4. Rational Maps

In this section we introduce the notions of rational map and birational equivalence, which are important for the classification of varieties. A rational map is a morphism which is only defined on some open subset. Since an open subset of a variety is dense, this already carries a lot of information. In this respect algebraic geometry is more "rigid" than differential topology or topology. In particular, the concept of birational equivalence is unique to algebraic geometry.

<u>LEMMA 4.1</u> Let X and Y be varieties, let φ and ψ be two morphisms from X to Y, and suppose there is a nonempty open subset $U \subseteq X$ s.t. $\varphi|_U = \psi|_U$. Then $\varphi = \psi$.

<u>PROOF</u> Since any variety is isomorphic to a quasi-projective variety (see our notes just before Ex 3.16), we may assume $Y \subseteq \mathbb{P}^n$ is quasi-projective. Then by composing with the inclusion morphism $Y \longrightarrow \mathbb{P}^n$ we reduce to the case $Y = \mathbb{P}^n$. We consider the product $\mathbb{P}^n \times \mathbb{P}^n$ which has a structure of projective variety given by its Segre embedding (Ex. 3.16). The morphisms φ, ψ determine a morphism $\varphi \times \psi : X \longrightarrow \mathbb{P}^n \times \mathbb{P}^n$. Let $\Delta = \{ P \times P \mid P \in \mathbb{P}^n \}$, be the <u>diagonal subset</u> of $\mathbb{P}^n \times \mathbb{P}^n$. It is defined by $\{ x_i y_j = x_j y_i \mid i,j = 0, \dots, n \}$ and so is a closed subset of $\mathbb{P}^n \times \mathbb{P}^n$ (i.e. $z_{ij} - z_{ji}$ intersected with $\mathbb{P}^n \times \mathbb{P}^n$. This is Δ since if $x_k \neq 0$ and $y_e \neq 0$ $x_k y_e = y_k x_e$ so $y_k \neq 0$ $x_e \neq 0$ also. Then $\forall j$ $x_k y_j = x_j y_k$ so $y_j = x_j \cdot y_k / x_k$ so $x = y$). By hypothesis $(\varphi \times \psi)(U) \subseteq \Delta$, so since $(\varphi \times \psi)^{-1} \Delta$ is closed, $U \subseteq (\varphi \times \psi)^{-1} \Delta$ and U is dense in X, $(\varphi \times \psi)(X) \subseteq \Delta$ and so $\varphi = \psi$. $\square$

<u>DEFINITION</u> Let X, Y be varieties. A <u>rational map</u> $\varphi : X \longrightarrow Y$ is an equivalence class of pairs (U, φ_U) where U is a nonempty open subset of X, φ_U is a morphism of U to Y, and where (U, φ_U) and (V, φ_V) are equivalent if φ_U and φ_V agree on $U \cap V$. The rational map φ is <u>dominant</u> if for some (and hence every) pair (U, φ_U), the image of φ_U is dense in Y.

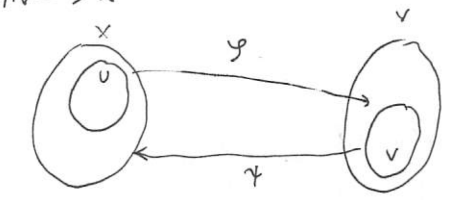
To clarify this last statement, suppose (U, φ_U) is dominant and $(V, \varphi_V) = (U, \varphi_U)$. Since U is an open subset of an irreducible space it is irreducible and dense in X. Moreover $U \cap V$ is an open subset of U, so is irreducible and dense in V. If $\varphi_U(U)$ is dense in Y and $Z \subseteq Y$ is a closed subset containing $\varphi_V(V)$, then

$$Z \supseteq \varphi_V(V) \supseteq \varphi_V(U \cap V) = \varphi_U(U \cap V)$$

Hence $U \cap V \subseteq \varphi_U^{-1} Z$. But this implies $U \subseteq \varphi_U^{-1} Z$ so $\varphi_U(U) \subseteq Z$, implying $Z = Y$. Hence $\varphi_V(V)$ is dense in Y.

