3. Morphisms

So far we have defined affine and projective varieties, but we have not discussed what mappings are allowed between them. We have not even said when two are isomorphic. In this section we will discuss the regular functions on a variety, and then define a morphism of varieties. Thus we will have a good category in which to work. Let Y be a quasi-affine variety in $\mathbb{A}^n$. We will consider functions f from Y to k.

DEFINITION A function $f : Y \longrightarrow k$ is <u>regular at a point</u> $P \in Y$ if there is an open neighborhood U with $P \in U \subseteq Y$ and polynomials $g, h \in A = k[x_1, \dots, x_n]$, such that h is nowhere zero on U, and $f = g/h$ on U. (Here of course we interpret the polynomials as functions on $\mathbb{A}^n$, hence on Y). We say that f is <u>regular on Y</u> if it is regular at every point of Y.

LEMMA 3.1 A regular function is continuous, when k is identified with $\mathbb{A}^1_k$ in its Zariski topology.

PROOF It is enough to show that f^{-1} of a closed set is closed. A closed set of $\mathbb{A}^1_k$ is a finite set of points, so it is sufficient to show that $f^{-1}(a) = \{P \in Y \mid f(P) = a\}$ is closed for any $a \in k$. This can be checked locally: a subset Z of a topological space Y is closed iff. Y can be covered by any number of open sets U such that $Z \cap U$ is closed in U for each U. (Since then $U - Z$ is open, so taking unions $Y - Z$ is open). So let U be an open set on which f can be represented by g/h, with $g, h \in A$ and h nowhere 0 on U. Then

$$\begin{aligned} f^{-1}(a) \cap U &= \{P \in U \mid g(P)/h(P) = a\} \\ &= \{P \in U \mid (g - ah)(P) = 0\} \\ &= Z(g - ah) \cap U \end{aligned}$$

which is closed in U. Hence $f^{-1}(a)$ is closed in Y. $\square$

Now let us consider a quasi-projective variety $Y \subseteq \mathbb{P}^n$:

DEFINITION A function $f : Y \longrightarrow k$ is <u>regular at a point</u> $P \in Y$ if there is an open neighborhood U with $P \in U \subseteq Y$, and homogenous polynomials $g, h \in S = k[x_0, \dots, x_n]$, of the same degree, such that h is nowhere zero on U, and $f = g/h$ on U. (Note that in this case, even though g and h are not functions on $\mathbb{P}^n$, their quotient is a well-defined function whenever $h \neq 0$, since they are homogenous of the same degree). We say that f is <u>regular on</u> Y if it is regular at every point.

REMARK 3.1.1 As in the quasi-affine case (and using essentially the same proof) a regular function is necessarily continuous. An important consequence of this is the fact that if f and g are regular functions on a variety X, and if $f = g$ on some nonempty open subset $U \subseteq X$, then $f = g$ everywhere. This follows since X is irreducible (either as an affine or projective variety or by Ex 1.6) and hence by Ex 1.6 U is dense and irreducible. Hence since if f is regular and g is regular, $f - g$ is regular, the points of X at which $f - g = 0$ is closed and contains the dense open set U, and is hence all of X.

Now we can define the category of varieties.

DEFINITION Let k be a fixed algebraically closed field. A <u>variety over k</u> (or simply <u>variety</u>) is any affine, quasi-affine, projective or quasi-projective variety as defined above. If X, Y are two varieties, a <u>morphism</u> $\varphi : X \longrightarrow Y$ is a continuous map such that for every open set $V \subseteq Y$, and for every regular function $V \longrightarrow k$, the function $f\varphi : \varphi^{-1}V \longrightarrow k$ is regular.

Clearly the composition of two morphisms is a morphism, so we have a category. In particular, we have the notion of isomorphism: an <u>isomorphism</u> $\varphi : X \longrightarrow Y$ of two varieties is a morphism which admits an inverse morphism $\psi : Y \longrightarrow X$ with $\psi\varphi = 1$ and $\varphi\psi = 1$. Note that an isomorphism is necessarily bijective and bicontinuous, but a bijective bicontinuous morphism need not be an isomorphism (Ex 3.2).

Now we introduce some rings of functions associated with any variety.
(Note if Y is any variety we consider the unique function $\emptyset \longrightarrow k$ to be regular on the open subset $\emptyset \subseteq Y$)

DEFINITION Let Y be a variety. We denote by $\mathcal{O}(Y)$ the ring of all regular functions on Y. If P is a point of Y, we define the local ring of P on Y, $\mathcal{O}_{P,Y}$ (or simply $\mathcal{O}_P$) to be the ring of germs of regular functions on Y, near P. In other words, an element of $\mathcal{O}_P$ is a pair (U, f) where U is an open subset of Y containing P, and f is a regular function on U, and where we identify two such pairs (U,f) and (V,g) if $f=g$ on $U\cap V$. (Note since $U\cap V$ is irred. $f=g$ on $U\cap V$ iff. $f=g$ on some open subset $U\cap V$)

To see that this germ relation is an equivalence relation, suppose $(U,f)\sim(V,g)$ and $(V,g)\sim(W,h)$. Then certainly $f=h$ on $U\cap V\cap W$, and since f,h restrict to regular functions on the irreducible space $U\cap W$, and since the closed set $(f-h)^{-1}0$ contains $(U\cap W)\cap V$, an open and hence dense subset of $U\cap W$, $f=h$ on all of $U\cap W$.

If $f,g: Y\longrightarrow k$ are two regular functions on an arbitrary variety Y, then $f+g$, $f-g$, fg are all regular on Y. Also, the constant functions $0: x\longmapsto 0$ and $1: x\longmapsto 1$ are clearly regular. (this for Y any variety). First, for Y quasi-affine, let $P\in Y$ and $P\in U_1\subseteq Y$, $P\in U_2\subseteq Y$ be s.t. $f=h/q$ on U_1 and $g=h'/q'$ on U_2, both q and q' nonzero on U_1, U_2 resp. Then on $U_1\cap U_2$, the polynomial qq' is nonzero, and

$$f+g=\frac{hq'+qh'}{qq'}$$

$$f-g=\frac{hq'-qh'}{qq'}$$

$$fg=\frac{hh'}{qq'}$$

Similarly if Y is quasi-projective, U_1, U_2, h, q, h', q' as above, all homogenous and h,q (resp. h',q') of the same degree, then $hq'\pm qh'$ is homogenous of degree equal to qq''s degree, and similarly for hh', qq'. It is now easy to see that the regular functions $Y\longrightarrow k$ form a ring. Note that $\mathcal{O}_P$ is indeed a local ring, with

$$(U,f)+(V,g)=(U\cap V, f+g)$$
$$(U,f)\cdot(V,g)=(U\cap V, fg)$$
$$1=(Y,1)$$
$$0=(Y,0)$$

Both these operations are well-defined, since if $(U,f)\sim(U',f')$ and $(V,g)\sim(V',g')$, $f=f'$ on $U\cap U'$ and $g=g'$ on $V\cap V'$, so $(U\cap V, f+g)=(U'\cap V', f'+g')$, since on $U\cap U'\cap V\cap V'$, $f=f'$ and $g=g'$, likewise for the product. The unique maximal ideal m is the set of germs of regular functions which vanish at P. Clearly the set of all (U,f), $f(P)=0$ form an ideal of $\mathcal{O}_P$, and if $f(P)\neq 0$ let $P\in\hat{U}\subseteq U$ be s.t. $f=g/h$ on $\hat{U}$, h nonzero on $\hat{U}$. Then $g(P)\neq 0$, and moreover $D(g)=\{P' \mid g(P')\neq 0\}$ is open in U, so we can find $V\subseteq U$ open containing P s.t. g is nonzero on V. Then let $q=h/g$ on V. Then

$$\begin{aligned}(U,f)(V,q)&=(U\cap V, fq)\\&=(\hat{U}\cap V, g/h\cdot h/g)\\&=(\hat{U}\cap V, 1)\\&=(Y,1)\end{aligned}$$

Hence anything outside of m is a unit. Hence $\mathcal{O}_P$ is local. We also claim that $\mathcal{O}_P/m\cong k$. This is defined in the obvious way

$$\phi: \mathcal{O}_P\longrightarrow k$$

by $\phi(U,f)=f(P)$. This is clearly a morphism of rings, and $\operatorname{Ker}\phi=m$. Since ϕ is surjective (for $e\in k$, (Y,e) is regular), $\mathcal{O}_P/m\cong k$.

DEFINITION If Y is a variety, we define the function field $K(Y)$ of Y as follows: an element of $K(Y)$ is an equivalence class of pairs (U,f) where U is a nonempty open subset of Y, f is a regular function on U, and where we identify two pairs (U,f) and (V,g) if $f=g$ on $U\cap V$. The elements of $K(Y)$ are called rational functions on Y.

Note that $K(Y)$ is in fact a field. Since Y is irreducible, any two nonempty open subsets have a nonempty intersection. Hence, as we already checked, the above defines an equivalence relation. Hence we can define addition and multiplication in $K(Y)$, making it into a ring. Then if $(U,f)\in K(Y)$ with $f\neq 0$, we can restrict f to the open set $V=U-U\cap Z(f)$ where it never vanishes (a regular function is continuous, so $f^{-1}0$ is closed, hence the complement of $Z(f)=f^{-1}0$ is open). Then $1/f$ is regular on V, since at $P\in V$ let $P\in W\subseteq V$ be s.t. $f=g/h$ on W. Then $f\neq 0$ on W implies $g\neq 0$ on W, so that $h/g: W\longrightarrow k$ is regular, so $1/f$ is regular. Hence $(V,1/f)$ is an inverse for (U,f).

Now we have defined, for any variety Y, the ring of global functions $\mathcal{O}(Y)$, the local ring $\mathcal{O}_P$ at a point of Y, and the function field $K(Y)$. By restricting functions we obtain natural maps

$$\mathcal{O}(Y) \longrightarrow \mathcal{O}_P \longrightarrow K(Y)$$

which in fact are injective by (3.1.1). Hence we will usually treat $\mathcal{O}(Y)$ and $\mathcal{O}_P$ as subrings of $K(Y)$. (Note that all three rings are k-algebras)

If we replace Y by an isomorphic variety W via $\varphi: Y \longrightarrow W$, $\varphi^{-1}: W \longrightarrow Y$, then φ sets up an isomorphism of rings $\mathcal{O}(Y) \cong \mathcal{O}(W)$, $\mathcal{O}_{P,Y} \cong \mathcal{O}_{P',W}$ where $P \in Y$ and $P' = \varphi(P) \in W$, and $K(Y) \cong K(W)$. This works because for $f: W \longrightarrow k$ regular, $f\varphi: V \longrightarrow k$ is regular, and $f \longmapsto f\varphi$ is a morphism of rings $\mathcal{O}(W) \longrightarrow \mathcal{O}(Y)$, which is clearly bijective. Likewise if $P \in Y$ and $P' = \varphi(P) \in W$, for $(U,f) \in \mathcal{O}_{P,Y}$ let $\Theta(U,f)$ denote the pair

$$(\varphi(U), f\varphi^{-1}) \qquad f\varphi^{-1}: \varphi(U) \longrightarrow k$$

This defines a morphism of rings $\mathcal{O}_{P,Y} \longrightarrow \mathcal{O}_{P',W}$ which is easily seen to be bijective. A similar argument shows $K(Y) \cong K(W)$. Thus we can say that $\mathcal{O}(Y)$, $\mathcal{O}_P$ and $K(Y)$ are <u>invariants</u> of the variety Y (and the point P) up to isomorphism.

Our next task is to relate $\mathcal{O}(Y)$, $\mathcal{O}_P$ and $K(Y)$ to the affine coordinate ring $A(Y)$ of an affine variety, and the homogenous coordinate ring $S(Y)$ of a projective variety, which were introduced earlier. We will find that for an affine variety Y, $A(Y) = \mathcal{O}(Y)$, so it is an invariant up to isomorphism. However, for a projective variety Y, $S(Y)$ is not an invariant: it depends on the embedding of Y in projective space (Ex 3.9).

<u>THEOREM 3.2</u> Let $Y \subseteq \mathbb{A}^n$ be an affine variety with affine coordinate ring $A(Y)$. Then:

(a) $\mathcal{O}(Y) \cong A(Y)$;

(b) for each point $P \in Y$, let $m_P \subseteq A(Y)$ be the ideal of functions vanishing at P. Then $P \longmapsto m_P$ gives a 1-1 correspondence between the points of Y and the maximal ideals of $A(Y)$;

(c) for each P, $\mathcal{O}_P \cong A(Y)_{m_P}$, and $\dim \mathcal{O}_P = \dim Y$;

(d) $K(Y)$ is isomorphic to the quotient field of $A(Y)$, and hence $K(Y)$ is a finitely generated extension field of k, of transcendence degree $\dim Y$.

<u>PROOF</u> We will proceed in several steps. First we define a map $\alpha: A(Y) \longrightarrow \mathcal{O}(Y)$. Every polynomial $f \in A = k[x_1, \dots, x_n]$ defines a regular function on $\mathbb{A}^n$ and hence on Y. Thus we have a homomorphism $A \longrightarrow \mathcal{O}(Y)$. Its kernel is just $I(Y)$, so we obtain an injective homomorphism $\alpha: A(Y) \longrightarrow \mathcal{O}(Y)$. From (1.4) we know there is a 1-1 correspondence (inclusion-reversing) between algebraic subsets of Y and prime ideals containing $I(Y)$. Hence the points (= minimal gebraic subsets) correspond to the maximal ideals of A containing $I(Y)$. The maximal ideal corresponding to $P \in Y$ is

$$m_P = \{ f \in A \mid f(P) = 0 \}$$

This proves (b). For each P there is a natural map $A(Y)_{m_P} \longrightarrow \mathcal{O}_P$, induced by $\mathcal{O}(Y) \longrightarrow \mathcal{O}_P$ and $A(Y) \longrightarrow \mathcal{O}(Y)$. It is injective because $A(Y)$ is a domain and if $f/g, f'/g' \in A(Y)_{m_P}$ are s.t. they determine the same regular function on some open $P \in U' \subseteq Y$, then $fg = f'g$ on U' and hence by 3.1.1 $fg = f'g$ on all of Y, hence in $A(Y)$, so that $f/g = f'/g'$ in $A(Y)_{m_P}$. To see that the morphism is surjective, let $(U,f) \in \mathcal{O}_P$, $f: U \longrightarrow k$ regular. In particular, f is regular at P, so let $U' \subseteq U$ be open and $g, h \in A$ be s.t. $f = g/h$ on U', $h \neq 0$ on U'. Then $(U,f) = (U', g/h)$ in $\mathcal{O}_P$, and $h(P) \neq 0$ implies that in $A(Y)$, $h \notin m_P$, so $g/h \in A(Y)_{m_P}$ and (U,f) is the image of g/h. Hence $A(Y)_{m_P} \cong \mathcal{O}$. Now $\dim \mathcal{O}_P = \operatorname{height} m_P$. Since $A(Y)/m_P \cong k$, we conclude from (1.7) and (1.8A) that $\dim Y = \dim \mathcal{O}_P$.

From (c) it follows that the quotient field of $A(Y)$ is isomorphic to the quotient field of $\mathcal{O}_P$ for every P, and this is equal to $K(Y)$, because every rational function is actually in some $\mathcal{O}_P$. That is, $\mathcal{O}_P \cong A(Y)_{m_P}$ which can be identified with a subring of $Q(A(Y))$, hence there is induced $Q(A(Y)) \longrightarrow K(Y)$ each $P \in Y$, defined uniquely s.t.

$$\begin{tikzcd} A(Y)_{m_P} \cong \mathcal{O}_P \arrow[r] \arrow[d] & K(Y) \\ Q(A(Y)) \arrow[ur, "\varphi"'] & \end{tikzcd}$$

commutes. We define $\varphi(f/g) = (D(g), f/g)$ where $D(g) = \{ P \in Y \mid g(P) \neq 0 \}$, which is open.

This is well-defined since if $f/g = f'/g'$ in $Q(A(Y))$, then $fg' = gf'$ in $A(Y)$ and so in $D(g) \cap D(g') = D(gg')$ both f/g and f'/g' agree. φ is injective since if $(D(g), f/g) = (D(g'), f'/g')$ then $f/g = f'/g'$ on $D(gg')$, and hence by 3.1.1 $fg = gf'$ on all of Y, hence $f/g = f'/g'$ in $A(Y)$. It is surjective since if (U, f) is in $K(Y)$, let $P \in U$ be s.t. $f(P) \neq 0$. Then on a neighborhood U' of P, $f = g/h$ some $g, h \in A$. Hence $(U, f) = (U', f) = (U', g/h) = (D(h), g/h) = \varphi(g/h)$. Since φ is clearly a morphism of rings, $Q(A(Y)) \cong K(Y)$, and φ is defined in such a way that the above diagram commutes for all P. Now $A(Y)$ is finitely generated as a k-algebra, so $K(Y)$ is a finitely generated field extension of k. Furthermore, the transcendence degree of $K(Y)/k$ is equal to $\dim Y$ by (1.7) and (1.8A). This proves (d).

To prove (a) we note that $\mathcal{O}(Y) \subseteq \bigcap_{P \in Y} \mathcal{O}_P$, where our rings are regarded as subrings of $K(Y)$. Using (b), (c), we have

$$A(Y) \subseteq \mathcal{O}(Y) \subseteq \bigcap_{m} A(Y)_m$$

where m runs over all the maximal ideals of $A(Y)$. Since if B is a domain B is equal to the intersection (inside its quotient field) of ~~its~~ localisations at all maximal ideals, we get equality. More explicitly, define

$$\alpha : A(Y) \longrightarrow \mathcal{O}(Y)$$

as above. We need only show that α is surjective. See Milne's notes. $\square$

PROPOSITION 3.3 Let $U_i \subseteq \mathbb{P}^n$ be the open set defined by the equation $x_i \neq 0$. Then the mapping $\varphi_i : U_i \longrightarrow \mathbb{A}^n$ of (2.2) above is an isomorphism of varieties.

PROOF We have already shown that it is a homeomorphism, so we need only check that the regular functions are the same on any open set. On U_i the regular functions are locally quotients of homogenous polynomials in $x_0, \dots, x_n$ of the same degree. On $\mathbb{A}^n$ the regular functions are locally quotients of polynomials in $y_1, \dots, y_n$. Let $P \in \mathbb{A}^n$ and suppose $P \in U \subseteq \mathbb{A}^n$ s.t. on U, $f = g/h$. Notice that for α, β as in the proof of (2.2), and $a_0, \dots, a_n \in k$, assuming wlog $i = 0$,

$$\frac{g(a_1/a_0, \dots, a_n/a_0)}{h(a_1/a_0, \dots, a_n/a_0)} = \frac{\beta(g)(a_0, \dots, a_n)\, a_0^{d}}{\beta(h)(a_0, \dots, a_n)\, a_0^{e}}$$

where the leading term of g has degree e and resp. d for h. Hence if we let $\hat{\beta}(g) = x_0^{d}\beta(g) = x_0^{d+e} g(x_1/x_0, \dots, x_n/x_0)$ and $\hat{\beta}(h) = x_0^{d+e} h(x_1/x_0, \dots, x_n/x_0)$ then $\hat{\beta}(g), \hat{\beta}(h)$ are both homogenous of degree $d+e$ and $\hat{\beta}(g)/\hat{\beta}(h)$ takes the same value on $(a_0, \dots, a_n)$ as g/h does on $(a_1/a_0, \dots, a_n/a_0)$, and for any $(b_1, \dots, b_n) \in U$, $\hat{\beta}(h)$ is nonzero on $(1, b_1, \dots, b_n)$ since h is nonzero at $(b_1, \dots, b_n)$. Hence if $f : \mathbb{A}^n \longrightarrow k$ is regular, so is $f\varphi_i$ so that φ_i is a morphism of varieties.

