2. PROJECTIVE VARIETIES

To define projective varieties, we proceed in a manner analgous to the definition of affine varieties, except that we work in projective space. Let k be our fixed algebraically closed field. We defined <u>projective n-space</u> as follows:

<u>DEFINITION</u> We define <u>projective n-space</u> over k, denoted by $\mathbb{P}^n_k$ or just $\mathbb{P}^n$, to be the set of equivalence classes of $(n+1)$-tuples $(a_0, \dots, a_n)$ of elements of k, not all zero, under the equivalence relation given by $(n \geq 1)$

$$(a_0, \dots, a_n) \sim (\lambda a_0, \dots, \lambda a_n) \qquad 0 \neq \lambda \in k.$$

equivalently, $\mathbb{P}^n$ is the quotient of $\mathbb{A}^{n+1} - \{(0, \dots, 0)\}$ under the equivalence relation which identifies points lying on the same line through the origin. An element of $\mathbb{P}^n$ is called a point, and if P is a point, then any $(n+1)$-tuple $(a_0, \dots, a_n)$ in the equivalence class P is called a set of <u>homogenous coordinates</u> for P.

<u>DEFINITION</u> (Graded Ring) A <u>graded ring</u> is a ring S, together with a decomposition

$$S = \bigoplus_{d \geq 0} S_d$$

of S into a direct sum of abelian groups S_d, such that for any $d, e \geq 0$, $S_d \cdot S_e \subseteq S_{d+e}$. An element of S_d is called a <u>homogenous element of degree d</u>. Thus any element of S can be written uniquely as a (finite) sum of homogenous elements. An ideal $\mathfrak{a} \subseteq S$ is a <u>homogenous ideal</u> if

$$\mathfrak{a} = \bigoplus_{d \geq 0} (\mathfrak{a} \cap S_d)$$

<u>PROPOSITION</u> Let $S = \bigoplus_{d \geq 0} S_d$ be a graded ring, then

(1) An ideal is homogenous if and only if it can be generated by homogenous elements

(i.e. if $f \in \mathfrak{a}$ all of fs homog. components are in $\mathfrak{a}$)

(2) Let $\mathfrak{a}, \mathfrak{b}$ be homogenous ideals, then

(i) $\mathfrak{a} + \mathfrak{b}$ is homogenous

(ii) $\mathfrak{a}\mathfrak{b}$ is homogenous

(iii) $\mathfrak{a} \cap \mathfrak{b}$ is homogenous

(iv) $\sqrt{\mathfrak{a}}$ is homogenous

(3) A homogenous ideal $\mathfrak{p}$ is prime iff. for any homogenous elements $f, g \in S$, if $fg \in \mathfrak{p}$ either $f \in \mathfrak{p}$ or $g \in \mathfrak{p}$.

<u>PROOF</u> (a) Suppose $\mathfrak{a}$ is homogenous. For any $f \in \mathfrak{a}$, the homogenous parts $f^{(d)} \in S_d$, $d \geq 0$, are in $\mathfrak{a}$, and the collection of all such homogenous parts generates $\mathfrak{a}$. Conversely suppose $\mathfrak{a} = (f_i)_{i \in I}$ with each f_i homogenous. Suppose $g \in \mathfrak{a}$, with say $g = g_1 f_1 + \dots + g_n f_n$, $g_i \in S$. Notice that generally if $x = y + z$, $x^{(d)} = y^{(d)} + z^{(d)}$, and so

$$g^{(d)} = (g_1 f_1)^{(d)} + \dots + (g_n f_n)^{(d)}$$

But for y homogenous, $(xy)^{(d)} = x^{(d-m)} y$ where $y \in S_m$. Hence

$$g^{(d)} = g_1^{(d-m_1)} f_1 + \dots + g_n^{(d-m_n)} f_n \qquad f_i \in S_{m_i}$$

But since $\mathfrak{a}$ is an ideal, this is in $\mathfrak{a}$. Hence $\mathfrak{a}$ is homogenous.

(2) (i) Let $f \in \mathfrak{a} + \mathfrak{b}$, say $f = a + b$. Then $f^{(d)} = a^{(d)} + b^{(d)} \in \mathfrak{a} + \mathfrak{b}$.

(ii) Notice that for $x, y \in S$

$$(xy)^{(d)} = \sum_{\substack{i,j \\ i+j=d}} x^{(i)} y^{(j)}$$

And hence if $f \in \mathfrak{a}\mathfrak{b}$, $f = ab$ say, since $\mathfrak{a}, \mathfrak{b}$ are both homogenous, $a^{(i)}, b^{(j)} \in \mathfrak{a}, \mathfrak{b}$ each i, j, so $f^{(d)} \in \mathfrak{a}\mathfrak{b}$.

(iii) If $f \in \mathfrak{a} \cap \mathfrak{b}$, say $f = \sum f^{(d)}$, then $f^{(d)} \in \mathfrak{a} \cap \mathfrak{b}$ each d.

(iv) We induct on the number n of nonzero homogenous parts of an element $f \in \sqrt{\mathfrak{a}}$. If $n=1$, the only nonzero component is $f^{(d)}=f$, so trivially $f^{(d)} \in \sqrt{\mathfrak{a}}$ all d. So now suppose the claim holds for all $n<k$. Let

$$f = f^{(d_1)} + \cdots + f^{(d_k)} \qquad f^{(d_i)} \in S_{d_i}$$

be in $\sqrt{\mathfrak{a}}$, so that $\exists m>0$, $f^m \in \mathfrak{a}$. We may suppose $d_1 < d_2 < \cdots < d_k$ and hence that $(f^m)^{(md_1)} = (f^{(d_1)})^m$. Hence $f^{(d_1)} \in \sqrt{\mathfrak{a}}$. But then $f^{(d_2)} + \cdots + f^{(d_k)} = f - f^{(d_1)} \in \sqrt{\mathfrak{a}}$, and so by the inductive hypothesis we are done.

(3) Necessity is clear. Suppose $\mathfrak{p}$ is s.t. whenever $f, g \in S$ are homogenous ($f=f^{(d)}$, $g=g^{(e)}$) and $fg \in \mathfrak{p}$, either f or g is in $\mathfrak{p}$. Suppose $ab \in \mathfrak{p}$, both $a, b \neq 0$, and $a = \sum a^{(d)}$, $b = \sum b^{(e)}$. Then

$$(ab)^{(i)} = \sum_{\substack{d,e \\ d+e=i}} a^{(d)} b^{(e)}$$

Since $\mathfrak{p}$ is homogenous, each of these sums is in $\mathfrak{p}$. Let d_0 and e_0 be the smallest degrees of components occurring in the expansion of a, b resp. Then $(ab)^{(d_0+e_0)} = a^{(d_0)} b^{(e_0)}$. Hence either $a^{(d_0)}$ or $b^{(e_0)}$ is in $\mathfrak{p}$. Suppose wlog that $a^{(d_0)} \in \mathfrak{p}$. Writing $a = a^{(d_0)} + a^{(d_1)} + \cdots + a^{(d_n)}$ we proceed by induction. Working modulo $\mathfrak{p}$, $(ab)^{(d_1+e_0)} = \sum'_{d+e=d_1+e_0} a^{(d)} b^{(e)}$ can only involve $d=d_0$ or d_1 (e_0 is minimal) so modulo $\mathfrak{p}$, $(ab)^{(d_1+e_0)}$ is $a^{(d_1)} b^{(e_0)}$. Hence $a^{(d_1)} \in \mathfrak{p}$. Proceeding in this fashion, each $a^{(d_i)}$ $i=1,\ldots,n$ is in $\mathfrak{p}$, and hence so is a. (N.B it is conceivable in the first step that $b^{(e_0)} \in \mathfrak{p}$ also. Just keep showing $a^{(d_i)}$ is in $\mathfrak{p}$, but replace $b^{(e_0)}$ by minimal degree comp not in $\mathfrak{p}$). $\square$

We make the polynomial ring $S = k[x_0, \ldots, x_n]$ into a graded ring by taking S_d to be the set of all linear combinations of monomials of total weight d in $x_0, \ldots, x_n$. If $f \in S$ is a polynomial, we cannot use it to define a function on $\mathbb{P}^n$, because of the nonuniqueness of the homogenous coordinates. However, if f is a homogenous polynomial of degree d, then

$$f(\lambda a_0, \ldots, \lambda a_n) = \lambda^d f(a_0, \ldots, a_n)$$

so that the property of f being zero or not depends only on the equivalence class of $(a_0, \ldots, a_n)$. Thus f gives a function from $\mathbb{P}^n$ to $\{0,1\}$ by $f(P)=0$ if $f(a_0,\ldots,a_n)=0$ and $f(P)=1$ if $f(a_0,\ldots,a_n) \neq 0$. Thus we can talk about the zeros of a homogenous polynomial, namely

$$Z(f) = \{ P \in \mathbb{P}^n \mid f(P) = 0 \}$$

If T is any set of homogenous elements of S, we define the zero set of T to be

$$Z(T) = \{ P \in \mathbb{P}^n \mid f(P) = 0 \text{ for all } f \in T \}$$

If $\mathfrak{a}$ is a homogenous ideal of S, we define $Z(\mathfrak{a}) = Z(T)$, where T is the set of all homogenous elements in $\mathfrak{a}$. Since S is a noetherian ring, any set of homogenous elements T has a finite subset $f_1, \ldots, f_r$ such that $Z(T) = Z(f_1, \ldots, f_r)$.

DEFINITION A subset Y of $\mathbb{P}^n$ is an algebraic set if there exists a set T of homogenous elements of S such that $Y = Z(T)$.

PROPOSITION 2.1 The union of any two algebraic sets is an algebraic set. The intersection of any family of algebraic sets is an algebraic set. The empty set and the whole space are algebraic sets.

PROOF If $Y = Z(T)$ and $Q = Z(S)$, $Y \cup Q = Z(TS)$ and if $Y_i = Z(T_i)$ all $i \in I$, $\bigcap Y_i = Z(\sum_{i \in I} T_i)$, where the sum denotes all possible homogenous sums $\sum f_i$, $f_i \in T_i$. Since 1 is homogenous, $\emptyset = Z(1)$, and $\mathbb{P}^n = Z(0)$. $\square$

DEFINITION We define the Zariski topology on $\mathbb{P}^n$ by taking the open sets to be the complements of algebraic sets.

Once we have a topological space, the notions of irreducible subset and the dimension of a subset, which were defined in §1, will apply.

DEFINITION A projective algebraic variety (or simply projective variety) (occasionally just variety) is an irreducible algebraic set in $\mathbb{P}^n$, with the induced topology. The dimension of a projective or quasi-projective variety is its dimension as a topological space.

If Y is any subset of $\mathbb{P}^n$, we define the homogenous ideal of Y in S, denoted $I(Y)$, to be the ideal generated by $\{ f \in S \mid f \text{ is homogenous and } f(P)=0 \text{ for all } P \in Y \}$. If Y is an algebraic set, we define the homogenous coordinate ring of Y to be $S(Y) = S/I(Y)$. We refer to (Ex. 2.1 — 2.7) below for various properties of algebraic sets in projective space and their homogenous ideals.

Our next objective is to show that projective n-space has an open covering by affine n-spaces, and hence that every projective (respectively, quasi-projective) variety has an open covering by affine (resp., quasi-affine) varieties. First we introduce some notation. If $f \in S$ is a nonzero linear homogenous polynomial, then the zero set of f is called a <u>hyperplane</u>. In particular, we denote the zero set of x_i by H_i, for $i = 0, \ldots, n$. Let U_i be the open set $\mathbb{P}^n - H_i$. Then $\mathbb{P}^n$ is covered by the open sets U_i, because if $P = (a_0, \ldots, a_n)$ is a point, then at least one $a_i \neq 0$, hence $P \in U_i$. We define a mapping $\varphi_i : U_i \longrightarrow \mathbb{A}^n$ as follows. If $P = (a_0, \ldots, a_n) \in U_i$, then $\varphi_i(P) = Q$, where Q is the point with affine coordinates

$$\left(\frac{a_0}{a_i}, \ldots, \frac{a_n}{a_i} \right)$$

with a_i/a_i omitted. Note that φ_i is well-defined since the ratios a_j/a_i are independent of the choice of homogenous coordinates.

<u>PROPOSITION 2.2</u> The map φ_i is a homeomorphism of U_i with its induced topology to $\mathbb{A}^n$ with the Zariski topology.

<u>PROOF</u> φ_i is clearly bijective, so it will be sufficient to show that the closed sets of U_i are identified with the closed sets of $\mathbb{A}^n$ by φ_i. We may assume $i = 0$, and we write simply U for U_0 and $\varphi : U \longrightarrow \mathbb{A}^n$ for φ_0. Let k be our alg. closed base field and set $A = k[y_1, \ldots, y_n]$. We define a map α from the set S^h of homogenous elements of S to A, and a map from A to S^h. Given $f \in S^h$, we set $\alpha(f) = f(1, y_1, \ldots, y_n)$. On the other hand, given $g \in A$ of degree e, then $x_0^e g(x_1/x_0, \ldots, x_n/x_0)$ is a homogenous polynomial of degree e in the x_i, which we call $\beta(g)$. Now let $Y \subseteq U$ be a closed subset. Let $\overline{Y}$ be its closure in $\mathbb{P}^n$. This is an algebraic set, so $\overline{Y} = Z(T)$ for some subset $T \subseteq S^h$. Let $T' = \alpha(T)$. Suppose $(a_1/a_0, \ldots, a_n/a_0) \in \varphi(Y)$, and $f \in T$. Then $\alpha(f)(a_1/a_0, \ldots, a_n/a_0) = f(1, a_1/a_0, \ldots, a_n/a_0) = 0$ iff. $f(a_0, a_1, \ldots, a_n) = 0$ iff. $(a_0, \ldots, a_n) \in Z(f)$. Hence $\varphi(Y) \subseteq Z(T')$. Conversely if $(x_1, \ldots, x_n) \in Z(T')$, so that for all $f \in T$, $\alpha(f)(x_1, \ldots, x_n) = 0 \iff f(1, x_1, \ldots, x_n) = 0$. Hence $(1, x_1, \ldots, x_n) \in \overline{Y}$, but it is also in U_0, so that $(1, x_1, \ldots, x_n) \in \overline{Y} \cap U = Y$, hence $(x_1, \ldots, x_n) \in \varphi(Y)$. Hence $Z(T') = \varphi(Y)$, so $\varphi(Y)$ is closed.

Conversely, let W be a closed subset of $\mathbb{A}^n$. Then $W = Z(T')$ for some subset T' of A, and we claim $\varphi^{-1}(W) = Z(\beta(T')) \cap U$. Let $(a_0, \ldots, a_n) \in \varphi^{-1}(W)$, so that $(a_1/a_0, \ldots, a_n/a_0) \in W$. Hence $\forall f \in T'$, $f(a_1/a_0, \ldots, a_n/a_0) = 0$. Hence

$$\begin{aligned} \beta(f)(a_0, \ldots, a_n) &= \{ x_0^e f(x_1/x_0, \ldots, x_n/x_0) \}(a_0, \ldots, a_n) \\ &= a_0^e f(a_1/a_0, \ldots, a_n/a_0) = 0. \end{aligned}$$

Hence $(a_0, \ldots, a_n) \in Z(\beta(T')) \cap U$. Conversely, suppose $(a_0, \ldots, a_n) \in Z(\beta(T')) \cap U$, so $a_0 \neq 0$ and for all $f \in T'$, $\{x_0^e f(x_1/x_0, \ldots, x_n/x_0)\}(a_0, \ldots, a_n) = 0$. Then

$$a_0^e f(a_1/a_0, \ldots, a_n/a_0) = 0$$

$$a_0 \neq 0 \Rightarrow f(a_1/a_0, \ldots, a_n/a_0) = 0$$

and hence $\varphi(a_0, \ldots, a_n) = (a_1/a_0, \ldots, a_n/a_0) \in Z(T') = W$, as required. Hence $\varphi^{-1}(W)$ is closed in U. Thus φ and φ^{-1} are both closed maps, so φ is a homeomorphism. $\square$

<u>COROLLARY 2.3</u> If Y is a projective (resp. quasi-projective) variety, then Y is covered by the open sets $Y \cap U_i$, $i = 0, \ldots, n$ which are homeomorphic to affine (resp. quasi-affine) varieties via the mapping φ_i defined above.

<u>PROOF</u> If Y is closed, irreducible in $\mathbb{P}^n$, then $Y \cap U_i$ is a closed and irreducible (Ex 1.6) subset of U_i. Hence under $\varphi_i : U_i \longrightarrow \mathbb{A}^n$ identifies $Y \cap U_i$ with a closed, irreducible subset of $\mathbb{A}^n$ — an affine variety. If Y is a projective variety and $Z \subseteq Y$ is an open subset, say $Z = Y \cap Q$, Q open in $\mathbb{P}^n$, then $Y \cap U_i$ is a closed irreducible subset of U_i, and $Z \cap U_i$ is an open subset of $Y \cap U_i$, since $(Q \cap U_i)$ is open in U_i and $Z \cap U_i = (Y \cap Q) \cap U_i = (Y \cap U_i) \cap (Q \cap U_i)$. Hence φ_i identifies Z with an quasi-affine variety. $\square$

<u>NOTE</u> Cor 2.3 is true <u>provided</u> $Y \cap U_i$ is <u>nonempty</u>!

<u>NOTE</u> Let T be a set of homogenous elements in $S = k[x_0, \ldots, x_n]$ and $\mathfrak{a} = (T)$ the homogenous ideal generated by T. Let T' be the set of homogenous elements of $\mathfrak{a}$. Clearly $T' \supseteq T$ so $Z(T') = Z(T)$. Hence $Z(\mathfrak{a}) = Z(T') \subseteq Z(T)$. Conversely if $(a_0, \ldots, a_n) \in Z(T)$ and $f \in T'$, then $f = \sum f_i t_i$ for $t_i \in T$.

$$f(a_0, \ldots, a_n) = \sum f_i(a_0, \ldots, a_n) t_i(a_0, \ldots, a_n) = 0$$

So $(a_0, \ldots, a_n) \in Z(T') = Z(\mathfrak{a})$. Hence $Z(\mathfrak{a}) = Z(T)$.

<u>NOTE</u> Points are closed in $\mathbb{P}^n$: given $(a_0, \ldots, a_n) \in \mathbb{P}^n$ with $a_i \neq 0$ this point is the only solution to $T = \{ x_0 - \frac{a_0}{a_i} x_i, \ldots, x_n - \frac{a_n}{a_i} x_i \}$ (exclude x_i)

NOTE (Laurent Polynomials)

Just as $R[x]$ is defined as a set of functions $f : \mathbb{Z}^+ \longrightarrow R$ $(\mathbb{Z}^+ = \{0,1,2,\ldots\})$, we define:

DEFINITION The ring $R[x,x^{-1}]$, called the ring of Laurent polynomials is the set of all functions

$$f : \mathbb{Z} \longrightarrow R$$

with finite support. Addition and multiplication are defined by

$$(f+g)(n) = f(n) + g(n)$$
$$(f \cdot g)(n) = \sum_{\substack{s,t \\ s+t=n}} f(s)g(t)$$
$$0(n) = 0$$
$$1(n) = \delta_{n0}$$

It is easy to see this makes $R[x,x^{-1}]$ into a commutative ring with unit.