Note that the Lemma implies that the relation on pairs (U, φ_U) just described is an equivalence relation. Note that in an irreducible space <u>any</u> two nonempty open subsets have a nonempty intersection. Note also that a rational map $\varphi : X \longrightarrow Y$ is <u>not</u> in general a map of the set X to Y. Clearly one can compose dominant rational maps $(U, \varphi_U) : X \longrightarrow Y$ and $(V, \psi_V) : Y \longrightarrow Z$ to form $(\varphi^{-1} V, \psi_V \varphi_U)$. The map $\psi_V \varphi_U$ is easily checked to be a morphism, and we show that $\psi_V \varphi_U(\varphi^{-1} V)$ is dense as follows. Since $\varphi^{-1} V$ is an open subset of U, $\varphi(\varphi^{-1} V) \subseteq V$ is a dense subset of Y and hence of V. Suppose $K \subseteq Z$ is closed with $\psi_V \varphi_U(\varphi^{-1} V) \subseteq K$. Then $\psi^{-1} K$ is closed and $\varphi(\varphi^{-1} V) \subseteq \psi^{-1} K$. Hence $\psi(V) \subseteq K$. Since $\psi(V)$ is dense in Z, $K = Z$ as required. Hence we can consider the category of varieties and dominant rational maps. An "isomorphism" in this category is called a birational map. One checks that the composition above is well-defined.

<u>DEFINITION</u> A <u>birational map</u> $\varphi : X \longrightarrow Y$ is a rational map which admits an inverse, namely a rational map $\psi : Y \longrightarrow X$ such that $\psi \varphi = 1_X$ and $\varphi \psi = 1_Y$. That is, if we represent φ by (U, φ_U), there is $V \subseteq Y$ and a morphism $\psi : V \longrightarrow X$



Such that $(V, \psi)(U, \varphi) = (X, 1_X)$ – that is, $(\varphi^{-1} V, \psi \varphi) \sim (X, 1_X)$, i.e. $\forall x \in \varphi^{-1} V$ $\psi \varphi(x) = x$. Similarly $(U, \varphi)(V, \psi) = (Y, 1_Y)$ equivalent to $\forall y \in \psi^{-1} U$ $\varphi \psi(y) = y$. If there is a birational map from X to Y we say that X and Y are <u>birationally equivalent</u>, or simply <u>birational</u>.

The main result of this section is that the category of varieties and dominant rational maps is equivalent to the category of finitely generated field extensions of k, with the arrows reversed. Before giving this result, we need a couple of lemmas which show that on any variety, the open affine subsets form a base for the topology. We say loosely that a variety is <u>affine</u> if it is isomorphic to an affine variety.

<u>LEMMA 4.2</u> Let Y be a hypersurface in $\mathbb{A}^n$ given by the equation $f(x_1,\ldots,x_n)=0$. Then $\mathbb{A}^n - Y$ is isomorphic to the hypersurface H in $\mathbb{A}^{n+1}$ given by $x_{n+1}f=1$. In particular, $\mathbb{A}^n - Y$ is affine, and its affine ring is $k[x_1,\ldots,x_n]_f$.

<u>PROOF</u> The case $f=0$ is vacuous. We have checked during Milne's notes that $\mathbb{A}^n - Y$ is isomorphic as a ringed space to H, where $H = Z(1-x_{n+1}f)$. In Milne's notes these are just algebraic sets – so the Lemma holds for <u>any</u> f, in the sense that $\mathbb{A}^n - Y$ is isomorphic as a ringed space to the ringed space arising from a closed set in $\mathbb{A}^{n+1}$.

Moreover as an open subset of an irreducible space, $\mathbb{A}^n - Y$ is irreducible. Hence $Z(1-x_{n+1}f)$ is irreducible, implying $1-x_{n+1}f$ is irreducible (whether f is or not). Hence $Z(1-x_{n+1}f)$ is an affine variety. So the Lemma is true for <u>arbitrary</u> $f \neq 0$ (not just f irred.). $\square$

<u>PROPOSITION 4.3</u> On any variety Y, there is a base for the topology consisting of open affine subsets.

