}$. Since $(f) + \mathfrak{p}$ is proper the irreducible components of $Y \cap H$ correspond to the minimal primes over $\sqrt{(f) + \mathfrak{p}}$ which are primes $\mathfrak{q}$ minimal over (f) in $A/\mathfrak{p}$. By (1.11A) such a prime has height 1 in $A/\mathfrak{p}$. Hence $1 + \operatorname{coht}\mathfrak{q} = \dim A/\mathfrak{p} = r$, so $\dim A/\mathfrak{q} = r - 1$. Hence every irreducible component of $Y \cap H$ has dimension $r - 1$.

<u>NOTE</u> If $r = 0$, Y is a point, so either Case I or II must prevail.

Q1.9 Let $\mathfrak{a} \subseteq A = k[x_1, \ldots, x_n]$ be an ideal which can be generated by r elements. By Krull's Generalised PID Theorem (see A&M notes) any prime minimal over $\mathfrak{a}$ has height $\leq r$. So

$\boxed{\text{Case I}}$ $\mathfrak{a} = A$. Then $Z(\mathfrak{a}) = \emptyset$.

$\boxed{\text{Case II}}$ $\mathfrak{a}$ is proper. Then an irreducible component of $\mathfrak{a}$ is $Z(\mathfrak{p})$, $\mathfrak{p}$ a minimal prime of $\mathfrak{a}$. Hence

$$\dim Z(\mathfrak{p}) = \dim A/\mathfrak{p} = n - \operatorname{ht}\mathfrak{p} \geq n - r$$

as required (using 1.8A).

NOTE Dimension and Q 1.8

Let $Y \subseteq \mathbb{A}^n$ be an affine variety of dimension r. Let $f_1, \dots, f_k$ be arbitrary polynomials. We claim that if $Z = Y \cap Z(f_1) \cap \cdots \cap Z(f_k)$ is nonempty, then every irreducible component of Z has dimension at least $r-k$. (The components may have different dimensions)

PROOF By induction on k. For $k=1$ either $Y \subseteq Z(f_1)$ and $\dim Y \cap Z(f_1) = r$ or $Y \nsubseteq Z(f_1)$ and every irreducible component has dimension $r-1$ ($Y \cap Z(f_1) \neq \emptyset$ by assumption) by Q1.8. So the result holds for $k=1$.

Assume the result is true for $k-1$. So by assumption $Y \cap Z(f_1) \cap \cdots \cap Z(f_k) \neq \emptyset$ which implies that $Y \cap Z(f_1) \cap \cdots \cap Z(f_{k-1}) \neq \emptyset$ and so if

$$Y \cap Z(f_1) \cap \cdots \cap Z(f_{k-1}) = Y_1 \cup \cdots \cup Y_n$$

is the irreducible decomposition then $\dim Y_i \geqslant r-(k-1)$ by the inductive hypothesis. So $Z = Y \cap Z(f_1) \cap \cdots \cap Z(f_k)$ is the union $Y_1 \cap Z(f_k) \cup \cdots \cup Y_n \cap Z(f_k)$. Some of these may be empty, but the nonempty ones have irreducible decompositions into a union of closed, irreducible sets with dimensions $\geqslant r-k$. Hence

$$Z = Q_1 \cup \cdots \cup Q_s$$

with the Q_i affine varieties of dimension $\geqslant r-k$. By throwing away redundant Q_i, we complete the proof. □

NOTE This is Q1.9

1.10 (a) Let Y be any subspace of a topological space X. If $S \subseteq Y$ is an irreducible closed subset of Y in the induced topology, then its closure in X is an irreducible closed subset $\bar{S} \subseteq X$, satisfying $\bar{S} \cap Y = S$. This works since if $\bar{S} = V \cup W$, with V and W closed in X, then $S = (V \cap Y) \cup (W \cap Y)$. Hence say $S = V \cap Y$, then $\bar{S} = V$.