Conversely, suppose $f : U_i \longrightarrow k$ is regular, where U_i is a quasi-projective variety. Let $P \in \mathbb{A}^n$, say $P = (q_1, \dots, q_n)$. Then $P' = (1, q_1, \dots, q_n)$ has an open neighborhood U' in U_i and homogenous polynomials $g, h \in k[x_0, \dots, x_n]$ of the same degree s.t. $f = g/h$ on U'. Then for $(p_1, \dots, p_n) \in \varphi_i(U')$,

$$\begin{aligned}
(f\varphi_i^{-1})(p_1, \dots, p_n) &= f(1, p_1, \dots, p_n) \\
&= \frac{g(1, p_1, \dots, p_n)}{h(1, p_1, \dots, p_n)} \\
&= \frac{\alpha(g)(p_1, \dots, p_n)}{\alpha(h)(p_1, \dots, p_n)}
\end{aligned}$$

hence $f\varphi_i^{-1}$ is also regular, as required. $\square$

NOTE $\mathcal{O}(Y) = \bigcap \mathcal{O}(U_i)$, $Y = \bigcup_i U_i$

Let Y be a variety, $Y = \bigcup_i U_i$ a cover of Y by nonempty open subsets. As usual, there is an injection of rings $\mathcal{O}(U_i) \longrightarrow K(Y)$ and we identify $\mathcal{O}(Y), \mathcal{O}(U_i)$ $i \in I$ with these subrings of $K(Y)$. Clearly in this sense $\mathcal{O}(Y) \subseteq \mathcal{O}(U_i)$ for each i and we claim that $\mathcal{O}(Y) = \bigcap_i \mathcal{O}(U_i)$.

For suppose $(U, f) \in K(Y)$ is in each $\mathcal{O}(U_i)$, say $(V_i, f_i) \in \mathcal{O}(U_i)$ s.t. $(U, f) = (V_i, f_i)$ in $K(Y)$, so $f = f_i$ on $U \cap V_i$. Now for $i \neq j$ $U \cap V_i \cap V_j$ is a nonempty open subset of $V_i \cap V_j$ and on this set $f_i = f = f_j$. Hence $f_i = f_j$ on $V_i \cap V_j$, so we can find a regular function $F : Y \longrightarrow k$ (the V_i cover Y) s.t. $F|_{V_i} = f_i$ for each i.

Since $U = \bigcup_i V_i \cap U$ and $F = F|_{V_i} = f_i = f$ on $V_i \cap U$ we see that $(U, f) = (Y, F) \in \mathcal{O}(Y)$ in $K(Y)$, as required.

NOTE Since $\mathcal{O}(U) \longrightarrow K(Y)$ is injective for any $U \subseteq Y$, $\mathcal{O}(U)$ is a domain.

Before stating the next result, we introduce some notation. If S is a graded domain, and $\mathfrak{p}$ a homogonous prime ideal in S, then let T be the multiplicatively closed set of homogenous elements of S not in $\mathfrak{p}$. Then we define a grading on $T^{-1}S$ by

$$T^{-1}S = \bigoplus_{d \geq 0} M_d$$

$$M_d = \{ f/g \mid f \text{ homogenous and } \deg f - \deg(g) = d \}$$

NO

not a +ve grading

This is well-defined since if $f/g = f'/g'$ with $f' = \sum_d f'^{(d)}$ then

$$g'f = \sum_d g f'^{(d)}$$

Since $g'f$ is homogenous all the $gf'^{(d)}$ are zero except for one. Since $g \neq 0$ this implies f' is homogenous. Moreover $g'f = gf'$ belongs to $S_{\deg f + \deg g'} \cap S_{\deg g + \deg f'}$. Hence either $f = f' = 0$ or $df - dg = df' - dg'$. The definition above should probably read $M_d = 0$ plus f/g $f \neq 0$ etc. It is easily checked that M_d are abelian groups and that $M_d \cap M_e = 0$ if $d \neq e$....

NOTE This isn't a positive grading because $\deg f - \deg g$ may be negative.

Nonetheless, the subring M_0 of $T^{-1}S$ is defined, and we denote it by $S_{(\mathfrak{p})}$. $S_{(\mathfrak{p})}$ is a local ring with maximal ideal $(\mathfrak{p} \cdot T^{-1}S) \cap S_{(\mathfrak{p})}$. (this is easily checked). In particular if $\mathfrak{p} = (0)$ then $S_{((0))}$ is a field. Similarly if $f \in S$ is a homogenous element, we denote by $S_{(f)}$ the subring of elements of degree 0 in the ring S_f. Note that if g is homogenous of the same degree as f^m, so $g/f^m \in S_{(f)} \subseteq S_f$, and if $g/f^m = g'/f^n$ in S_f, it is easy to check that g' is also homogenous and the same degree as f^n. (Note if $S = k[-]$ that elements of k are hom. of degree 0). So $\ell/1 \in k[-]_{(f)}$ for $\ell \in k$.

THEOREM 3.4 Let $Y \subseteq \mathbb{P}^n$ be a projective variety with homogenous coordinate ring $S(Y)$. Then

(a) $\mathcal{O}(Y) = k$;

(b) for any point $P \in Y$, let $\mathfrak{m}_P \subseteq S(Y)$ be the ideal generated by the set of homogenous $f \in S(Y)$ such that $f(P) = 0$. Then $\mathcal{O}_P = S(Y)_{(\mathfrak{m}_P)}$;

(c) $K(Y) \cong S(Y)_{((0))}$

PROOF To begin with, let $U_i \subseteq \mathbb{P}^n$ be the open set $x_i \neq 0$ and let $Y_i = Y \cap U_i$. Then U_i is isomorphic to $\mathbb{A}^n$ by the isomorphism φ_i of (3.3), so we can consider Y_i as an affine variety. There is a natural isomorphism φ_i^* of the affine coordinate ring $A(Y_i)$ with the localisation $S(Y)_{(x_i)}$ of the homogenous coordinate ring of Y. We first make an isomorphism of $k[y_1, \dots, y_n]$ with $k[x_0, \dots, x_n]_{(x_i)}$ by sending $f(y_1, \dots, y_n)$ to $f(x_0/x_i, \dots, x_n/x_i)$, leaving out x_i/x_i, as in the proof of (2.2). (see the following note). This isomorphism sends $I(Y_i)$ to $I(Y)S_{(x_i)}$, since $\forall (a_0, \dots, a_n) \in Y_i$,

$$f\left(\frac{a_0}{a_i}, \dots, \frac{a_n}{a_i}\right) = 0 \text{ iff. } f\left(\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\right)(a_0, \dots, a_n) = 0$$

$\left[ev_{(a_0, \dots, a_n)} : k[x_0, \dots, x_n] \to k \right.$ induces w.d. $k[x_0, \dots, x_n]_{x_i} \to k$

$$\text{iff. } \frac{g(x_0, \dots, x_n)}{x_i^m}(a_0, \dots, a_n) = 0$$

$$\text{where } f\left(\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\right) = \frac{g}{x_i^m} \quad (1)$$

in $k[x_0, \dots, x_n]_{(x_i)}$. Not unique.

$$\text{iff. } g(a_0, \dots, a_n) = 0.$$

Note that if $g \in k[x_0, \dots, x_n]$ is zero on $Y_i = Y \cap U_i$, then since Y_i is open and hence dense in Y, g is zero on all of Y.

[IMPORTANT NOTE We are trying to show $A(Y_i) \cong S(Y)_{(x_i)}$. In case $Y \cap U_i = \emptyset$, which can only happen if $x_i \in I(Y)$, $Y_i = \emptyset$ and one could say $A(Y_i) = 0$. Certainly since $x_i = 0$ in $S(Y)$, $S(Y)_{(x_i)} = 0$. So we may as well assume $Y \cap U_i \neq \emptyset$, and in this case we can apply the above logic to see that g is zero on Y iff. it is zero on Y_i]

Since any g/x_i^n in $I(Y)S_{(x_i)}$ has the form $f\left(\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\right)$ for some f, the above shows that the iso identifies $I(Y_i)$ with $I(Y)S_{(x_i)}$ (which is thus a prime ideal). Hence

$$\varphi_i^* : A(Y_i) = \frac{k[y_1, \dots, y_n]}{I(Y_i)} \cong \frac{k[x_0, \dots, x_n]_{(x_i)}}{I(Y)S_{(x_i)}} \cong \left(S/I(Y) \right)_{(x_i)} = S(Y)_{(x_i)}$$

(see note)

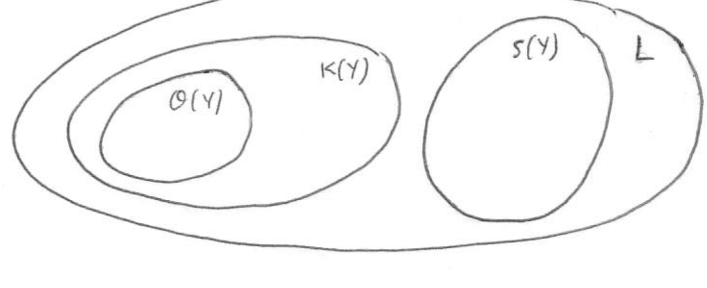
as k-algebras. This map is $f(y_1, \dots, y_n) + I(Y_i) \mapsto f\left(\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\right) + I(Y)S_{(x_i)} \mapsto \dfrac{g(x_0, \dots, x_n) + I(Y)}{x_i^n + I(Y)}$ where g is as in (1).

Now to prove (b), let $P \in Y$ be any point, and choose i s.t. $P \in Y_i$. Then by (3.2) $\mathcal{O}_P \cong A(Y_i)_{\mathfrak{m}'_P}$ where $\mathfrak{m}'_P$ is the maximal ideal of $A(Y_i)$ corresponding to P. Equation (1) on the previous page shows that $\varphi_i^*(\mathfrak{m}'_P) = \mathfrak{m}_P \cdot S(Y)_{(x_i)}$ (note $x_i \notin \mathfrak{m}_P$). Hence $A(Y_i)_{\mathfrak{m}'_P}$ is isomorphic as a k-algebra to the localisation of $S(Y)_{(x_i)}$ at $\mathfrak{m}_P \cdot S(Y)_{(x_i)}$. But the canonical morphism of k-algebras $S(Y)_{x_i} \longrightarrow S(Y)_{\mathfrak{m}_P}$ restricts to $S(Y)_{(x_i)} \longrightarrow S(Y)_{(\mathfrak{m}_P)}$ and this induces a morphism from the aforementioned localisation to $S(Y)_{(\mathfrak{m}_P)}$

$$\frac{\bar{f}/\bar{x}_i^{n}}{\bar{g}/\bar{x}_i^{m}} \longmapsto \frac{\overline{f x_i^{m}}}{\overline{x_i^{n} g}} \qquad \begin{array}{l} g \notin \mathfrak{m}_P \\ \text{hom. deg } m. \end{array}$$

It is easy enough to check that this is an isomorphism of k-algebras. Hence $\mathcal{O}_P \cong A(Y_i)_{\mathfrak{m}'_P} \cong S(Y)_{(\mathfrak{m}_P)}$, which proves b). One can use the same argument to see that localising $S(Y)_{(x_i)}$ at all the nonzero things (which is well-defined since $S(Y)_{(x_i)} \cong A(Y_i)$ is a domain) is isomorphic to $S(Y)_{((0))}$. But then $K(Y_i) \cong S(Y)_{((0))}$ by (3.2). Pick an i so that $Y_i \neq \emptyset$, then $K(Y) \cong K(Y_i) \cong S(Y)_{((0))}$, proving c).

To prove a), we first do some legwork. We consider $S(Y)$, $\mathcal{O}(Y)$, $K(Y)$ as sub k-algebras of the quotient field L of $S(Y)$



The inclusion $S(Y) \hookrightarrow L$ is obvious, as is $\mathcal{O}(Y) \hookrightarrow K(Y)$. The inclusion $K(Y) \hookrightarrow L$ is defined via (choose $Y_i \neq \emptyset$)

$$K(Y) \cong K(Y_i) \cong Q(A(Y_i)) \cong Q(S(Y)_{(x_i)}) \cong S(Y)_{((0))} \hookrightarrow L$$

which works as follows: let (V, f) be a rational function on Y. Then $(V \cap Y_i, f|_{V \cap Y_i})$ is a rational function on Y_i. There are $h, g \in k[y_1, \dots, y_n]$ with $g \notin I(Y_i)$ such that

$$f(a_0, \dots, a_n) = \frac{h\left(\frac{a_0}{a_i}, \dots, \frac{a_n}{a_i}\right)}{g\left(\frac{a_0}{a_i}, \dots, \frac{a_n}{a_i}\right)} \qquad \forall (a_0, \dots, a_n) \in D(g) \cap V \cap Y_i$$

Under the correspondence (1) on the previous page, $h/g \in Q(A(Y_i))$ maps to $\dfrac{h\left(\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\right)}{g\left(\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\right)} = \dfrac{h'(x_0, \dots, x_n)\, x_i^{m}}{g'(x_0, \dots, x_n)\, x_i^{n}}$

where $h\left(\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\right) = h'/x_i^{n}$ and $g\left(\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\right) = g'/x_i^{m}$. This element of $S(Y)_{((0))} \in L$ defines a rational map on Y with domain $D(g' x_i^{n}) = D(g') \cap Y_i = D(g) \cap Y_i$, defined by

$$(a_0, \dots, a_n) \longmapsto \frac{h'(a_0, \dots, a_n)\, a_i^{m}}{g'(a_0, \dots, a_n)\, a_i^{n}} = \frac{h\left(\frac{a_0}{a_i}, \dots, \frac{a_n}{a_i}\right)}{g\left(\frac{a_0}{a_i}, \dots, \frac{a_n}{a_i}\right)}$$

which agrees with $f : V \to k$ on $D(g) \cap V \cap Y_i$, an open subset of Y_i. In the case where $V = Y$ and so $f \in \mathcal{O}(Y)$, f is identified with a polynomial $h/1 \in Q(A(Y_i))$ and hence with $h'(x_0, \dots, x_n)/x_i^{m} \in L$ s.t. on all of Y_i f and $\frac{h'}{x_i^{m}}$ agree. This holds for any i with $Y_i \neq \emptyset$. Hence to show that f takes a constant value on Y, it suffice to show that for each i with $Y_i \neq \emptyset$ the polynomial h'/x_i^{m} takes a constant value on Y_i – but this holds if h'/x_i^{m} is an element of $k \subseteq L$. (let $Y_{i_1}, \dots, Y_{i_k}$ be s.t. $Y_{i_j} \neq \emptyset$. Then each Y_{i_j} is a nonempty open subset of Y $\therefore$ so is $\bigcap_j Y_{i_j}$).

For each i with $Y_i \neq \emptyset$ find $h_i/x_i^{N_i}$ s.t. f agrees with this polynomial on Y_i. Thus in L, $h_i/x_i^{N_i} = h_j/x_j^{N_j}$ for each i, j, since $h_i x_j^{N_j} - h_j x_i^{N_i} \in k[x_0, \dots, x_n]$ is zero on the nonempty open subset $Y_i \cap Y_j \subseteq Y$ and hence is zero on all of Y. That is,

$$h_i x_j^{N_j} - h_j x_i^{N_i} \in I(Y) \Rightarrow \frac{h_i}{x_i^{N_i}} = \frac{h_j}{x_j^{N_j}}$$

Denote this element of L by g. So either $Y_i = \phi$, in which case $x_i \in I(Y)$ and $x_i = 0$ in L, so $gx_i = 0$, or $Y_i \neq \phi$ and $g x_i^{N_i} = (h_i / x_i^{N_i}) x_i^{N_i} = h_i \in S(Y)_{N_i}$ (which means the homog. elts. of $S(Y)$ of degree N_i). Choose $N \geq \sum_i N_i$. Then $S(Y)_N$ is spanned as a k-vector space by monomials of degree N in $x_0, \dots, x_n$, and in any such monomial, at least one x_i occurs to a power $\geq N_i$. Thus we have $S(Y)_N \cdot g \subseteq S(Y)_N$. Iterating, we have $S(Y)_N \cdot g^q \subseteq S(Y)_N$ for all $q > 0$. In particular $x_0^N g^q \in S(Y)$ for all $q > 0$. This shows that the subring $S(Y)[g] \subseteq L$ is contained in $x_0^{-N} S(Y)$, which is a finitely generated $S(Y)$-module. (if $x_0 = 0$ in $S(Y)$, pick some other index) Since we can then apply Atiyah & Macdonald 5.1 iii) to see that $g \in L$ is integral over $S(Y)$, there are elements $a_1, \dots, a_m \in S(Y)$ s.t.

$$g^m + a_1 g^{m-1} + \dots + a_m = 0$$

Pick any i with $Y_i \neq \phi$. Then

$$\left(\frac{h_i}{x_i^{N_i}}\right)^m + a_1 \left(\frac{h_i}{x_i^{N_i}}\right)^{m-1} + \dots + a_m = 0$$

$$h_i^m + a_1 (h_i)^{m-1} x_i^{N_i} + \dots + a_m x_i^{mN_i} = 0$$

This is an eq^n in $S(Y)$, and if an element of $S(Y)$ is zero, its homogenous component of order mN_i is zero. Hence

$$h_i^m + \lambda_1 (h_i)^{m-1} x_i^{N_i} + \dots + \lambda_m x_i^{mN_i} = 0 \qquad \lambda_1, \dots, \lambda_m \in k$$

where λ_i is the 0-th degree hom. part of $a_i \in S(Y)$. Considering this as an equation in L and dividing through by $x_i^{mN_i}$ we find an equation of algebraic dependence of g over $k \subseteq L$. Since k is algebraically closed, we conclude that g is equal to a constant in L, as required. $\square$

<u>NOTE</u> Any variety is an irreducible, noetherian topological space in which points are closed.

<u>NOTE</u> If Y is any variety, $U \subseteq Y$ nonempty open, then U is a variety and $K(Y) \cong K(U)$. Regular maps are restrictions from Y (equiv. embed U in affine or proj. space and define directly)

<u>COROLLARY</u> If Y is any variety then $K(Y)$ is a finitely generated field extension of k, and

$$\dim Y = \text{tr.deg.}\, K(Y)/k$$

<u>PROOF</u> For affine varieties by Theorem 3.2. If Y is quasi-affine, then $\overline{Y}$ is an affine variety and Y is an open subset of $\overline{Y}$, so $K(Y)$ is isomorphic as a k-algebra to $K(\overline{Y})$. By Prop 1.10 $\dim Y = \dim \overline{Y} = \text{tr.deg.}\, K(\overline{Y})/k = \text{tr.deg.}\, K(Y)/k$. Let Y be a projective variety in $\mathbb{P}^n$, $U_i \subseteq \mathbb{P}^n$ a canonical open with $Y \cap U_i = Y_i \neq \phi$. By Ex 2.6 $\dim Y = \dim Y_i$. Here Y_i is an affine variety, so $\dim Y_i = \text{tr.deg}\, K(Y_i)/k$. But again $K(Y_i)$ is k-isomorphic to $K(Y)$, so $\dim Y = \dim Y_i = \text{tr.deg}\, K(Y_i)/k = \text{tr.deg}\, K(Y)/k$. If Y is quasi-projective then by Ex 2.7 b) $\dim Y = \dim \overline{Y}$ and we proceed as for quasi-affine case. $\square$

<u>LEMMA</u> If Y is any variety and $P \in Y$, then $\mathcal{O}_{P,Y}$ is a Noetherian local domain, and $\mathcal{O}(Y)$ is a Noetherian domain provided Y is affine or projective. Moreover for any variety Y and $P \in Y$, $\dim Y = \dim \mathcal{O}_{P,Y}$.

<u>PROOF</u> For affine (hence also quasi-affine) varieties this follows from Theorem 3.2 (b). If $Y \subseteq \mathbb{P}^n$ is projective let i be s.t. $Y_i = Y \cap U_i$ is nonempty. Then for any $P \in Y_i$, since Y_i is isomorphic as a variety to an affine variety and $\mathcal{O}_{P,Y} \cong \mathcal{O}_{P,Y_i}$, it follows that $\mathcal{O}_{P,Y}$ is a Noetherian local domain. The quasi-projective case is now immediate. If Y is affine $\mathcal{O}(Y)$ is a Nothenian domain by 3.2 (a) and if Y is projective by 3.4 (a). For the dimension claim use (1.10), (Ex 2.7), the above arguments. $\square$
(Ex 2.6)

<u>NOTE</u> Any variety of dimension 0 is a point with $\mathcal{O}(Y) = k$ and local ring $\mathcal{O}_P = k$ at the single point $P \in Y$.