PROPOSITION There is an isomorphism of rings

$$R[x,x^{-1}] \cong R[x,y]/(xy-1) \cong R[x]_x$$

PROOF We already know that $R[x]_x \cong R[x,y]/(xy-1)$. Since x^{-1} is clearly a unit in $R[x,y]$, we get $\varphi : R[x]_x$ defined for $f \in R[x]$ and $n \geq 0$ by

$$\varphi(f/x^n) = \phi(f)(x^{-1})^n$$

where $\phi : R[x] \longrightarrow R[x,x^{-1}]$ is the obvious morphism $\phi(f)(n) = f(n)$, $n \geq 0$ $\phi(f)(n) = 0$, $n<0$. φ is injective since $\phi(f)(x^{-1})^n = \phi(g)(x^{-1})^m$ implies that for $x \in \mathbb{Z}^+$,

$$\begin{aligned} f(x) &= \phi(f)(x) \\ &= (\phi(f)(x^{-1})^n)(x-n) \\ &= (\phi(g)(x^{-1})^m)(x-n) \\ &= \phi(g)(x+m-n) \\ &= g(x+m-n) \end{aligned}$$

Hence $x^n g = x^m f$, so $f/x^n = g/x^m$ in $R[x]_x$. And φ is surjective since if $f \in R[x,x^{-1}]$, let $a \in \mathbb{Z}$ be least s.t. $f(a) \neq 0$. Then if b is largest s.t. $f(b) \neq 0$, and we presume $a<0$

$$\begin{aligned} f &= f(a)x^a + \cdots + f(b)x^b \\ &= \varphi\left(\frac{f(a)}{x^a}\right) + \cdots + \varphi\left(\frac{f(-1)}{x}\right) + \varphi(f(0)) + \cdots + \varphi\left(f(b)x^b\right) \in \operatorname{Im}\varphi \end{aligned}$$

since $(x^{-1})^n = x^{-n}$. Hence φ is an isomorphism. $\square$

$\mathbb{Z}$ - Graded Rings and their Localisations

If we invert an element of a graded ring, even a homogenous element, we usually do not get a graded ring in the sense of Chapter 1: Negative degrees will occur in the obvious grading. Thus we introduce the notion of a $\mathbb{Z}$-graded ring:

DEFINITION A $\mathbb{Z}$-graded ring R is a ring R such that

$$R = \cdots R_{-2} \oplus R_{-1} \oplus R_0 \oplus R_1 \oplus R_2 \cdots$$

as abelian groups and $R_i R_j \subseteq R_{i+j}$. The elements of R_i are called homogenous elements of degree i. A homogenous ideal in a $\mathbb{Z}$-graded ring is simply an ideal generated by homogenous elements.

The case of ordinary graded rings is the case where $R_i = 0$ for $i<0$.

EXERCISE 2.14 (Characterisation of homogenous ideals)

(a) An ideal I of a $\mathbb{Z}$-graded ring is homogenous iff. $\forall f \in I$, all the homogenous components of f are in I.

PROOF: See Hartshorne notes. Same as graded case

(b) $\sqrt{I}$ is homogenous.

PROOF: Again, same proof.

(c) If I, J are homogenous, so is $(I:J) = \{f \in R \mid fJ \subseteq I\}$

PROOF Suppose $f = f^{(d_1)} + \cdots + f^{(d_k)} \in (I:J)$, $d_i \in \mathbb{Z}$ and $f^{(d_i)} \in S_{d_i}$. Then since J is generated by homogenous elements, to show $f^{(d_i)} \in (I:J)$ it suffices to show $f^{(d_i)} q \in I$, $q \in J$ is homogenous. We suppose $d_1 < \cdots < d_k$, and that q has degree s. Then

$$fq \in I \Rightarrow f^{(d_1)}q + \cdots + f^{(d_k)} q \in I$$
$$\Rightarrow f^{(d_i)} = (fq)^{(s+d_i)} \in I, \text{ since } I \text{ homogenous.}$$

Hence $(I:J)$ is homogenous.

(d) A homogenous ideal I is prime iff. for f, g homogenous, $fg \in I \Rightarrow f \in I$ or $g \in I$.

PROOF The condition is clearly necessary. The proof for the graded case works again.

Given a projective variety $X \subseteq \mathbb{P}^r_k$, it is very useful to be able to write the localisations of the affine coordinate rings of the affine open pieces of X directly in terms of the homogenous coordinate ring of X. The following exercise explains how to do this, in a form that works for arbitrary $\mathbb{Z}$-graded rings.

EXERCISE 2.17 (Localisation of graded rings) Suppose R is a $\mathbb{Z}$-graded ring and $0 \neq f \in R_1$. Then R_f is again a $\mathbb{Z}$-graded ring, where the grading of R_f is inherited from that of $R[x, x^{-1}]$, which is canonically identified with R_f. That is, the grading of r/f^n is $d-n$ where r has grade d. To see this is well-defined, suppose $r/f^n = s/f^m$, so that $f^q(rf^m - sf^n) = 0$, some $q \geq 0$. Then $f^{m+q} r$ and $f^{n+q} s$ have the same grade, but $f \in R_1$ means the grade of s is $d+m+q-(n+q) = d+m-n$. Hence s/f^m has grade $d+m-n-m = d-n$, as required. (Above obviously r, s refer to monomials). That is, for $d \in \mathbb{Z}$, $(R_f)_d$ consists of all elements of the form r/f^n where $r \in R_{d+n}$ (hence $d+n \geq 0$). for $n \geq 0$. The above shows these groups are disjoint. Suppose

$$r/f^n \in (R_f)_d \cap (R_{f,d_1} + \cdots + R_{f,d_m}) \qquad d, d_i \text{ all distinct}$$

Then $r/f^n \in (R_f)_d$ implies $r \in R_{d+n}$. Then if $r_i \in R_{d_i+n_i}$ and

$$r/f^n = r_1/f^{n_1} + \cdots + r_m/f^{n_m}$$

Let $k \geq \max\{\{n_i\}_i, n\}$, so that $f^{k-n} r = f^{k-n_1} r_1 + \cdots + f^{k-n_m} r_m$. Then $f^{k-n} r \in R_{d+k}$ and $f^{k-n_i} r_i \in R_{d_i+k}$. Hence $f^{k-n} r = 0$, so that $r/f^n = 0$ in R_f. Hence we have defined a grading.

NOTE In a graded domain, principal ideals which are homogenous are generated by a homogenous elt.

Let $S = \bigoplus_{d \geq 0} S_d$ be a graded domain, $\mathfrak{a} \subseteq S$ a homogenous ideal, $f \in S$ with $\mathfrak{a} = (f)$. Let

$$f = f_n + \cdots + f_m \qquad \begin{array}{l} 0 \neq f_n \in S_n \\ 0 \neq f_m \in S_m \text{ etc.} \end{array}$$

Since $\mathfrak{a}$ is homogenous $f_n \in (f)$, so $f_n = gf$ for some $g = g_e + \cdots + g_f \in S$. Then

$$f_n = gf = g_e f_n + (g_e f_{n+1} + g_{e+1} f_n) + \cdots + g_f f_m$$

Clearly we must have $f + m = n$ since $g_f \neq 0$ and $f_m \neq 0$. But $n \leq m$ so this implies $m = n$ and f is homogenous.

NOTE Projective Closure

Let $Y \subseteq \mathbb{P}^n$ be a projective variety, and suppose $Y \cap U_0$ is nonempty. Then $\varphi_0(Y \cap U_0)$ is affine and its projective closure is Y.

NOTE If $\mathfrak{a}_i \subseteq k[x_0, \ldots, x_n]$ are homogenous ideals it is not hard to check that $Z(\sum \mathfrak{a}_i) = \bigcap Z(\mathfrak{a}_i)$

NOTE Generating ideals in a Noetherian ring

Let R be a nonzero Noetherian ring, $\mathfrak{a}$ an ideal. Suppose $\{f_i\}_{i \in I}$ is a set of elements with $\mathfrak{a} = (\{f_i\}_{i \in I})$. We claim that $\mathfrak{a}$ can be generated by a finite subset of the f_i. If I is finite this is trivial. So assume I is infinite, and

$$S = \{ J \subseteq I \mid J \neq \emptyset \text{ and } J \text{ is finite} \} \qquad (\text{assume } I \neq \emptyset)$$

For each $J \in S$ let $\mathfrak{a}_J = (\{f_j\}_{j \in J})$. Then $\{\mathfrak{a}_J \mid J \in S\}$ is a nonempty set of ideals in a Noetherian ring, hence has a maximal element $\mathfrak{a}_K$. Clearly $\mathfrak{a} = \mathfrak{a}_K$ and we are done.

NOTE Dimension and irreducible components

Let X be a nonempty Noetherian topological space, $Y \subseteq X$ a nonempty closed subset and

$$Y = Y_1 \cup \cdots \cup Y_n$$

the decomposition of Y into its irreducible components. Then $\dim Y = \sup_{1 \leq i \leq n} \dim Y_i$. Any chain of distinct irreducible closed sets in a Y_i becomes such a chain in Y, so clearly $\sup \dim Y_i \leq \dim Y$ (in particular, if this supremum is ∞ then $\dim Y = \infty$). If $Z_0 \subset Z_1 \subset \cdots \subset Z_m$ is a chain in Y then

$$Z_m = Z_m \cap Y = Z_m \cap Y_1 \cup \cdots \cup Z_m \cap Y_n$$

Since Z_m is irreducible it is contained in some Y_i, so the whole chain belongs in some Y_i. This shows the reverse inequality, so

$$\dim Y = \sup \dim Y_i$$

NOTE Morphisms $\mathbb{P}^n \longrightarrow \mathbb{P}^m$

Let $f_0(x_0,\dots,x_n), f_1(x_0,\dots,x_n),\dots, f_m(x_0,\dots,x_n)$ be homogenous polynomials all of the same degree d, admitting no common solution in $\mathbb{P}^n$ (so we assume all are nonzero, i.e. $d \geq 0$). We define

$$\varphi : \mathbb{P}^n \longrightarrow \mathbb{P}^m$$
$$\varphi(a_0,\dots,a_n) = (f_0(a_0,\dots,a_n),\dots,f_m(a_0,\dots,a_n))$$

$$\phi : k[y_0,\dots,y_m] \longrightarrow k[x_0,\dots,x_n]$$
$$\phi(y_i) = f_i(x_0,\dots,x_n)$$

⌜Note ϕ preserves homogenous polynomials If degree e, $\phi(g)$ is degree de⌟

Then ϕ is a morphism of k-algebras, and φ is a well-defined map of sets. We now show it is a morphism of varieties. Let $g(y_0,\dots,y_m)$ be homogenous. Then

$$\begin{aligned} Z(\phi(g)) &= \{(a_0,\dots,a_n) \mid \phi(g)(a_0,\dots,a_n) = 0\} \\ &= \{(a_0,\dots,a_n) \mid g(\varphi(a_0,\dots,a_n)) = 0\} \\ &= \varphi^{-1}(Z(g)) \end{aligned}$$

So φ is continuous. To see it is a morphism, let $f: U \longrightarrow k$ be regular, $V \subseteq \mathbb{P}^m$ open and let $x \in \varphi^{-1}U$. Say $\varphi(x) \in V \subseteq U$ with $g,h \in k[y_0,\dots,y_m]$ s.t. $\forall v \in V$, $f(v) = g(v)/h(v)$. Then for $y \in \varphi^{-1}V$ $y = (a_0,\dots,a_n)$

$$\begin{aligned} f\varphi(a_0,\dots,a_n) &= f(f_0(a_0,\dots,a_n),\dots,f_m(a_0,\dots,a_n)) \\ &= \frac{\phi(g)(a_0,\dots,a_n)}{\phi(h)(a_0,\dots,a_n)} \end{aligned}$$

Since g,h are homogenous of the same degree, so are $\phi(g),\phi(h)$. Hence $f\varphi$ is regular and φ is a continuous morphism of varieties.

NOTE Linear Morphisms $\mathbb{P}^{n-1} \longrightarrow \mathbb{P}^n$ $(n \geq 2)$

Consider the following special case. Let $(a_0,\dots,a_n) \in \mathbb{P}^n$, so some element, say a_i, is nonzero. Put

$$\begin{aligned} f_0(x_0,\dots,x_{n-1}) &= x_0 \\ &\vdots \\ f_{i-1}(x_0,\dots,x_{n-1}) &= x_{i-1} \\ f_i(x_0,\dots,x_{n-1}) &= \frac{-1}{a_i}(a_0x_0 + a_1x_1 + \cdots + a_{i-1}x_{i-1} + a_{i+1}x_i + \cdots + a_nx_{n-1}) \\ f_{i+1}(x_0,\dots,x_{n-1}) &= x_i \\ &\vdots \\ f_n(x_0,\dots,x_{n-1}) &= x_{n-1} \end{aligned}$$

Then $\varphi : \mathbb{P}^{n-1} \longrightarrow \mathbb{P}^n$ is a morphism of varieties, $\varphi(b_0,\dots,b_{n-1}) = (b_0,\dots,b_{i-1}, \frac{-1}{a_i}(a_0b_0 + \cdots + a_{i-1}b_{i-1} + a_{i+1}b_i + \cdots + a_nb_{n-1}), \dots b_{n-1})$.

$$a_0x_0 + a_1x_1 + \cdots + a_nx_n = 0$$

This projective variety is precisely the image of φ. Clearly φ is injective, and $K: Z \longrightarrow \mathbb{P}^{n-1}$ given by $(b_0,\dots,b_n) \longmapsto (b_0,\dots,b_{i-1},b_{i+1},\dots,b_n)$ is a morphism of varieties (use $\eta: k[x_0,\dots,x_{n-1}] \longrightarrow k[y_0,\dots,y_n]$ which leaves out y_i). Hence

⌜The hyperplane $a_0x_0 + \dots + a_nx_n = 0$ is isomorphic as a variety to $\mathbb{P}^{n-1}$⌟

Automorphisms of $\mathbb{P}^n$

Daniel Murfet

June 7, 2004

Let k be a field and $A \in GL_{n+1}(k)$ be $A = (a_{ij})$ and define

$$\varphi : \mathbb{P}^n \longrightarrow \mathbb{P}^n$$
$$\varphi(x_1, \ldots, x_{n+1}) = (\ldots, \sum_j a_{ij} x_j, \ldots)$$

This is well-defined since if $\varphi(x_1, \ldots, x_{n+1}) = 0$ then since A is invertible all the x_i are zero. Clearly $\varphi(\lambda x_1, \ldots, \lambda x_{n+1}) = \varphi(x_1, \ldots, x_n)$. As we will show, φ is an automorphism of varieties, and any morphism determined by an invertible matrix in this way is called a *projective transformation*.

First we check continuity. Define a morphism of k-algebras

$$\theta : k[x_1, \ldots, x_{n+1}] \longrightarrow k[x_1, \ldots, x_{n+1}]$$
$$\theta(x_i) = \sum_j a_{ij} x_j$$

This is clearly an automorphism of k-algebras. Note that for a homogenous polynomial $f \in k[x_1, \ldots, x_{n+1}]$ we have

$$\begin{aligned}\varphi^{-1}(Z(f)) &= \{(a_1, \ldots, a_{n+1}) \,|\, f(\ldots, \sum_j a_{ij} a_j, \ldots) = 0\} \\ &= \{(a_1, \ldots, a_{n+1}) \,|\, \theta(f)(a_1, \ldots, a_{n+1}) = 0\} \\ &= Z(\theta(f))\end{aligned}$$

Since θ maps a homogenous polynomial of degree e to another homogenous polynomial of degree e, we see immediately that φ is continuous.

Next we check that φ is a morphism of varieties. Let $f : U \longrightarrow k$ be regular, where U is an open subset of $\mathbb{P}^n$. Let x be an element of the open set $\varphi^{-1}U$ and let V be an open neighborhood of $\varphi(x)$ in U, g, h homogenous polynomials of the same degree such that $f(v) = g(v)/h(v)$ for all $v \in V$. Then for $(a_1, \ldots, a_{n+1}) \in \varphi^{-1}V$

$$\begin{aligned}f\varphi(a_1, \ldots, a_{n+1}) &= \frac{g(\ldots, \sum_j a_{ij} a_j, \ldots)}{h(\ldots, \sum_j a_{ij} a_j, \ldots)} \\ &= \frac{\theta(g)(a_1, \ldots, a_{n+1})}{\theta(h)(a_1, \ldots, a_{n+1})}\end{aligned}$$

Hence $f\varphi$ is regular and so φ is a morphism of varieties. Since the morphism induced by A^{-1} is clearly inverse to φ, we have shown that φ is an automorphism of the variety $\mathbb{P}^n$.

It is clear that if φ, ϕ are projective transformations determined by respective matrices A, B then the composition $\varphi\phi$ is the projective transformation determined by the product AB. So the composition of projective transformations is a projective transformation.

Lemma 1. *Given two sets of three distinct points in* $\mathbb{P}^1$

$$(P_1, P_2, P_3) \quad \text{and} \quad (Q_1, Q_2, Q_3)$$

there is a unique projective transformation φ *of* $\mathbb{P}^1$ *such that*

$$\varphi(P_i) = Q_i \qquad i = 1, 2, 3$$

Proof. First consider the case where $P_1 = (1,0), P_2 = (1,1), P_3 = (0,1)$. Let $Q_i = (a_i, b_i), i = 1,2,3$. Since the points Q_1 and Q_3 are distinct the matrix $A = \left(\begin{smallmatrix} a_1 & a_3 \\ b_1 & b_3 \end{smallmatrix}\right)$ has column rank 2 and is thus invertible. Let (α, β) be such that

$$\begin{pmatrix} a_1 & a_3 \\ b_1 & b_3 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} a_3 \\ b_3 \end{pmatrix}$$

By scaling Q_1 by α and Q_3 by β we may assume that $\alpha = \beta = 1$. It is then easily checked that the automorphism φ determined by A has the required property.

In the general case, let ϕ be the transformation taking $(1,0), (1,1), (0,1)$ to P_1, P_2, P_3 and φ the transformation taking $(1,0), (1,1), (0,1)$ to Q_1, Q_2, Q_3. Then $\varphi\phi^{-1}$ has the required property. If ψ is another transformation with the property that $\psi(P_i) = Q_i, i = 1,2,3$ then $\varphi^{-1}\psi\phi$ maps the elements $(1,0), (1,1), (0,1)$ to themselves. This implies that the matrix determining the composite must be a scalar multiple of the identity, so $\psi = \varphi\phi^{-1}$, proving uniqueness. $\square$

Definition 1. Points $P_1, \ldots, P_s$ of $\mathbb{P}^n$ are *collinear* if there is a linear polynomial $a_0x_0 + \ldots + a_nx_n$ which admits each P_i as a solution.