If $Z_0 \subset Z_1 \subset \cdots \subset Z_r$ is a chain of irreducible closed subsets of Y, then we have $\bar{Z}_0 \subset \bar{Z}_1 \subset \bar{Z}_2 \subset \cdots \subset \bar{Z}_r$. Inclusion follows since $\bar{Z}_i \cap Y = Z_i \supseteq Z_{i-1}$ so that by def^N of closure $\bar{Z}_{i-1} \subseteq \bar{Z}_i$. If we were to have $\bar{Z}_i = \bar{Z}_{i+1}$, then $Z_i = \bar{Z}_i \cap Y = \bar{Z}_{i+1} \cap Y = Z_{i+1}$, which is a contradiction. Hence there is a chain of irreducible closed subsets of length r in X. The same argument implies that an infinite chain in Y produces an infinite chain in X. Hence $\dim Y \leq \dim X$.

(b) Let X be covered by $\{U_i\}$. We claim $\dim X = \sup \dim U_i$. By (a), $\dim U_i \leq \dim X \; \forall i$, so that $\sup \dim U_i \leq \dim X$. Let $Z_0 \subset Z_1 \subset Z_2 \subset \cdots$ be a strictly increasing chain of irreducible closed subsets of X. Since $\{U_i\}$ is a cover, let U_0 be s.t. $Z_0 \cap U_0 \neq \phi$. Then U_0 has a nonempty intersection with all Z_i, and by 1.6 these intersections are dense and irreducible. We certainly have $Z_i \cap U_0 \subseteq Z_{i+1} \cap U_0$, and since $U_0 \cap Z_{i+1}$ is dense in Z_{i+1}, and $Z_i \subset Z_{i+1}$ is a proper closed subset, we cannot have $Z_{i+1} \cap U_0 \subseteq Z_i$. That is, there is a proper inclusion $Z_i \cap U_0 \subset Z_{i+1} \cap U_0$. Hence we have produced a chain of irreducible closed subsets $Z_0 \cap U_0 \subset Z_1 \cap U_0 \subset \cdots$ of U_0, so that $\dim X \leq \sup \dim U_i$ and hence $\dim X = \sup \dim U_i$.

(c) Let $X = \{0,1\}$ with topology $\mathcal{J} = \{\phi, \{0\}, \{0,1\}\}$ so that ϕ, $\{1\}$ and X are closed and $\{1\}, X$ are irreducible closed. Let $U = \{0\}$. Clearly $\bar{U} = X$ so U is dense, and while $\{1\} \subset X$ shows $\dim X = 1$, $\{0\}$ contains no irreducible closed subsets at all, other than itself (we are interested only in the induced topology on $\{0\}$ when calculating dimension). Hence $\dim U = 0 < 1 = \dim X$.

(d) Let Y be a closed subset of an irreducible finite-dimensional space X. If $Z_0 \subset Z_1 \subset \cdots \subset Z_r$ is any chain of irreducible closed subsets of Y, it is also such a chain in X, and hence since X is irreducible $Z_0 \subset Z_1 \subset \cdots \subset Z_r \subset X$ is a chain of closed irreducibles in X. Hence the only way for $\dim Y = \dim X$ is to have $Y = X$.

(e) Since spectrums of Noetherian rings are noetherian spaces and Krull dimension = combinatorial dimension of spectrum, it clearly suffices to find a Noetherian ring of infinite dimension — this example is from Atiyah and Macdonald. Let $A = k[x_1, x_2, \ldots, x_n, \ldots]$, and let $m_1, m_2, \ldots$ be an increasing sequence of natural numbers so that $m_{i+1} - m_i > m_i - m_{i-1}$ for all $i > 1$. We have prime ideals $(x_{m_i+1}, \ldots, x_{m_{i+1}})$ {prime since $k[x_1, \ldots]/(x_{i_1}, \ldots, x_{i_n})$ is the polynomial ring in remaining x_j, hence domain}, and hence proper chains (devide $(x_{m_i+1}, \ldots, x_{m_{i+1}})$ by $\mathfrak{p}_i$).

$$\mathfrak{p}_i = (x_{m_i+1}, \ldots, x_{m_{i+1}}) \supset (x_{m_i+2}, \ldots, x_{m_{i+1}}) \supset \cdots \supset (x_{m_{i+1}}) \supset (0)$$

hence $\operatorname{ht} \mathfrak{p}_i = m_{i+1} - m_i$ (we can apply the Hauptidealsatz to see in the above the chain is maximal, but we only need ht $\geq$) Now A is not noetherian, but if $S = A - \bigcup_i \mathfrak{p}_i$, then $A[S^{-1}]$ is Noetherian (Ex. 9, Ch. 7, Atiyah–Macdonald) and in this ring $\mathfrak{p}_i$ has height $m_{i+1} - m_i$, so that $\dim A[S^{-1}] = \infty$, as required.