<u>PROOF</u> Any Noetherian topological space in which points are closed (i.e. any variety) is a point if it is irreducible and of dimension 0 (see Note in Ch 1.1). Conversely any point in affine or projective space is a variety. Say $Y = \{P\}$ is a variety. It is easy to check directly that $k = \mathcal{O}(Y) = \mathcal{O}_P$. $\square$

<u>NOTE</u> If Y is any variety, $P \in Y$, then the residue field of $\mathcal{O}_{P,Y}$ is k, since in the quotient $\mathcal{O}_{P,Y}/\mathfrak{m}$ the regular map (U, f) is identified with $f(P)$.

NOTE Y any variety $Q(\mathcal{O}_{P,Y}) \cong K(Y)$

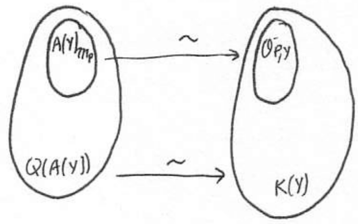
PROPOSITION Let Y be a variety. There is an injective homomorphism of rings

$$\varphi : \mathcal{O}_{P,Y} \longrightarrow K(Y)$$

for every $P \in Y$. We claim that $K(Y)$ is the quotient field of $\mathcal{O}_{P,Y}$, so every $(V,g) \in K(Y)$ can be written as $(U,f)(U',f')^{-1}$ for (U,f) and $(U',f') \in \mathcal{O}_{P,Y}$.

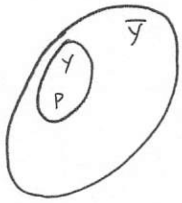
(First assume Y is affine)

PROOF There is an isomorphism of k-algebras $A(Y)_{m_P} \longrightarrow \mathcal{O}_{P,Y}$ defined by $f/g \mapsto (D(g), f/g)$. Identifying $A(Y)_{m_P}$ with a subring of the quotient field $Q(A(Y))$, this isomorphism extends to an isomorphism $Q(A(Y)) \cong K(Y)$ (see Theorem 3.2) so we have a commutative diagram



Since $Q(A(Y))$ is the quotient field of $A(Y)_{m_P}$ the result follows for affine varieties.

If Y is quasi-affine, $Y \subseteq \bar{Y} \subseteq \mathbb{A}^n$ and $P \in Y$ then we have a commutative diagram



$$\begin{tikzcd} \mathcal{O}_{P,Y} \arrow[r, "\sim"] \arrow[d] & \mathcal{O}_{P,\bar{Y}} \arrow[d] \\ K(Y) \arrow[r, "\sim"] & K(\bar{Y}) \end{tikzcd}$$

So the result holds for Y since it holds for $\bar{Y}$. A similar argument works for projective and quasi-projective varieties. $\square$

<u>NOTE</u> $k[y_1,\dots,y_n] \cong k[x_0,\dots,x_n]_{(x_i)}$

Let k be any commutative ring, and $n \geq 1$.
Here $k[x_0,\dots,x_n]_{(x_i)}$ is a sub k-algebra of $k[x_0,\dots,x_n]_{x_i}$ containing things like

$$\frac{e}{1}\ \ e\in k, \qquad \frac{a_0x_0+\cdots+a_nx_n}{x_i}, \qquad \frac{a\,x_0x_1+\cdots+x_1x_2+\cdots+x_3^2}{x_i^2}, \ \dots$$

Define $\varphi : k[y_1,\dots,y_n] \longrightarrow k[x_0,\dots,x_n]_{(x_i)}$ by

$$\begin{aligned}
y_1 &\longmapsto x_0/x_i\\
y_2 &\longmapsto x_1/x_i\\
&\ \ \vdots\\
y_i &\longmapsto x_{i-1}/x_i\\
y_{i+1} &\longmapsto x_{i+1}/x_i \qquad\qquad (\text{exclude } x_i/x_i)\\
&\ \ \vdots\\
y_n &\longmapsto x_n/x_i
\end{aligned}$$

Now define $\theta : k[x_0,\dots,x_n] \longrightarrow k[y_1,\dots,y_n]$ by $x_0 \mapsto y_1,\dots, x_{i-1}\mapsto y_i,\ x_i \mapsto 1,\dots, x_n \mapsto y_n$.
This induces a morphism of k-algebras $\theta' : k[x_0,\dots,x_n]_{x_i} \longrightarrow k[y_1,\dots,y_n]$ with

$$\frac{f(x_0,\dots,x_n)}{x_i^n} \longmapsto f(y_1,\dots,y_i,1,y_{i+1},\dots,y_n)$$

This restricts to a morphism of k-algebras $\phi : k[x_0,\dots,x_n]_{(x_i)} \longrightarrow k[y_1,\dots,y_n]$. Now $\phi\varphi(y_i) = y_i$, so $\phi\varphi = 1$. Similarly one checks that

$$\begin{aligned}
\varphi\phi\left(\frac{f(x_0,\dots,x_n)}{x_i^n}\right) & \qquad f \text{ hom. degree } n\\
&= \varphi f(y_1,\dots,y_i,1,y_{i+1},\dots,y_n)\\
&= f\left(\frac{x_0}{x_i},\dots,\frac{x_{i-1}}{x_i},1,\frac{x_{i+1}}{x_i},\dots,\frac{x_n}{x_i}\right)\\
&= \frac{f(x_0,\dots,x_n)}{x_i^n}
\end{aligned}$$

Hence $\varphi\phi = 1$ and φ is an iso of k-algebras. In the case $n=0$, $A[x]\longrightarrow A$, $x \mapsto 1$ induces $A[x]_{(x)} \longrightarrow A$
$f(x)/x^n \longmapsto f(1)$ which is an isomorphism of A-algebras (f hom. deg $n \Rightarrow f = ax^n$).

<u>NOTE</u> Grading on Quotients

Let S be a graded domain, $I \subseteq S$ homogenous ideal. Define a grading on S/I by (if $S = \bigoplus_{d\geq 0} S_d$)

$$S/I = \bigoplus_{d\geq 0} \hat{S}_d$$

where

$$\hat{S}_d = \{ f + I \mid f \in S_d \}$$

The details are all easily checked.

<u>NOTE</u> It is not difficult to check $\left(S/I(Y)\right)_{(x_i)} \cong \dfrac{S_{(x_i)}}{I(Y)S_{(x_i)}}$ via $\dfrac{f+I(Y)}{x_i^n + I(Y)} \to \dfrac{f}{x_i^n} + I(Y)S_{(x_i)}$ is w.d. and iso of k-algebras.
for $x_i \notin I(Y)$, $I(Y)$ prime homogenous, $S = k[x_0,\dots,x_n]$

NOTE $S_{(x)}[z,z^{-1}] \cong S_x$

Let S be a graded domain $S = S_0 \oplus S_1 \oplus S_2 \oplus \cdots$ and $x \in S_1$ a nonzero homogenous element of degree 1. We have already defined the subring $S_{(x)}$ of S_x. Note that $x = x/1 \in S_x$ does not belong to $S_{(x)}$. We claim that S_x is isomorphic to the ring of Laurent polynomials with coefficients in $S_{(x)}$. Recall that

$$S_{(x)}[z,z^{-1}] \cong S_{(x)}[z]_z$$

(as S_0 algebras, $S_0 \subseteq S_{(x)}$) and so there is a homomorphism of S_0-algebras

$$\varphi : S_{(x)}[z,z^{-1}] \longrightarrow S_x$$
$$f(z,z^{-1}) \longmapsto f(x,x^{-1})$$

This is surjective since an element $g/x^n \in S_x$ is a sum of terms g_i/x^n with $g_i \in S_i$. But $g_i/x^n = x^{i-n} g_i/x^i$ is in the image of φ. So it only remains to show that φ is injective. Consider an element $f(z,z^{-1})$ of $S_{(x)}[z,z^{-1}]$

$$f(z,z^{-1}) = a_{-n}z^{-n} + \cdots + a_0 + a_1 z + \cdots + a_m z^m \qquad a_i = f_i/x^{n_i} \in S_{(x)}$$
$$f_i \in S_{n_i}$$

Suppose $\varphi(f) = 0$. We show that $f_{-n} = \cdots = f_0 = \cdots = f_m = 0$ so that $f = 0$. Let N be a positive integer large enough that $N + k - n_k \geq 0$ for all $-n \leq k \leq m$. Then since S is a domain, multiplying by x^N gives

$$f_{-n} x^{N-n-n_n} + \cdots + f_0 x^{N-n_0} + \cdots + f_m x^{N+m-n_m} = 0$$

But x is of degree 1 so each of these summands is homogenous of distinct degrees $N-n, N-n+1, \ldots, N, N+1, \ldots, N+m$. Since S is graded, each summand is zero, and since $x \neq 0$ it follows that $f_{-n} = \cdots = f_0 = \cdots = f_m = 0$, as required.

Our next result shows that if X and Y are affine varieties, then X is isomorphic to Y iff. $A(X)$ is isomorphic to $A(Y)$ as a k-algebra. Actually, the proof gives more, so we state the stronger result.

<u>PROPOSITION 3.5</u> Let X be any variety and Y an affine variety. Then there is a natural bijective mapping of sets:

$$\alpha : Hom(X,Y) \xrightarrow{\sim} Hom(A(Y), \mathcal{O}(X))$$

where the left Hom means morphisms of varieties, and the right Hom means homomorphisms of k-algebras.

<u>PROOF</u> Given a morphism $\varphi: X \longrightarrow Y$, φ carries regular functions on Y to regular functions on X. Hence φ induces a map $\mathcal{O}(Y) \longrightarrow \mathcal{O}(X)$, which is clearly a homomorphism of k-algebras. But we have seen (3.2) that $\mathcal{O}(Y) \cong A(Y)$, so we get a homomorphism $A(Y) \longrightarrow \mathcal{O}(X)$. This defines α.

Conversely, suppose we are given a homomorphism $h: A(Y) \longrightarrow \mathcal{O}(X)$ of k-algebras. Suppose that Y is a closed subset of $\mathbb{A}^n$, so that $A(Y) = k[x_1, \dots, x_n]/I(Y)$. Let $\bar{x}_i$ be the image of x_i in $A(Y)$, and consider the elements $\xi_i = h(\bar{x}_i) \in \mathcal{O}(X)$. These are global functions on X, so we can use them to define a mapping $\psi: X \longrightarrow \mathbb{A}^n$ by $\psi(P) = (\xi_1(P), \dots, \xi_n(P))$ for $P \in X$. We next show that the image of ψ is contained in Y. Since $Y = Z(I(Y))$, it is sufficient to show that for any $P \in X$ and $f \in I(Y)$, $f(\psi(P)) = 0$. But

$$f(\psi(P)) = f(\xi_1(P), \dots, \xi_n(P))$$

Now f is a polynomial, and h is a morphism of k-algebras, so we have

$$f(\xi_1(P), \dots, \xi_n(P)) = h(f(\bar{x}_1, \dots, \bar{x}_n))(P) = 0$$

since $f \in I(Y)$. So ψ defines a map from X to Y, which induces the given homomorphism h. To complete the proof, we must show that ψ is a morphism. This is a consequence of the following Lemma. (Naturality is easy to check).

<u>LEMMA 3.6</u> Let X be any variety, and let $Y \subseteq \mathbb{A}^n$ be an affine variety. A map of sets $\psi: X \longrightarrow Y$ is a morphism iff. $x_i\psi$ is a regular function on X for each i, where $x_1, \dots, x_n$ are the coordinate functions on $\mathbb{A}^n$.

<u>PROOF</u> If ψ is a morphism, the $x_i\psi$ must be regular functions, by definition of a morphism. Conversely, suppose the $x_i\psi$ are regular. Then for any polynomial $f = f(x_1, \dots, x_n)$, $f\psi$ is also regular on X. Since the closed sets of Y are defined by the vanishing of polynomial functions, and since regular functions are continuous, we see that ψ^{-1} takes closed sets to closed sets, so ψ is continuous. Finally, since regular functions on open subsets of Y are locally quotients of polynomials, $g\psi$ is regular for any regular function g on any open subset of Y. Hence ψ is a morphism.

In detail, say $g: U \rightarrow k$ is regular and $x \in \psi^{-1}U$. Then $\psi(x) \in V \subseteq U$, V open, and we have $f, h \in k[x_1, \dots, x_n]$ s.t. h is nowhere zero on V and $\forall y \in V$ $g(y) = f(y)/h(y)$. Then $\forall k \in \psi^{-1}V$, $(g\psi)(k) = g(\psi(k)) = f(\psi(k))/h(\psi(k))$. But $f\psi$ is regular on X, hence on $\psi^{-1}V$, and similarly $h\psi$ is regular and nonzero on $\psi^{-1}V$, so $1/h\psi$ is regular on $\psi^{-1}V$. Hence $g\psi$ is regular on $\psi^{-1}V$, as required. □

<u>COROLLARY 3.7</u> If X and Y are two affine varieties, then X and Y are isomorphic if and only if $A(X)$ and $A(Y)$ are isomorphic as k-algebras.

<u>PROOF</u> We need only check that for $X = Y$ α identifies $1_X : X \longrightarrow X$ and $1: A(X) \longrightarrow A(X)$. But this is easy. □

NOTE In particular Lemma 3.6 implies that there is no difference between a regular map $U \longrightarrow k$ and a morphism of varieties $U \longrightarrow \mathbb{A}^1$.

VARIETIES AS RINGED SPACES

A variety (by which we mean an affine, quasi-affine, projective or quasi-projective variety) consists of two pieces of data: a topological space, and information about which functions on the space are regular. Given this, we can forget about any polynomials or affine-spaces. These two pieces of information are sufficient to define morphisms of varieties and hence their category. Recall:

Affine variety: A subspace $X \subseteq \mathbb{A}^n$ s.t. there is a prime ideal $\mathfrak{p} \subseteq k[x_1, \dots, x_n]$ with $X = V(\mathfrak{p})$. ($\mathbb{A}^n$ Zariski top)
Quasi-affine variety: An open subspace of an affine variety.
Projective variety: A subspace $X \subseteq \mathbb{P}^n$ s.t. there is a homogenous prime ideal $\mathfrak{p} \subseteq k[x_0, \dots, x_n]$ with $X = V(\mathfrak{p})$. Here $\mathbb{P}^n$ has the Zariski topology.

Quasi-Projective variety An open subspace of a projective variety.

We then define what a "regular" function $Y \to k$ is for a quasi-affine and quasi-projective X. Note that if Y is quasi-affine (or quasi-projective) then an open subset of Y is an open subset of the containing variety.

DEFINITION Let $X \subseteq \mathbb{A}^n$ be a subspace. A function $f: X \to k$ is regular if $\forall x \in X$ there is an open (for Z. top.) neighborhood $x \in U \subseteq X$ and elements $h, g \in k[x_1, \dots, x_n]$ such that h is nonzero on U and $f = g/h$ on U.

Let $X \subseteq \mathbb{P}^n$ be a subspace ($\mathbb{P}^n$ with the Zariski topology). A function $f: X \to k$ is regular if $\forall x \in X$ there is an open neighborhood $x \in U \subseteq X$ and homogenous polynomials of the same degree $h, g \in k[x_0, \dots, x_n]$, such that h is nowhere zero on U and $f = g/h$ on U.

(*) Let X be any variety and for $U \subseteq X$ open let $\mathcal{O}(U)$ be all the regular maps $U \to k$. It is easy to check that $\mathcal{O}(U)$ is a ring. In fact, it is a k-algebra. And it is easily seen that $\mathcal{O}$ defines a sheaf of k-algebras on X. The ringed space $\mathcal{O}$ contains all necessary information about the variety. Of course if $X = V(\mathfrak{p})$ is affine and $A(X) = k[x_1, \dots, x_n]/\mathfrak{p}$ then

$$\mathcal{O}(X) = A(X)$$
$$\mathcal{O}(X)_P \cong A(X)_{\mathfrak{m}_P} \qquad P \in X$$

NOTES If $X \subseteq \mathbb{A}^n$ (or $\mathbb{P}^n$) is a subspace, $X' \subseteq X$ and $f: X \to k$ is regular, then $f|_{X'}: X' \to k$ is also regular. If $f: X \to k$ is regular and everywhere nonzero, then $1/f: X \to k$, $x \mapsto f(x)^{-1}$ is also regular.

Hence all of Hartshorne's varieties correspond to sheaves of k-algebras. We now define a class of sheaves of k-algebras which includes Hartshorne's varieties, so that we can define products etc. for affine and projective varieties simultaneously. Notice that any affine or projective variety is noetherian ($\mathbb{A}^n$, $\mathbb{P}^n$ are noetherian, subspaces of a noetherian space are noetherian) (see Exs 1, 2), hence any of Hartshorne's variet are noetherian and thus quasi compact.

(*) In fact any subspace of $\mathbb{A}^n$, $\mathbb{P}^n$ gives rise to a sheaf of k-algebras.

DEFINITION An <u>abstract prevariety</u> (Milne's algebraic prevariety) consists of a noetherian space V and a sheaf of k-algebras $\mathcal{O}_V$ on V, such that $\mathcal{O}_V(U)$ is a k-subalgebra of the algebra of all maps $U \longrightarrow k$, and the sheaf action is restriction, subject to the condition: Each point has a neighborhood U s.t. $(U, \mathcal{O}_V|_U)$ is isomorphic to the sheaf of k-algebras arising from an algebraic set.

Here a <u>morphism</u> of Milne's "ringed spaces" is $\psi : X \longrightarrow Y$ s.t. for $f \in \mathcal{O}_Y(U)$, $f\psi \in \mathcal{O}_X(\psi^{-1}U)$.

An abstract prevariety is <u>separated</u> or is an <u>abstract variety</u> if

<u>separation axiom</u>: An abstract prevariety $(Y, \mathcal{O}_Y)$ is <u>separated</u> if for any morphisms

$$\psi, \varphi : X \longrightarrow Y$$

from $\underset{\text{ab. prevar.}}{(X, \mathcal{O}_X)}$ to the abstract prevar $(Y, \mathcal{O}_Y)$, the set $\{z \in Z \mid \varphi(z) = \psi(z)\}$ is closed.

See p46 of Milne for justification. His algebraic prevarieties say "quasicompact" instead of "noetherian". From now on "algebraic set" means an alg. prevariety isomorphic to an alg. set.

Firstly if V is quasicompact and covered by noetherian spaces, it will be noetherian any way (see Exs), so this part is equivalent.

LEMMA Let $\varphi, \psi : Z \longrightarrow V$ be morphisms of algebraic sets. The subset of Z on which φ, ψ and agree is closed.

PROOF The set where ψ and φ agree is

$$\bigcap_{i=1}^{n} V(x_i\varphi - x_i\psi) \qquad (1)$$

where we identify V with an algebraic set $V \subseteq \mathbb{A}^n$ to produce x_i. Then $x_i\varphi - x_i\psi$ is regular so (1) is closed. $\square$ (Here we use Z affine to show $x_i\varphi - x_i\psi$ is continuous)

REMARK In order to check that an abstract prevariety is separated, it suffices to check that for every pair of morphisms $\varphi, \psi : Z \longrightarrow V$ with Z alg. set that $\{z \in Z \mid \varphi(z) = \psi(z)\}$ is closed. To prove this remark, cover Z with open algebraic sets.

$$\{z \in Z \mid \varphi(z) = \psi(z)\} = \bigcup_i \underbrace{\{z \in U_i \mid \varphi(z) = \psi(z)\}}_{\substack{\text{all closed} \\ \text{finite since } Z \text{ quasicompact.}}}$$

COROLLARY An affine variety is an abstract variety.

LEMMA A quasi-affine variety is an abstract variety.