Lemma 2. *Given two sets of four distinct points in* $\mathbb{P}^2$

$$(P_1, P_2, P_3, P_4) \quad \text{and} \quad (Q_1, Q_2, Q_3, Q_4)$$

which satisfy the condition that no three points in the set are collinear, there is a unique projective transformation φ *of* $\mathbb{P}^2$ *such that*

$$\varphi(P_i) = Q_i \qquad i = 1, 2, 3, 4$$

Proof. First consider the case where

$$\begin{aligned} P_1 &= (1,0,0) \\ P_2 &= (0,1,0) \\ P_3 &= (0,0,1) \\ P_4 &= (1,1,1) \end{aligned}$$

Let $Q_i = (a_i, b_i, c_i)$. Since no three of the Q_i are collinear, the matrix

$$A = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$$

has column rank 3 and is thus invertible. As before, by scaling the Q_i if necessary, we can assume that

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} a_4 \\ b_4 \\ c_4 \end{pmatrix}$$

If φ is the projective transformation determined by A, then φ has the required property. The general case and uniqueness follow in the same way as before. $\square$

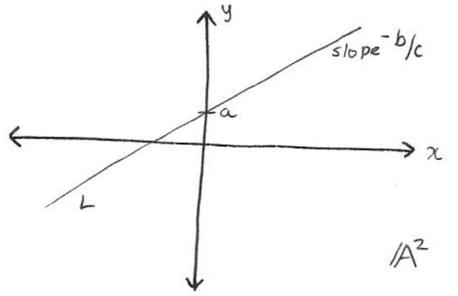
UNDERSTANDING PROJECTIVE SPACE

For $n \geq 1$ we identify $\mathbb{A}^n$ with the tuples $(1, a_1, \ldots, a_n) \in \mathbb{P}^n$. This is an isomorphism of varieties. The points not belonging to $\mathbb{A}^n$ are of the form $(0, a_1, \ldots, a_n)$, which we identify with the line

$$a_1 x_1 + \cdots + a_n x_n = 0$$

in $\mathbb{A}^n$, and think of as being an ∞ in the direction of this line. So $\mathbb{P}^n$ is $\mathbb{A}^n$ together with an ∞ for every line through the origin in $\mathbb{A}^n$. Now consider Ex 2.9 and the Note which shows that the projective closure of a hypersurface is a hypersurface. If we work now with $n=2$ a line in $\mathbb{A}^2$ is the affine variety (assume $b, c \neq 0$) L:

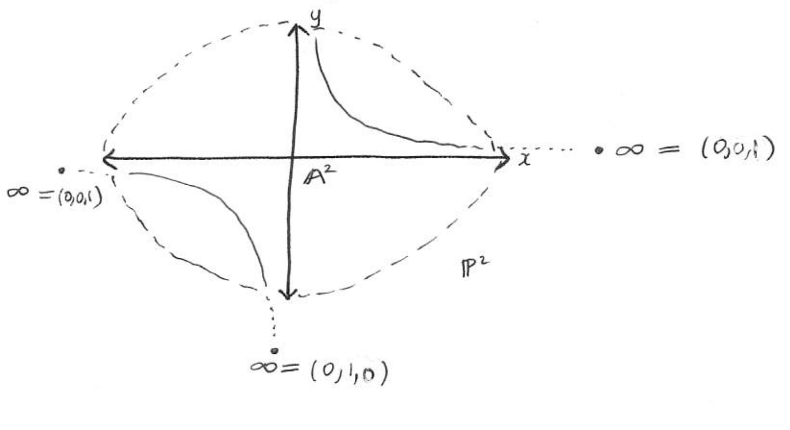
$$a + bx + cy = 0 \qquad \text{coordinates } x, y$$



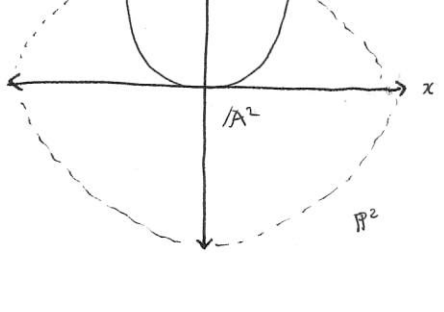
The projective closure of this line is the hypersurface $aw + bx + cy = 0$ (w the projective coordinate). The solutions with $w = 1$ are precisely L, so the only new solution is $(0, 1, -b/c)$ which corresponds to the ∞ in the direction of L. (We order coordinates w, x, y)

We have numerous other examples:

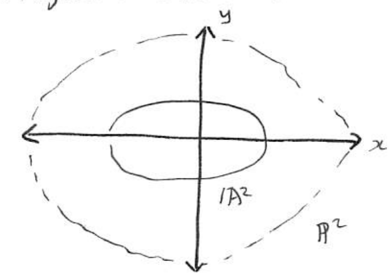
Hyperbola $xy = 1$. Projective closure is $xy - w^2 = 0$ which adds the infinities in the direction of the x and y axis:



Parabola $y = x^2$. Projective closure is $wy - x^2 = 0$, which adds the infinity $(0,1,0)$ in the direction of the y-axis.



$\boxed{\text{Ellipse}}$ $x^2/a^2 + y^2/b^2 = 1$. Projective closure is $x^2/a^2 + y^2/b^2 = w^2$, which adds <u>no</u> infinities.



From Exercise 1.1 we know that in $\mathbb{A}^2$ all conics are isomorphic either to $y - x^2$ or $xy - 1$. Since the discriminant of the ellipse is $-1/a^2b^2 \neq 0$ any ellipse is isomorphic to the hyperbola. But we will see in Ex 3.1 that conics in $\mathbb{P}^2$ are all isomorphic. (to $\mathbb{P}^1$ no less). Visually, the ∞s make this apparent.

Any tuple $(a_0, a_1, a_2) \in \mathbb{P}^2$ determines a well-defined line ℓ_{a_0,a_1,a_2} and $\ell_{a_0,a_1,a_2} = \ell_{b_0,b_1,b_2}$ iff $(a_0,a_1,a_2) = (b_0,b_1,b_2)$ in $\mathbb{P}^2$ (since $\ell_{a_0,a_1,a_2} = Z(a_0x_0 + a_1x_1 + a_2x_2)$). Moreover

<u>LEMMA</u> Two distinct lines in $\mathbb{P}^2$ meet at a point

<u>PROOF</u> Consider two lines

$$\begin{aligned} \alpha x_0 + \beta x_1 + \gamma x_2 &= 0 \\ \alpha' x_0 + \beta' x_1 + \gamma' x_2 &= 0 \end{aligned} \qquad (0)$$

Since the lines are distinct the tuples (α, β, γ), $(\alpha', \beta', \gamma')$ are linearly independent in $\mathbb{A}^3$. That is, the matrix

$$\begin{pmatrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{pmatrix} \qquad (1)$$

has row rank (hence also col-rank) of 2. Note that two columns are linearly dependent iff the determinant of the associated 2×2 matrix is 0. Since col-rank $= 2$, the tuple

$$\left(\begin{vmatrix} \beta & \gamma \\ \beta' & \gamma' \end{vmatrix}, \begin{vmatrix} \gamma & \alpha \\ \gamma' & \alpha' \end{vmatrix}, \begin{vmatrix} \alpha & \beta \\ \alpha' & \beta' \end{vmatrix} \right) \neq (0,0,0) \qquad (2)$$

One checks that this point of $\mathbb{P}^2$ lies on both lines. The matrix in (1) represents a linear map $\phi : k^3 \longrightarrow k^2$ whose image has dimension $=$ col-rank $= 2$. Hence the kernel has dimension 1, implying that any other solution to (0) is equal to (2) in $\mathbb{P}^2$. $\square$

<u>LEMMA 2.5</u> Given two distinct points in $\mathbb{P}^2$, there is a unique line going through them.

<u>PROOF</u> Let (a_0, a_1, a_2) and (b_0, b_1, b_2) be distinct points. We want to find a line

$$\alpha x_0 + \beta x_1 + \gamma x_2$$

such that

$$\begin{aligned} \alpha a_0 + \beta a_1 + \gamma a_2 &= 0 \\ \alpha b_0 + \beta b_1 + \gamma b_2 &= 0 \end{aligned}$$

That is, (α, β, γ) should be a nonzero solution of $\begin{pmatrix} a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \end{pmatrix}$ which again has col-rank 2, hence a kernel of dimension 1, and so a unique solution in $\mathbb{P}^2$. This solution is

$$(\alpha, \beta, \gamma) = \left(\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}, \begin{vmatrix} a_2 & a_0 \\ b_2 & b_0 \end{vmatrix}, \begin{vmatrix} a_0 & a_1 \\ b_0 & b_1 \end{vmatrix} \right). \quad \square$$

EXERCISES Ch.1.2 Hartshorne

Q2.1 Let us survey the homogenous ideals of $S = k[x_0,..,x_n]$. We have the improper homogenous ideal S, and proper homogenous ideals $\mathfrak{a}$. For a proper hom. ideal $\mathfrak{a}$, we cannot have $\mathfrak{a} \not\subseteq (x_0,\dots,x_n)$, because this would imply that $f(0,..,0) \neq 0$, some $f \in \mathfrak{a}$, implying that f has nonzero component homogenous of degree 0. Since $\mathfrak{a}$ is homogenous, this component would belong to $\mathfrak{a}$, contradicting the assumption that $\mathfrak{a}$ is proper. Further, if $\mathfrak{a} \subseteq (x_0 - a_0, \dots, x_n - a_n)$ then $\mathfrak{a}$ belongs to $(x_0 - \lambda a_0, \dots, x_n - \lambda a_n)$ for $\lambda \in k$, since if $f \in \mathfrak{a}$, $f = \sum_{i \geq 1} f_i$, then $f_i \in \mathfrak{a}$, so

$$\begin{aligned} f(\lambda a_0, \dots, \lambda a_n) &= \sum_{i\geq 1} f_i(\lambda a_0, \dots \lambda a_n) \\ &= \sum_{i \geq 1} \lambda^i f_i(a_0, \dots, a_n) = 0. \end{aligned}$$

Hence for $\mathfrak{a}$ proper and homogenous, the variety $V(\mathfrak{a})$ contains O and for every point $(a_0,\dots,a_n) \in V(\mathfrak{a})$, also $(\lambda a_0, \dots, \lambda a_n) \in V(\mathfrak{a})$, $\lambda \in k$. Hence $V(\mathfrak{a}) - O$ modulo the "ray" relation partitions $V(\mathfrak{a}) - O$ into elements of $\mathbb{P}^n$, and this set is the variety of $\mathbb{P}^n$ determined by $\mathfrak{a}$, as defined earlier. So $\mathfrak{a} \subseteq (x_0 - a_0, \dots, x_n - a_n)$ iff. $\forall f \in \mathfrak{a}$ homogenous $f(a_0, \dots, a_n) = 0$.

Hence if $f \in S$, either $\deg f = 0$ and the result is trivial, or $\deg f > 0$. Assume $f(P) = 0$, $\forall P \in Z(\mathfrak{a})$. ($f$ homogenous, $\mathfrak{a}$ homogenous, $Z(\mathfrak{a}) \subseteq \mathbb{P}^n$). Then by the above f belongs to all the maximal ideals containing $\mathfrak{a}$ (including $(x_0,..,x_n)$ since $\deg f > 0$). Hence $f \in \sqrt{\mathfrak{a}}$ by the normal Nullstellensatz.

Q2.2 Let $\mathfrak{a} \subseteq S$ be homogenous. Suppose $Z(\mathfrak{a}) = \phi$. Then the only maximal ideal of S containing $\mathfrak{a}$ is $(x_0, .., x_n)$, if $\mathfrak{a}$ is proper, and otherwise $\mathfrak{a}$ is S. Hence $\sqrt{\mathfrak{a}}$ is S or $(x_0,..,x_n) = S_+ = \bigoplus_{d>0} S_d$. Now, if $\mathfrak{a}$ hom. is given s.t. $\sqrt{\mathfrak{a}} = S$ or $\sqrt{\mathfrak{a}} = S_+$, then either $\mathfrak{a} = S$ (in which case all $S_d \subseteq \mathfrak{a}$, $d > 0$) or $\sqrt{\mathfrak{a}} = S_+$. Hence let m_i, $0 \leq i \leq n$ be s.t. $x_i^{m_i} \in \mathfrak{a}$. Then $S_d \subseteq \mathfrak{a}$ where $d = \sum m_i \cdot (n+1)$. Finally, if $d > 0$ is s.t. $S_d \subseteq \mathfrak{a}$ ($\Rightarrow$ then $\bigoplus_{e \geq d} S_e \subseteq \mathfrak{a}$). Then for $0 \leq i \leq n$, $x_i^d \in \mathfrak{a}$. If $m = (x_0 - a_0, .., x_n - a_n)$ is a maximal ideal of S containing $\mathfrak{a}$, then $a_i^d = 0$, $0 \leq i \leq n$, so each $a_i = 0$. Hence either $\mathfrak{a} = S$ or $\mathfrak{a}$ is contained only in $(x_0,\dots,x_n)$, so in $\mathbb{P}^n$, $Z(\mathfrak{a}) = \phi$.

Q2.3 (a) Suppose $T_1 \subseteq T_2$ are subsets of S^h, and that $(a_0,\dots,a_n) \in Z(T_2)$. Then for $f \in T_1$, $f \in T_2$ implies $f(a_0,..,a_n) = 0$, so $(a_0,..,a_n) \in Z(T_1)$.

(b) Suppose $Y_1 \subseteq Y_2$ are subsets of $\mathbb{P}^n$, and that $f \in I(Y_2)$ is a homogenous element of S s.t. $f(a_0,\dots,a_n) = 0$ for $(a_0,..,a_n) \in Y_2$. Then also this holds for Y_1, so $f \in I(Y_1)$. Since $I(Y_2)$ is generated by such f, $I(Y_2) \subseteq I(Y_1)$.

(c) Let $Y_1, Y_2 \subseteq \mathbb{P}^n$. If $f \in I(Y_1 \cup Y_2)$ is homogenous, then clearly $f \in I(Y_1)$ and $f \in I(Y_2)$, so $f \in I(Y_1) \cap I(Y_2)$. If $g \in I(Y_1) \cap I(Y_2)$ is homogenous, then clearly $g \in I(Y_1 \cup Y_2)$. Hence both ideals contain the same homogenous elements, and are thus equal.

(d) Let $\mathfrak{a} \subseteq S$ be homogenous with $Z(\mathfrak{a}) \neq \phi$. Then for homogenous $g \in I(Z(\mathfrak{a}))$ iff. $g \in \sqrt{\mathfrak{a}}$ by 2.1. Hence since both ideals are homogenous, they are equal.

(e) Let $Y \subseteq \mathbb{P}^n$. Clearly $Z(I(Y))$ is a closed set containing Y. Suppose $Y \subseteq Y' \subseteq Z(I(Y))$, $Y' = Z(T')$ a closed subset. Then $Y \subseteq Z(T')$ implies that for $f \in T'$, $f(a_0, \dots, a_n) = 0$ for $(a_0, \dots, a_n) \in Y$. Hence since f is homogenous, $f \in I(Y)$. Hence $T' \subseteq I(Y)$ so $Y' = Z(T') \supseteq Z(I(Y))$. Hence $Z(I(Y)) = \bar{Y}$.

Q2.4 (a) Let $Y \subseteq \mathbb{P}^n$ be an algebraic set. Then $\sqrt{I(Y)} = I(Z(I(Y))) = I(\bar{Y}) = I(Y)$, so $I(Y)$ is radical. (If $Y = \phi$, $I(Y) = S$)
Similarly if $I \subseteq S$ is a homogenous radical ideal, not equal to S_+, then

$$\begin{aligned} \overline{Z(I)} &= Z(I(Z(I))) \\ &= Z(\sqrt{I}) = Z(I) \end{aligned}$$

[i.e. S S^+ $I(Y)$
ϕ Y]

Notice that for $Y \subseteq \mathbb{P}^n$, $I(Y)$ is never S_+, since if it were the only point that could be in Y is $(0,\dots,0)$, but this is impossible. The above corresponce is inclusion reversing and bijective by 2.3 (c), (d), (e), (a), (b), since $I(Z(S)) = I(\phi) = S$.

(b) Suppose $I(Y)$ is prime and that $Y = Y_1 \cup Y_2$, Y_1, Y_2 proper closed subsets. Then $I(Y) = I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$. Since the ideal $I(Y)$ is prime, wlog $I(Y_1) \subseteq I(Y)$. Hence $Y = Z(I(Y)) \subseteq Z(I(Y_1)) = Y_1$. ($Y_1$ is closed in $\mathbb{P}^n$). This shows that Y is irreducible. Conversely, suppose Y is irreducible and that f, g are homogenous with $fg \in I(Y)$. Then $Y \subseteq Z(fg) = Z(f) \cup Z(g)$. Hence $Y = (Z(f) \cap Y) \cup (Z(g) \cap Y)$, a union of two closed subsets. Since Y is irreducible, wlog $Y = Y \cap Z(f)$, so that $Y \subseteq Z(f)$. Hence $f \in I(Y)$, and this shows that $I(Y)$ is prime.

(c) This is the old problem that says "there is no polynomial zero everywhere except $(0,\dots,0)$" (see p4 of these notes). Given this, it follows that $I(\mathbb{P}^n) = \{f \in S \mid f(P) = 0,\ P \in \mathbb{P}^n\} = \{f \in S \mid f \in (x_0 - a_0, \dots, x_n - a_n),\ \text{not all } a_i = 0\} = \bigcap m$, where the intersection is over all maximal ideals other than $(x_0,\dots,x_n)$. But by assumption, this is 0, since S is Hilbert implies the Jacobson radical is 0, and the above claim "..." implies we cannot have f , $f \notin (x_0,\dots,x_n)$. and $f \in I(\mathbb{P}^n)$. Hence since 0 is prime, $I(\mathbb{P}^n) = 0$, $\mathbb{P}^n$ is irreducible. (or $Z(0) = \mathbb{P}^n$

2.5 (a) $\mathbb{P}^n$ can be covered by the U_i, which are homeomorphic to $\mathbb{A}^n$ and are thus noetherian. Hence the result (and (b) using Prop 1.5) follow from: If X is a topological space with a finite open cover $X = \bigcup_i V_i$ with each V_i noetherian in its induced topology, then X is noetherian. Say $Q_1 \supseteq Q_2 \supseteq \cdots$ is a descending chain of closed sets in X. Then

$$Q_1 \cap U_i \supseteq Q_2 \cap U_i \supseteq \cdots$$

is a descending chain of closed sets in U_i. Let N_i be s.t. $Q_j \cap U_i = Q_{N_i} \cap U_i$ $\forall j > N_i$. If $N = \max\{N_i\}$ then $Q_j \cap U_i = Q_N \cap U_i$ $\forall i$, $\forall j \geq N$. Thus

$$Q_j = \bigcup_i Q_j \cap U_i = \bigcup_i Q_N \cap U_i = Q_N \quad \forall j \geq N$$

Q2.6 Let $Y \subseteq \mathbb{P}^n$ be a projective variety with homogenous coordinate ring $S(Y) = S/I(Y)$. We show in our Section 3 (Morphism notes) (see Proof of Theorem 4.3 and subsequent Notes) that if $\varphi_i : U_i \longrightarrow \mathbb{A}^n$ are the usual isomorphisms and Y_i is the affine variety $\varphi_i(Y \cap U_i)$ [for $Y \cap U_i \neq \emptyset$] then $A(Y_i)$ is isomorphic as a k-algebra to the ring $S(Y)_{(x_i)}$, which is the subring of $S(Y)_{x_i}$ consisting of pairs g/x_i^n with g of deg. i. (Note that $Y \cap U_i \neq \emptyset$ insures $x_i \notin I(Y)$ so $x_i \neq 0$ in $S(Y)$). A further note shows that $S(Y)_{x_i}$ is isomorphic as a k-algebra to $S(Y)_{(x_i)}[z,z^{-1}]$ (Laurent polynomials). As we have noted previously, $R[z,z^{-1}] \cong R[z]_z$, so

$$S(Y)_{x_i} \cong S(Y)_{(x_i)}[z,z^{-1}] \cong A(Y_i)[z]_z$$

as k-algebras. But if R is a domain, Q its quotient field $k \subseteq R$ a field, and R' any subring $R \subseteq R' \subseteq Q$ then the quotient field of R' is isomorphic as a k-algebra to Q (get $Q' \to Q$ must be injective since Q' field, and image is a field containing R, must be Q). Hence the quotient field Q of $S(Y)$ is isomorphic as a k-algebra to the quotient field of $A(Y_i)[z]$, call it Q'. But both $S(Y)$ and $A(Y_i)[z]$ are domains and f.g. k-algebras, so by (1.8A)

$$\dim S(Y) = \text{tr.deg.}\, Q/k = \text{tr.deg.}\, Q'/k = \dim(A(Y_i)[z]) = 1 + \dim A(Y_i)$$

where the last step follows from our Atiyah & Macdonald Notes. But by (1.7) this gives $\dim S(Y) = 1 + \dim Y_i$. This shows that whenever $Y \cap U_i$ is nonempty, $\dim Y_i$ agrees for all i. But $Y \cap U_i$ forms an open cover of Y, so by Ex 1.10 $\dim Y = \sup\{\dim(Y \cap U_i) \mid Y \cap U_i \neq \emptyset\}$. But $Y \cap U_i$ is homeomorphic to Y_i, so $\dim Y = \dim Y_i$, whenever $Y \cap U_i \neq \emptyset$. Hence

$$\dim S(Y) = 1 + \dim Y$$

as required.