1.11 Let $Y \subseteq \mathbb{A}^3$ be given parametrically by $x = t^3, y = t^4, z = t^5$. Then $f(x,y,z) \in k[x,y,z]$ in $I(Y)$ implies that $f(\alpha^3, \alpha^4, \alpha^5) = 0$, $\forall \alpha \in k$. That is, letting $\varphi : k[x,y,z] \longrightarrow k[t]$ be induced by $x \mapsto t^3, y \mapsto t^4, z \mapsto t^5$, and letting $ev_\alpha : k[t] \longrightarrow k$ denote evaluation at $\alpha \in k$

$$\begin{array}{ccc} k[x,y,z] & \xrightarrow{\varphi} & k[t] \\ & & \downarrow ev_\alpha \\ & & k \end{array}$$

we note that $f(\alpha^3, \alpha^4, \alpha^5) = 0$ iff. $ev_\alpha \varphi(f) = 0$, so iff. $\varphi(f) \in \mathfrak{m}_\alpha$ (k alg. closed). But in $k[t]$, $\bigcap_{\alpha \in k} \mathfrak{m}_\alpha = 0$, so f is in $I(Y)$ iff. $\varphi(f) = 0$, so $I(Y) = \operatorname{Ker} \varphi$. Since $\operatorname{Im} \varphi = k[t^3, t^4, t^5]$ is a subring of $k[t]$ and is hence a domain, $I(Y)$ is prime. Now

$$\operatorname{ht} I(Y) + \dim \frac{k[x,y,z]}{I(Y)} = \dim k[x,y,z] = 3$$

Hence, since $\dim k[x,y,z]/I(Y) = \dim k[t^3, t^4, t^5] = k[t]$ (since $k[t]$ is an integral extension of $k[t^3, t^4, t^5]$) and $\dim k[t] = 1$, so $\operatorname{ht} I(Y) = 3 - 1 = 2$. ($k[t]$ is integral over $k[t^3, t^4, t^5]$ since it is generated as a $k[t^3, t^4, t^5]$-module by $1, t, t^2$).
Now we have to show that $I(Y)$ cannot be generated by two elements. Let

$$f(x,y,z) = \sum_{i,j,k} \alpha_{ijk} x^i y^j z^k \qquad (1)$$

Then a polynomial f as in (1) is in $I(Y)$ iff. $\varphi(f) = 0$, so iff. $\sum_{i,j,k} \alpha_{ijk} t^{3i+4j+5k} = 0$. Hence, iff. for each $n \geq 0$,

$$\sum_{\substack{i,j,k \geq 0 \\ \text{s.t. } 3i+4j+5k = n}} \alpha_{ijk} = 0 \qquad \text{for each } n \geq 0.$$

Let us consider some values of n:

$$\begin{array}{lll}
n=0 & (i,j,k)=(0,0,0) & \alpha_{0,0,0}=0 \\
n=1 & (i,j,k) \text{ does not exist} & \\
n=2 & (i,j,k) \text{ does not exist} & \\
n=3 & (i,j,k)=(1,0,0) & \alpha_{1,0,0}=0 \\
n=4 & (i,j,k)=(0,1,0) & \alpha_{0,1,0}=0 \\
n=5 & (i,j,k)=(0,0,1) & \alpha_{0,0,1}=0 \\
n=6 & (i,j,k)=(2,0,0) & \alpha_{2,0,0}=0 \\
n=7 & (i,j,k)=(1,1,0) & \alpha_{1,1,0}=0 \\
n=8 & (i,j,k)=(1,0,1) \text{ OR } (i,j,k)=(0,2,0) \text{ hence } & \alpha_{1,0,1}+\alpha_{0,2,0}=0. \\
n=9 & (i,j,k)=(3,0,0) \text{ OR } (i,j,k)=(0,1,1) \text{ hence } & \alpha_{3,0,0}+\alpha_{0,1,1}=0 \\
n=10 & (i,j,k)=(2,1,0) \text{ OR } (i,j,k)=(0,0,2) \text{ hence } & \alpha_{2,1,0}+\alpha_{0,0,2}=0 \\
n=11 & (i,j,k)=(1,2,0) \text{ OR } (i,j,k)=(2,0,1) \text{ hence } & \alpha_{1,2,0}+\alpha_{2,0,1}=0
\end{array}$$