PROOF It suffices to show that if $(X, \mathcal{O}_X)$ is an abstract variety and $U \subseteq X$ is open then $(U, \mathcal{O}_X|_U)$ is an abstract variety. $\square$

LEMMA If $(X, \mathcal{O}_X)$ is an abstract variety and $U \subseteq X$ is open, $(U, \mathcal{O}_X|_U)$ is an abstract variety.

PROOF First recall Prop 2.19 of Milne's notes: If $(V, \mathcal{O}_V)$ is an algebraic set, and we identify $V \subseteq \mathbb{A}^n$, for $f \in A(V)$ the sheaf $(D(f), \mathcal{O}_V|_{D(f)})$ is an algebraic set. Such $D(f)$ are a basis for V. Now cover X in algebraic sets $X = \bigcup_i V_i$ s.t. $(V_i, \mathcal{O}_V)$ alg. set. Let $U \subseteq X$ be open. Let $x \in U$, say $x \in V_i$. Then $U \cap V_i$ is an open subset of V_i. Hence $\exists$ $D(f) \subseteq U \cap V_i \subseteq V_i$ s.t. $(D(f), \mathcal{O}_{V_i}|_{D(f)})$ is an algebraic set. But $\mathcal{O}_{V_i}|_{D(f)} = \mathcal{O}_X|_{D(f)} = \mathcal{O}_U|_{D(f)}$. Hence $(U, \mathcal{O}_X|_U)$ is an abstract prevariety. Clearly $(X, \mathcal{O}_X)$ separated $\Rightarrow$ $(U, \mathcal{O}_X|_U)$ separated. $\square$

COROLLARY 3.8 We claim that the following functor:

$$X \longmapsto A(X)$$

$$X \longrightarrow Y \longmapsto A(Y) \cong \mathcal{O}(Y) \longrightarrow \mathcal{O}(X) \cong A(X)$$

is an arrow-reversing equivalence of the category of affine varieties over k, and the category of finitely generated integral domains over k.

PROOF This certainly describes a functor. The Prop. shows H is fully faithful, and H is clearly representative. $\square$

NOTE Consider Prop 3.5 in the case where $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$ are both affine. Let $h: A(Y) \longrightarrow A(X)$ be a morphism of k-algebras and $\varphi: X \longrightarrow Y$ the induced morphism. Then for $P = (a_1, \dots, a_n) \in X$,

$$\begin{aligned} h^{-1}\mathfrak{m}_P &= \{ f \in A(Y) \mid h(f) \in \mathfrak{m}_P \} \\ &= \{ f \in A(Y) \mid h(f)(a_1, \dots, a_n) = 0 \} \\ &= \{ f \in A(Y) \mid f(h(x_1)(a_1, \dots, a_n), \dots, h(x_m)(a_1, \dots, a_n)) = 0 \} \\ &= \mathfrak{m}_{\varphi(P)} \end{aligned}$$

Also,

$$\varphi(P) = (h(x_1 + I(Y))(P), \dots, h(x_m + I(Y))(P))$$

THEOREM 3.9A (Finiteness of Integral Closure) Let A be an integral domain which is a finitely generated algebra over a field k. Let K be the quotient field of A, and let L be a finite algebraic extension of K. Then the integral closure A' of A in L is a finitely-generated A-module, and is also a finitely generated domain over k.

PROOF See our EFT notes. This is Theorem 9 Vol 1 Ch. V of Z+S p. 267. $\square$

EXERCISES I.3.

$\boxed{3.10}$ Subvarieties A subset of a topological space is locally closed if it is an open subset of its closure, or, equivalently, if it is the intersection of an open set with a closed set.

If X is a quasi-affine or quasi-projective variety and Y is an irreducible locally closed subset, then Y is also a quasi-affine (resp. quasi-projective) variety, by virtue of being a locally closed subset of the same affine or projective space. We call this the induced structure on Y, and we call Y a subvariety of X.

Let $\varphi: X \longrightarrow Y$ be a morphism of varieties, $X \subseteq X'$ and $Y' \subseteq Y$ irreducible locally closed subsets such that $\varphi(X') \subseteq Y'$. Then we claim $\varphi|_{X'}: X' \longrightarrow Y'$ is a morphism. This follows from a more general result (see Note: Isos of subspaces of varieties)

NOTE A subvariety of $\mathbb{A}^n$ is a quasi-affine variety, and any quasi-affine variety is a subvariety. Hence a closed subvariety is precisely an affine variety.

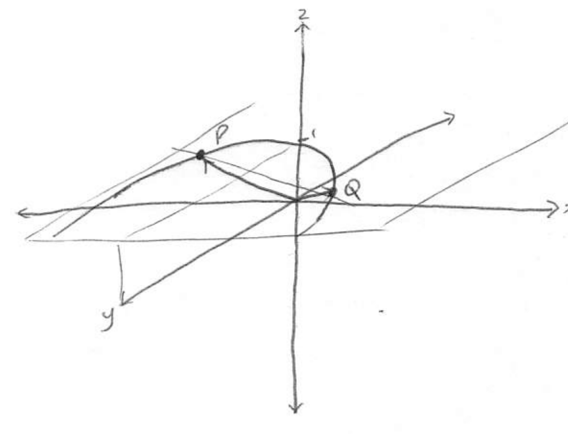
$\boxed{3.1}$ a) Any conic in $\mathbb{A}^2$ is isomorphic either to $y-x^2$ or $xy-1$. By Ex1.1 the former is isomorphic to $\mathbb{A}^1$. Also by Ex1.1 $xy-1$ has coordinate ring $k[x,x^{-1}]$. By Lemma 4.2 $\mathbb{A}^1 - 0 = D(x)$ is isomorphic to the affine variety $xy-1$ in $\mathbb{A}^2$, as required.

b) Any proper open subset of $\mathbb{A}^1$ is $D(f)$ for some nonconstant $f \in k[x]$. But then $D(f)$ is isomorphic to $yf-1$ in $\mathbb{A}^2$ which has coordinate ring $k[x,y]_f$. Since $k[x]$ cannot be isomorphic as a k-algebra to $k[x,y]_f$, we are done.

c) Let C be a conic in $\mathbb{P}^2$ defined by (See proof in §1.1 exercises also)

$$C: \ ax^2 + by^2 + cz^2 + dxy + eyz + gxz$$

Identify $\mathbb{A}^2$ with $U_2 = \mathbb{P}^2 - V(y_2)$. Then $C \cap \mathbb{A}^2$ is the conic $ax^2+by^2+dxy+ey+gx+c$ in $\mathbb{A}^2$. We know the solutions to this conic in $\mathbb{A}^2$ are an infinite set (bijective to $y-x^2$ or $xy-1$). Pick any $P \in C \cap \mathbb{A}^2$, say $(p_1,p_2,1)$. Let K be the line joining $O=(0,0,1) \in \mathbb{A}^2$ and P in $\mathbb{A}^2$. This line meets $C \cap \mathbb{A}^2$ in at most 2 places, so pick $Q \in C \cap \mathbb{A}^2$ not on this line, say $Q = (q_1, q_2, 1)$. Pick another point $T \in C \cap \mathbb{A}^2$ not on PQ, OP or OQ, say $T = (t_1, t_2, 1)$. We claim these tuples are linearly independent considered as elements of $\mathbb{A}^3$.



The reason is that if T were $\lambda P + \mu Q$ for some μ, λ then since $T = (t_1, t_2, 1)$ it would be on PQ, which is impossible. Put the three vectors P, Q, T into the columns of a matrix B

$$B = \begin{pmatrix} p_1 & q_1 & t_1 \\ p_2 & q_2 & t_2 \\ 1 & 1 & 1 \end{pmatrix}$$

By construction B is invertible, so let $A = B^{-1}$. Let $\varphi: \mathbb{P}^2 \longrightarrow \mathbb{P}^2$ be the automorphism of $\mathbb{P}^2$ induced by A. See NOTE Autos of $\mathbb{P}^n$ to see that φ maps C to another conic. Moreover $\varphi(C)$ contains $\varphi(P), \varphi(Q), \varphi(T)$, which are

moreover the iso of $\mathbb{P}^2$ restricts to $C \cong \varphi(C)$

$$\varphi(Q) = (0,1,0) \quad \varphi(P) = (1,0,0) \quad \varphi(T) = (0,0,1)$$

If the conic $\varphi(C)$ is defined by

$$a'x^2 + b'y^2 + c'z^2 + d'xy + e'yz + g'xz$$

Then the fact that $\varphi(P), \varphi(Q), \varphi(T) \in \varphi(C)$ implies $a' = b' = c' = 0$, so $\varphi(C)$ is defined by

$$d'xy + e'yz + g'xz$$

Consider the matrix

$$A = \begin{pmatrix} 1/d' & 0 & 0 \\ 0 & 1/e' & 0 \\ 0 & 0 & 1/g' \end{pmatrix}$$

(argument easily modified if d', e', g' are 0)

Let $\phi: \mathbb{P}^2 \longrightarrow \mathbb{P}^2$ be the induced automorphism, then $\phi\varphi(C)$ is defined by $xy + yz + xz = 0$ (see the <u>Note</u>). In particular, <u>all conics in $\mathbb{P}^2$ are isomorphic</u>. So we need only show that one of them is isomorphic to $\mathbb{P}^1$. Consider the 2-uple embedding $\mathbb{P}^1 \longrightarrow \mathbb{P}^2$ $(a,b) \longmapsto (a^2, ab, b^2)$ whose image is the conic $XZ - Y^2$. By Ex 3.4 this is an isomorphism. To check that the image of $\rho: \mathbb{P}^1 \longrightarrow \mathbb{P}^2$ is indeed $Z(xz - y^2)$, note that clearly

$$\operatorname{Im}\rho \subseteq Z(xz - y^2)$$

Now suppose $(e,f,g) \in Z(xz - y^2)$ so $eg = f^2$. Either $f = 0$, in which case $eg = 0$ (say $e = 0$) and then $(e,f,g) = \rho(0, \sqrt{g})$ where $\sqrt{g}$ exists since k alg. closed. If $f \neq 0$ we may assume $f = 1$. Let us note that $eg = 1$ implies $e \neq 0$ and $g \neq 0$. Let $\sqrt{e}, \sqrt{g}$ be s.t. $\sqrt{e}\sqrt{g} = 1$. Then $(e,f,g) = \rho(\sqrt{e}, \sqrt{g})$. Hence $\operatorname{Im}\rho = Z(xz - y^2)$ as required.

d) Any two 1-dimensional subspaces of $\mathbb{P}^2$ intersect (see 3.7a) but $\mathbb{A}^2$ does not have this property

e) By Theorem 3.4 the ring of regular functions on a projective variety is k, which is only possible for an affine variety if it is a point.

$\boxed{\text{Q3.2}}$ A morphism whose underlying map on the topological spaces is a homeomorphism need not be an isomorphism

(a) Let $\varphi: \mathbb{A}^1 \longrightarrow \mathbb{A}^2$ be $\varphi(t) = (t^2, t^3)$. This is induced by $h: k[x,y] \longrightarrow k[x]$ with $x \longmapsto x^2, y \longmapsto x^3$, so is a morphism of varieties. Its image is $V(y^2 - x^3)$ (if $(a,b) \in V(y^2 - x^3)$ let $Y^2 = a$. Then $(Y^3)^2 = Y^6 = a^3 = b^2$ so $Y^3 = \pm b$. If nec. change signs on Y so $Y^2 = a$ and $Y^3 = b$). The usual techniques (see end of §1) show $y^2 - x^3$ irred.
if char=2 By the Note at the end of this set of exercises, φ restricts to a bijective morphism of varieties $\phi: \mathbb{A}^1 \longrightarrow Y$
$(Y-b)^2 = 0$ where $Y = V(y^2 - x^3)$. Injective since if $u^3 = v^3$ and $u^2 = v^2$, either $u = v$ or $u = -v$. If $u = -v$ and char $= 2$
$\therefore Y = b.$ then $u = v$, otherwise $u^3 = v^3 = -v^3 \Rightarrow v = 0$, and $u = 0$.

The map $\psi: Y \longrightarrow \mathbb{A}^1$ defined on $Y - (0,0)$ by $(a,b) \longmapsto b/a$ and on $(0,0)$ by $\psi(0,0) = 0$, is an inverse to ϕ and clearly $\psi|_{Y-(0,0)}$ is regular and hence a morphism from $U = Y - (0,0)$ to $\mathbb{A}^1$. Hence U is isomorphic as a variety to $\mathbb{A}^1 - 0$. The closed sets in $\mathbb{A}^1$ are just finite sets of points, so ψ is continuous and hence $\mathbb{A}^1$ is homeomorphic to Y. If φ were an isomorphism h would be surjective, but this is a contradiction since there is no polynomial f with $f(x^2, x^3) = x$. Hence

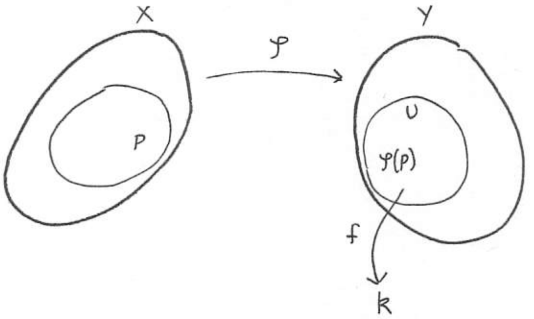
$\mathbb{A}^1$ is homeomorphic to Y via φ

$\mathbb{A}^1$ is birational to Y via φ

<u>But</u> $\mathbb{A}^1$ is <u>not</u> isomorphic to Y via φ

(b) For another example let the characteristic of the base field k be $p > 0$ and define a map $\varphi: \mathbb{A}^1 \longrightarrow \mathbb{A}^1$ by $a \longmapsto a^p$. This is injective since $a^p = b^p \Leftrightarrow (a-b)^p = 0 \Rightarrow a = b$ and surjective since k is alg. closed. Clearly φ is the morphism induced by $k[x] \longrightarrow k[x]$, $x \longmapsto x^p$. Since the topology on $\mathbb{A}^1$ is given by the finite sets as closed, φ is a homeomorphism. But it cannot be an isomorphism because there is no polynomial with $f(x^p) = x$. This morphism is the Frobenius morphism.

$\boxed{Q3.3}$ (a) Let $\varphi : X \longrightarrow Y$ be a morphism of varieties and let $P \in X$.



We define a map $\varphi_P^* : \mathcal{O}_{\varphi(P),Y} \longrightarrow \mathcal{O}_{P,X}$ as follows. Let $(U,f) \in \mathcal{O}_{\varphi(P),Y}$, that is, $f : U \longrightarrow k$ is regular. Then $f\varphi : \varphi^{-1}U \longrightarrow k$ is regular and $P \in \varphi^{-1}U$ so $(\varphi^{-1}U, f\varphi) \in \mathcal{O}_{P,X}$. So we define

$$\varphi_P^*(U,f) = (\varphi^{-1}U, f\varphi)$$

One checks this well-defined, and it is clearly a morphism of k-algebras. Let $\mathfrak{m}_P \subseteq \mathcal{O}_{P,X}$ and $\mathfrak{m}_{\varphi(P)} \subseteq \mathcal{O}_{\varphi(P),Y}$ be the unique maximal ideals. Then

$$(U,f) \in \mathfrak{m}_{\varphi(P)} \iff f\varphi(P) = 0 \iff \varphi_P^*(U,f) \in \mathfrak{m}_P$$

So $\varphi_P^{*\,-1}\mathfrak{m}_P = \mathfrak{m}_{\varphi(P)}$ and φ_P^* is a local morphism of local rings.

(b) It is clear that if φ is an isomorphism all the φ_P^* are isomorphisms, for $P \in X$. So suppose φ is a homeomorphism and φ_P^* is an isomorphism for all $P \in X$. We need only show that the inverse ϕ of φ is a morphism (we know it is also a homeomorphism). Let $V \subseteq X$ be open and $g : V \longrightarrow k$ regular. We need to show that $g\phi : \phi^{-1}V \longrightarrow k$ is regular. It suffices to show that each point of $\phi^{-1}V = \varphi(V)$ has an open neighborhood on which $g\phi$ is regular. Let $P \in V$ be given. Then $\varphi_P^* : \mathcal{O}_{\varphi(P),Y} \longrightarrow \mathcal{O}_{P,X}$ is surjective, so there is an open neighborhood Q of $\varphi(P)$ in Y and a regular map $m : Q \longrightarrow k$ with $(\varphi^{-1}Q, m\varphi) = (V, g)$, so $m\varphi = g$ on $V \cap \varphi^{-1}Q$. Hence $m = g\phi$ on $Q \cap \phi^{-1}V$, which is an open neighborhood of $\varphi(P)$ in $\phi^{-1}V$. So $g\phi$ is regular, as required.

(c) Suppose $\varphi(X)$ is dense in Y and let $P \in X$. Suppose $\varphi_P^* : \mathcal{O}_{\varphi(P),Y} \longrightarrow \mathcal{O}_{P,X}$ were not injective, say $\varphi_P^*(V,g) = \varphi_P^*(W,h)$. So g and h agree on the set $\varphi(X) \cap V \cap W$, and the set of <u>all points</u> where g,h agree is a closed subset of $V \cap W$ – say $V \cap W \cap Z$ where $Z \subseteq Y$ is closed. Then $(V \cap W)^c$ is a closed subset of Y and $\varphi(X) \subseteq (V \cap W)^c \cup Z$. Since $\varphi(X)$ is dense, this implies $Z \supseteq V \cap W$ and hence $(V,g) = (W,h)$ as required.

$\boxed{Q3.4}$ Done in our Ex 2.12 solution

$\boxed{Q3.5}$ By abuse of language, we say that a variety is "affine" if it is isomorphic to an affine variety. We claim that if $H \subseteq \mathbb{P}^n$ is any hypersurface then $\mathbb{P}^n - H$ is affine. First we prove the case where $H = Z(f)$ is a <u>hyperplane</u>. That is, f is a nonzero linear polynomial. By our linear variety notes (see Exs §2) there is an automorphism of $\mathbb{P}^n$ identifying H with $Z(x_0)$. Hence $\mathbb{P}^n - H \cong \mathbb{P}^n - Z(x_0) = V_0$ is affine.

In the general case let $H = Z(f)$ where f is irreducible and homogeneous of degree $d > 1$. Write f as a linear combination of ordered monomials: $f = \sum_{i=1}^{s} e_i M_i$. Order <u>all</u> monomials of degree d in $x_0, \dots, x_n$ so that the list begins with $M_1, \dots, M_s, \dots$ and using this order let $\rho_d : \mathbb{P}^n \longrightarrow \mathbb{P}^N$ be the d-uple embedding. Then ρ_d gives an isomorphism of $\mathbb{P}^n$ with the projective variety $\operatorname{Im}\rho_d \subseteq \mathbb{P}^N$ by Ex 3.4. Let $g = \sum_{i=1}^{s} e_i z_i$ be the polynomial in $k[z_0, \dots, z_N]$ corresponding to f: identifying $\mathbb{P}^n$ with $\operatorname{Im}\rho_d$ we have

$$\mathbb{P}^n \cap Z(g) = \{(a_0, \dots, a_n) \in \mathbb{P}^n \mid \textstyle\sum_{i=1}^{s} e_i M_i(a_0, \dots, a_n) = 0\}$$
$$= H$$

Since $Z(g)$ is a hyperplane $\mathbb{P}^N - Z(g)$ is affine. The space $\mathbb{P}^n - H$ is irreducible, hence a closed irreducible subset of $\mathbb{A}^N \cong \mathbb{P}^N - Z(g)$. Hence $\mathbb{P}^n - H$ is isomorphic to an affine variety and is thus affine.

Q3.6 There are quasi-affine varieties which are not affine. Let $X = \mathbb{A}^2 - (0,0)$. Then the inclusion $\varphi : X \longrightarrow \mathbb{A}^2$ is a morphism of varieties. Suppose X is isomorphic to an affine variety Z. The isomorphism of k-algebras $\mathcal{O}(\mathbb{A}^2) \longrightarrow \mathcal{O}(X)$ would lead to an isomorphism $\mathcal{O}(\mathbb{A}^2) \cong \mathcal{O}(Z)$ which by (3.8) would imply that the composite $Z \to X \to \mathbb{A}^2$ was an isomorphism of varieties, implying that $X \to \mathbb{A}^2$ is surjective, which is clearly a contradiction.