Q2.7 (a) Consider the open cover $\mathbb{P}^n = U_0 \cup U_1 \cup \cdots \cup U_n$. Each U_i is homeomorphic to $\mathbb{A}^n$, so by Ex 1.10 $\dim \mathbb{P}^n = \sup \dim U_i = \dim \mathbb{A}^n = n$.

(b). Let $Z \subseteq \mathbb{P}^n$ be a projective variety and $Y \subseteq Z$ a nonempty open set. Let $J = \{0 \leq j \leq n \mid Y \cap U_j \neq \emptyset\}$. For each $j \in J$ $Y \cap U_j$ is homeomorphic to the open subset $\varphi_j(Y \cap U_j)$ of the affine variety $\varphi_j(Z \cap U_j) \subseteq \mathbb{A}^n$. By Ex 2.6 this implies $\dim Y \cap U_j = \dim Z \cap U_j = \dim Z$ (by 2.6)(and Prop 1.10). But then by Ex 1.10 $\dim Y = \sup \dim Y \cap U_j = \dim Z$, as required. (Note that $\overline{Y} = Z$)(using Ex 1.6)

Q2.8 Let $Y \subseteq \mathbb{P}^n$ be a projective variety with ideal $\mathfrak{p} = I(Y) \subseteq k[x_0, \ldots, x_n]$. Let $Z \subseteq \mathbb{A}^{n+1}$ be the affine variety determined by $\mathfrak{p}$. By Ex 2.6

$$\dim Y = \dim S(Y) - 1 = \dim \frac{k[x_0, \ldots, x_n]}{\mathfrak{p}} - 1$$
$$= \dim A(Z) - 1$$

So $\dim Y = n - 1$ iff. $\dim A(Z) = n = (n+1) - 1$ and by (Prop 1.13) this is iff. $\mathfrak{p} = (f)$ where f is a nonconstant irreducible polynomial. But since $\mathfrak{p}$ is homogenous, f must be homogenous (see an earlier note). This completes the proof. (Note the irrelevant prime $(x_0, \ldots, x_n)$ can never be principal so we never have $Z(f) = \emptyset$ in $\mathbb{P}^n$ for an irreducible homogenous f)

Q2.9 *Projective Closure of an Affine Variety* Let $Y \subseteq \mathbb{A}^n$ be an affine variety, and use $\varphi_0 : \mathbb{A}^n \longrightarrow U_0$, $(a_1,\dots,a_n) \longmapsto (1, a_1, \dots, a_n)$ to identify Y with a subset W of $\mathbb{P}^n$. We call the closure $\overline{W}$ of W in $\mathbb{P}^n$ the projective closure of Y. Since φ_0 is a homeomorphism W is irreducible, so its closure is a projective variety.

(a) We claim that $I(\overline{W})$, which is the ideal generated by all the homogenous polynomials $f \in S = k[x_0,\dots,x_n]$ which are zero on $\overline{W}$ (since $Z(f)$ is closed, this is iff. f is zero on W), is generated by the collection

$$\beta(I(Y))$$

of homogenous polynomials (notation of the proof of (2.2)). Let J denote this ideal – i.e. $J = (\beta(I(Y)))$. Since for $f \in I(Y) \subseteq k[y_1,\dots,y_n]$, with order e, $\beta(f) = x_0^e f(x_1/x_0, \dots, x_n/x_0)$, and hence for $(1, a_1, \dots, a_n) \in W$, since $(a_1,\dots,a_n) \in Y$, $\beta(f)(1, a_1, \dots, a_n) = f(a_1,\dots,a_n) = 0$. Hence $Z(\beta(f)) \supseteq W$. Since $\beta(f)$ is homogenous $Z(\beta(f))$ is closed, hence $Z(\beta(f)) \supseteq \overline{W}$, and so $f \in I(Y) \Rightarrow \beta(f) \in I(\overline{W})$. It follows that, as claimed, $\beta(I(Y)) \subseteq I(\overline{W})$.

Now suppose $f \in I(\overline{W})$ – that is, there are homogenous polynomials $p_i \in S$ and arbitrary polynomials $a_i \in S$ with p_i zero on $\overline{W}$, s.t. $f = \sum a_i p_i$. Suppose p_i is homogenous of degree d. Then $\alpha(p_i)$ is $p_i(1, y_1, \dots, y_n)$, and since p_i is zero on W, $\alpha(p_i)$ is zero on Y. Hence $\alpha(p_i) \in I(Y)$. Now $\beta(\alpha(p_i))$ is formed from $\alpha(p_i)$ by letting f be the largest order of a monomial in $\alpha(p_i)$, and multiplying monomials by x_0 until we get a polynomial homogenous of order f. Since the monomial of p_i which becomes of order f in $\alpha(p_i)$ must have the lowest power of x_0,

$$x_0^k \beta(\alpha(p_i)) = p_i$$

for some k. Hence $p_i \in J$, and so $f \in J$ – so $I(\overline{W}) = J$ as required.

(b) Firstly, we claim that $I(Y) = (z - x^3, y - x^2)$. Clearly we have $\supseteq$. Suppose that $f \in I(Y)$, and

$$f = f_1(z - x^3) + f_2(y - x^2) + f_3(x)$$

then $f(t, t^2, t^3) = 0 \Rightarrow f_3(t) = 0$, $t \in k$. Hence $f_3 = 0$, and $f \in (z - x^3, y - x^2)$. If $W = \varphi_0(Y)$, then by (a), $I(\overline{W})$ is generated by $\beta(I(Y))$. We need

LEMMA Let $f, g \in k[y_1,\dots,y_n]$ where f has order e and g has order d, $d > e$. Then

$$\beta(f+g) = x_0^{d-e}\beta(f) + \beta(g)$$

This cannot be relaxed to $\geq$, because then leading (highest order) monomials would cancel: consider $xz - y^2 = xz - x^4 + x^4 - y^2$.

PROOF Firstly, we can apply β without collecting terms, since we are just multiplying each monomial by sufficient powers of x_0 to make $f+g$ be homogenous of degree d. Each monomial in g gets multiplied by $d -$ its order, so we just get $\beta(g)$. Each monomial in f has its order increased to $d \geq e$ – this can be achieved by getting them all to e (this is $\beta(f)$) and then multiplying everything by x_0^{d-e}. $\square$

Note also that if $f \in k[y_1,\dots,y_n]$ and x^α is a monomial that $\beta(f x^\alpha) = \beta(f) x^\alpha$.

LEMMA Let $f_1, \dots, f_n \in k[y_1,\dots,y_n]$ with the order of $f_i = e_i$ and $e_i < e_{i+1}$ $1 \leq i \leq n-1$. Then

$$\beta(f_1 + \dots + f_n) = x_0^{e_n - e_1}\beta(f_1) + x_0^{e_n - e_2}\beta(f_2) + \dots + x_0^{e_n - e_{n-1}}\beta(f_{n-1}) + \beta(f_n)$$
$$= \sum_{i=1}^{n} x_0^{e_n - e_i}\beta(f_i)$$

PROOF By induction on n. The above Lemma handles $n = 2$. Suppose it holds for $n = k-1$. Then

$$\beta(f_1 + \dots + f_k) = \beta\left(\sum_{i=1}^{k-1} f_i + f_k\right)$$
$$= x_0^{e_k - e_{k-1}}\beta\left(\sum_{i=1}^{k-1} f_i\right) + \beta(f_k)$$
$$= x_0^{e_k - e_{k-1}} \sum_{i=1}^{k-1} x_0^{e_{k-1} - e_i}\beta(f_i) + \beta(f_k)$$
$$= \sum_{i=1}^{k} x_0^{e_k - e_i}\beta(f_i). \quad \square$$

and finally,

LEMMA If $f, g \in k[y_1, \dots, y_n]$ then if the monomials in g all have distinct degrees,

$$\beta(fg) = \beta(f)\beta(g)$$

PROOF Let $g = \sum_{i=1}^{n} x_{\alpha_i} x^{\alpha_i}$ with the order of g being b. Then, assuming $|\alpha_i| < |\alpha_{i+1}|$, $1 \le i \le n$, (so $b = |\alpha_n|$)

$$\begin{aligned}
\beta(fg) &= \beta\left(f\left\{ \textstyle\sum_{i=1}^{n} x_{\alpha_i} x^{\alpha_i} \right\} \right) \\
&= \beta\left(\textstyle\sum_{i=1}^{n} x_{\alpha_i} f x^{\alpha_i} \right) \\
&= \textstyle\sum_{i=1}^{n} x_{\alpha_i} x_0^{b-|\alpha_i|} \beta(f) x^{\alpha_i} \\
&= \textstyle\sum_{i=1}^{n} x_0^{b-|\alpha_i|} \beta(f) x_{\alpha_i} x^{\alpha_i} \\
&= \beta(f)\beta(g)
\end{aligned}$$

Now let $f \in I(Y)$, $f = f_1(x,y,z)(z-x^3) + f_2(x,y,z)(y-x^2)$. Then let $e = \text{order } f_1$, $d = \text{order } f_2$

$$\beta(f) = \begin{cases} \beta(f_1(z-x^3)) + x_0^{d-e-1}\beta(f_2(y-x^2)) & e+3 > d+2 \\ x_0^{e-d+1}\beta(f_1(z-x^3)) + \beta(f_2(y-x^2)) & e+3 < d+2 \end{cases}$$

$$= \begin{cases} \beta(f_1)(zx_0^2 - x^3) + x_0^{d-e-1}\beta(f_2)(yx_0 - x^2) & e+3 > d+2 \\ \beta(f_1)x_0^{e-d+1}(zx_0^2 - x^3) + \beta(f_2)(yx_0 - x^2) & e+3 < d+2 \end{cases}$$

Notice we do not consider the case $e+3 = d+2$, because then we cannot apply $\beta(f+g) = x_0^{d-e}\beta(f) + \beta(g)$. So provided f can be written as above, with $e+3 \neq d+2$, $\beta(f)$ is a member of $(zx_0^2 - x^3, yx_0 - x^2)$. What happens when $e+3 = d+2$? For example, consider $xz - y^2 \in I(Y)$,

$$xz - y^2 = \underbrace{x}_{f_1}(z-x^3) + \underbrace{\{-(x^2+y)\}}_{f_2}(y-x^2)$$

then $e = o(f_1) = 1$, $d = o(f_2) = 2$, so $d = e+1$. Notice that $\beta(xz-y^2) = xz-y^2$, but $\beta(f_1(z-x^3)) + \beta(f_2(y-x^2))$ is $x_0^2(xz-y^2)$. (Thus in particular we have answered the question, since $xz - y^2 \notin (zx_0^2 - x^3, yx_0 - x^2)$ – to get z we would have to pick up x_0 as well, since everything else has a factor of x^2). We finish with the claim that <u>that's all</u>! – that is, we claim that $I(\overline{W}) = (zx_0^2 - x^3, yx_0 - x^2, xz - y^2)$. Consider $f \in I(Y)$, $f = f_1(z-x^3) + f_2(y-x^2)$. Then $\beta(f)$ will be in $(zx_0^2 - x^3, yx_0 - x^2)$ provided the highest order bits of $f_1(z-x^3)$ and $f_2(y-x^2)$ don't cancel (although we required $d > e$ in the first Lemma, the proof gives more). Let $f_1 = f_1' + h$, where h is homogenous and contains of all the monomials of f of the highest order $e = \text{order } f_1$. Similarly $f_2 = f_2' + g$. Then we are assuming $hx^3 = -gx^2$, or $hx = -g$. This means that

$$\begin{aligned}
f = f_1(z-x^3) + f_2(y-x^2) &= f_1'(z-x^3) + f_2'(y-x^2) + h(z-x^3) - hx(y-x^2) \\
&= f_1'(z-x^3) + f_2'(y-x^2) + h(z - x^3 - xy + x^3) \\
&= f_1'(z-x^3) + f_2'(y-x^2) + h(z-xy)
\end{aligned}$$

and we just repeat the process on $f_1'(z-x^3) + f_2'(y-x^2)$ (which have strictly smaller orders).

⌜NOTE: Above shows that the ideal of the twisted cubic in $\mathbb{P}^3$ is $(zx_0^2 - x^3, yx_0 - x^2, xz - y^2)$⌟

NOTE The Projective Closure of a Hypersurface is a Hypersurface

Let $Y \subseteq \mathbb{A}^n$ be the hypersurface $Y = Z(f)$ for a nonconstant irreducible polynomial $f \in k[x_1, \dots, x_n]$. We claim that the projective closure $Z \subseteq \mathbb{P}^n$ of Y is the hypersurface $Z(\beta(f))$ (in particular, we will show $\beta(f)$ is irreducible).

<u>LEMMA</u> If $f, g \in k[x_1, \dots, x_n]$ and $\beta : k[x_1, \dots, x_n] \longrightarrow k[x_0, x_1, \dots, x_n]$ is as before, then

$$\beta(fg) = \beta(f)\beta(g)$$

<u>PROOF</u> Write $f = f_0 + \dots + f_d$, $g = g_0 + \dots + g_e$, assuming $f_d \neq 0$, $g_e \neq 0$ (trivial if $f = 0$ or $g = 0$). Then

$$\begin{aligned} \beta(fg) &= \beta\Big(\sum_{i,j} f_i g_j\Big) = \sum_{i,j} x_0^{d+e-i-j} f_i g_j \\ &= \sum_{i,j} (x_0^{d-i} f_i)(x_0^{e-j} g_j) \\ &= \beta(f)\beta(g). \quad \square \end{aligned}$$

It follows from Ex 2.9 that $I(Z) = (\beta(f))$. Since $I(Z)$ is prime, $\beta(f)$ is irreducible (directly, there is at least one monomial in $\beta(f)$ not involving x_0, so if $\beta(f) = HG$, $f = \alpha\beta(f) = \alpha H \alpha G \Rightarrow$ say αH involves only x_0 and therefore is a unit (otherwise contradict homog. of $\beta(f)$)). This proves that the projective closure of a hypersurface is a hypersurface.

NOTE Projective Closure preserves dimension

Let $Y \subseteq \mathbb{A}^n$ be affine of dimension r. Let Z be the projective closure of Y. Then $Z \cap U_0 = \varphi_0(Y)$, so since Y is homeomorphic to $\varphi_0(Y)$ we have by Q2.7

$$\dim Z = \dim Z \cap U_0 = \dim Y$$

So the above note also follows by dimension calculations and Ex 2.8.

NOTE Alternative Generators of the Twisted Cubic in $\mathbb{P}^3$

Let $Y \subseteq \mathbb{A}^3$ be the twisted cubic $Y = \{(t, t^2, t^3) \mid t \in k\}$, Z the projective closure of Y. We have shown in Ex 2.9 that $I(Z)$ is the prime ideal $(wy - x^2, zw^2 - x^3, xz - y^2)$. (the projective coordinates being w, x, y, z). We claim that

$$I(Z) = (x^2 - wy,\ xy - wz,\ y^2 - xz) \qquad (1)$$

Clearly $xy - wz$ is zero on the tuples $(1, t, t^2, t^3)$ which are the image of Y in $U_0 \subseteq \mathbb{P}^3$. But $Z(xy - wz)$ is closed and thus contains Z, so a power of $xy - wz$ belongs to $I(Z)$. Hence $Z \subseteq Z(x^2 - wy, xy - wz, y^2 - xz)$. In the other direction, $x^3 - zw^2 = x(x^2 - wy) + w(xy - zw)$, so $I(Z) \subseteq (x^2 - wy, xy - wz, y^2 - xz)$, and $Z(x^2 - wy, xy - wz, y^2 - xz) \subseteq Z$, as required.

But then $I(Z) = \sqrt{(x^2 - wy, xy - wz, y^2 - xz)}$, so we have the equality in (1).

<u>NOTE</u> We have defined the projective closure of an affine variety $Y \subseteq \mathbb{A}^n$ using the isomorphism $\mathbb{A}^n \cong U_0 \subseteq \mathbb{P}^n$, but one may also use $\mathbb{A}^n \cong U_i$ for $1 \leq i \leq n$. Let $\varphi : \mathbb{P}^n \longrightarrow \mathbb{P}^n$ be the automorphism defined by the matrix

$$\begin{array}{cc} & \begin{matrix} & i & \end{matrix} \\ \begin{matrix} \\ \\ \\ i \\ \\ \\ \\ \end{matrix} & \begin{pmatrix} 0 & \cdots & 1 & \cdots & 0 \\ 1 & 0 & \cdots & \cdots & 0 \\ & & \vdots & & \\ 0 & \cdots & 1 & 0 & 0 \cdots \\ 0 & \cdots & 0 & 1 & \cdots \\ & & \vdots & & \\ 0 & \cdots & \cdots & \cdots & 1 \end{pmatrix} \end{array} \qquad \begin{aligned} x_0 &\longmapsto x_i \\ x_1 &\longmapsto x_0 \\ &\vdots \\ x_i &\longmapsto x_{i-1} \\ x_{i+1} &\longmapsto x_{i+1} \\ &\vdots \\ x_n &\longmapsto x_n \end{aligned}$$

Then φ identifies $Y \subseteq U_0$ with $Y \subseteq U_i$, so everything we've proved about the projective closure is independent of i. Including the proj. closure of $Z(f)$ being $Z(\beta(f))$ where β beefs up using x_i.

Linear Varieties

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A *linear* polynomial in $k[x_0, \dots, x_n]$ is a nonzero homogenous polynomial of degree 1:

$$f = a_0 x_0 + \dots + a_n x_n$$

Let $f_0, \dots, f_n$ be linear polynomials with associated tuples $(a_{i0}, \dots, a_{in})$ for $0 \leq 1 \leq n$. Assume that these tuples are linearly independent in k^{n+1} so that the matrix $A = (a_{ij})$ has row rank $n+1$, hence column rank $n+1$ and so the associated endomorphism of k^{n+1} is an isomorphism, implying that A is invertible. Let $B = (b_{ij})$ be the inverse of A. We define morphisms of k-algebras:

$$\varphi : k[x_0, \dots, x_n] \longrightarrow k[x_0, \dots, x_n]$$
$$\varphi(x_i) = f_i = \sum_j a_{ij} x_j$$

and

$$\phi : k[x_0, \dots, x_n] \longrightarrow k[x_0, \dots, x_n]$$
$$\phi(x_i) = \sum_k b_{ik} x_k$$

It is easy enough to check that $\varphi\phi = \phi\varphi = 1$. This isomorphism identifies the ideal $(f_0, \dots, f_n)$ with $(x_0, \dots, x_n)$, which is a prime ideal. We can now prove

Lemma 1. *Any ideal in $k[x_0, \dots, x_n]$ generated by linear polynomials is prime.*

Proof. Let $\mathfrak{a}$ be an ideal generated by linear polynomials. Clearly $\mathfrak{a}$ is a proper homogenous ideal. Even if $\mathfrak{a}$ is generated by an infinite number of linear polynomials, we can find a finite subset which generate $\mathfrak{a}$ (see an earlier Note). If linear polynomials $g_1, \dots, g_s$ generate $\mathfrak{a}$ but are linearly dependent, then we can omit one of the g_i and still have a generating set. In this way we produce a set of generators $f_0, \dots, f_r$ for $\mathfrak{a}$ which are linearly independent (since the coefficients form tuples in k^{n+1} it is clear that $r \leq n$). Say

$$f_i = a_{i0} x_0 + \dots + a_{in} x_n$$

for $0 \leq i \leq r$. Then the set of linearly independent vectors $\mathbf{a}_i = (a_{i0}, \dots, a_{in})$ $(0 \leq i \leq r)$ can be extended to a basis $\mathbf{a}_0, \dots, \mathbf{a}_r, \mathbf{a}_{r+1}, \dots, \mathbf{a}_n$ for k^{n+1} and we define linear polynomials $f_{r+1}, \dots, f_n$ using these new tuples. Then $f_0, \dots, f_n$ induces the isomorphism φ of the above discussion, under which the ideal $\mathfrak{a} = (f_0, \dots, f_r)$ corresponds to the prime ideal $(x_0, \dots, x_r)$. Hence $\mathfrak{a}$ is prime. $\square$

The above proof also shows the following:

Corollary 2. *If $\mathfrak{a}$ is an ideal in $k[x_0, \ldots, x_n]$ generated by k linearly independent linear polynomials, then the height of $\mathfrak{a}$ is k. In particular, the number of elements in any set of linearly independent linear generators is the same.*

Definition 1. A *linear variety* in $\mathbb{P}^n$ is a projective variety of the form $Y = Z(\mathfrak{a})$ where $\mathfrak{a}$ is an ideal generated by linear polynomials.