For $n \geqslant 12$ we have to have at least $i+j+k \geqslant 3$, since $2\cdot 5=10<12$. Now, we can write for any $f \in k[x,y,z]$

$$f = \alpha_{0,0,0}+\alpha_{1,0,0}x+\alpha_{0,1,0}y+\alpha_{0,0,1}z+\alpha_{2,0,0}x^2+\alpha_{1,1,0}xy+\alpha_{1,0,1}xz+\alpha_{0,2,0}y^2$$
$$+\alpha_{3,0,0}x^3+\alpha_{0,1,1}yz+\alpha_{2,1,0}x^2y+\alpha_{0,0,2}z^2+\alpha_{1,2,0}xy^2+\alpha_{2,0,1}x^2z+\sum_{\substack{i,j,k \\ 3i+4j+5k\geqslant 12}}\alpha_{i,j,k}x^iy^jz^k$$

If $f \in I(Y)$, then, using the above we can simply to

$$f = \alpha_{1,0,1}(xz-y^2)+\alpha_{3,0,0}(x^3-yz)+\alpha_{2,1,0}(x^2y-z^2)+\alpha_{1,2,0}(xy^2-x^2z)+g \qquad (2)$$

where g is a polynomial whose monomials all have total degree $\geqslant 3$. Suppose $I(Y)=(g_1,g_2)$ could be generated by two polynomials g_1 and g_2. Then $g_1,g_2 \in I(Y)$ and so have expansions like (2). But $xz-y^2 \in (g_1,g_2)$ implies $xz-y^2=p_1g_1+p_2g_2 = p_1(\beta_{1,0,1}(xz-y^2)+\cdots)+p_2(\gamma_{1,0,1}(xz-y^2)+\cdots)$. We show that there is no way to produce p_1,p_2 and q_1,q_2 and m_1,m_2 s.t.

$$xz-y^2=p_1g_1+p_2g_2 \qquad (3)$$
$$x^3-yz=q_1g_1+q_2g_2 \qquad (4)$$
$$x^2y-z^2=m_1g_1+m_2g_2 \qquad (5)$$

We write $p_i=a_i+p_i'$, where $a_i \in k$ and p_i' has no constant term. Likewise $q_i=b_i+q_i'$, $m_i=c_i+m_i'$. Then, if

$$g_1 = \beta_{1,0,1}(xz-y^2)+\beta_{3,0,0}(x^3-yz)+\beta_{2,1,0}(x^2y-z^2)+g_1'$$
$$g_2 = \gamma_{1,0,1}(xz-y^2)+\gamma_{3,0,0}(x^3-yz)+\gamma_{2,1,0}(x^2y-z^2)+g_2' \qquad (6)$$

where g_1' and g_2' are such that each monomial has degree $\geqslant 3$. Then (3) becomes

$$xz-y^2=(a_1\beta_{1,0,1}+a_2\gamma_{1,0,1})(xz-y^2)+(a_1\beta_{3,0,0}+a_2\gamma_{3,0,0})(x^3-yz)+(a_1\beta_{2,1,0}+a_2\gamma_{2,1,0})(x^2y-z^2)$$
$$+a_1g_1'+a_2g_2'+p_1'g_1+p_2'g_2.$$

since we have seperated monomials of order <3 and $\geqslant 3$ (with the exception of x^2y, but by def$^{\sim}$ in (2) no monomial occurs twice, so x^2y cannot occur in a_1g_1' or a_2g_2', and there is no monomial in g_1 or g_2 of lower degree which divides x^2y, so it cannot occur in $p_1'g_1$ or $p_2'g_2$ either), we can equate coefficients to find

$$a_1\beta_{1,0,1}+a_2\gamma_{1,0,1}=1$$
$$a_1\beta_{3,0,0}+a_2\gamma_{3,0,0}=0 \qquad a_1g_1'+a_2g_2'+p_1'g_1+p_2'g_2=0 \qquad (7)$$
$$a_1\beta_{2,1,0}+a_2\gamma_{2,1,0}=0$$