So it suffices to show that φ
The map $\mathcal{O}(\mathbb{A}^2) \longrightarrow \mathcal{O}(X)$ is clearly injective. To see that it is surjective, note that X is covered by the open subsets $D(x) = \mathbb{A}^2 - Z(x)$ and $D(y) = \mathbb{A}^2 - Z(y)$. Hence in $K(X)$, $\mathcal{O}(X) = \mathcal{O}(D(x)) \cap \mathcal{O}(D(y))$ (see an earlier note). Using the canonical isomorphism $K(X) \to K(\mathbb{A}^2)$ we see that the same formula holds in $K(\mathbb{A}^2) \cong k(x,y)$. Under this latter isomorphism $\mathcal{O}(D(x))$ and $\mathcal{O}(D(y))$ are identified with $k[x,y]_x$ and $k[x,y]_y$ respectively (see our Milne notes), so $\mathcal{O}(X)$ is k-isomorphic to $k[x,y]_x \cap k[x,y]_y$. Let

$$f = \frac{g(x,y)}{x^N} = \frac{h(x,y)}{y^M}$$

be in this intersection. Since $k[x,y]$ is a UFD, we can assume the fractions are in their lowest terms. Thus $y^M g(x,y) = x^N h(x,y)$ implies $N = M = 0$ so $f \in k[x,y]$. Hence $\mathcal{O}(X)$ is identified with $k[x,y]$ via $K(\mathbb{A}^2) \cong k(x,y)$. Considering the commutative diagram

$$\begin{array}{ccccc} k[x,y] & \xrightarrow{\sim} & \mathcal{O}(\mathbb{A}^2) & \longrightarrow & \mathcal{O}(X) \\ \downarrow & & \downarrow & & \downarrow \\ k(x,y) & \xrightarrow[\sim]{} & K(\mathbb{A}^2) & \xrightarrow[\sim]{} & K(X) \end{array}$$

and the fact that $\mathcal{O}(X) \longrightarrow K(X)$ is injective we see that $\mathcal{O}(\mathbb{A}^2) \longrightarrow \mathcal{O}(X)$ is surjective, as required.

Q3.7 a) See §7 b) See §7

Q3.8 Let H_i, H_j be the hyperplanes in $\mathbb{P}^n$ defined by $x_i = 0$ and $x_j = 0$ with $i \neq j$. That is, $\mathbb{P}^n - H_i = U_i$ and $\mathbb{P}^n - H_j = U_j$. Then $\mathbb{P}^n - (H_i \cap H_j)$ is the open set $U_i \cup U_j$. We claim that the only regular functions on $U_i \cup U_j$ are constant. Let $f : U_i \cup U_j \longrightarrow k$ be regular. Then $f|_{U_i} : U_i \longrightarrow k$ is regular. Since $U_i \cong \mathbb{A}^n$ there is a polynomial $g(x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$ s.t. $f|_{U_i} = \beta(g)/x_i^d$ where d is the degree of g (i.e. highest degree monomial in g — $\beta(g)$ is g with every monomial in g multiplied by a power of x_i so that $\beta(g)$ is hom. degree d). Similarly there is $h(x_0, \dots, x_{j-1}, x_{j+1}, \dots, x_n)$ s.t. $f|_{U_j} = \beta(h)/x_j^e$. But these regular maps agree on $U_i \cap U_j$. Hence (looking at $U_i \cap U_j \subseteq U_j \cong \mathbb{A}^n$)

$$\beta(g)(x_0, \dots, x_{j-1}, 1, x_{j+1}, \dots, x_n) = x_i^d h \qquad (1)$$

(Using $K(\mathbb{A}^n) \cong k(x_0, \dots, x_{j-1}, x_{j+1}, \dots, x_n)$). But the polynomial on the left is the polynomial g with x_j replaced by 1 and every monomial multiplied by a suitable power of x_i. But (1) implies x_i^d divides every monomial — but there is at least one monomial in g we didn't multiply by x_i at all (the highest degree term). Hence $d = 0$ and g is a constant. Similarly $e = 0$ and h is a constant, so $g = h$ and f is a constant function, as desired.

Q3.9 The homogenous coordinate ring of a projective variety is not invariant under isomorphism (as a graded k-algebra). For example, let $X = \mathbb{P}^1$, and let Y be the 2-uple embedding of $\mathbb{P}^1$ in $\mathbb{P}^2$. Ie $\rho : \mathbb{P}^1 \longrightarrow \mathbb{P}^2$ is

$$\rho(a_0, a_1) = (a_0^2, a_0 a_1, a_1^2)$$

Then $Y = \operatorname{Im}\rho = Z(\mathfrak{a})$ where $\mathfrak{a} = \operatorname{Ker}\psi$ and

$$\psi : k[z_0, z_1, z_2] \longrightarrow k[x_0, x_1]$$
$$z_0 \mapsto x_0^2 \quad z_1 \mapsto x_0 x_1 \quad z_2 \mapsto x_1^2$$

Therefore $S(Y) = k[z_0, z_1, z_2]/\mathfrak{a}$. The graded part of degree 1 is a 3-dimensional k-vector space, since ψ is injective on linear polynomials. Since $k[x_0, x_1] = S(X)$ has graded 1-degree part of dimension 2, we cannot have $S(X) \cong S(Y)$ as graded rings.

Q3.11 Let X be a variety, $P \in X$. We claim there is a bijection between the prime ideals of the local ring $\mathcal{O}_P$ and the closed subvarieties of X containing P. (a closed subvariety being a closed, irreducible subset of X).
Using (4.3) or commonsense, let U be an open affine neighborhood of P. We claim that intersecting with U gives a bijection between closed subvarieties of X containing P and closed subvarieties of U containing P. Let $Y \subseteq X$ be a closed subvariety, then $Y \cap U$ is clearly a closed subvariety of U. Since $Y = Cl_X(Y \cap U)$ the map is injective. Finally if $P \in Z \subseteq U$ is a closed subvariety $Y = Cl_X(Z)$ is a closed subvariety of X and $Y \cap U = Z$ (since Z closed $\Rightarrow \exists Q$ closed in X $Z = U \cap Q$ $\therefore Y \cap U = Z$).

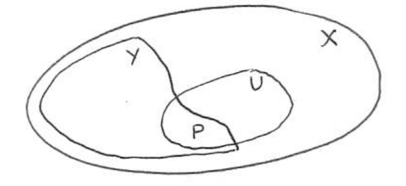
So it suffices to prove the result for X affine, say $X \subseteq \mathbb{A}^n$. Then $\mathcal{O}_P \cong A(Y)_{m_P}$ and the prime ideals of $\mathcal{O}_P$ are in bijection with the prime ideals of $k[x_1, \dots, x_n]$ contained in m_P and containing $I(Y)$. But these correspond precisely to the closed irreducible subsets of Y containing P.

Q3.12 See Notes following Theorem 3.4

Q3.13 <u>The Local Ring of a Subvariety</u> Let X be a variety, $Y \subseteq X$ a subvariety (i.e. an irreducible locally closed subset). Let $\mathcal{O}_{Y,X}$ be the set of equivalence classes (U, f) where $U \subseteq X$ is open, $U \cap Y \neq \emptyset$ and f is a regular function on U. We say (U, f) is equivalent to (V, g) iff $f = g$ on $U \cap V$. Hence there is an injective map of sets $\mathcal{O}_{Y,X} \longrightarrow K(X)$, the image of which contains k and is closed under the addition and multiplication: if $U \cap Y \neq \emptyset$ and $V \cap Y \neq \emptyset$ then $U \cap Y$ and $V \cap Y$ are nonempty open subsets of the irreducible topological space Y, hence have a nonempty intersection: so $(U \cap V) \cap Y \neq \emptyset$.

Hence $\mathcal{O}_{Y,X}$ is a k-algebra. To see it is a local ring, let $m \subseteq \mathcal{O}_{Y,X}$ be all (U, f) with $f = 0$ on $U \cap Y$. Then it is easy to check that m is an ideal. To see that $(\mathcal{O}_{Y,X}, m)$ is a local ring, say $(V, g) \in \mathcal{O}_{Y,X} - m$. Then if $W = X - g^{-1}0$, $W \cap Y \neq \emptyset$ and (W, g^{-1}) is thus an inverse for (V, g) in $\mathcal{O}_{Y,X}$. We claim that $\mathcal{O}_{Y,X}$ is a Noetherian local domain of dimension $\dim X - \dim Y$.

To prove this, let $P \in Y$ and let U be an open affine neighborhood of P in X (see (4.2))



Since Y is irreducible, $Y \cap U$ a nonempty open subset of Y, the topological space $Y \cap U \subseteq U$ is irreducible. Since Y is locally closed there are sets $V, Q \subseteq X$ open and closed resp. s.t. $Y = V \cap Q$. Hence $Y \cap U = (V \cap U) \cap (Q \cap U)$ is a locally closed subset of U: hence $Y \cap U$ is a <u>subvariety</u> of U. Since both Y, X are varieties we have $\dim X = \dim U$ and $\dim Y = \dim Y \cap U$. It is easy to check that there is an isomorphism of k-algebras $\mathcal{O}_{Y \cap U, U} \longrightarrow \mathcal{O}_{Y,X}$, so we may reduce to the case where X is affine.

Say $X \subseteq \mathbb{A}^n$ and let $X = Z(\mathfrak{p})$ where $\mathfrak{p} \subseteq k[x_1, \dots, x_n]$ is a prime ideal. Since Y is a locally closed subset of X it is a locally closed subset of $\mathbb{A}^n$, hence is an open subset of its closure $\overline{Y} \subseteq X$, which is an affine variety: say $\overline{Y} = Z(\mathfrak{q})$ where $\mathfrak{q}$ is prime and $\mathfrak{p} \subseteq \mathfrak{q}$. By (1.10) $\dim Y = \dim \overline{Y}$. It is clear that for an open $U \subseteq X$ $U \cap Y \neq \emptyset$ iff $U \cap \overline{Y} \neq \emptyset$ and hence there is an isomorphism of k-algebras $\mathcal{O}_{Y,X} \cong \mathcal{O}_{\overline{Y},X}$.

So finally we reduce to the case of $X = Z(\mathfrak{p}) \subseteq \mathbb{A}^n$ and $Y = Z(\mathfrak{q}) \subseteq X$ where $\mathfrak{p}, \mathfrak{q}$ are prime ideals of $k[x_1, \dots, x_n]$ with $\mathfrak{p} \subseteq \mathfrak{q}$. Consider the commutative diagram

$$\begin{tikzcd} \mathcal{O}(X) \arrow[d] & A(X) \arrow[l, "\sim"'] \arrow[d] & (= A/\mathfrak{p}) \\ K(X) & Q(A(X)) \arrow[l, "\varphi"', "\sim"] & \end{tikzcd}$$

We claim that this isomorphism $K(X) \cong Q(A(X))$ identifies $\mathcal{O}_{Y,X}$ and $A(X)_{\mathfrak{q}}$ ($\mathcal{O}_{Y,X}$ being a subring of $K(X)$ in the obvious way). Say $f/g \in Q(A(X))$ with $g \notin \mathfrak{q}$. That is, g is not zero on all of Y. Then $\varphi(f/g) = (D(g), f/g)$ is clearly a member of $\mathcal{O}_{Y,X} \subseteq K(X)$. To see that φ maps $A(X)_{\mathfrak{q}}$ <u>onto</u> $\mathcal{O}_{Y,X}$ let $(U, F) \in \mathcal{O}_{Y,X}$ be given, and let $P \in U \cap Y$. Then f is regular so there is an open neighborhood $P \in W \subseteq U$ and $g, h \in k[x_1, \dots, x_n]$ s.t. $h \neq 0$ on W and $f = g/h$ on W. Hence $h \notin \mathfrak{q}$ and $f = \varphi(g/h)$, as required.

Hence $\dim \mathcal{O}_{Y,X}$ is equal to the height of $\mathfrak{q}/\mathfrak{p}$ in $A(X) = A/\mathfrak{p}$. But $A/\mathfrak{p}$ is affine, so

$$\operatorname{ht} \mathfrak{q}/\mathfrak{p} + \operatorname{coht} \mathfrak{q}/\mathfrak{p} = \dim A/\mathfrak{p} = \operatorname{coht} \mathfrak{p}$$

$$\dim \mathcal{O}_{Y,X} + \operatorname{coht} \mathfrak{q} = \operatorname{coht} \mathfrak{p}$$

$$\begin{aligned} \therefore \quad \dim \mathcal{O}_{Y,X} &= \operatorname{coht} \mathfrak{p} - \operatorname{coht} \mathfrak{q} \\ &= \dim A(X) - \dim A(Y) \\ &= \dim X - \dim Y \end{aligned}$$

as required. Returning to the general case, as a subring of $K(X)$, $\mathcal{O}_{Y,X}$ contains all the local rings $\mathcal{O}_{P,X}$ $P \in Y$, so clearly $K(X)$ is the quotient field of $\mathcal{O}_{Y,X}$ (<u>not</u> $K(Y)$ as Hartshorne says – if $Y = P \in X$ then $K(Y)$ is $k \neq K(X)$) Clearly $\mathcal{O}_{Y,X} = \mathcal{O}_P$ if Y is the point P and $K(X)$ if $Y = X$.

$\boxed{Q3.14}$ <u>Projection from a Point</u> Let $H \subseteq \mathbb{P}^{n+1}$ $(n \geqslant 1)$ be a hyperplane and $P \in \mathbb{P}^{n+1}$ a point not on H. We seek to identify H with $\mathbb{P}^n$ and define a morphism $\varphi : \mathbb{P}^{n+1} - P \longrightarrow \mathbb{P}^n$ by $\varphi(Q) =$ the intersection of the unique line containing P, Q with $\mathbb{P}^n$.

Life would be simpler if $P = (1, 0, \dots, 0)$ and $H = Z(x_0)$, and we claim there is an automorphism $\psi : \mathbb{P}^{n+1} \longrightarrow \mathbb{P}^{n+1}$ which maps $P \longmapsto (1, 0, \dots, 0)$ and $H \longmapsto Z(x_0)$, whatever P, H may be (provided $P \notin H$). Let $H = Z(f)$. Then (see our Linear Variety Notes), $I(P) = (b_i x_0 - b_0 x_i, \dots, b_i x_n - b_n x_i)$ where $P = (b_0, \dots, b_n)$ and $b_i \neq 0$. Since $P \notin H$ the linear polynomials $f, b_i x_0 - b_0 x_i, \dots, b_i x_n - b_n x_i$ are linearly independent (and there are $n+1$ of them). By our Linear variety notes there is an automorphism

$$\phi : k[x_0, \dots, x_n] \longrightarrow k[x_0, \dots, x_n]$$

mapping $f \longmapsto x_0$ and $b_i x_0 - b_0 x_i \longmapsto x_1, \dots, b_i x_n - b_n x_i \longmapsto x_n$. Then ϕ induces an automorphism $\psi : \mathbb{P}^{n+1} \longrightarrow \mathbb{P}^{n+1}$ with $\psi^{-1}(Z(g)) = Z(\phi^{-1}(g))$, or $\psi(Z(g)) = Z(\phi(g))$ for all polynomials g. In particular ψ identifies H with $Z(x_0)$ and

$$\begin{aligned} \psi(P) &= \psi(Z(b_i x_0 - b_0 x_i) \cap \dots \cap Z(b_i x_n - b_n x_i)) \\ &= Z(x_1) \cap \dots \cap Z(x_n) = (1, 0, \dots, 0) \end{aligned}$$

Clearly ϕ maps linear polynomials to linear polynomials, hence maps linear varieties to linear varieties since a linear variety is just a projective variety which can be written as the intersection of hyperplanes. Hence in particular ϕ maps lines to lines.

Let H, P be arbitrary $(P \notin H)$. Given a point $Q \neq P$ we have already shown in our Linear Variety notes that there is a unique line containing P, Q and that this line meets H at precisely one point. So we can define a function $\varphi : \mathbb{P}^{n+1} - P \longrightarrow H$ in this way. Let ψ be the above automorphism and suppose the corresponding map $\varphi' : \mathbb{P}^{n+1} - (1, 0, \dots, 0) \longrightarrow Z(x_0)$ is a morphism of varieties. Since ψ maps lines to lines it is straightforward to check that $\varphi'(\psi(Q)) = \psi(\varphi(Q))$ for $Q \neq P$ in $\mathbb{P}^{n+1}$. Restricting gives an isomorphism $\mathbb{P}^{n-1} - P \cong \mathbb{P}^{n+1} - (1, 0, \dots, 0)$ so the diagram

$$\begin{tikzcd} \mathbb{P}^{n+1} - (1,0,\dots,0) \arrow[r, "\varphi'"] & Z(x_0) \\ \mathbb{P}^{n+1} - P \arrow[u, "\psi"] \arrow[r, "\varphi"'] & H \arrow[u, "\psi"'] \end{tikzcd}$$

It follows that if φ' is a morphism, so is φ, so we reduce to the case of $P = (1, 0, \dots, 0)$ and $H = Z(x_0)$.

Let $\varphi : \mathbb{P}^{n+1} - P \longrightarrow Z(x_0)$ be defined as above, and let $Q \neq P$ with $Q = (b_0, \dots, b_n)$ be given. Say $b_i \neq 0$. Then

$$L = Z\left(x_1 - \frac{b_1}{b_i} x_i, \dots, x_n - \frac{b_n}{b_i} x_i, x_{n+1} - \frac{b_{n+1}}{b_i} x_i\right)$$

is a line in $\mathbb{P}^{n+1}$ passing through both $P=(1,0,\dots,0)$ and Q. To find the point $\varphi(Q)\in L\cap Z(x_0)$ we are looking for a nonzero solution to the matrix

$$\begin{array}{c}
\begin{array}{ccccccccc} 0 & 1 & & & i & & & & n+1 \end{array}\\
\begin{pmatrix}
1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0\\
0 & 1 & 0 & \cdots & -b_1/b_i & 0 & \cdots & 0\\
0 & 0 & 1 & \cdots & -b_2/b_i & 0 & \cdots & 0\\
\vdots & & & \ddots & \vdots & & & \vdots\\
0 & \cdots & 0 & 1 & -b_{i-1}/b_i & 0 & \cdots & 0\\
0 & \cdots & & 0 & -b_{i+1}/b_i & 1 & \cdots & 0\\
\vdots & & & & \vdots & & \ddots & 0\\
0 & 0 & \cdots & 0 & -b_{n+1}/b_i & 0 & \cdots & 1
\end{pmatrix}
\end{array}$$

The first row corresponds to x_0, the second to $x_1 - b_1/b_i x_i$ and so on. If $(x_0,\dots,x_{n+1})$ were a solution, we would need $x_0=0$, $x_1 = b_1/b_i x_i, \dots, x_{n+1} = b_{n+1}/b_i x_i$ (x_i free). Putting $x_i = 1$ we find that

$$\begin{aligned}\varphi(Q) &= (0, b_1/b_i, b_2/b_i, \dots, b_{i-1}/b_i, 1, b_{i+1}/b_i, \dots, b_{n+1}/b_i)\\ &= (0, b_1, b_2, \dots, b_{i-1}, b_i, b_{i+1}, \dots, b_{n+1})\end{aligned}$$

Note that this definition is independent of i. Let $\theta: k[x_0,\dots,x_{n+1}] \to k[x_0,\dots,x_{n+1}]$ be $x_0\mapsto 0, x_i \mapsto x_i$ $i\neq 0$. Let g be a homogenous polynomial, then $\theta(g)$ is also homogenous (of the same degree, provided $\theta(g)\neq 0$) and

But then technically 0 is hom. of every degree.

$$\varphi: \mathbb{P}^{n+1} - P \longrightarrow H$$
$$\varphi^{-1}(Z(g)\cap H) = Z(\theta(g)) - P$$

Hence φ is a continuous mapping. To see that φ is a morphism let $V\subseteq H$ be open and $f: V\to k$ regular. Say $Q\in\varphi^{-1}V$ and W is an open neighborhood of $\varphi(Q)$ s.t. $\forall z\in W$ $f(z) = g(z)/h(z)$ where $g,h\in k[x_0,\dots,x_{n+1}]$ are homogenous of the same degree and $h\neq 0$ on W. Then $\forall x\in\varphi^{-1}W$ $(f\varphi)(x) = \theta(g)(x)/\theta(h)(x)$, so $f\varphi$ is regular and hence φ is a morphism.

b) Let $Y\subseteq\mathbb{P}^3$ be the twisted cubic curve which is the image of the 3-uple embedding of $\mathbb{P}^1$ in $\mathbb{P}^3$

$$\begin{aligned}\rho: \mathbb{P}^1 &\longrightarrow \mathbb{P}^3\\ (a,b) &\longmapsto (a^3, a^2b, ab^2, b^3)\end{aligned}$$

If t,u are the homogenous coordinates of $\mathbb{P}^1$ we say that Y is the curve given parametrically by $(x,y,z,w) = (t^3, t^2u, tu^2, u^3)$. Let $P=(0,0,1,0)$, and let $\mathbb{P}^2$ be the hyperplane $z=0$. The projection of Ex 3.14 is easily checked to be

$$\begin{aligned}\varphi: \mathbb{P}^3 - P &\longrightarrow \mathbb{P}^2\\ (b_0,b_1,b_2,b_3) &\longmapsto (b_0,b_1,b_3)\end{aligned}$$

Hence $\varphi(Y) = \{(t^3, t^2u, u^3) \mid (t,u)\in\mathbb{P}^1\}$. This is clearly contained in the projective curve $Z = Z(x_1^3 - x_2x_0^2)$, which is the projective closure of the cuspidal cubic $y^3 - x^2$ using $\mathbb{A}^2\cong U_2\subseteq\mathbb{P}^2$. We claim that $Z=\varphi(Y)$. Since k is algebraically closed, let $x^3 - 1 = (x-a)(x-b)(x-c)$ be the factorisation of x^3-1. If $\gamma,\alpha\in k$ and $\gamma^3=\alpha^3$ then (provided $\alpha\neq 0$) $(\gamma/\alpha)^3 = 1$ so $\gamma/\alpha\in\{a,b,c\}$ so $\gamma = a\alpha$ (or $b\alpha$, $c\alpha$). Suppose $(b_0,b_1,b_2)\in Z$. Then $b_1^3 = b_2b_0^2$. Put $t^3 = b_0$ and $u^3 = b_2$ (if $b_0=0$, $t=0$ and clearly $(0,\ ,b_2) = (0,0,u^3)$, similarly if $b_2=0$. So assume $b_0,b_2\neq 0$). Then $t,u\neq 0$, and $(ut^2)^3 = u^3t^6 = b_2b_0^2 = b_1^3$ so $ut^2 = ab_1$ say. Then $a\neq 0$ so replace $u\leftrightarrow a^{-1}u$ and $(t^3,t^2u,u^3) = (b_0,b_1,b_2)$ as required. Hence $\varphi(Y) = Z$.