Since any ideal generated by linear polynomials is prime homogenous, if $Y = Z(\mathfrak{a})$ is a linear variety then $I(Y) = \mathfrak{a}$. So a projective variety Y is linear if and only if $I(Y)$ can be generated by linear polynomials.

Definition 2. A *hyperplane* in $\mathbb{P}^n$ is a projective variety of the form $Y = Z(f)$ where f is a linear polynomial. By Corollary 2 or Exercise 2.8 of Hartshorne, any hyperplane has dimension $n - 1$.

Note that if f is any linear polynomial, $Z(f)$ is empty iff. (f) is the irrelevant maximal ideal $(x_0, \ldots, x_n)$ which is impossible since f cannot be a unit multiple of $x_0, \ldots, x_n$. So $Z(f)$ is a hyperplane.

The following gives our solution to Exercise 2.11(a) of Hartshorne.

Lemma 3. *Any linear variety in $\mathbb{P}^n$ is an intersection of hyperplanes. Conversely, any nonempty intersection of hyperplanes is a linear variety.*

Proof. Let Y be a linear variety, and suppose $\mathfrak{a} = (f_0, \ldots, f_r)$ where the f_i are linear polynomials. Then

$$Y = Z(\mathfrak{a}) = Z(\sum_i (f_i)) = \bigcap_i Z(f_i)$$

So Y is an intersection of hyperplanes. Conversely if $f_0, \ldots, f_r$ are linear polynomials and $Q = Z(f_0) \cap \ldots \cap Z(f_r)$ is nonempty, then $Q = Z(f_0, \ldots, f_r)$ and since the ideal $(f_0, \ldots, f_r)$ is prime we have $I(Q) = (f_0, \ldots, f_r)$. Hence Q is a linear variety. $\square$

For general homogenous irreducible polynomials $p_0, \ldots, p_r$ it is *not true* that

$$I(Z(p_0) \cap \ldots \cap Z(p_r)) = (p_0, \ldots, p_r)$$

See Exercise 2.16 for a counterexample. But if the p_i are *linear* and the intersection is nonempty, then this equality holds, since $(p_0, \ldots, p_r)$ is prime:

$$I(Z(p_0) \cap \ldots \cap Z(p_r)) = I(Z(p_0, \ldots, p_r)) = \sqrt{(p_0, \ldots, p_r)} = (p_0, \ldots, p_r)$$

Lemma 4. *A linear variety Y in $\mathbb{P}^n$ has dimension r if and only if $I(Y)$ is minimally generated by $n - r$ linear polynomials (equivalently, is generated by $n - r$ linearly independent linear polynomials).*

Proof. By Exercise 2.6 we have

$$r = dimY = dimS(Y) - 1 = n - \text{height} I(Y)$$

But by Corollary 2 the height of $I(Y)$ is the unique integer k for which there exists a set of linearly independent linear generators. Any minimal generating set consisting of linear polynomials must be linearly independent, so this completes the proof. $\square$

Suppose $\mathfrak{a}$ can be generated by $n+1$ linearly independent linear polynomials $f_0, \ldots, f_n$. Then the associated tuples $\mathbf{a}_0, \ldots, \mathbf{a}_n$ must span k^{n+1}, implying that

$$\mathfrak{a} = (f_0, \ldots, f_n) = (x_0, \ldots, x_n)$$

And hence $Z(\mathfrak{a}) = \emptyset$ in $\mathbb{P}^n$. This proves

Lemma 5. *If $f_0, \ldots, f_r$ are linear polynomials and $Z(f_0) \cap \ldots \cap Z(f_r)$ is empty then $r \geq n$.*

Proof. By assumption $\mathfrak{a} = (f_0, \ldots, f_r) = (x_0, \ldots, x_n)$. The set $f_0, \ldots, f_r$ can be refined to a linearly independent set of linear generators for $\mathfrak{a}$, which must have $n+1$ elements. Hence $r \geq n$. □

Finally we answer part (d) of the Exercise.

Lemma 6. *If Y, Z are linear varieties in $\mathbb{P}^n$ of respective dimensions r, s and $r+s \geq n$, then $Y \cap Z$ is nonempty and is a linear variety of dimension $\geq r+s-n$.*

Proof. We can write $Y = Z(f_1) \cap \ldots \cap Z(f_{n-r})$ and $Z = Z(g_1) \cap \ldots \cap Z(g_{n-s})$ where the f_i and g_j are linearly independent generators for the ideals $I(Y), I(Z)$ respectively. Thus $Y \cap Z$ is the intersection of $2n - r - s$ hyperplanes. Provided $r + s \geq n$ it follows from the previous Lemma that this intersection is nonempty. Then we can refine the list $f_1, \ldots, f_{n-r}, g_1, \ldots, g_{n-s}$ to find a set of linearly independent generators of $I(Y \cap Z) = I(Y) + I(Z)$ with $q \leq 2n - r - s$ elements. Then

$$dim(Y \cap Z) = n - q \geq r + s - n$$

□

This completes Exercise 2.11. We keep our old proofs because they use different techniques which may be useful at some point. Note only that the proof of (a) part (ii) implies (i) is incorrect in the written notes. Other than that, the solutions are valid.

The ideal of a point in affine space is a maximal ideal in $k[x_1, \ldots, x_n]$, which must have the form $(x_1 - a_1, \ldots, x_n - a_n)$ for some a_i since k is algebraically closed. These maximal ideals are certainly not homogenous! So no maximal ideal of $k[x_0, \ldots, x_n]$ can occur as the ideal of any algebraic set in $\mathbb{P}^n$. Instead, the ideals of projective points are prime ideals of coheight 1:

Lemma 7. *If $P = (a_0, \ldots, a_n)$ is a point of $\mathbb{P}^n$ with $a_i \neq 0$, then*

$$I(P) = (a_i x_0 - a_0 x_i, \ldots, a_i x_n - a_n x_i)$$

Moreover $I(P)$ is a prime ideal of coheight 1 in $k[x_0, \ldots, x_n]$.

Proof. The ideal $\mathfrak{a} = (a_i x_0 - a_0 x_i, \ldots, a_i x_n - a_n x_i)$ is homogenous and $Z(\mathfrak{a}) = \{P\}$. Since $\mathfrak{a}$ is generated by n linearly independent linear polynomials it is a prime ideal of height n. Hence $\mathfrak{a} = I(P)$ and $I(P)$ has coheight 1. □

The maximal ideals containing $I(P)$ are those ideals corresponding to tuples $(b_0, \ldots, b_n)$ which are equal to P in $\mathbb{P}^n$, together with the irrelevant maximal ideal which corresponds to $(0, \ldots, 0)$.

The following is proved earlier in our notes:

Lemma 8. *Any hyperplane in $\mathbb{P}^n$ is isomorphic to $\mathbb{P}^{n-1}$.*

Proof. Let the hyperplane be $Z(f)$ where $f = a_0x_0 + \ldots + a_nx_n$ is a linear polynomial. We select some i with $a_i \neq 0$ and define the isomorphism $\varphi : \mathbb{P}^{n-1} \longrightarrow Z(f)$ by mapping $(c_0, \ldots, c_{n-1})$ to

$$(c_0, \ldots, c_{i-1}, -\frac{1}{a_i}(c_0a_0 + \cdots + c_{i-1}a_{i-1} + c_ia_{i+1} + \cdots + a_nc_{n-1}), c_i, \ldots, c_{n-1})$$

□

Lemma 9. *Let two hyperplanes H, K in $\mathbb{P}^n$ have nonempty intersection. Identifying H with $\mathbb{P}^{n-1}$ induces an isomorphism of $H \cap K$ with a hyperplane of $\mathbb{P}^{n-1}$.*

Proof. Let $H = Z(f)$ and $K = Z(g)$ where

$$\begin{aligned} f &= a_0x_0 + \cdots + a_nx_n \\ g &= b_0x_0 + \cdots + b_nx_n \end{aligned}$$

Select some i with $a_i \neq 0$ and let $\varphi : \mathbb{P}^{n-1} \longrightarrow H$ be the isomorphism of Lemma 8. It is not difficult to check that φ identifies $H \cap K$ with the hyperplane

$$\begin{aligned} &(b_0 - \frac{b_i}{a_i}a_0)x_0 + \cdots + (b_{i-1} - \frac{b_i}{a_i}a_{i-1})x_{i-1} + \\ &(b_{i+1} - \frac{b_i}{a_i}a_{i+1})x_i + \cdots + (b_n - \frac{b_i}{a_i}a_n)x_{n-1} = 0 \end{aligned}$$

□

Corollary 10. *A linear variety of dimension $r \geq 1$ in $\mathbb{P}^n$ is isomorphic to $\mathbb{P}^r$.*

Proof. By induction on n. If $n = 1$ then this is trivial, since there can be no such linear variety. If $n = 2$ then we need only consider linear varieties of dimension $r = 1$. But these are hyperplanes in $\mathbb{P}^2$, so we use Lemma 8. So assume the result is true for $n - 1$ where $n > 2$ and let $Y \subseteq \mathbb{P}^n$ be a linear variety of dimension $r \geq 1$. Note that $r \leq n - 1$ by Ex 1.10. If $r = n - 1$ then we are in the situation of Lemma 8. So assume $I(Y)$ is generated by $n - r \geq 2$ linearly independent linear polynomials

$$I(Y) = (f_1, \ldots, f_{n-r})$$

Then $Z(f_1)$ is a hyperplane in $\mathbb{P}^n$ and is thus isomorphic to $\mathbb{P}^{n-1}$. Now

$$Y = \bigcap_{i=1}^{n-r} Z(f_i) = \bigcap_{i=2}^{n-r} Z(f_1) \cap Z(f_i)$$

Considered as closed subsets of $\mathbb{P}^{n-1}$ the $n - r - 1$ sets $Z(f_1) \cap Z(f_i)$ are hyperplanes by Lemma 9, and it follows that Y is a linear variety in $\mathbb{P}^{n-1}$. Hence by the inductive hypothesis Y is isomorphic to $\mathbb{P}^r$. □

Definition 3. A *line* in $\mathbb{P}^n$ is a linear variety of dimension 1.

So lines arise as the zero sets of ideals $\mathfrak{a}$ generated by $n-1$ linearly independent linear polynomials, and any line is isomorphic to $\mathbb{P}^1$. There are no lines in $\mathbb{P}^1$.

Proposition 11. *Two distinct lines in $\mathbb{P}^n$ meet at most one point ($n \geq 2$).*

Proof. Let H, K be two distinct lines in $\mathbb{P}^n$ (we must have $n \geq 2$). Then $I(H) = (f_1, \dots, f_{n-1})$ and $I(K) = (g_1, \dots, g_{n-1})$. Write

$$f_i = a_{i0}x_0 + \cdots + a_{in}x_n$$
$$g_i = b_{i0}x_0 + \cdots + a_{in}x_n$$

The fact that the lines H, K are distinct means that the matrix

$$A = \begin{pmatrix} a_{10} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{(n-1)0} & \cdots & a_{(n-1)n} \\ b_{10} & \cdots & b_{1n} \\ \vdots & & \vdots \\ b_{(n-1)0} & \cdots & b_{(n-1)n} \end{pmatrix}$$

has row-rank $\geq n$. Hence the kernel of A has dimension ≤ 1 and so the two lines meet at most one point of $\mathbb{P}^n$. $\square$

Proposition 12. *Given two distinct points in $\mathbb{P}^n$ ($n \geq 2$) there is a unique line going through them.*

Proof. Let $(a_0, \dots, a_n)$ and $(b_0, \dots, b_n)$ be distinct points. We want to find a linear variety of dimension 1 in $\mathbb{P}^n$ containing both points. That is, we need $n-1$ linearly independent linear polynomials

$$f_i = c_{i0}x_0 + \cdots + c_{in}x_n \qquad 1 \leq i \leq n-1$$

with

$$c_{i0}a_0 + \cdots + c_{in}a_n = 0$$
$$c_{i0}b_0 + \cdots + c_{in}b_n = 0$$

for $1 \leq i \leq n-1$. That is, the tuples $(c_{i0}, \dots, c_{in})$ should be nonzero solutions of the matrix

$$A = \begin{pmatrix} a_0 & \cdots & a_n \\ b_0 & \cdots & b_n \end{pmatrix}$$

Since the two points are distinct this matrix has col-rank 2 and hence a kernel of dimension $n-1$. Take a basis for the kernel and use these vectors to produce the required polynomials f_i. This gives a line in $\mathbb{P}^n$ going through both points, and this line is unique by the previous Proposition. $\square$

Let H be a hyperplane in $\mathbb{P}^n$ and L a line. Then H, L are linear varieties of respective dimensions $n-1, 1$ so that the intersection $H \cap L$ is nonempty by Lemma 6. If $I(H) = (f)$ and $I(L) = (g_1, \dots, g_{n-1})$ then

$$H \cap L = Z(f) \cap Z(g_1) \cap \ldots \cap Z(g_{n-1})$$
$$I(H \cap L) = (f, g_1, \dots, g_{n-1})$$

Provided L is not contained in H, the polynomials $f, g_1, \ldots, g_{n-1}$ are linearly independent, so $H \cap L$ has dimension 0. Since $H \cap L$ is a projective variety, it is irreducible and hence $H \cap L$ is a single point. So provided the line is not contained in the hyperplane, the two meet at precisely one point.

Q2.11 Linear Varieties in $\mathbb{P}^n$ A linear polynomial in $k[x_0,\dots,x_n]$ is a homogenous polynomial (nonzero) $a_0x_0 + \cdots + a_nx_n$ of degree 1. A hypersurface defined by a linear polynomial is called a <u>hyperplane</u> (Clearly any linear polynomial is irreducible if it is nonzero). Hence any hyperplane has dimension $n-1$ by Ex 2.8.

> Our proof of (ii) $\Rightarrow$ (i) here is incorrect. See the TeX notes.

(a) Let $Y \subseteq \mathbb{P}^n$ be a nonempty closed set. We claim the following are equivalent:

(i) $I(Y)$ can be generated by linear polynomials

(ii) Y can be written as an intersection of hyperplanes

<u>PROOF</u> (i)$\Rightarrow$(ii) Say $I(Y) = (f_1,\dots,f_m)$ where f_i are nonzero linear polynomials (we may assume the list is finite, since if an ideal is generated by an infinite list it is generated by a finite subset — see an earlier note). Then $Y = Z(I(Y)) = Z(\sum_i (f_i)) = Z(f_1) \cap \cdots \cap Z(f_m)$.

(ii)$\Rightarrow$(i) If $Y = \bigcap_i Z(f_i)$ then $I(Y) = (\{f_i\})$ so we may assume $Y = Z(f_1) \cap \cdots \cap Z(f_m)$, i.e. $I(Y) = (f_1,\dots,f_m)$. $\square$

In either case, we call Y a <u>linear algebraic set</u>, or <u>linear variety</u> if Y is irreducible (In what follows we say "linear variety" but proofs are all true for "linear alg set")

<u>WARNING</u> Our definition of a linear algebraic set includes the "nonempty" condition. For example $\mathfrak{a} = (x_0,\dots,x_n) = S_+$ is clearly generated by linear polynomials but $Z(\mathfrak{a}) = \emptyset$. <u>However</u> if $Y \subseteq \mathbb{P}^n$ is <u>any</u> closed set and $I(Y)$ (which is S if $Y=\emptyset$) is generated by linear polynomials, then Y cannot be empty (since you can't gen. S with linear polynomials)

<u>NOTE</u> Every $(a_0,\dots,a_n) \in \mathbb{P}^n$ determines a hyperplane (this is well-defined) and this assignment is injective.

(b) First see our Note: Morphisms $\mathbb{P}^n \longrightarrow \mathbb{P}^m$. First we prove

<u>LEMMA</u> Let $f = a_0y_0 + \cdots + a_ny_n$ for $(a_0,\dots,a_n) \in \mathbb{P}^n$, and let $\varphi: \mathbb{P}^{n-1} \longrightarrow Z(f) \subseteq \mathbb{P}^n$ be the isomorphism discussed earlier. If $(b_0,\dots,b_n) \in \mathbb{P}^n$ determines another linear polynomial g then φ identifies the closed set $Z(f) \cap Z(g) \subseteq \mathbb{P}^n$ with the following hyperplane in $\mathbb{P}^{n-1}$:

$$\begin{aligned}\left(b_0 - \tfrac{b_i}{a_i}a_0\right)x_0 + \cdots + \left(b_{i-1} - \tfrac{b_i}{a_i}a_{i-1}\right)x_{i-1} + \left(b_{i+1} - \tfrac{b_i}{a_i}a_{i+1}\right)x_i \\ + \cdots \left(b_n - \tfrac{b_i}{a_i}a_n\right)x_{n-1} = 0\end{aligned} \tag{1}$$

(Assuming $Z(f) \cap Z(g) \neq \emptyset$)(and $(a_0,\dots,a_n) \neq (b_0,\dots,b_n)$ in $\mathbb{P}^n$)

<u>PROOF</u> Assuming $Z(f) \cap Z(g) \neq \emptyset$, $Z(f) \cap Z(g)$ is a nonempty closed subset of $Z(f)$. Since $Z(f) \cong \mathbb{P}^{n-1}$ under φ, we can write (taking $a_i \neq 0$)

$$\begin{aligned}\varphi^{-1}(Z(f) \cap Z(g)) &= \{(c_0,\dots,c_{n-1}) \mid \varphi(c_0,\dots,c_{n-1}) \in Z(g)\} \\ &= \{(c_0,\dots,c_{n-1}) \mid (c_0,\dots,c_{i-1}, \tfrac{1}{a_i}(c_0a_0 + \cdots + c_{i-1}a_{i-1} + c_ia_{i+1} + \\ &\qquad \cdots + a_nc_{n-1}), c_i,\dots,c_{n-1}) \in Z(g)\} \\ &= Z(h) \quad \text{where } h \text{ is the polynomial (1)}\end{aligned}$$

Notice that all the coefficients in (1) are zero iff. $(a_0,\dots,a_n) \sim (b_0,\dots,b_n)$. $\square$

Let $Y \subseteq \mathbb{P}^n$ be a linear variety. Then we can write $Y = Z(f_1) \cap \cdots \cap Z(f_r)$ for a finite collection $\{f_i\}$ of linear polynomials. Note that for nonzero linear polynomials f,g determined by $(a_0,\dots,a_n)$ and $(b_0,\dots,b_n)$, $Z(f) = Z(g)$ iff. $(a_0,\dots,a_n) = (b_0,\dots,b_n)$ in $\mathbb{P}^n$. So we may assume all f_i are determined by tuples distinct in $\mathbb{P}^n$. Moreover we can refine the intersection to ensure that $Z(f_i) \not\supseteq \bigcap_{j\neq i} Z(f_j)$ $\forall i$. Call such a decomposition a <u>minimal</u> decomposition of Y (i.e. with all f_i associated with distinct elements of $\mathbb{P}^n$ and the intersection condition).