doing the same with $x^3 - yz$ and $x^2y - z^2$ we find

$$\begin{aligned} b_1\beta_{1,0,1} + b_2\gamma_{1,0,1} &= 0 \\ b_1\beta_{2,1,0} + b_2\gamma_{2,1,0} &= 1 \\ b_1\beta_{3,0,0} + b_2\gamma_{3,0,0} &= 0 \end{aligned} \qquad (8) \qquad\qquad \begin{aligned} c_1\beta_{1,0,1} + c_2\gamma_{1,0,1} &= 0 \\ c_1\beta_{2,1,0} + c_2\gamma_{2,1,0} &= 0 \\ c_2\beta_{3,0,0} + c_2\gamma_{3,0,0} &= 1 \end{aligned} \qquad (9)$$

and (7), (8), (9) give us our contradiction, since they imply (supposing $\gamma_{3,0,0}, \gamma_{2,1,0}, \gamma_{1,0,1}$ nonzero)

$$\frac{\beta_{3,0,0}}{\gamma_{3,0,0}} = -\frac{a_2}{a_1} = \frac{-b_2}{b_1} \qquad \frac{\beta_{2,1,0}}{\gamma_{2,1,0}} = \frac{-c_2}{c_1} = -\frac{a_2}{a_1} \qquad \frac{\beta_{1,0,1}}{\gamma_{1,0,1}} = \frac{-c_2}{c_1} = \frac{-b_2}{b_1} \qquad (10)$$

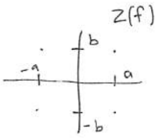
hence all of these quotients are equal – say to $\alpha \in k$. But then $\beta_{1,0,1} = \alpha\gamma_{1,0,1}$ and $-a_2 = \alpha a_1$ show, using (7)

$$1 = a_1\beta_{1,0,1} + a_2\gamma_{1,0,1} = a_1\alpha\gamma_{1,0,1} + a_2\gamma_{1,0,1} = (a_1\alpha + a_2)\gamma_{1,0,1} = 0$$

Using (7), (8), (9) and (10), this contradiction will work whenever more than one of $\gamma_{3,0,0}, \gamma_{2,1,0}, \gamma_{1,0,1}$ is nonzero (resp. using the reciprocals of (10), more than one of $\beta_{3,0,0}, \beta_{2,1,0}, \beta_{1,0,1}$ nonzero). But the fact that (3), (4), (5) hold (hence (7), (8), (9)) means that $\beta_{1,0,1}$ and $\gamma_{1,0,1}$ (resp. $\beta_{2,1,0}$ and $\gamma_{2,1,0}$, $\beta_{3,0,0}$ and $\gamma_{3,0,0}$) cannot be zero at the same time. Hence we have our contradiction, since for example $\beta_{2,1,0} = \beta_{1,0,1} = 0$ implies both $\gamma_{2,1,0}$ and $\gamma_{1,0,1}$ are nonzero. Hence $I(Y)$ cannot be generated by 2 elements.

$\boxed{\text{Q1.12}}$ We are looking for $f \in \mathbb{R}[x,y]$ irreducible, s.t. $Z(f) \subseteq \mathbb{R}^2$ is <u>not</u> irreducible. Obvious choice is to go for $Z(f)$ as the union of some points. Then f would be

$$(x^2 - a^2)^2 + (y^2 - b^2)^2$$



to get four points. Then $Z(f) = Z(x-a, y-b) \cup Z(x+a, y+b) \cup Z(x-a, y+b) \cup Z(x+a, y-b)$, so is not irreducible. For simplicity we take $a=1$, $b=0$, so $f = (x^2-1)^2 + y^2$, which factorises as

$$f = (x^2 + iy - 1)(x^2 - iy - 1) \in \mathbb{C}[x,y]$$

so as $\mathbb{R}[x,y] \subseteq \mathbb{C}[x,y]$, both UFDs, f is irreducible in $\mathbb{R}[x,y]$. (multiplying by units in $\mathbb{C}$ can never turn a real polynomial $h(x,y)$ into $x^2 + iy - 1$.), but as we have already seen, $Z(f)$ is <u>not</u> irreducible.