EXERCISES I.3

$\boxed{3.15}$ Products Let $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$ be affine varieties.

(a) We claim that $X \times Y \subseteq \mathbb{A}^{n+m}$ with its induced topology (not the product topology) is irreducible. Suppose that $X \times Y = Z_1 \cup Z_2$ where Z_1 and Z_2 are proper closed subsets of $X \times Y$. Let

$$X_i = \{x \in X \mid x \times Y \subseteq Z_i\} \qquad i = 1,2$$

To see that X_1 and X_2 are closed, define for each $y \in Y$ the k-algebra homomorphism (Put $y = (b_1, \ldots, b_m)$)

$$\begin{aligned} h_y : k[x_1,\ldots,x_n,y_1,\ldots,y_m] &\longrightarrow k[x_1,\ldots,x_n] \\ x_i &\longmapsto x_i \\ y_i &\longmapsto b_i \end{aligned}$$

Then the induced morphism $\varphi_y : \mathbb{A}^n \longrightarrow \mathbb{A}^{n+m}$ is defined by $\varphi_y(P) = (P, y)$. Hence $\varphi_y^{-1} Z_i$ is a closed set in $\mathbb{A}^n$ for $i=1,2$ and $y \in Y$. But

$$\begin{aligned} \bigcap_{y \in Y} \varphi_y^{-1} Z_i &= \{P \in \mathbb{A}^n \mid \varphi_y(P) \in Z_i \ \forall y \in Y\} \\ &= \{P \in \mathbb{A}^n \mid (P,y) \in Z_i \ \forall y \in Y\} \\ &= X_i \end{aligned}$$

This proves that X_i is closed $(i=1,2)$. To prove that $X = X_1 \cup X_2$ is a bit of a chore. Let $x \in X$ - we must show either $x \times Y \subseteq Z_1$ or $x \times Y \subseteq Z_2$. Let h be

$$\begin{aligned} h : k[x_1,\ldots,x_n,y_1,\ldots,y_m] &\longrightarrow k[y_1,\ldots,y_m] \\ x_i &\longmapsto a_i \\ y_i &\longmapsto y_i \end{aligned}$$

Firstly, $x \times Y = Z(\{x_1 - a_1, \ldots, x_n - a_n\} \cup I(Y))$, so $x \times Y$ is closed, in $\mathbb{A}^{n+m}$. Similarly $X \times Y = Z(I(X) \cup I(Y))$, so $X \times Y$ is closed. Note

$$\begin{aligned} I(x \times Y) &= \{f(x_1,\ldots,x_n,y_1,\ldots,y_m) \mid h(f) \in I(Y)\} \\ &= h^{-1} I(Y). \end{aligned}$$

Since $I(Y)$ is prime (Y is an affine variety), it follows that $I(x \times Y)$ is prime and hence $x \times Y$ is irreducible. But $x \times Y = (x \times Y \cap Z_1) \cup (x \times Y \cap Z_2)$ expresses $x \times Y$ as the union of two closed subsets - hence $x \times Y \subseteq Z_i$ for some i. Hence $X = X_1 \cup X_2$ as required.

This completes the proof that $X \times Y$ is an affine variety, called the product of X and Y. (Note we use both that X, Y are irreducible, despite focusing on X)

(b) &(c) We have already noted that $X \times Y \subseteq \mathbb{A}^{n+m}$ is the locus of $(I(X), I(Y)) \subseteq k[x_1,\ldots,x_n,y_1,\ldots,y_m]$. We claim the following is a coproduct of k-algebras

$$\begin{tikzcd}[column sep=small] k[x_1,\ldots,x_n]/I(X) \arrow[dr] & & k[y_1,\ldots,y_m]/I(Y) \arrow[dl] \\ & \dfrac{k[x_1,\ldots,x_n,y_1,\ldots,y_m]}{(I(X),I(Y))} & \end{tikzcd} \qquad (1)$$

For suppose R is another k-algebra, $\varphi : k[x_1,\ldots,x_n]/I(X) \to R$, $\psi : k[y_1,\ldots,y_m]/I(Y) \to R$. These two morphisms correspond to elements of $V(I(X)) \subseteq R^n$ and $V(I(Y)) \subseteq R^m$ resp. It is not hard to check that the morphism

$$\phi : \frac{k[x_1, \dots, x_n, y_1, \dots, y_m]}{(I(X), I(Y))} \longrightarrow R$$
$$\bar{x}_i \longmapsto \varphi(\bar{x}_i)$$
$$\bar{y}_i \longmapsto \psi(\bar{y}_i)$$

is well-defined and has the necessary property. Hence there is a canonical isomorphism of k-algebras

$$\frac{k[x_1, \dots, x_n, y_1, \dots, y_m]}{(I(X), I(Y))} \cong \frac{k[x_1, \dots, x_n]}{I(X)} \otimes_k \frac{k[y_1, \dots, y_m]}{I(Y)} \qquad \begin{array}{c} (f + I(X)) \otimes (g + I(Y)) \\ \downarrow \\ fg + (I(X), I(Y)) \end{array}$$

Since X, Y are arbitrary, and part (a) shows that $X \times Y$ is irreducible and $(I(X), I(Y)) = I(X \times Y)$, we have coincidentally proven:

<u>LEMMA</u> The tensor product of two finitely generated k-domains is a finitely generated k-domain.

Since (1) is a coproduct in <u>kAlg</u> it is a coproduct in the category of f.g k-domains. Hence the associated diagram in the category of affine varieties is a product. But this is

$$\begin{tikzcd} X & & Y \\ & X \times Y \arrow[ul] \arrow[ur] & \end{tikzcd}$$

The projection maps are easily checked to be $(x,y) \mapsto x$ and $(x,y) \mapsto y$.

(d) In (c) we showed that $A(X \times Y) \cong A(X) \otimes_k A(Y)$ as k-algebras. So it suffices to show that for affine k-algebras R, S (affine = f.g. k-domain)

$$\dim(R \otimes_k S) = \dim R + \dim S \qquad (1)$$

it does ... see ⊛

For convenience we assume k alg-closed (perhaps the proof below still works), so that if $\dim R = 0$ then $R = k$ (obvious... see p84 A&M notes). Then $R \otimes_k S \cong S$, so (1) is immediate. (uses commutativity.. of course we assume R, S comm.). So we may assume R, S have positive dimensions d, e resp. Since dimension is preserved by integral extensions, R and S are <u>not</u> integral extensions of k, and by Noether's Normalisation Lemma (p81 A&M notes) there are $x_1, \dots, x_d \in R$ and $y_1, \dots, y_e \in S$ algebraically independent over k with R integral over $k[x_1, \dots, x_d]$ and S integral over $k[y_1, \dots, y_e]$ (d, e occur because integral ext. pres. dimension).

We know that $R \otimes_k S$ is an affine k-algebra, and it is clear that for $r \in R$ and $s \in S$ the elements $r \otimes 1$ and $1 \otimes s$ are integral over the subring $k[x_1 \otimes 1, \dots, x_d \otimes 1, 1 \otimes y_1, \dots, 1 \otimes y_e]$ of $R \otimes_k S$. Since the integral closure of this subring contains $r \otimes 1, 1 \otimes s$ $\forall r \in R, s \in S$, it is all of $R \otimes_k S$. That is, $R \otimes_k S$ is integral over $k[x_1 \otimes 1, \dots, x_d \otimes 1, 1 \otimes y_1, \dots, 1 \otimes y_e]$.

ASIDE: If M' is a submodule of M, N' a submodule of N (over a field k) then the canonical map $M' \otimes_k N' \longrightarrow M \otimes_k N$ is injective, since we can write $M = M' \oplus M''$ and $N = N' \oplus N''$, so since tensor products preserve direct sums $M' \otimes N \longrightarrow M \otimes N$ is injective, and so is $M' \otimes N' \to M' \otimes N$.

We have already noted that (by ASIDE) the map

$$k[x_1, \dots, x_d, y_1, \dots, y_e] \cong k[x_1, \dots, x_d] \otimes_k k[y_1, \dots, y_e] \longrightarrow R \otimes_k S$$

is injective, and clearly has image $k[x_1 \otimes 1, \dots, x_d \otimes 1, 1 \otimes y_1, \dots, 1 \otimes y_e]$, so we have immediately

$$\dim R \otimes_k S = d + e$$

(If k is not alg. closed, just means you have to consider $\dim R = 0$ or $\dim S = 0$, so R say is integral over k, and the same proof still works) ⊛

NOTE Affine Cover of Projective Space

For $1 \le i \le n$ let $U_i = \mathbb{P}^n - V(x_i)$ be the canonical cover of $\mathbb{P}^n$. Then U_i is homeomorphic to $\mathbb{A}^n$ via

$$\begin{aligned} \varphi : U_i &\longrightarrow \mathbb{A}^n \\ (a_0, \dots, a_n) &\longmapsto \left(\tfrac{a_0}{a_i}, \dots, \tfrac{a_{i-1}}{a_i}, \tfrac{a_{i+1}}{a_i}, \dots, \tfrac{a_n}{a_i}\right) \end{aligned}$$

by Proposition 2.2 of Hartshorne. But U_i is a quasi-affine variety, and we claim φ is an isomorphism of varieties.

Let $U \subseteq \mathbb{A}^n$ be open, $f : U \longrightarrow k$ a regular map. We must show $f\varphi : \varphi^{-1}U \longrightarrow k$ is regular. To this end, let $x \in \varphi^{-1}U$. Wlog we assume $i = 0$, so that if $x = (a_0, \dots, a_n)$, $a_0 \neq 0$. Let $\varphi(x) \in V \subseteq U$, where V is open, and there are polynomials $g, h \in k[y_1, \dots, y_n]$ s.t. h is nonzero on V and

$$f(v) = g(v)/h(v) \qquad \forall v \in V$$

Using the notation of Prop 2.2, consider the homogenous polynomials $\beta(g), \beta(h) \in k[x_0, \dots, x_n]$. If the highest degree monomials in g, h resp. have degrees e, f, then

$$\begin{aligned} \beta(g) &= x_0^e \, g\left(\tfrac{x_1}{x_0}, \dots, \tfrac{x_n}{x_0}\right) \\ \beta(h) &= x_0^f \, h\left(\tfrac{x_1}{x_0}, \dots, \tfrac{x_n}{x_0}\right) \end{aligned}$$

That is, each monomial in g has x_0 tacked on until it has degree e, and similarly for h. Assume $e \geq f$ (the case $f < e$ is similar). Then $\beta(g)$ and $x_0^{e-f}\beta(h)$ are homogenous polynomials of degree e. For any $y \in \varphi^{-1}V$ with $y = (b_0, \dots, b_n)$ we have $(x_0^{e-f}\beta(h))(b_0, \dots, b_n) = b_0^e \, h\left(\tfrac{b_1}{b_0}, \dots, \tfrac{b_n}{b_0}\right) \neq 0$. Further,

$$\begin{aligned} \frac{\beta(g)(b_0, \dots, b_n)}{x_0^{e-f}\beta(h)(b_0, \dots, b_n)} &= \frac{b_0^e \, g\left(\tfrac{b_1}{b_0}, \dots, \tfrac{b_n}{b_0}\right)}{b_0^{e-f} \cdot b_0^f \, h\left(\tfrac{b_1}{b_0}, \dots, \tfrac{b_n}{b_0}\right)} \\ &= \frac{g\varphi(y)}{h\varphi(y)} = f\varphi(y) \end{aligned}$$

Hence $f\varphi$ is regular and φ is a morphism of varieties. Now consider $\phi = \varphi^{-1}$

$$\begin{aligned} \phi : \mathbb{A}^n &\longrightarrow U_0 \\ (a_1, \dots, a_n) &\longmapsto (1, a_1, \dots, a_n) \end{aligned}$$

Let $W \subseteq U_0$ be open and $f : W \longrightarrow k$ regular. We must show that $f\phi : \phi^{-1}W \longrightarrow k$ is regular. Let $x = (a_1, \dots, a_n) \in \phi^{-1}W$, and let $(1, a_1, \dots, a_n) \in V \subseteq W$ where V is open and there are homogenous polynomials $g, h \in k[x_0, \dots, x_n]$ of the same degree s.t. h is nonzero on V, and

$$f(v) = \frac{g(v)}{h(v)} \quad \forall v \in V$$

Then $\phi^{-1}V$ is open, $x \in \phi^{-1}V \subseteq \phi^{-1}W$ and $\forall y \in \phi^{-1}V$

$$\begin{aligned} f\phi(y) &= f(1, b_1, \ldots, b_n) \\ &= \frac{g(1, b_1, \ldots, b_n)}{h(1, b_1, \ldots, b_n)} \end{aligned}$$

Since $h(1, y_1, \ldots, y_n) \in k[y_1, \ldots, y_n]$ is nonzero on $\phi^{-1}V$, this shows that $f\phi$ is regular. Hence ϕ is a morphism of varieties, and it follows that U_i and $\mathbb{A}^n$ are isomorphic as varieties.

It is easy enough to check that for any subspace $X \subseteq \mathbb{A}^n$ and open set (in X's subspace topology) $U \subseteq X$ that φ, ϕ identify the regular maps $U \longrightarrow k$ with regular maps $\varphi^{-1}U \longrightarrow k$ (even though $\varphi^{-1}U$ may not be open).

Projective Closure

Let $Y \subseteq \mathbb{A}^n$ be an affine variety, $\varphi : \mathbb{A}^n \longrightarrow U_0 \subseteq \mathbb{P}^n$ the standard homeomorphism, which identifies Y with a subset of U_0. Let $\overline{Y}$ denote the closure of this subset in $\mathbb{P}^n$. In Ex 2.9 we showed that $I(\overline{Y})$ is the homogenous ideal generated by $\beta(I(Y))$. We claim that $I(\overline{Y})$ is prime, so that $\overline{Y}$ is a projective variety.

NOTE This follows directly from Example I.1.4, as I later realised.

Suppose $f, g \in k[x_0, \ldots, x_n]$ are homogenous and $fg \in I(\overline{Y})$. This is equivalent to fg being zero on Y, which is equivalent to $f(1, x_1, \ldots, x_n)g(1, x_1, \ldots, x_n)$ being zero on Y. Since Y is irreducible $I(Y)$ is prime, so either $f(1, x_1, \ldots, x_n) \in I(Y)$ or $g(1, x_1, \ldots, x_n) \in I(Y)$. Then either $f \in I(\overline{Y})$ or $g \in I(\overline{Y})$, as required.

Now $Y = U_0 \cap \overline{Y}$ (clearly $Y \subseteq U_0 \cap \overline{Y}$, and $Y = U_0 \cap Z$ for some closed $Z \subseteq \mathbb{P}^n$, so $\overline{Y} \subseteq Z$), and the ringed space structure on $Y \subseteq \mathbb{A}^n$ is isomorphic to the ringed space structure on $Y \subseteq U_0$, which is now a quasi-projective variety. Hence any affine variety is isomorphic (as a variety) to a quasi-projective variety. It follows immediately that any quasi-affine variety is also quasi-projective. Hence

PROPOSITION Any variety is isomorphic (as a variety) to a quasi-projective variety.

3.16 <u>Products of Quasi-Projective Varieties</u> Let

$$\phi : \mathbb{P}^n \times \mathbb{P}^m \longrightarrow \mathbb{P}^N \qquad N = nm + n + m$$

be the Segre embedding $(a_0, \dots, a_n, b_0, \dots, b_m) \mapsto (c_{ij})$ $c_{ij} = a_i b_j$ (see Ex 2.14) and use it to give $\mathbb{P}^n \times \mathbb{P}^m$ the structure of a projective variety. Let $k[z_{ij}]$ be the coordinate ring of $\mathbb{A}^N$ with $0 \le i \le n$, $0 \le j \le m$ and let $U_{ij} \subseteq \mathbb{P}^N$ be defined by $z_{ij} \neq 0$. For $0 \le c \le n$ and $0 \le d \le m$ define

$$\varphi_{cd} : U_{cd} \longrightarrow \mathbb{P}^n \times \mathbb{P}^m$$
$$(a_{00}, \dots, \underset{cd}{1}, \dots, a_{nm}) \longmapsto (a_{0d}, \dots, a_{nd}, a_{c0}, \dots, a_{cm})$$

Using the fact that $\operatorname{Im}\phi = Z(\mathfrak{a})$, where $\mathfrak{a}$ contains $z_{ij} z_{cd} - z_{id} z_{cj}$ $0 \le i, c \le n$ $0 \le j, d \le m$, we see that $\phi\varphi = 1$. Finally, let $f \in k[x_0, \dots, x_n]$ and $g \in k[y_0, \dots, y_m]$ be homogenous polynomials. Then (that is, $\phi\varphi(x) = x$ for $x \in U_{cd} \cap \operatorname{Im}\phi$)

$$\phi(Z(f) \times Z(g)) \cap U_{cd} = \{ (a_{ij}) \mid a_{cd} \neq 0 \text{ and } \varphi(a_{ij}) \in Z(f) \times Z(g) \text{ and } (a_{ij}) \in \operatorname{Im}\phi \}$$
$$= \{ (a_{ij}) \mid a_{cd} \neq 0 \text{ and } f(a_{0d}, \dots, a_{nd}) = 0,\ g(a_{c0}, \dots, a_{cm}) = 0 \text{ and } (a_{ij}) \in \operatorname{Im}\phi \}$$

Let $\theta : k[x_0, \dots, x_n] \longrightarrow k[x_0, \dots, x_n, y_0, \dots, y_m]$ be $x_i \mapsto z_{id}$ and $\theta' : k[y_0, \dots, y_m] \longrightarrow k[x_0, \dots, x_n, y_0, \dots, y_m]$ be $y_i \mapsto z_{ci}$. Then, continuing

$$= \{ (a_{ij}) \mid a_{cd} \neq 0 \text{ and } \theta(f)(a_{ij}) = 0 = \theta'(g)(a_{ij}) \text{ and } (a_{ij}) \in \operatorname{Im}\phi \}$$
$$= U_{cd} \cap Z(\theta(f)) \cap Z(\theta'(g)) \cap \operatorname{Im}\phi$$

Since $\theta(f), \theta'(g)$ are homogenous and the U_{cd} cover $\mathbb{P}^N$, this shows that $Z(f) \times Z(g)$ is closed in $\mathbb{P}^n \times \mathbb{P}^m$.