<u>LEMMA</u> If $Y \subseteq \mathbb{P}^n$ is a linear variety with minimal decomposition $Y = Z(f_1) \cap \cdots \cap Z(f_r)$ then $r \leq n$ and $\dim Y = n - r$.

<u>PROOF</u> By induction on $n \geq 1$. In $\mathbb{P}^1$ a linear variety is a point (and all points are linear varieties). So the intersection of two hyperplanes (distinct) is empty. So if $Y = Z(f_1) \cap \cdots \cap Z(f_r)$ is minimal $r=1$ and clearly $\dim Y = 0 = 1-1$. Now assume the Lemma holds for $n-1$, with $n > 1$. Let $Y = Z(f_1) \cap \cdots \cap Z(f_r)$ be a minimal decomposition. Then $Z(f_1) \cap Z(f_2), \dots, Z(f_1) \cap Z(f_r)$ is a list of $r-1$ distinct, nonempty closed subsets of $Z(f_1)$, (distinct by minimality of the decomposition), each identified with a hyperplane in $\mathbb{P}^{n-1}$. Identify Y with a closed subset of $\mathbb{P}^{n-1}$. Then the above list forms a minimal decomposition of Y in $\mathbb{P}^{n-1}$. By the inductive hypothesis $r-1 \leq n-1$, so $r \leq n$, and

$$\dim Y = (n-1) - (r-1) = n - r \quad \square$$

Now to the actual Exercise! Let $Y \subseteq \mathbb{P}^n$ have dimension $n-r$ $(0 \le r \le n)$. If Y is a linear variety we can write $Y = Z(f_1) \cap \cdots \cap Z(f_t)$ for some linear polynomials $f_1, \dots, f_t$. Assume the $f_1, \dots, f_t$ are a minimal generating set of $I(Y)$. Then their decomposition of Y must also be minimal, for otherwise we could omit terms and generate Y with fewer of the f_i. (We may assume $r > 0$ and $I(Y) \neq 0$ since hyperplanes are proper). Then by the Lemma $t \le n$ and $n - r = \dim Y = n - t$ so $t = r$. So $I(Y)$ is minimally generated by r elements. (Any generating set contains a minimal one)

For (b) we need the following Lemmas

dim must be

NOTE The above shows that a linear variety of dimension r in $\mathbb{P}^n$ $(0 \le r < n)$ has a minimal decomposition as the intersection of $n-r$ hyperplanes.

LEMMA Let $f = a_0x_0 + \cdots + a_nx_n$ be a linear polynomial (nonzero). Then $Z(f)$ is nonempty, in $\mathbb{P}^n$.

PROOF The element f is irreducible, so $\mathfrak{p} = (f)$ is prime and $Z(\mathfrak{f}) = \emptyset$ iff. $\mathfrak{p} = S_+$ by Ex 2.2. But this impossible, since f cannot be a unit multiple of $x_0, \dots, x_n$ $(n \geq 1)$. Hence $\mathfrak{p} \neq S_+$ and $Z(f)$ is nonempty. □

LEMMA If $f = a_0x_0 + \cdots + a_nx_n$ and $g = b_0x_0 + \cdots + b_nx_n$ are linear polynomials, then $Z(f) \cap Z(g)$ is nonempty (two hyperplanes meet) in $\mathbb{P}^n$.

PROOF If $(a_0, \dots, a_n)$ and $(b_0, \dots, b_n)$ determine the same tuple in $\mathbb{P}^n$ then $Z(f) = Z(g)$ so the result follows from the previous Lemma. Otherwise there are two cases:

Ⓐ There is no $0 \le i \le n$ with both $a_i \neq 0$ and $b_i \neq 0$. If $a_i = b_i = 0$ for some i then $(0, \dots, \overset{i}{1}, 0, \dots, 0) \in Z(f) \cap Z(g)$ and we are done. Otherwise not all the b_i are nonzero, so the nonzero coefficients define a linear polynomial in $\le n-1$ variables, which admits a nonzero solution by the above. Pad this solution with zeros (in all the spots $a_i \neq 0$) and you have an element of $Z(f) \cap Z(g)$.

Ⓑ $a_i \neq 0$ and $b_i \neq 0$ for some $0 \le i \le n$. Since $(a_0, \dots, a_n) \neq (b_0, \dots, b_n)$ in $\mathbb{P}^n$ the linear polynomial in $n-1$ variables

$$\left(b_0 - \frac{b_i}{a_i}a_0\right)y_0 + \left(b_1 - \frac{b_i}{a_i}a_0\right)y_1 + \cdots$$

$$\cdots + \left(b_{i-1} - \frac{b_i}{a_i}a_{i-1}\right)y_{i-1} + \left(b_{i+1} - \frac{b_i}{a_i}a_{i+1}\right)y_{i+1} + \cdots$$

$$\cdots + \left(b_n - \frac{b_i}{a_i}a_n\right)y_n$$

is nonzero, hence has a nonzero solution $c_0, c_1, \dots, c_{i-1}, c_{i+1}, \dots, c_n$. Put

$$c_i = -1/a_i(a_0c_0 + a_1c_1 + \cdots + a_{i-1}c_{i-1} + a_{i+1}c_{i+1} + \cdots + a_nc_n)$$

and we have an element in $Z(f) \cap Z(g)$. □

LEMMA If $f_1, \dots, f_r$ are linear polynomials and $Z(f_1) \cap \cdots \cap Z(f_r) = \emptyset$ in $\mathbb{P}^n$ then $r \geq n+1$.

PROOF By induction on $n \geq 1$. If $n = 1$ then for any linear polynomial f $Z(f)$ is a point, so to get $\emptyset$ you must intersect $2 \geq 2 = n+1$ of them. Assume it is true for $n-1$. If $Z(f_1) \cap \cdots \cap Z(f_r) = \emptyset$ we may assume this is a minimal intersection (otherwise through out terms and get $s \geq n$ with $s \le r$). The intersections $Z(f_1) \cap Z(f_2), \dots, Z(f_1) \cap Z(f_r)$ are nonempty (by previous Lemma) closed, distinct hyperplanes in $\mathbb{P}^{n-1} \cong Z(f_1)$, which intersect to give $\emptyset$ in $\mathbb{P}^{n-1}$. By the inductive hypothesis $r - 1 \geq n - 1 + 1$ so $r \geq n+1$ as required. □ (Ex. $n=3$ $\emptyset = Z(x_0) \cap Z(x_1) \cap Z(x_2) \cap Z(x_3)$)

(b) Let Y, Z be linear varieties in $\mathbb{P}^n$, of dimensions r, s respectively. Then Y is the intersection of $n-r$ hyperplanes, Z the intersection of $n-s$ hyperplanes, so $Y \cap Z$ is the intersection of $2n-r-s$ hyperplanes. By the previous Lemma if $Y \cap Z = \emptyset$ then $2n - r - s \geq n+1$ so $r + s \le n - 1 < n$. Hence if $r + s - n \geq 0$ then $Y \cap Z \neq \emptyset$. If $Y \cap Z$ is nonempty we can refine the intersection to get a minimal decomposition of $Y \cap Z$ with $q \le 2n - r - s$ hyperplanes. Then

$$\dim Y \cap Z = n - q \geq r + s - n. \quad \square$$

The d-Uple Embedding

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For given $n, d > 0$ let $M_0, M_1, \ldots, M_N$ be the monomials of degree d in the $n+1$ variables $x_0, \ldots, x_n$ where $N = \binom{n+d}{n} - 1$ (this is justified at the end of our Section 1.1 solutions). For a given monomial f of degree d let $[f]$ denote the index $0 \le [f] \le N$ with $M_{[f]} = f$. The assignment of indices to the monomials may be completely arbitrary.

Given $n, d > 0$ and the ordering $[-]$ on the monomials, we define a map

$$\rho_d : \mathbb{P}^n \longrightarrow \mathbb{P}^N$$
$$\rho_d(a_0, \ldots, a_n) = (M_0(a_0, \ldots, a_n), \ldots, M_N(a_0, \ldots, a_n))$$

This is called the *d-Uple embedding* of $\mathbb{P}^n$ in $\mathbb{P}^N$. It is easily seen that this map is well-defined. The map is also injective, since if $\rho_d(a_0, \ldots, a_n) = \rho_d(b_0, \ldots, b_n)$ then there is $0 \neq \lambda \in k$ such that

$$(M_0(a_0, \ldots, a_n), \ldots, M_N(a_0, \ldots, a_n)) = (\lambda M_0(b_0, \ldots, b_n), \ldots, \lambda M_N(b_0, \ldots, b_n))$$

Let $0 \le i \le n$ be such that $a_i \neq 0$. Then $a_i^d = M_{[x_i^d]}(a_0, \ldots, a_n) = \lambda b_i^d$, so $a_i = \mu b_i$ where $\mu = \lambda a_i^{1-d} b_i^{d-1}$. Then for any $0 \le j \le n$ we have

$$\begin{aligned} \mu b_j &= \lambda a_i^{1-d} b_i^{d-1} b_j \\ &= a_i^{1-d} \lambda M_{[x_i^{d-1} x_j]}(b_0, \ldots, b_n) \\ &= a_i^{1-d} M_{[x_i^{d-1} x_j]}(a_0, \ldots, a_n) \\ &= a_i^{1-d} a_i^{d-1} a_j \\ &= a_j \end{aligned}$$

so $(a_0, \ldots, a_n) = (b_0, \ldots, b_n)$ and ρ_d is injective. We claim that the image of ρ_d in $\mathbb{P}^N$ is a projective variety. To prove this, first consider the morphism of k-algebras

$$\theta : k[y_0, \ldots, y_N] \longrightarrow k[x_0, \ldots, x_n]$$
$$y_i \mapsto M_i$$

Let $\mathfrak{a}$ be the prime ideal $Ker\theta$. This ideal is homogenous since if f is homogenous of degree e then $\theta(f)$ is homogenous of degree de. It is easy to see that $Im(\rho_d) \subseteq Z(\mathfrak{a})$ since if $g \in k[y_0, \ldots, y_n]$,

$$\begin{aligned} g(\rho_d(a_0, \ldots, a_n)) &= g(M_0(a_0, \ldots, a_n), \ldots, M_N(a_0, \ldots, a_n)) \\ &= \theta(g)(a_0, \ldots, a_n) \end{aligned}$$

In particular $Z(\mathfrak{a})$ is nonempty, so $Z(\mathfrak{a})$ is a projective variety in $\mathbb{P}^N$. The hard part is to show that $Z(\mathfrak{a}) \subseteq Im(\rho_d)$. We proceed as follows: assume $(b_0, \dots, b_N) \in Z(\mathfrak{a})$, and let $b_i \neq 0$. Let h be the monomial with $[h] = i$. Then the polynomial h^d can be written as a product $h^d = x_{i_1}^d \dots x_{i_s}^d$ (where we may have $x_{i_k} = x_{i_j}$ for $j \neq k$). Thus

$$y_i^d - y_{[x_{i_1}^d]} \dots y_{[x_{i_s}^d]} \in \mathfrak{a}$$

and since $(b_0, \dots, b_N) \in Z(\mathfrak{a})$ we see that $b_i^d = b_{[x_{i_1}^d]} \dots b_{[x_{i_s}^d]}$. The fact that $b_i \neq 0$ implies that $b_{[x_{i_j}^d]} \neq 0$ for all $0 \leq j \leq s$. We have just shown that for any $(b_0, \dots, b_N) \in Z(\mathfrak{a})$ there is $0 \leq K \leq n$ with $b_{[x_K^d]} \neq 0$.

Given such K, suppose we could find $(a_0, \dots, a_n) \in \mathbb{P}^n$ with $b_i = \lambda M_i(a_0, \dots, a_n)$ for all $0 \leq i \leq N$. In particular this would imply

$$b_{[x_K^d]} = a_K^d, \quad b_{[x_K^{d-1} x_i]} = a_K^{d-1} a_i \quad i \neq K$$

So that for $i \neq K$

$$a_i = \frac{a_K b_{[x_K^{d-1} x_i]}}{a_K^d} = \frac{a_K b_{[x_K^{d-1} x_i]}}{b_{[x_K^d]}}$$

So the obvious plan of attack is to try putting $a_K = 1$ and $a_i = b_{[x_K^{d-1} x_i]} / b_{[x_K^d]}$ and try to show that

$$\rho_d(a_0, \dots, a_n) = (b_0, \dots, b_N)$$

And this is precisely what we are going to do. For any $0 \leq j \leq N$ there are nonnegative integers $m_0, \dots, m_n$ with $m_0 + \dots + m_n = d$ and

$$M_j = x_0^{m_0} \dots x_n^{m_n}$$

Then

$$x_K^{d(d-1)} M_j = (x_K^{d-1} x_0)^{m_0} \dots (x_K^{d-1} x_n)^{m_n}$$

This implies that

$$y_{[x_K^d]}^{d-1} y_j - \prod_{i=0}^{n} y_{[x_K^{d-1} x_i]}^{m_i} \in \mathfrak{a}$$

Since $(b_0, \dots, b_N) \in Z(\mathfrak{a})$ we can replace "y"s by "b"s in the above polynomial and divide through by $b_{[x_k^d]}$ to obtain (using the fact that $m_0 + \dots + m_n = d$)

$$\frac{b_j}{b_{[x_K^d]}} = \prod_{i=0}^{n} \frac{b_{[x_K^{d-1} x_i]}^{m_i}}{b_{[x_K^d]}^{m_i}}$$

Since j was arbitrary, we have for any $0 \leq j \leq n$

$$\begin{aligned} b_j &= b_{[x_K^d]} M_j \left(\frac{b_{[x_K^{d-1} x_0]}}{b_{[x_K^d]}}, \dots, \frac{b_{[x_K^{d-1} x_n]}}{b_{[x_K^d]}} \right) \\ &= b_{[x_K^d]} M_j(a_0, \dots, a_n) \end{aligned}$$

Since $b_{[x_K^d]} \neq 0$ it follows that $\rho_d(a_0, \dots, a_n) = (b_0, \dots, b_N)$ in $\mathbb{P}^N$, as required. Hence $Im(\rho_d) = Z(\mathfrak{a})$ and the map ρ_d gives a bijection of $\mathbb{P}^n$ and $Z(\mathfrak{a})$

In fact, the above argument defines a map $\psi : Z(\mathfrak{a}) \longrightarrow \mathbb{P}^n$ in which $(b_0, \ldots, b_N) \in Z(\mathfrak{a})$ is mapped to

$$\psi(b_0, \ldots, b_N) = (b_{[x_K^{d-1}x_0]}, \ldots, b_{[x_K^{d-1}x_n]})$$

where $0 \leq K \leq n$ is such that $b_{[x_K^d]} \neq 0$. Notice that the definition is actually independent of K, since ρ_d is injective and K was chosen arbitrarily in the proof that $Im(\rho_d) = Z(\mathfrak{a})$. By construction $\rho_d \psi = 1$ and it is easily seen that $\psi \rho_d = 1$. We claim that ρ_d defines a morphism of varieties $\mathbb{P}^n \longrightarrow Z(\mathfrak{a})$ and that $\psi : Z(\mathfrak{a}) \longrightarrow \mathbb{P}^n$ is also a morphism.

Continuity of ρ_d is immediate, for if $g(y_0, \ldots, y_N)$ is a homogenous polynomial

$$\rho_d^{-1}(Z(g)) = Z(\theta(g))$$

where $\theta(g)$ is also homogenous. To show continuity of ψ, consider the homomorphisms of k-algebras defined for $0 \leq K \leq n$ by

$$\theta'_K : k[x_0, \ldots, x_n] \longrightarrow k[y_0, \ldots, y_N]$$
$$x_i \mapsto y_{[x_K^{d-1}x_i]}$$

Given a homogenous polynomial $f(x_0, \ldots, x_n)$ to show that $\psi^{-1}Z(f)$ is closed it suffices to show that $U_{[x_K^d]} \cap \psi^{-1}Z(f)$ is closed in $U_{[x_K^d]}$ for all $0 \leq K \leq n$, since we have already shown that every $(b_0, \ldots, b_N) \in Z(\mathfrak{a})$ has some $b_{[x_K^d]} \neq 0$. But $U_{[x_K^d]} \cap \psi^{-1}Z(f)$ is the set

$$\{(b_0, \ldots, b_N) \in Z(\mathfrak{a}) \,|\, b_{[x_K^d]} \neq 0 \text{ and } f(b_{[x_K^{d-1}x_0]}, \ldots, b_{[x_K^{d-1}x_n]}) = 0\}$$

which is the intersection of the closed set $Z(\theta'_K(f))$ with $U_{[x_K^d]}$. Hence ψ is also continuous.

A standard argument using the morphism θ shows that ρ_d is a morphism of varieties, and by using the morphisms θ'_K and considering the restrictions $\psi|_{U_{[x_K^d]}}$ it is also straightforward to check that ψ is a morphism of varieties.

Hence ρ_d gives rise to an isomorphism of varieties $\mathbb{P}^n \cong Z(\mathfrak{a})$. In particular, the d-Uple embedding of $\mathbb{P}^n$ in $\mathbb{P}^N$ is a projective variety of dimension n.

Example 1. With $n = 1$ and $d = 2$ the relevant monomials are x_0^2, x_0x_1, x_1^2. Depending on the way we order the monomials, we obtain 6 embeddings of $\mathbb{P}^1$ in $\mathbb{P}^2$. For example, the following 2-Uple embedding

$$\rho_d(a, b) = (a^2, ab, b^2)$$

gives an isomorphism of $\mathbb{P}^1$ with the conic $xz - y^2$ in $\mathbb{P}^2$ (see our typed notes on conics for a proof).

Example 2. Recall the twisted cubic curve in $\mathbb{A}^3$ is the set of all tuples (t, t^2, t^3) with $t \in k$, which is equal to the affine variety $Z(y^2 - x, z^3 - x)$. If we identify $\mathbb{A}^3$ with the open set $U_0 \subseteq \mathbb{P}^3$ and take the closure W of these points, we obtain the *twisted cubic curve in* $\mathbb{P}^3$. In Exercise 2.9 we showed that

$$I(W) = (wy - x^2, zw^2 - x^3, xz - y^2)$$

where the coordinates of $\mathbb{P}^3$ are w, x, y, z. The claim that W is the image of the 3-Uple embedding of $\mathbb{P}^1$ in $\mathbb{P}^3$, given by

$$\rho_d(a, b) = (a^3, a^2b, ab^2, b^3)$$

By setting $a = 1$ and noting that $\rho_d(1, b) = (1, b, b^2, b^3)$ see that $Im(\rho_d)$ contains the twisted cubic curve of $\mathbb{A}^3$ and hence contains the closure W of these points. To prove the reverse inclusion $Im(\rho_d) \subseteq W$ we note that $Im(\rho_d) = Z(\mathfrak{a})$ and $W = Z(I(W))$, so it would suffice to show that θ maps the polynomials of $I(W)$ to zero. Here $\theta : k[w, x, y, z] \longrightarrow k[t, u]$ is the map $w \mapsto t^3, x \mapsto t^2u, y \mapsto tu^2, z \mapsto u^3$ and by considering the generators of $I(W)$ it is clear that $I(W) \subseteq \mathfrak{a}$.

So the twisted cubic curve in $\mathbb{P}^3$ is of dimension 1 and is isomorphic to $\mathbb{P}^1$.