Let $X \subseteq \mathbb{P}^n$ and $Y \subseteq \mathbb{P}^m$ be projective varieties. Then X (resp. Y) is the intersection of all the loci of the homogenous polynomials vanishing on X (resp. Y). Hence

$$X \times Y = \{ \cap Z(f) \} \times \{ \cap Z(g) \}$$
$$= \cap Z(f) \times Z(g)$$

Hence $X \times Y$ is closed. Our next task is to prove that $X \times Y$ is irreducible.

Suppose Z_1, Z_2 are closed sets in $\mathbb{P}^n \times \mathbb{P}^m$ with $X \times Y = Z_1 \cup Z_2$. Let

$$X_i = \{ x \in X \mid \phi(x \times Y) \subseteq Z_i \} \qquad i = 1, 2$$

To show that X_i is closed, we introduce for $y \in Y$ the homomorphism

$$g_y : k[z_{ij}] \longrightarrow k[x_0, \dots, x_n]$$
$$z_{ij} \longmapsto x_i b_j$$

(select a rep. $y = (b_0, \dots, b_m)$). Let $T_i \subseteq I(Z_i) \subseteq k[z_{ij}]$ be the homogenous elements of $I(Z_i)$. (considering Z_i as a closed subset of $\mathbb{P}^N$). Then $g_y(T_i) \subseteq k[x_0, \dots, x_n]$ is a collection of homogenous elements, since g_y preserves monomial degrees. Now,

$$
\begin{aligned}
Z(g_y(T_i)) &= \{(a_0,\dots,a_n) \mid g_y(f)(a_0,\dots,a_n) = 0 \quad \forall f \in T\} \\
&= \{(a_0,\dots,a_n) \mid f(a_0b_0,\dots,a_nb_m) = 0 \ \forall f \in T\} \\
&= \{x \in \mathbb{P}^n \mid f(\phi(x,y)) = 0 \quad \forall f \in T\} \\
&= \{x \in \mathbb{P}^n \mid \phi(x,y) \in Z(T) = Z_i\}
\end{aligned}
$$

Since $Z_i \subseteq X \times Y$, $\phi(x,y) \in Z_i \Rightarrow x \in X$. Hence $X_i = \bigcap_{y \in Y} Z(g_y(T_i))$, and hence X_i is closed in $\mathbb{P}^n$. Next we show that $X = X_1 \cup X_2$. To this end, let $x \in X$ be given. Since $\{x\} \subseteq \mathbb{P}^n$ is closed (it is the locus of $a_1x_0 - x_1, a_2x_0 - x_2, \dots, a_nx_0 - x_n$ assuming $a_0 = 1$), earlier results in this exercise show $x \times Y \subseteq \mathbb{P}^n \times \mathbb{P}^m$ is closed. We claim it is also irreducible: define $h: k[z_{ij}] \to k[y_0,\dots,y_m]$ by $z_{ij} \mapsto a_iy_j$ where $x = (a_0,\dots,a_n)$.

Since $I(Y) \subseteq k[y_0,\dots,y_m]$ is a prime ideal, $h^{-1}I(Y) \subseteq k[z_{ij}]$ is also prime. Notice that $I(x \times Y)$ is the homogenous radical ideal generated by those homogenous $f \in k[z_{ij}]$ for which

$$f(a_0b_0,\dots,a_nb_m) = 0 \quad \forall (b_0,\dots,b_m) \in Y$$

But $f(a_0b_0,\dots,a_nb_m) = h(f)(b_0,\dots,b_m)$, so this condition says precisely that $f \in h^{-1}I(Y)$. Clearly any homogenous $g \in h^{-1}I(Y)$ belongs to $I(x \times Y)$. Since $h^{-1}I(Y)$ is a homogenous ideal (let $f \in h^{-1}I(Y)$ be divided as $f = \sum_i m_i$. Then $h(f) \in I(Y)$ means $\sum_i h(m_i) \in I(Y)$, and since $I(Y)$ is homogenous, each $h(m_i) \in I(Y)$. Thus $m_i \in h^{-1}I(Y)$.) we have $I(x \times Y) = h^{-1}I(Y)$. Hence $x \times Y$ is irreducible.

Now we can write $x \times Y = (x \times Y \cap Z_1) \cup (x \times Y \cap Z_2)$ as the union of two closed subsets. Since $x \times Y$ is irreducible we must have $x \times Y \subseteq Z_i$ for some i, and hence $x \in X_i$. It follows that $X = X_1 \cup X_2$.

Since X is irreducible, either $X = X_1$ or $X = X_2$. Then respectively $X \times Y = Z_1$ or Z_2, so $X \times Y$ is irreducible.

(c) Let $X \subseteq \mathbb{P}^n$ and $Y \subseteq \mathbb{P}^m$ be projective varieties. We know now that $X \times Y$ is also projective. The projection maps $X \times Y \to X$, $X \times Y \to Y$ are continuous since the inverse image of a closed set $Z \subseteq Y$ is $X \times Z$, which is closed by earlier results in this exercise. Now let us show $X \times Y \to Y$ is a morphism of varieties.

Let $U \subseteq Y$ be open and $g: U \to k$ regular. Denoting the projection $p: X \times Y \to Y$ we claim that $gp: X \times U \to k$ is regular. Let $(x,u) \in X \times U$. Suppose $x = (a_0,\dots,a_n)$ with $a_k \neq 0$. That is, $x \in X \cap U_k$, where U_k is $\mathbb{P}^n - Z(x_k)$. Also find V open s.t. $x \in V \subseteq U$ and $h_1, h_2 \in k[y_0,\dots,y_n]$ s.t. h_1, h_2 are homogenous of the same degree, h_2 is nowhere zero on V, and

$$g(v) = h_1(v)/h_2(v) \qquad \forall v \in V$$

Let $h: k[y_0,\dots,y_n] \to k[z_{ij}]$ be $y_i \mapsto z_{ki}$. Then $(X \cap U_k) \times V$ is an open subset of $X \times U$, and $h(h_1), h(h_2) \in k[z_{ij}]$ are homogenous polynomials of the same degree, and

$$
\begin{aligned}
gp(a_0,\dots,a_n,v) &= g(v) \\
&= h_1(v)/h_2(v) \\
&= \frac{h(h_1)(a_0,\dots,a_n,v)}{h(h_2)(a_0,\dots,a_n,v)}
\end{aligned}
$$

Note $h(h_2)$ is nonzero on $(X \cap U_k) \times V$. Hence p is a morphism. The other projection is similarly shown to be a morphism.

NOTE If $U \subseteq X$ and $V \subseteq Y$ are quasi-projective varieties, it now follows that

$$U \times V = (U \times Y) \cap (X \times V)$$

is open, and hence the product of quasi-projective varieties is quasi-projective.

We have just shown that the maps $X \times Y \longrightarrow X$, $X \times Y \longrightarrow Y$ are morphisms of varieties. Now $U \times V \subseteq X \times Y$ is an open subset, and the ringed space structure is the induced one. So it is easily seen that $U \times V \longrightarrow U$ and $U \times V \longrightarrow V$ are also morphisms of varieties.

Finally we show that this defines a product in the category of varieties. Let $X \subseteq \mathbb{P}^n$, $Y \subseteq \mathbb{P}^m$ be projective varieties, $U \subseteq X$ and $V \subseteq Y$ open subsets. Let Z be any variety and suppose we have morphisms of varieties

$$\alpha : Z \longrightarrow U \qquad \beta : Z \longrightarrow V$$

Let $\gamma : Z \longrightarrow U \times V$ be $z \longmapsto (\alpha(z), \beta(z))$. We must show that γ is a morphism of varieties, which will prove $U \times V \longrightarrow U$, $U \times V \longrightarrow V$ is a product.

Let $W_i \subseteq \mathbb{P}^n$ and $Q_j \subseteq \mathbb{P}^m$ be $\mathbb{P}^n - V(x_i)$ and $\mathbb{P}^m - V(y_j)$ resp. $(0 \le i \le n, 0 \le j \le m)$. Introduce the following open subsets of Z:

$$K_{ij} = \alpha^{-1}(U \cap W_i) \cap \beta^{-1}(V \cap Q_j) \qquad \begin{matrix} 0 \le i \le n \\ 0 \le j \le m \end{matrix}$$

Note that the K_{ij} cover Z. Now, if we take an open subset of any variety and induce the ringed space structure on it, the result is still a variety (this is clear for affine and projective varieties, and is easily checked in the other two cases). Hence each K_{ij} can be regarded as a variety. It is straightforward to check that γ will be a morphism of varieties if and only if $\gamma_{ij} = \gamma|_{K_{ij}}$ is a morphism of varieties $\forall i, j$. So to complete the proof, we need to show γ_{ij} is a morphism of varieties.

Let $U_{ij} \subseteq \mathbb{P}^N$ be $\mathbb{P}^N - V(z_{ij})$. It is clear that $\operatorname{Im} \gamma_{ij} \subseteq U_{ij}$. The set $X \times Y \cap U_{ij} \subseteq X \times Y$ is an open subset of $X \times Y$, and $\operatorname{Im} \gamma_{ij} \subseteq X \times Y \cap U_{ij}$. Giving $X \times Y \cap U_{ij}$ the structure of a quasi-projective variety, suppose we could show that $\theta_{ij} : K_{ij} \longrightarrow X \times Y \cap U_{ij}$, $k \longmapsto \gamma(k)$ was a morphism of varieties. Let $T \subseteq U \cap V$ be open. Since $\operatorname{Im} \gamma_{ij} \subseteq U_{ij}$:

$$\begin{aligned} \gamma_{ij}^{-1}(T) &= \{x \in K_{ij} \mid \gamma(x) \in T\} \\ &= \{x \in K_{ij} \mid \gamma(x) \in T \cap X \times Y \cap U_{ij}\} \\ &= \theta_{ij}^{-1}(T \cap X \times Y \cap U_{ij}) \end{aligned}$$

Since $T \cap X \times Y \cap U_{ij}$ is an open subset of $X \times Y \cap U_{ij}$ ($T \subseteq X \times Y$ is open in $X \times Y$), $\gamma_{ij}^{-1}(T)$ is open and γ_{ij} will be continuous. Suppose $f : T \longrightarrow k$ is regular. Then $T \cap X \times Y \cap U_{ij}$ is an open subset of T, so $f|_{T \cap X \times Y \cap U_{ij}}$ is regular. But $T \cap X \times Y \cap U_{ij}$ is an open subset of $X \times Y \cap U_{ij}$ and we are assuming θ_{ij} is a morphism. Hence $f\theta_{ij} : \theta_{ij}^{-1}(T \cap X \times Y \cap U_{ij}) \longrightarrow k$ is regular. But $\theta_{ij}^{-1}(T \cap X \times Y \cap U_{ij}) = \gamma_{ij}^{-1}(T)$ and on this open set $f\theta_{ij} = f\gamma_{ij}$. Hence if θ_{ij} is a morphism, γ_{ij} is a morphism. So it only remains to show θ_{ij} is a morphism $\forall i, j$.

The homeomorphism $\varphi : U_{ij} \longrightarrow \mathbb{A}^N$ establishes an isomorphism of varieties between the quasi-projective variety $X \times Y \cap U_{ij}$ and the affine variety $\varphi(X \times Y \cap U_{ij}) \subseteq \mathbb{A}^N$ (see our earlier notes). Hence by Lemma 3.6 the map $\Theta_{ij} : K_{ij} \longrightarrow X \times Y \cap U_{ij}$ will be a morphism provided the map T

we may assume $X \times Y \cap U_{ij} \neq \phi$ since we need only consider $K_{ij} \neq \phi$

$$K_{ij} \longrightarrow \varphi(X \times Y \cap U_{ij})$$
$$x \longmapsto \varphi Y(x)$$

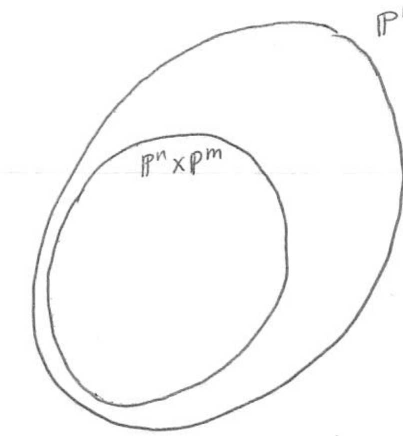
yields a regular map on K_{ij} when composed with all the projections $\mathbb{A}^N \longrightarrow k$. An element of $\mathbb{A}^N$ consists of a sequence of N elements of k. For notational purposes we index these by the pairs $(0,0), (0,1), \ldots, (1,0), \ldots, (n,m)$ – omitting (i,j). Similarly $A(\mathbb{A}^N) = k[x_{00}, \ldots, x_{nm}]$ with x_{ij} omitted. Then for $(c,d) \neq (i,j)$

$$\begin{aligned} x_{cd}\, \varphi\, Y(x) &= x_{cd}\, \varphi(\alpha(x), \beta(x)) \\ &= x_{cd}\, \varphi(\alpha(x)_0 \beta(x)_0, \ldots, \alpha(x)_n \beta(x)_m) \\ &= \frac{\alpha(x)_c\, \beta(x)_d}{\alpha(x)_i\, \beta(x)_j} \end{aligned} \qquad (1)$$

This map $K_{ij} \longrightarrow k$ is the product of two maps $x \longmapsto \frac{\alpha(x)_c}{\alpha(x)_i}$, $x \longmapsto \frac{\beta(x)_d}{\beta(x)_j}$ both of which are well-defined and regular. The first is $\alpha : Z \longrightarrow U$ composed with the regular map $x_c/x_i : W_i \longrightarrow k$, the second is β composed with the regular map $y_d/y_j : Q_j \longrightarrow k$. As the product of two regular maps the map in (1) is regular, which completes the entire proof! Party time!

<u>NOTE</u> Since any variety is isomorphic to a quasi-projective variety, this shows that the category of varieties has finite products.

NOTE Explicit Topology on $\mathbb{P}^n \times \mathbb{P}^m$



Let $k[\{z_{ij}\}]$, $k[x_0,\dots,x_n]$ and $k[y_0,\dots,y_m]$ be the relevant coordinate rings,

$$\begin{aligned} \theta : k[z_{ij}] &\longrightarrow k[x_0,\dots,x_n,y_0,\dots,y_m] \\ z_{ij} &\longmapsto x_i y_j \end{aligned}$$

$$\begin{aligned} \beta_{cd} : k[x_0,\dots,x_n,y_0,\dots,y_m] &\longrightarrow k[z_{ij}] \\ x_i &\longmapsto z_{id} \\ y_j &\longmapsto z_{cj} \end{aligned}$$

Let $f \in k[x_0,\dots,x_n,y_0,\dots,y_m]$ be seperately homogenous in the xs and ys – say every monomial in f has degree u in the xs and degree v in the ys. Then it is easy to see that

$$\theta\beta_{cd}(f) = x_c^{\,v} y_d^{\,u} f$$

For any such f it makes sense to say if $f(a_0,\dots,a_n,b_0,\dots,b_m) = 0$ for $(a_0,\dots,a_n) \in \mathbb{P}^n$ and $(b_0,\dots,b_m) \in \mathbb{P}^m$. Let $Z(f) \subseteq \mathbb{P}^n \times \mathbb{P}^m$ denote the set of such pairs. One checks that

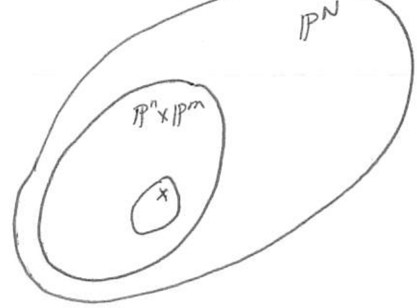
$$Z(f) = (\mathbb{P}^n \times \mathbb{P}^m) \cap \bigcap_{c,d} Z(\beta_{cd} f)$$

Note that $\beta_{cd}(f)(a_0b_0,\dots,a_nb_m) = \theta\beta_{cd}(f) = a_c^{\,v} b_d^{\,u} f(a_0,\dots,a_n,b_0,\dots,b_m)$. Hence $Z(f) \subseteq \mathbb{P}^n \times \mathbb{P}^m$ is closed. Clearly for any $g \in k[z_{ij}]$, $\theta(g)$ is seperately homogenous, and in the above sense $Z(\theta(g)) \subseteq \mathbb{P}^n \times \mathbb{P}^m$ is

$$Z(\theta(g)) = (\mathbb{P}^n \times \mathbb{P}^m) \cap Z(g)$$

Hence any closed set in $\mathbb{P}^n \times \mathbb{P}^m$ can be written as an intersection $\bigcap_i Z(f_i)$ where $f_i \in k[x_0,\dots,x_n,y_0,\dots,y_m]$ are seperately homogenous in each variable, and moreover all the $Z(f)$ for such f are closed.

NOTE Explicit Ringed Space Structure on $\mathbb{P}^n \times \mathbb{P}^m$



Let $X \subseteq \mathbb{P}^n \times \mathbb{P}^m$ be any subspace, $f: X \longrightarrow k$ any map. Let $\Theta: k[z_{ij}] \longrightarrow k[x_0, \ldots, x_n, y_0, \ldots, y_m]$ be as usual. Then

f regular $\iff$ $\forall x \in X$ there is $U \subseteq X$ open, $x \in U$, and $g, h \in k[z_{ij}]$ s.t. g, h homogenous same degree and $\forall (a_0, \ldots, a_n, b_0, \ldots, b_m) \in \mathbb{P}^n \times \mathbb{P}^m$

$$\begin{aligned} f(a_0, \ldots, a_n, b_0, \ldots, b_m) &= \frac{g(a_0 b_0, \ldots, a_n b_m)}{h(a_0 b_0, \ldots, a_n b_m)} \leftarrow \neq 0 \text{ of course} \\ &= \frac{\Theta(g)(a_0, \ldots)}{\Theta(h)(a_0, \ldots)} \end{aligned}$$

If $q, z \in k[x_0, \ldots, x_n, y_0, \ldots, y_m]$ is seperately homogenous in the xs and ys of resp. degrees e, f and e', f', then q and z have the same "degree" if $e = e'$, $f = f'$.

PROPOSITION Let $X \subseteq \mathbb{P}^n \times \mathbb{P}^m$ be any subspace. A map $f: X \longrightarrow k$ is regular if and only if every point $x \in X$ admits an open neighborhood U and seperately homogenous polynomials of the same degree $g, h \in k[x_0, \ldots, x_n, y_0, \ldots, y_m]$ such that for $(a_0, \ldots, a_n) \times (b_0, \ldots, b_m) \in X$

$$f(a_0, \ldots, a_n, b_0, \ldots, b_m) = \frac{g(a_0, \ldots, a_n, b_0, \ldots, b_m)}{h(a_0, \ldots, a_n, b_0, \ldots, b_m)} \qquad (1)$$

Where of course $h \neq 0$ on U.