Q2.10 <u>The Cone over a Projective Variety</u> Let $Y \subseteq \mathbb{P}^n$ be a nonempty algebraic set, and let $\theta : \mathbb{A}^{n+1} - \{(0,\dots,0)\} \longrightarrow \mathbb{P}^n$ be the map which sends the point with affine coordinates $(a_0, \dots, a_n)$ to the point with homogenous coordinates $(a_0, \dots, a_n)$. θ is continuous, since for $g \in k[x_0, \dots, x_n]$ homogenous, $\theta^{-1}(Z(g)) = Z(g) \cap \mathbb{A}^{n+1} \setminus \{(0,\dots,0)\}$. The Zariski topology on $\mathbb{P}^n$ is the quotient topology, which follows from

<u>PROPOSITION</u> Let k be an algebraically closed field. Then an ideal $\mathfrak{a} \subseteq k[x_0, \dots, x_n]$ is homogenous iff.

$$\mathfrak{a} \subseteq (x_0 - a_0, \dots, x_n - a_n) \Rightarrow \mathfrak{a} \subseteq (x_0 - \lambda a_0, \dots, x_n - \lambda a_n) \quad \lambda \neq 0$$

for all $a_0, \dots, a_n \in k$.

NO

<u>PROOF</u> We have already noted that this condition is necessary. We use induction on n to show that if a sum of ... Clearly $n = 1$ holds.
If $m_1 + m_2 \in \mathfrak{a}$, let $\mathfrak{a} \subseteq (x_0 - a_0, \dots, x_n - a_n)$ (if $\mathfrak{a}$ is improper, it is homogenous trivially), so that $m_1(a_0, \dots, a_n) = -m_2(a_0, \dots, a_n)$. Then for $0 \neq \lambda \in k$, we also have by hypothesis

$$\lambda^{d_1} m_1(a_0, \dots, a_n) = -\lambda^{d_2} m_2(a_0, \dots, a_n)$$

where $m_1 \in S_{d_1}$ and $m_2 \in S_{d_2}$. Either $m_i(a_0, \dots, a_n) = 0$, $i = 1, 2$ or one of them is nonzero – wlog suppose $m_1(a_0, \dots, a_n) \neq 0$. Then

$$\lambda^{d_1 - d_2} = \frac{-m_2(a_0, \dots, a_n)}{m_1(a_0, \dots, a_n)}$$

no. what about roots of unity?

since this holds for any λ, we must have $d_1 = d_2$, which is a contradiction.

We define the <u>affine cone</u> over Y to be

$$C(Y) = \theta^{-1} Y \cup \{(0, \dots, 0)\}$$

(a) Let $I(Y)$ be the homogenous ideal of Y. Then $I(Y) \subseteq (x_0, \dots, x_n)$ and for $(a_0, \dots, a_n) \in Y$, $I(Y) \subseteq (x_0 - a_0, \dots, x_n - a_n)$. $C(Y)$ is precisely $Z(I(Y))$ (see preamble to Q2.1).

(b) By Q2.4 (b) Y is irreducible iff. $I(Y)$ is prime, so iff. $Z(I(Y))$ is irreducible.

(c) By def^N, $S(Y) = k[x_0, \dots, x_n]/I(Y)$, so by Q2.6 $\dim C(Y) = \dim k[x_0, \dots, x_n]/I(Y) = \dim S(Y) = \dim Y + 1$.

Q2.12 <u>The d-Uple Embedding</u> For given $n, d > 0$, let $M_0, M_1, \dots, M_N$ be all the monomials of degree d in the $n+1$ variables $x_0, \dots, x_n$, where $N = \binom{n+d}{n} - 1$. We define a mapping (see the Note at the end of these exercises to see why $N = \binom{n+d}{n} - 1$)

$$\rho_d : \mathbb{P}^n \longrightarrow \mathbb{P}^N$$

by sending the point $P = (a_0, \dots, a_n)$ to the point $\rho_d(P) = (M_0(a), \dots, M_N(a))$ obtained by substituting the a_i in the monomials M_j. This is called the d-Uple <u>embedding</u> of $\mathbb{P}^n$ in $\mathbb{P}^N$. For example, if $n = 1$, $d = 2$ then $N = 2$ (the three monomials are $x_0^2, x_1^2, x_0 x_1$), and the image Y of the 2-uple embedding of $\mathbb{P}^1$ in $\mathbb{P}^2$ is a conic

$$(a_0, a_1) \longmapsto (a_0^2, a_1^2, a_0 a_1)$$

since the M_j are monomials, ρ_d is always well defined, since $\rho_d(\lambda a_0, \dots, \lambda a_n) = (M_0(\lambda a_0, \dots, \lambda a_n), \dots, M_N(\lambda a_0 \dots))$ $= (\lambda^d M_0(a), \dots, \lambda^d M_N(a)) = (M_0(a), \dots, M_N(a))$. Since x_i^d, $0 \leq i \leq n$ is a monomial of degree d, ρ_d is always injective : if $\rho_d(a_0, \dots, a_n) = \rho_d(b_0, \dots, b_n)$ then there is $0 \neq \lambda \in k$ s.t.

$$M_j(a_0, \dots, a_n) = \lambda M_j(b_0, \dots, b_n) \qquad 0 \leq j \leq N$$

in particular, let $0 \le i \le n$ be s.t. $a_i \neq 0$, and hence $a_i^d = \lambda b_i^d$ implies $a_i = \{ a_i^{-(d-1)} \lambda b_i^{d-1} \} b_i$. Put $\lambda' = a_i^{-(d-1)} \lambda b_i^{d-1}$ ($b_i \neq 0$ since $a_i \neq 0$, hence $\lambda' \neq 0$). Then for any $0 \le j \le n$,

$$\begin{aligned} \lambda' b_j &= a_i^{-(d-1)} \lambda b_i^{(d-1)} b_j \\ &= a_i^{-(d-1)} a_i^{(d-1)} a_j \qquad \{ x_i^{d-1} x_j \text{ is a monomial deg } d \} \\ &= a_j \end{aligned}$$

hence $(a_0, \dots, a_n) = (b_0, \dots, b_n)$.

(a) Let $\theta : k[y_0, \dots, y_N] \longrightarrow k[x_0, \dots, x_n]$ be $y_i \longmapsto M_i$, and $\mathfrak{a} = \operatorname{Ker} \theta$. Then $\mathfrak{a}$ is prime, and if

$$f = \sum_e m_e \qquad m_e \in S_e$$

is the unique expansion of f as a sum of homogenous polynomials, then $f \in \mathfrak{a}$ implies

$$0 = \theta(f) = \sum_e \theta(m_e) \tag{1}$$

now any monomial of order e in $k[y_0, \dots, y_N]$ will be taken by θ to a monomial of order de. Hence the $\theta(m_e)$ are all of distinct degrees, and so $\theta(m_e) = 0$, each e. Hence $m_e \in \mathfrak{a}$, and $\mathfrak{a}$ is homogenous. Hence $Z(\mathfrak{a})$ is a projective variety in $\mathbb{P}^N$.

(b) Let $g(y_0, \dots, y_N) \in \mathfrak{a}$. If $(a_0, \dots, a_n) \in \mathbb{P}^n$, then $\rho_d(a_0, \dots, a_n) = (M_0(a_0, \dots, a_n), \dots, M_N(a_0, \dots, a_n))$, and

$$\begin{aligned} g(\rho_d(a_0, \dots, a_n)) &= g(M_0(a), \dots, M_N(a)) \\ &= \theta(g)(a_0, \dots, a_n) = 0 \qquad \text{since } \mathfrak{a} = \ker \theta \end{aligned}$$

hence $\operatorname{Im} \rho_d \subseteq Z(\mathfrak{a})$. To prove that $Z(\mathfrak{a}) \subseteq \operatorname{Im} \rho_d$, note that since $\mathfrak{a} = \operatorname{Ker} \theta$, and hence $k[y_0, \dots, y_N]/\mathfrak{a}$ is isomorphic to a subring of $k[x_0, \dots, x_n]$, $\mathfrak{a}$ is a radical ideal. Hence $\operatorname{Im} \rho_d \subseteq Z(\mathfrak{a})$ implies $\mathfrak{a} \subseteq I(\operatorname{Im} \rho_d)$. We prove the reverse inclusion (and hence that $Z(\mathfrak{a}) = \overline{\operatorname{Im} \rho_d}$) by showing that if $f \in I(\operatorname{Im} \rho_d)$, then $\theta(f) = 0$. But this is obvious, since

$$\begin{aligned} \theta(f)(a_0, \dots, a_n) &= f(M_0(a_0, \dots, a_n), \dots, M_N(a_0, \dots, a_n)) \\ &= f(\rho_d(a_0, \dots, a_n)) = 0 \end{aligned}$$

and hence $\theta(f) = 0$.

<u>NOTE</u> Previous paragraph is irrelevant – can do it directly. Suppose $(b_0, \dots, b_N) \in Z(\mathfrak{a})$. For a monomial M of degree d, let $g(M)$ denote the index it is assigned – so $M = M_{g(M)}$. Then some b_i is nonzero. The key is to notice that if we raise the corresponding monomial to the dth power, it can be written as the product of x_j^d's for $0 \le j \le n$. Say

$$M_i^d = \prod_{p=1}^{k} x_{j_p}^d$$

then this means that $y_i^d - \prod y_{g(x_{j_p}^d)} \in \mathfrak{a}$, and so $b_i^d = \prod b_{g(x_{j_p}^d)}$. Hence since $b_i \neq 0$, some $b_{g(x_{j_p}^d)} \neq 0$. Relabel if necessary to arrange $j_p = 0$, so $b_{g(x_0^d)} \neq 0$. If indeed we find $(a_0, \dots, a_n) \in \mathbb{P}^n$ s.t. $(b_0, \dots, b_N) = \rho_d(a_0, \dots, a_n)$, then we would have $b_{g(x_0^d)} = a_0^d$, and $b_{g(x_0^{d-1} x_i)} = a_0^{d-1} a_i$ for $1 \le i \le n$. Hence we would have $\left(a_0^{d-1} a_i = \frac{a_0^d a_i}{a_0} \right)$

$$a_i = \frac{a_0 b_{g(x_0^{d-1} x_i)}}{a_0^d} = \frac{a_0 b_{g(x_0^{d-1} x_i)}}{b_{g(x_0^d)}}$$

since we're working in projective coordinates, we can arrange for $a_0 = 1$. Hence, we <u>claim</u>, setting $a_0 = 1$, $a_i = \frac{b_{g(x_0^{d-1} x_i)}}{b_{g(x_0^d)}}$

$$\rho_d(1, a_1, \dots, a_n) = (b_0, \dots, b_N).$$

we need to find $\lambda \in k$ s.t.

$$b_i = \lambda M_i(1, a_1, \dots, a_n) \qquad 0 \le i \le N$$

To this end, let $0 \le i \le N$ and suppose

$$M_i = x_0^{m_0} x_1^{m_1} \cdots x_n^{m_n} \qquad m_0 + \cdots + m_n = d$$

then $x_0^{d(d-1)} M_i = (x_0^{d-1} x_0)^{m_0} (x_0^{d-1} x_1)^{m_1} \cdots (x_0^{d-1} x_n)^{m_n}$. Hence the polynomial

$$y_{g(x_0^d)}^{d-1} y_i - \prod_{j=0}^{n} y_{g(x_0^{d-1} x_j)}^{m_j} \in \mathfrak{a}$$

since by assumption $(b_0, \ldots, b_N) \in Z(\mathfrak{a})$, we have

$$b_{g(x_0^d)}^{d-1} b_i = b_{g(x_0^{d-1} x_0)}^{m_0} \cdots b_{g(x_0^{d-1} x_n)}^{m_n}$$

dividing both sides by $b_{g(x_0^d)}^d$, gives

$$b_i / b_{g(x_0^d)} = \left\{ \frac{b_{g(x_0^{d-1} x_0)}^{m_0}}{b_{g(x_0^d)}^{m_0}} \right\} \cdots \left\{ \frac{b_{g(x_0^{d-1} x_n)}^{m_n}}{b_{g(x_0^d)}^{m_n}} \right\}$$

since $m_0 + \cdots + m_n = d$. Hence $b_i = b_{g(x_0^d)} M_i\left(1, \frac{b_{g(x_0^{d-1} x_1)}}{b_{g(x_0^d)}}, \ldots, \frac{b_{g(x_0^{d-1} x_n)}}{b_{g(x_0^d)}}\right) = b_{g(x_0^d)} M_i(1, a_1, \ldots, a_n)$.
hence we put $\lambda = b_{g(x_0^d)}$, and we're done!

(c) We already know that p_d is a bijection — we first show that it is also continuous. Since $Z(\mathfrak{a})$ is covered by the open subsets $Z(\mathfrak{a}) \cap U_i$ $i = 0, \ldots, N$, it suffices to see that

$$p_d^{-1}(Z(\mathfrak{a}) \cap U_i) = \{ (a_0, \ldots, a_n) \mid (M_0(a), \ldots, M_n(a)) \in Z(\mathfrak{a}) \cap U_i \}$$
$$= \{ (a_0, \ldots, a_n) \mid M_i(a_0, \ldots, a_n) \neq 0 \}$$
$$= \mathbb{P}^n \setminus Z(M_i)$$

Eh? Note that $p_d^{-1}(Z(\mathfrak{g})) = Z(\theta(\mathfrak{g}))$ and don't bother with

which is open. Hence p_d is continuous. We define the inverse $\hat{\theta} : Z(\mathfrak{a}) \longrightarrow \mathbb{P}^n$ to p_d in part (b). For each $(b_0, \ldots, b_N)$ in $Z(\mathfrak{a})$ it selected some j, $0 \le j \le n$ s.t. $b_{g(x_j^d)} \neq 0$ (recall $g(x_j^d)$ is an integer $0 \le \cdot \le n$ telling us which index in the N-tuple is assigned to the monomial x_j^d), then mapped $(b_0, \ldots, b_N)$ to

$$\left(\frac{b_{g(x_j^{d-1} x_0)}}{b_{g(x_j^d)}}, \ldots, \frac{b_{g(x_j^{d-1} x_n)}}{b_{g(x_j^d)}} \right)$$

to see that this is continuous, notice that for $0 \le i \le n$,

$$\hat{\theta}^{-1}(U_i) = \left\{ (b_0, \ldots, b_N) \mid \frac{b_{g(x_j^{d-1} x_i)}}{b_{g(x_j^d)}} \neq 0 \right\}$$
$$= Z(\mathfrak{a}) \cap U_{g(x_j^{d-1} x_i)}$$

so $\hat{\theta}$ is also continuous, and thus p_d is a homeomorphism.

(d) Here we want $p_d : \mathbb{P}^1 \longrightarrow \mathbb{P}^N$, so $n = 1$ and $N = 3$, the variables x_0, x_1, and all monomials of deg. $d = 3$, with the following order:

0	1	2	3
x_0^3	$x_0^2 x_1$	$x_0 x_1^2$	x_1^3

Hence $\operatorname{Im} p_d = Z(\mathfrak{a})$, $p_d : \mathbb{P}^1 \longrightarrow \mathbb{P}^N$ defined by $p_d(a_0, a_1) = (a_0^3, a_0^2 a_1, a_0 a_1^2, a_1^3)$ and $\mathfrak{a} = \operatorname{Ker} \theta$,
$\theta : k[y_0, y_1, y_2, y_3] \longrightarrow k[x_0, x_1]$ given by

$$y_0 \longmapsto x_0^3$$
$$y_1 \longmapsto x_0^2 x_1$$
$$y_2 \longmapsto x_0 x_1^2$$
$$y_3 \longmapsto x_1^3$$

By putting $a_0 = 1$ we see that $\rho_3(1,t) = (1,t,t^2,t^3)$, so since the twisted cubic curve in $\mathbb{P}^3$ (henceforth denoted C) is the closure of all these points, $C \subseteq Z(\mathfrak{a})$. To show $Z(\mathfrak{a}) \subseteq C$ we simply show that $I(C) \subseteq \mathfrak{a}$, that is, that the generators

$$y_3 y_0^2 - y_1^3, \quad y_2 y_0 - y_1^2, \quad y_1 y_3 - y_2^2$$

for $I(C)$ (obtained in Q2.9) are carried to 0 by θ. But this obvious by inspection — hence $C = Z(\mathfrak{a})$, as required.† $\square$

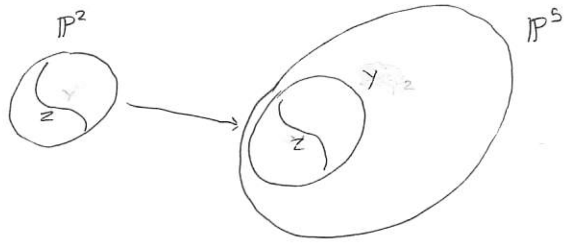
NOTE All the above remains true no matter how we order the monomials.

†It follows from an earlier note that the twisted cubic curve in $\mathbb{P}^3$ has dimension 1.

$\boxed{\text{Q2.13}}$ Let Y be the image of the 2-uple embedding $\rho_2 : \mathbb{P}^2 \longrightarrow \mathbb{P}^5$ given by

$$\rho_2(a_0,a_1,a_2) = (a_0^2, a_1^2, a_2^2, a_0a_1, a_0a_2, a_1a_2)$$

$$\theta : k[y_0,\dots,y_5] \longrightarrow k[x_0,x_1,x_2]$$
$$y_0 \longmapsto x_0^2$$
$$y_1 \longmapsto x_1^2$$
$$y_2 \longmapsto x_2^2$$
$$y_3 \longmapsto x_0x_1$$
$$y_4 \longmapsto x_0x_2$$
$$y_5 \longmapsto x_1x_2$$



This is the Veronese surface. If $Z \subseteq Y$ is a closed curve (since ρ_2 is a homeomorphism, we can unambiguously talk also of $Z \subseteq \mathbb{P}^2$ as a closed curve), where a curve is a variety of dimension 1, then since $\dim \mathbb{P}^2 = 2$, by Q2.8 $Z = Z(f)$ for an irreducible homogenous $f \in k[x_0,x_1,x_2]$. Since $Z(f) = Z(f^2)$, and f^2 will be a polynomial in the monomials $x_0^2, x_1^2, x_2^2, x_0x_1, x_0x_2, x_1x_2$ (call this polynomial $h \in k[y_0,\dots,y_5]$), we have $Z \subseteq Z(h)$ in $\mathbb{P}^5$, since

$$h(\rho_2(a_0,a_1,a_2)) = h(a_0^2,a_1^2,a_2^2,a_0a_1,a_0a_2,a_1a_2)$$
$$= f^2(a_0,a_1,a_2) = 0 \qquad (1)$$

If we write $Z(h) = \bigcup_{i=1}^n Y_i$ as a finite union of irreducible algebraic sets, then since Z is closed irreducible, it must be contained in some Y_i (else we can add it to the list and contradict uniqueness in Q2.5). Since we can assume the decomposition of $Z(h)$ is given by the irreducible factors of h, this shows that there is a hypersurface $V \subseteq \mathbb{P}^5$ with $Z \subseteq V$. Since $Z(h) \cap Y = Z$ by Eq (1), we have shown that $Z = Y \cap V$, as required. $\square$

$\boxed{\text{Q2.14}}$ The Segre embedding Let $\psi : \mathbb{P}^r \times \mathbb{P}^s \longrightarrow \mathbb{P}^N$ be the map

$$(a_0,\dots,a_r)\times(b_0,\dots,b_s) \longmapsto (a_0b_0, a_0b_1,\dots,a_ib_j,\dots,a_rb_s) \qquad (1)$$

in lexicographic order, $N = rs+r+s$. ψ is well-defined since if $(\lambda a_0,\dots,\lambda a_r) = (a_0,\dots,a_r)$ and $(\mu b_0,\dots,\mu b_s) = (b_0,\dots,b_s)$ then the RHS of (1) becomes $(\lambda\mu a_0b_0,\dots,\lambda\mu a_ib_j,\dots,\lambda\mu a_rb_s)$, which is $\psi((a_0,\dots,a_r)\times(b_0,\dots,b_s))$. (also RHS is nonzero since we pair up nonzero coordinates of LHS) It is injective since if

$$(a_0b_0, a_0b_1,\dots,a_0b_s,\dots\dots,a_rb_s)$$
$$= (a_0'b_0',\dots\dots,a_0'b_s',\dots, a_r'b_s')$$

Then $a_0b_0 = \lambda a_0'b_0', \dots, a_0b_s = \lambda a_0'b_s'$ so either $a_0 = 0$ or $b_0 = \lambda a_0'/a_0\, b_0', \dots, b_s = \lambda a_0'/a_0\, b_s'$ so $(b_0,\dots,b_s) = (b_0',\dots,b_s')$. If $a_0 = 0$ keep going till some $a_i \neq 0$ and do the same thing. Same deal shows $(a_0,\dots,a_r) = (a_0',\dots,a_r')$.

Let the homogenous coordinates of $\mathbb{P}^N$ be z_{ij} $0 \le i \le r$, $0 \le j \le s$, and θ the ring morphism

$$\theta : k[\{z_{ij}\}] \longrightarrow k[x_0,\dots,x_r,y_0,\dots,y_s]$$
$$z_{ij} \longmapsto x_iy_j$$

and $\mathfrak{a} = \operatorname{Ker}\theta$, which is then prime homogenous. We claim that $\operatorname{Im}\psi = Z(\mathfrak{a})$. If $f \in \mathfrak{a}$ then

$$f(\psi((a_0,\dots,a_r)\times(b_0,\dots,b_s))) = f(a_0b_0,\dots,a_ib_j,\dots,a_rb_s)$$
$$= \theta(f)(a_0,\dots,a_r,b_0,\dots,b_s) = 0$$

Hence $\operatorname{Im}\psi \subseteq Z(\mathfrak{a})$. For the the converse inclusion, note that for $0 \le i, c \le r$ and $0 \le j, d \le s$ we have $z_{ij}z_{cd} - z_{id}z_{cj} \in \mathfrak{a}$.

Suppose that $(a_{00}, a_{01}, \dots, a_{ij}, \dots, a_{rs}) \in Z(\mathfrak{a})$. Suppose that $a_{cd} \neq 0$. Then since

$$a_{ij}a_{cd} = a_{id}a_{cj}$$

$$\therefore \quad a_{ij} = \frac{1}{a_{cd}} a_{id}a_{cj}$$

we put $a_i = a_{id}$ and $b_j = a_{cj}$ so that in $\mathbb{P}^N$ we have $\psi((a_0, \dots, a_r) \times (b_0, \dots, b_s)) = (a_{00}, \dots, a_{rs})$ as required. Hence $\operatorname{Im}\psi = Z(\mathfrak{a})$, so $\operatorname{Im}\psi$ is a subvariety of $\mathbb{P}^N$. $\square$ ($\operatorname{Im}\psi$ is a projective variety)

$\boxed{Q\,2.15}$ The Quadric Surface in $\mathbb{P}^3$ consider the surface Q (a surface is a variety of dimension 2) in $\mathbb{P}^3$ defined by the equation $xy - zw = 0$, where we order the variables w, x, y, z.

(a) Consider the segre embedding $\psi : \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^3$ which is defined by

$$\psi((a_0, a_1) \times (b_0, b_1)) = (a_0b_0, a_0b_1, a_1b_0, a_1b_1)$$

By 2.14 the image of ψ is a projective variety in $\mathbb{P}^3$. Obviously $\operatorname{Im}\psi \subseteq Z(xy - zw)$. To show the reverse inclusion, suppose $(a_{00}, a_{01}, a_{10}, a_{11}) \in Z(xy - zw)$. Then $a_{01}a_{10} = a_{00}a_{11}$ and by cases

Ⓐ If $a_{11} = 0$, then ⓘ $a_{01} = 0$ put $(a_{00}, a_{10}) \times (1, 0)$ ⓘⓘ $a_{10} = 0$ put $(1, 0)(a_{00}, a_{01})$

Ⓑ If $a_{11} \neq 0$, use $(a_{01}, a_{11})(a_{10}, a_{11})$.

In any case the constructed pair map onto $(a_{00}, a_{01}, a_{10}, a_{11})$, so $\operatorname{Im}\psi = Z(xy - zw)$. Since $Q = Z(xy - zw)$ is the locus of a nonconstant, irreducible homogenous polynomial, by Ex 2.8 $\dim Q = 2$.

(b) For $t \in \mathbb{P}^1$ let

$$L_t = \psi(t \times \mathbb{P}^1)$$
$$M_t = \psi(\mathbb{P}^1 \times t)$$

If $t = (a_0, a_1)$ then we claim $L_t = Z(a_1 w - a_0 y) \cap Z(a_1 x - a_0 z)$. It is clear that $\subseteq$. For the reverse inclusion, if $(c_{00}, c_{01}, c_{10}, c_{11}) \in \mathbb{P}^3$ belongs to the RHS, then either $a_0 \neq 0$ or $a_1 \neq 0$. Wlog $a_0 \neq 0$ (case $a_1 \neq 0$ is similar), then

$$\begin{aligned} a_0 c_{01} c_{10} &= (a_0 c_{10}) c_{01} \\ &= (a_1 c_{00}) c_{01} \\ &= c_{00}(a_0 c_{11}) = a_0 c_{00} c_{11} \end{aligned}$$

$$\therefore \quad c_{01}c_{10} = c_{00}c_{11}$$

and $(c_{00}, c_{01}, c_{10}, c_{11}) \in Q$.

In any case, $(c_{00}, c_{01}, c_{10}, c_{11}) \in Q$, say $\psi((d_0, d_1) \times (b_0, b_1)) = (c_{00}, c_{01}, c_{10}, c_{11})$. This implies $(d_0, d_1) = t$ (since $a_1 d_0 b_0 = a_0 d_1 b_0$) so $(c_{00}, c_{01}, c_{10}, c_{11}) \in L_t$. Similarly M_t is the linear variety $Z(a_1 w - a_0 x) \cap Z(a_1 y - a_0 z)$. So both M_t, L_t are linear varieties for $t \in \mathbb{P}^1$. Both intersections are "minimal" in the sense used in our soln to Ex 2.11, so our notes there show that

$$\dim L_t = \dim M_t = 1$$

If $t, u \in \mathbb{P}^1$ are distinct then the fact that ψ is injective implies that $L_t \cap L_u = \emptyset$ and $M_t \cap M_u = \emptyset$. Clearly $L_t \cap M_u = \{t \times u\}$.

(c) The curve $Q \cap Z(x - y)$ is not one of these lines, and the topology on Q is ~~not~~ the one included by ψ and the product topology on $\mathbb{P}^1 \times \mathbb{P}^1$, because the set $Q \cap Z(x - y) \subseteq Q$ corresponds to the diagonal in $\mathbb{P}^1 \times \mathbb{P}^1$ - which is closed in the product topology iff. $\mathbb{P}^1$ is Hausdorff. But by Ex 1.7 d) this is impossible.

Q2.16 (a) The intersection of two varieties need not be a variety. For example consider the following quadrics in $\mathbb{P}^3$

$$Q_1 = Z(x^2 - yw)$$
$$Q_2 = Z(xy - zw)$$

Both polynomials are irreducible, so Q_1, Q_2 are quadric surfaces in $\mathbb{P}^3$. Let T be the twisted cubic curve in $\mathbb{P}^3$ — T is a projective variety of dimension 1,

$$I(T) = (zw^2 - x^3,\ yw - x^2,\ xz - y^2)$$

(see Q2.9), and let L be the line $Z(w) \cap Z(x)$ (order w,x,y,z) (this is a minimal decomp so by 2.11 L has dimension 1, so really is a line). Note that L is a projective variety since $(x,w) = I(L)$ is prime. It is clear that $L \subseteq Q_1 \cap Q_2$ and $T \subseteq Q_1$ since $x^2 - yw \in I(T)$. Suppose $(a_0, a_1, a_2, a_3) \in T$. If $a_0 = a_1 = 0$ then the point is in Q_2 since $L \subseteq Q_2$. Otherwise wlog assume $a_0 \neq 0$ (case $a_1 \neq 0$ similar). Then we may assume $a_0 = 1$, so

$$\begin{aligned} a_1 a_2 = a_1 a_2 a_0 &= a_1 a_1^2 \quad (x^2 = yw) \\ &= a_1^3 \\ &= a_3 a_0^2 \quad (x^3 = zw^2) \\ &= a_3 a_0 \quad (a_0 = 1) \end{aligned}$$

Hence $(a_0, a_1, a_2, a_3) \in Q_2$. Hence $L \cup T \subseteq Q_1 \cap Q_2$. We claim this is an equality. Suppose $(a_0, a_1, a_2, a_3) \in Q_1 \cap Q_2$ and say $a_0 \neq 0$ ($a_1 \neq 0$ similar). It is easy to check that $a_3 a_0^2 = a_1^3$, $a_0 a_2 = a_1^2$, $a_1 a_3 = a_2^2$. So

$$Q_1 \cap Q_2 = L \cup T$$

This is a decomposition of $Q_1 \cap Q_2$ into its irreducible components. So $Q_1 \cap Q_2$ is not irreducible, hence not a variety.

(b) Let C be the conic in $\mathbb{P}^2$ given by the equation $x^2 - yz = 0$ and L the line $y = 0$. Then clearly

$$C \cap L = \{(0,0,1)\} = \{P\}$$

We know that $P = Z(x^2 - yz) \cap Z(y) = Z(y, x^2 - yz) = Z(y, x^2)$. But $I(P) \neq (y, x^2)$ since (x^2, y) is not radical ($x^2 \in (x,y)$ but $x \notin (x^2, y)$). So even if the intersection of two varieties is a variety, the ideal of the intersection may not be the sum of the ideals.

Q2.17 Complete Intersections A projective variety Y of dimension r in $\mathbb{P}^n$ is a (strict) complete intersection if $I(Y)$ can be generated by $n-r$ elements. Y is a set-theoretic complete intersection if Y can be written as the intersection of $n-r$ homogenous hypersurfaces. (Hypersurface in the loose sense, $Z(f)$ where f need not be prime)

(a) Let $Y \subseteq \mathbb{P}^n$ be a projective variety, $Y = Z(\mathfrak{a})$. If $\mathfrak{a}$ can be generated by q elements then by Krull's PI Theorem (p72 A&M notes) $\operatorname{ht}\mathfrak{a} \leq q$ so $\dim Y = \dim k[x_0, \ldots, x_n]/\mathfrak{a} - 1 \geq n+1-q-1 = n-q$.

(b) If Y is a strict complete intersection with $I(Y) = (f_1, \ldots, f_{n-r})$ then $Y = Z(f_1) \cap \cdots \cap Z(f_{n-r})$ so Y is a set-theoretic complete intersection.

(c) Let Z be the twisted cubic curve in $\mathbb{P}^3$. By a Note following Ex 2.9

$$I(Z) = (x^2 - wy,\ xy - wz,\ y^2 - xz)$$

We claim that Z is a set-theoretic complete intersection but not a strict complete intersection. Since $\dim Z = 1$ (in Ex1.2 it is shown that the coordinate ring of the normal twisted cubic is 1, and projective closure preserves dimension) it suffices to show that Z is the intersection of 2 hypersurfaces but $I(Y)$ cannot be generated by 2 elements.

We claim $Z = Z(x^2 - wy) \cap Z(y^3 + wz^2 - 2xyz)$. This will follow from $\sqrt{\mathfrak{b}} = I(Z)$ where $\mathfrak{b} = (x^2 - wy,\ y^3 + wz^2 - 2xyz)$, which will follow from $\mathfrak{b} \subseteq I(Z)$ and $I(Z) \subseteq \sqrt{\mathfrak{b}}$ since $I(Z)$ is radical. But these inclusions follow from

$$y^3 + wz^2 - 2xyz = y(y^2 - xz) + z(wz - xy)$$
$$(xy - wz)^2 = w(y^3 + wz^2 - 2xyz) + y^2(x^2 - yw)$$
$$(y^2 - xz)^2 = y(y^3 + wz^2 - 2xyz) + z^2(x^2 - wy)$$

Since $I(Z)$ contains no homogenous elements of degree 0 or 1, if two homogenous elements generated $I(Z)$ they would be of order 2. But in the k-vector space of all homogenous order 2 polynomials in w, x, y, z $x^2 - wy, xy - wz, y^2 - xz$ are LI, and this space cannot be spanned by two elements.

i.e. k-module gen by. $x^2 - wy, xy - wz, y^2 - xz$

NOTE (Counting Monomials)

We claim that

LEMMA The number of monomials of weight d in $x_1,\dots,x_n$ is $\binom{n+d-1}{n-1} = \binom{n+d-1}{d}$ $(n\geq 1, d\geq 1)$

PROOF Given a monomial $x_1^2 x_2 x_3^3$ for example $(n=3, d=6)$ we can write

$$x_1 x_1 x_2 x_3 x_3 x_3$$
$$* \; * \,|\, * \,|\, * \; * \; *$$

In this way we set up a bijection between monomials in n variables of weight d and arrangements of d stars and $n-1$ bars. But such an arrangement is defined by choosing from $d+n-1$ "spots" where d stars will go : this is $\binom{n+d-1}{d}$. $\square$

LEMMA The number of monomials of weight $\leq d$ in $x_1,\dots,x_n$ is $\binom{n+d}{d}$

(that is, we allow the single mon. of wt. 0 $1 = x_1^0 \cdots x_n^0$)

PROOF So the weight is $0 \leq \cdot \leq d$. We define a bijection between monomials of weight $\leq d$ in $x_1,\dots,x_n$ and monomials of weight d in $x_0, x_1, \dots, x_n$ by

$$x_1^{\alpha_1}\cdots x_n^{\alpha_n} \longleftrightarrow x_0^{d-|\alpha|} x_1^{\alpha_1}\cdots x_n^{\alpha_n}$$

$$1 \longleftrightarrow x_0^d$$

Hence the Lemma follows immediately. $\square$

NOTE Testing for Irreducibility

THEOREM (Eisenstein's Criterion) Let R be a UFD with quotient field F. Let $f = a_0 + a_1 x + \cdots + a_n x^n$ ($a_n \neq 0$) be in $R[x]$ and suppose that $p \in R$ is a prime such that

p does not divide a_n
$p \mid a_i$ for $0 \leq i \leq n-1$
p^2 does not divide a_0

Then $f(x)$ is irreducible in $F[x]$, hence in $R[x]$.

PROOF See Adkins p 97. □

If k is a field $k[x_1, \ldots, x_n] \cong k[x_1, \ldots, x_{n-1}][x_n]$ is a UFD, so put $R = k[x_1, \ldots, x_{n-1}]$ in the above.

EXAMPLE ① $y^2 + x^2 + x$ is irreducible in $k[x,y]$. Consider $(x + x^2) + y^2 \in k[x][y]$ and $p = x \in k[x]$

② This works for any $y^n + a_1 x + \cdots + a_k x^k$ with $a_1 \neq 0$ and $n \geq 1$.

③ In fact it works for $y^n + f_1(x) y^{n-1} + \cdots + f_{n-1}(x) y + f_n(x)$ provided $x \mid f_i(x)$ and $f_i(x)$ has a nonzero x term. So for example

$$y^{120} - x^3 y^5 + x y^5 + 3x^4 y - 17x^2 + x$$
$$= y^{120} + (x - x^3) y^5 + (3x^4) y + x - 17x^2$$

is irreducible in $k[x,y]$.

④ So provided $f(x,y)$ has

ⓘ no constant term

ⓘⓘ Let $g(x)$ be the collection of x only terms. Then $x \mid g(x)$ but $x^2 \nmid g(x)$ (or the same with $x \longleftrightarrow y$)

Then f is irreducible.

Of course, this doesn't work for $y^2 - x^2(x+1)$. So here's the tough way. Suppose $f(x,y) \in k[x,y]$ is nonzero, nonconstant and involves no monomials of order 4 or more – so $f = f_0 + f_1 + f_2 + f_3$. To be explicit, say

$$f_1 = f_x \, x + f_y \, y$$
$$f_2 = f_{x^2} x^2 + f_{y^2} y^2 + f_{xy} xy$$
$$f_3 = f_{x^3} x^3 + f_{y^3} y^3 + f_{xy^2} x y^2 + f_{yx^2} y x^2$$

Now suppose $f = GH$. Then $G = G_0 + G_1$ and $H = H_0 + H_1 + H_2$ $(0 = f_4 = G_2 H_2$ so wlog $G_2 = 0)$

$$f_0 = G_0 H_0$$
$$f_x x + f_y y = G_0 H_1 + G_1 H_0$$
$$= G_0 (H_x x + H_y y) + H_0 (G_x x + G_y y)$$
$$= \{G_0 H_x + H_0 G_x\} x + \{G_0 H_y + H_0 G_y\} y$$

$$f_{x^2} x^2 + f_{y^2} y^2 + f_{xy} xy = G_0 H_2 + G_1 H_1$$
$$= G_0 (H_{x^2} x^2 + H_{xy} xy + H_{y^2} y^2)$$
$$+ (G_x x + G_y y)(H_x x + H_y y)$$

$$f_{x^3}x^3 + f_{y^3}y^3 + f_{xy^2}xy^2 + f_{yx^2}yx^2$$
$$= G_1H_2$$
$$= + (G_x x + G_y y)(H_{x^2}x^2 + H_{xy}xy + H_{y^2}y^2)$$

This leads to:

① $f_0 = G_0H_0$
② $f_x = G_0H_x + H_0G_x$
③ $f_y = G_0H_y + H_0G_y$
④ $f_{x^2} = G_0H_{x^2} + G_xH_x$
⑤ $f_{xy} = G_0H_{xy} + H_yG_x + H_xG_y$
⑥ $f_{y^2} = G_0H_{y^2} + G_yH_y$
⑦ $f_{x^3} = G_xH_{x^2}$
⑧ $f_{y^3} = G_yH_{y^2}$
⑨ $f_{xy^2} = G_xH_{y^2} + G_yH_{xy}$
⑩ $f_{yx^2} = G_xH_{xy} + G_yH_{x^2}$

<u>EXAMPLES</u> (i) $y^2 - x^2(x+1) = y^2 - x^3 - x^2$. The equations become

$0 = G_0H_0$ (wlog put $G_0 = 0$)

$0 = H_0G_x$
$0 = H_0G_y$ (since $f \neq 0$, $G \neq 0$ so $G_1 \neq 0$ $\therefore$ $H_0 = 0$ also)

$-1 = G_xH_x$
$0 = H_yG_x + H_xG_y$
$1 = G_yH_y$
$-1 = G_xH_{x^2}$
$0 = G_yH_{y^2}$
$0 = G_xH_{y^2} + G_yH_{xy}$
$0 = G_xH_{xy} + G_yH_{x^2}$

Case Ⓐ $G_y = 0$. Then $G = G_1 = G_xx$ with $G_x \neq 0$. But $x \nmid f$, so this is impossible

Ⓑ $G_y \neq 0$, and $H_{y^2} = 0$. Then $G_yH_{xy} = 0 \Rightarrow H_{xy} = 0$, and $G_yH_{x^2} = 0 \Rightarrow H_{x^2} = 0$ so $H_2 = 0$ which is impossible since x^3 has to come from somewhere! (ie. $1 = G_xH_{x^2} = 0$)