PROOF Any regular map $f: X \longrightarrow k$ satisfies this condition. For the converse, recall the following morphism:

$$\begin{aligned} \beta_{cd}: k[x_0, \ldots, x_n, y_0, \ldots, y_m] &\longrightarrow k[z_{ij}] \\ x_i &\longmapsto z_{id} \\ y_j &\longmapsto z_{cj} \end{aligned}$$

$$\Theta \beta_{cd}(z) = x_c^v y_d^u z \qquad \text{degree } z = (u, v)$$

We can cover X by open sets as follows:

$$X = \bigcup_{cd} X \cap (V_c \times W_d) \qquad \begin{aligned} V_c &= \mathbb{P}^n - V(x_c) \\ W_d &= \mathbb{P}^m - V(y_d) \end{aligned}$$

It would suffice to show $f|_{X \cap V_c \times W_d}$ regular for any c, d. So let $x \in X \cap V_c \times W_d$. By assumption we can find $x \in U \subseteq X \cap V_c \times W_d$ with U open in $X \cap V_c \times W_d$ and g, h seperately homogenous of the same degree s.t. $\forall u \in U$, $f(u) = g(u)/h(u)$.

Now $\beta_{cd}(g), \beta_{cd}(h)$ are homogenous polynomials and for $(a_0,\dots,a_n,b_0,\dots,b_m) \in U$ (so $a_c b_d \neq 0$)

$$\frac{\theta\beta_{cd}(g)(a_0,\dots,b_m)}{\theta\beta_{cd}(h)(a_0,\dots,b_m)} = \frac{a_c^{\,v} b_d^{\,u}\, g(a_0,\dots,b_m)}{a_c^{\,v} b_d^{\,u}\, h(a_0,\dots,b_m)}$$
$$= f(a_0,\dots,b_m)$$

Hence $f|_{X\cap V_c\times W_d}$ is regular, as required. □

NOTE Explicit Details of $\mathbb{A}^n \times \mathbb{P}^m$

By identifying $\mathbb{A}^n$ with the quasi-projective variety $U_0 \subseteq \mathbb{P}^n$ we can define the product $\mathbb{A}^n \times \mathbb{P}^m$ of varieties. The underlying set is the cartesian product $\mathbb{A}^n \times \mathbb{P}^m$. The topology on $\mathbb{P}^n \times \mathbb{P}^m$ is determined by the sets $Z(f)$ where $f \in k[x_0,\dots,x_n,y_0,\dots,y_m]$ is separately homogenous in the xs and ys. By intersecting these sets with U_0, using the maps $k[x_0,\dots,x_n,y_0,\dots,y_m] \longrightarrow k[x_1,\dots,x_n,y_0,\dots,y_m]$, $x_0 \mapsto 1$ and β which maps a polynomial $g(x_1,\dots,x_n,y_0,\dots,y_m)$ which is homogenous in the ys to a polynomial $g'(x_0,\dots,x_n,y_0,\dots,y_m)$ homogenous in the xs and ys (multiply each monomial by a suitable power of x_0), we see that:

TOPOLOGY The identification $\mathbb{A}^n \times \mathbb{P}^m = U_0 \times \mathbb{P}^m$ yields the following topology on $\mathbb{A}^n \times \mathbb{P}^m$. A subset $Z \subseteq \mathbb{A}^n \times \mathbb{P}^m$ is closed iff. it is an intersection of sets of the form

$$Z(f) = \{(a_1,\dots,a_n,b_0,\dots,b_m) \mid f(a_1,\dots,a_n,b_0,\dots,b_m) = 0\}$$

where $f \in k[x_1,\dots,x_n,y_0,\dots,y_m]$ is homogenous in the ys.

Using similar tricks (patching with x_0) and the previous Note:

VARIETY STRUCTURE Let $X \subseteq \mathbb{A}^n \times \mathbb{P}^m$ be any subspace. A map $f: X \longrightarrow k$ is regular if and only if every point $x \in X$ admits an open neighborhood U and polynomials $g, h \in k[x_1,\dots,x_n,y_0,\dots,y_m]$ homogenous of the same degree in the ys, s.t. for $P \in U$ $h(P) \neq 0$ and

$$f(P) = g(P)/h(P)$$

Define an open subset of $\mathbb{A}^n \times \mathbb{P}^m$, $V_i = \mathbb{A}^n \times \mathbb{P}^m - Z(y_i)$ $i = 0,\dots,m$. Clearly the V_i cover $\mathbb{A}^n \times \mathbb{P}^m$. We claim each V_i is isomorphic as a variety to $\mathbb{A}^{n+m}$. We may as well assume $i=0$. Then define

$$\varphi : \mathbb{A}^{n+m} \longrightarrow V_0$$
$$(a_1,\dots,a_{n+m}) \longmapsto (a_1,\dots,a_n,1,a_{n+1},\dots,a_{n+m})$$

$$\phi : V_0 \longrightarrow \mathbb{A}^{n+m}$$
$$(a_1,\dots,a_n,b_0,\dots,b_m) \longmapsto \left(a_1,\dots,a_n,\frac{b_1}{b_0},\dots,\frac{b_m}{b_0}\right)$$

Both these maps are well-defined and $\phi\varphi = \varphi\phi = 1$. Let $\theta: k[x_1,\dots,x_n,y_0,\dots,y_m] \longrightarrow k[x_1,\dots,x_n,y_1,\dots,y_m]$ be the map $y_0 \longmapsto 1$. If $g \in k[x_1,\dots,x_n,y_0,\dots,y_m]$ is homogenous in the ys, then

$$\varphi^{-1}(Z(g) \cap V_0) = Z(\theta(g))$$

So φ is continuous. If $h \in k[x_1,\dots,x_n,y_1,\dots,y_m]$ is any polynomial, multiply each monomial in h by a suitable power of y_0 until h is homogenous in the ys. Call this polynomial H. Note that

$$H(a_1,\dots,a_n,b_0,\dots,b_m) = b_0^{\,e}\, h\left(a_1,\dots,a_n,\frac{b_1}{b_0},\dots,\frac{b_m}{b_0}\right) \quad \text{for some } e \geq 0$$

Hence $\phi^{-1}(Z(h)) = Z(H)$, so both φ and ϕ are continuous. Hence V_0 is homeomorphic to $\mathbb{A}^{n+m}$. So it only remains to show that both φ and ϕ are variety morphisms.

φ is a morphism Suppose $f: V \longrightarrow k$ is regular, where $V \subseteq U_0 \subseteq \mathbb{A}^n \times \mathbb{P}^m$. From the details on the previous page, if $x \in \varphi^{-1}V$ then there is an open neighborhood $\varphi(x) \in W \subseteq V$ and polynomials $g, h \in k[x_1, \dots, x_n, y_0, \dots, y_m]$ homogenous of the same degree in the ys, such that $f(v) = g(v)/h(v)$ $\forall v \in W$ ($h \neq 0$ on W). If G, H are g, h with $1 \longleftrightarrow y_0$ then for all $z \in \varphi^{-1}W$ $f\varphi(z) = G(z)/H(z)$. Hence $f\varphi$ is regular and φ is a morphism.

ϕ is a morphism Suppose $f: V \longrightarrow k$ is regular, where $V \subseteq \mathbb{A}^{n+m}$. For $x \in \phi^{-1}V$ let $W \ni \phi(x)$ be open and $g, h \in k[x_1, \dots, x_n, y_1, \dots, y_m]$ s.t. $f(v) = g(v)/h(v)$ on W. Multiply the monomials of g, h by powers of y_0 to make polynomials G, H homogenous of the same degree in the ys, with

$$G(a_1, \dots, a_n, b_0, \dots, b_m) = b_0^e g(a_1, \dots, a_n, \tfrac{b_1}{b_0}, \dots, \tfrac{b_m}{b_0})$$
$$H(a_1, \dots, a_n, b_0, \dots, b_m) = b_0^e h(a_1, \dots, a_n, \tfrac{b_1}{b_0}, \dots, \tfrac{b_m}{b_0})$$

Then $\forall z \in \phi^{-1}W$, $f\phi(z) = g\phi(z)/h\phi(z) = G(z)/H(z)$, so $f\phi$ is regular and ϕ is a morphism.

NOTE Dimension of Products

Since $\mathbb{A}^n \times \mathbb{P}^m$ has an open subset isomorphic to $\mathbb{A}^{n+m}$, and dimension is local, we have

$$\dim \mathbb{A}^n \times \mathbb{P}^m = n+m$$

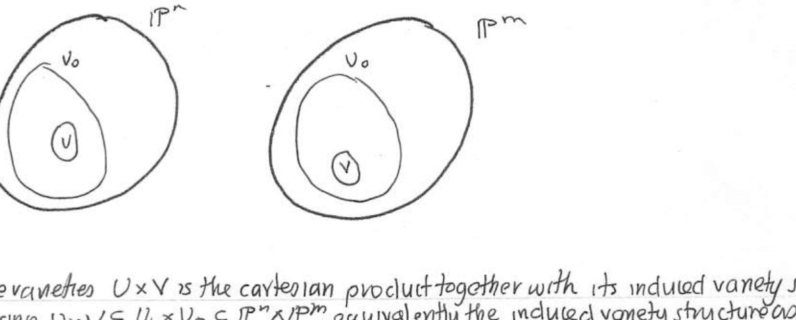
Now $\mathbb{A}^n \times \mathbb{P}^m$ is isomorphic to the open subset $V_0 \times \mathbb{P}^m \subseteq \mathbb{P}^n \times \mathbb{P}^m$ by construction. Hence

$$\dim \mathbb{P}^n \times \mathbb{P}^m = n+m$$

What we have actually shown above is that the open subset $U_i \times U_j \subseteq \mathbb{P}^n \times \mathbb{P}^m$ is isomorphic to $\mathbb{A}^{n+m}$, the iso being

$$\mathbb{A}^{n+m} \longrightarrow \mathbb{P}^n \times \mathbb{P}^m$$
$$(a_1, \dots, a_n, b_1, \dots, b_m) \longmapsto (a_1, \dots, a_{i-1}, 1, a_i, \dots, a_n, b_1, \dots, b_{j-1}, 1, b_j, \dots, b_n)$$

Let $U \subseteq \mathbb{A}^n$ and $V \subseteq \mathbb{A}^m$ be quasi-affine varieties. If we identify $\mathbb{A}^n \cong V_0 \subseteq \mathbb{P}^n$ and $\mathbb{A}^m \cong U_0 \subseteq \mathbb{P}^m$ and thus U with a quasi-projective variety in $\mathbb{P}^n$, V with a quasi-projective variety in $\mathbb{P}^m$;



As a product of quasi-projective varieties $U \times V$ is the cartesian product together with its induced variety structure as a subspace of $\mathbb{P}^n \times \mathbb{P}^m$. Or since $U \times V \subseteq V_0 \times V_0 \subseteq \mathbb{P}^n \times \mathbb{P}^m$, equivalently the induced variety structure as an open subset of $V_0 \times V_0$. But $V_0 \times V_0 \cong \mathbb{A}^{n+m}$ in such a way that $U \times V$ is identified with the product $U \times V \subseteq \mathbb{A}^{n+m}$. So the product $U \times V$ in the category of varieties is $U \times V$ with the structure induced by $U \times V \subseteq \mathbb{A}^{n+m}$.

Let $X \subseteq \mathbb{P}^n$ and $Y \subseteq \mathbb{P}^m$ be projective varieties and $X \times Y \subseteq \mathbb{P}^n \times \mathbb{P}^m$ their product. Let i, j be s.t. $X_i = X \cap U_i \neq \phi$ in $\mathbb{P}^n$ and $Y_j = Y \cap V_j \neq \phi$ in $\mathbb{P}^m$. Then $(X \times Y)_{ij} = X \times Y \cap U_i \times U_j = X_i \times Y_j$ is an open subset of $X \times Y$ (nonempty). It has the same underlying set as the product $X_i \times Y_j$ of affine varieties, and by the above comments the same variety structure as well, so $X \times Y$ has an open subset isomorphic to $X_i \times Y_j$. Since dimension is local (i.e. $\dim ? = \dim \mathcal{O}_{P,?}$) $\dim X = \dim X_i$, $\dim Y = \dim Y_j$ and

$$\dim(X \times Y) = \dim((X \times Y)_{ij}) = \dim X_i + \dim Y_j = \dim X + \dim Y.$$

(3.15 c)

Let $U \subseteq X \subseteq \mathbb{A}^n$ and $V \subseteq Y \subseteq \mathbb{A}^m$ be quasi-affine varieties. The product $X \times Y \subseteq \mathbb{A}^{n+m}$ comes with continuous projections, so $U \times V = (U \times Y) \cap (X \times V) \subseteq X \times Y$ is an open subset. Hence

$$\begin{aligned} \dim(U \times V) &= \dim(X \times Y) = \dim X + \dim Y \\ &= \dim U + \dim V \end{aligned}$$

Using the above, the same is true of quasi-projective varieties. In summary:

<u>THEOREM</u> The category of varieties has products. If X, Y are varieties then the underlying set of the product is the cartesian product $X \times Y$ and the projections are the canonical ones

$$X \times Y \longrightarrow X \qquad (x,y) \longmapsto x$$
$$X \times Y \longrightarrow Y \qquad (x,y) \longmapsto y$$

If X, Y are both quasi-affine the structure of $X \times Y$ comes from $\mathbb{A}^{n+m}$ $(X \subseteq \mathbb{A}^n, Y \subseteq \mathbb{A}^m)$ and if they are both quasi-projective from $\mathbb{P}^n \times \mathbb{P}^m$ (whose structure we described earlier). Finally, for any varieties X, Y

$$\dim(X \times Y) = \dim X + \dim Y$$

NOTE Dominant dual to Injective

In Proposition 3.5 we established a natural bijection for any variety X and an affine variety $Y \subseteq \mathbb{A}^n$:

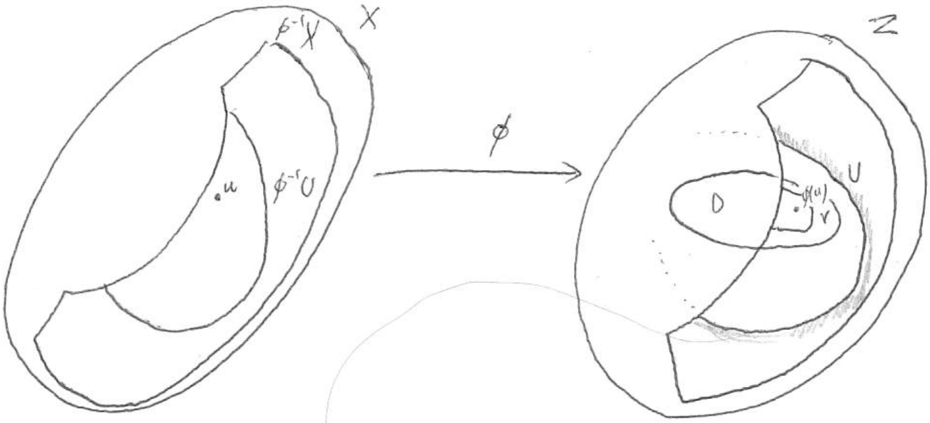
$$\alpha : \mathrm{Hom}(X,Y) \xrightarrow{\sim} \mathrm{Hom}(A(Y), \mathcal{O}(X))$$

We claim that α identifies dominant maps $X \longrightarrow Y$ (i.e. dense images) with injective maps $A(Y) \longrightarrow \mathcal{O}(X)$.
Let $\varphi : X \longrightarrow Y$ be a morphism of varieties

$$\begin{aligned}
\mathrm{Im}\,\varphi \text{ dense} &\iff Z \subseteq Y \text{ closed and } \mathrm{Im}\,\varphi \subseteq Z \text{ implies } Z = Y \\
&\iff \text{If } f \in A(Y) \text{ and } \mathrm{Im}\,\varphi \subseteq Z(f) \text{ then } Z(f) = Y \\
&\iff \text{If } f \in A(Y) \text{ and } \forall x \in X \;\; f(\varphi(x)) = 0 \text{ then } f = 0 \text{ (in } A(Y)) \\
&\iff \text{If } f \in A(Y) \text{ and } \forall x \in X \; \alpha(\varphi)(f)(x) = 0 \text{ then } f = 0 \\
&\iff \forall f \in A(Y) \;\; \alpha(\varphi)(f) = 0 \Rightarrow f = 0 \\
&\iff \alpha(\varphi) \text{ is injective.}
\end{aligned}$$

NOTE Isomorphisms of subspaces of Varieties

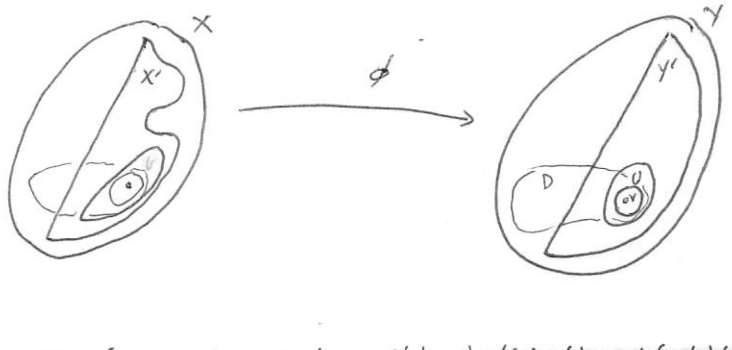
Let $\phi: Y \longrightarrow Z$ be an isomorphism of varieties, and let $X \subseteq Z$ be any subspace. The subspace X obtains a ringed structure as a subspace of $\mathbb{A}^n$ (resp. $\mathbb{P}^n$). Similarly $\phi^{-1}X$ obtains a ringed space structure. Suppose $U \subseteq X$ is an open subset of U, and $u \in \phi^{-1}U$. Then there is an open subset V of X s.t. $\phi(u) \in V \subseteq U$, and there is some open set D (open in Z) and a regular map $p: D \longrightarrow k$ s.t. $V \subseteq D$ and $p|_V = f|_V$ (in the affine case take $g, h \in k[x_1, \dots, x_n]$ s.t. $V \subseteq D(h)$ and put $p = g/h : D(h) \longrightarrow k$. Similarly in the projective case).



Then $\phi^{-1}V$ is an open subset of $\phi^{-1}U$, $\phi^{-1}D$ is open and $p\phi: \phi^{-1}D \longrightarrow k$ is regular since ϕ is a morphism. The topology of $\phi^{-1}V$ as a subspace of $\phi^{-1}U$ is the same as a subspace of D, and it is easily checked that for <u>any</u> subspaces $Q \subseteq P$ of affine or projective space, if $f: P \longrightarrow k$ is regular so is $f|_Q$. Hence $p\phi|_{\phi^{-1}V}$ is a regular map $\phi^{-1}V \longrightarrow k$. Since u was arbitrary, this proves that $p\phi: \phi^{-1}U \longrightarrow k$ is regular. By symmetry the induced map of spaces $\hat{\phi}: \phi^{-1}X \longrightarrow X$ is also an isomorphism of ringed spaces.

NOTE Induced morphisms More Generally

Let $\phi: X \longrightarrow Y$ be a morphism of varieties, $X' \subseteq X$ $Y' \subseteq Y$ arbitrary subspaces s.t. $\phi(X') \subseteq Y'$. Both X', Y' acquire ringed space structures by considering them as subspaces of affine or projective space. The restriction $\phi|_{X'}: X' \longrightarrow Y'$ is clearly continuous, and we claim it is a morphism.



Let $f: U \longrightarrow k$ bre regular for $V \subseteq Y'$ open. Let $u \in X'$ be s.t. $\phi(x) \in U$, and find $V \subseteq U$ open s.t. $\phi(x) \in V$ and $D \subseteq Y$ open and $z: D \longrightarrow k$ regular s.t. $z|_V = f|_V$ (assume $V \subseteq D$). Then $z\phi: \phi^{-1}D \longrightarrow k$ is regular since ϕ is a morphism, hence $z\phi|_{\phi^{-1}V \cap X'}: \phi^{-1}V \cap X' \longrightarrow k$ is regular. But this is $f \circ \phi|_{X'}$, so $\phi|_{X'}$ is a morphism.