<u>PROOF</u> We must show that for any point $P \in Y$ and any open set U containing P, that there exists an open affine set V with $P \in V \subseteq U$. First, since U is also a variety, we may assume $U = Y$. Secondly, since any variety is covered by quasi-affine varieties (2.3), we may assume that Y is quasi-affine in $\mathbb{A}^n$. Let $Z = \bar{Y} - Y$, which is a closed set in $\mathbb{A}^n$ (say Q affine $Y \subseteq Q \subseteq \mathbb{A}^n$ and $T \subseteq \mathbb{A}^n$ is open s.t. $Y = Q \cap T$. Then $Y = \bar{Y} \cap T$ and so $\bar{Y} - Y = \bar{Y} \cap Y^c = \bar{Y} \cap \{\bar{Y}^c \cup T^c\} = \bar{Y} \cap T^c$ which is closed in $\mathbb{A}^n$). Let $\mathfrak{a} \subseteq A = k[x_1,\ldots,x_n]$ be the ideal of Z. Then, since Z is closed, and $P \notin Z$, we can find a polynomial $f \in \mathfrak{a}$ such that $f(P) \neq 0$. Let H be the hypersurface $f=0$ in $\mathbb{A}^n$. Then $Z \subseteq H$ but $P \notin H$. Thus $P \in Y - Y \cap H$, which is an open subset of Y. Furthermore $Y - Y \cap H$ is a closed subset of $\mathbb{A}^n - H$, which is affine by (4.2), hence $Y - Y \cap H$ is affine. (since $Z = \bar{Y} - Y$, and $Z \subseteq H$, $Y - Y \cap H = Y \cap (\mathbb{A}^n - H) = \bar{Y} \cap (\mathbb{A}^n - H)$). Note $Y - Y \cap H$ is irreducible by virtue of being open in Y. $\square$ (see Note: Isos of subspaces for some details)

Now we come to the main result of this section. Let $\varphi : X \longrightarrow Y$ be a dominant rational map, represented by (U, φ_U). Let $f \in K(Y)$ be a rational function, represented by (V, f) where V is an open set in Y, and f is a regular function on V. Since $\varphi_U(U)$ is dense in Y, $\varphi_U^{-1}(V)$ is a nonempty open subset of X, so $f\varphi_U$ is a regular function on $\varphi_U^{-1}(V)$. This gives us a rational function on X, and in this manner we have defined a homomorphism of k-algebras from $K(Y)$ to $K(X)$. These definitions are independent of the various choices made.

<u>THEOREM 4.4</u> For any two varieties X and Y, the above construction gives a bijection between

(i) the set of dominant rational maps from X to Y

(ii) the set of k-algebra homomorphisms from $K(Y)$ to $K(X)$

<u>PROOF</u> We will construct an inverse to the mapping given by the construction above. Let $\theta : K(Y) \longrightarrow K(X)$ be a homomorphism of k-algebras (which is injective since domain and codomain are fields). We wish to define a rational map from X to Y. By (4.3) Y is covered by affine varieties U_i. Since $K(Y) \cong K(U_i)$ we may assume Y affine (since if $X \to U_i$ has dense image in U_i, the image is dense in Y also). Let $A(Y)$ be its affine coordinate ring, and let $y_1,\ldots,y_n$ be generators for $A(Y)$ as a k-algebra. Then $\theta(y_1),\ldots,\theta(y_n)$ are rational functions on X. We can find an open set U such that the $\theta(y_i)$ are all regular on U. This induces immediately a morphism of k-algebras $A(Y) \longrightarrow \mathcal{O}(U)$. This map is injective since if $f \in A(Y)$ then $(Y,f) \in K(Y)$ so $\theta(Y,f) = (V,f')$ for some open $V \subseteq X$ and regular $f' : V \longrightarrow k$, and if $f'|_{U \cap V}$ is zero, then $(V,f')=0$, so since θ is injective $(Y,f)=0$, so $f=0$ on Y. Hence $f=0$ in $A(Y)$. By (3.5) the injective map $A(Y) \longrightarrow \mathcal{O}(U)$ corresponds to a dominant morphism $U \longrightarrow Y$ (see our notes on dominant vs. injective). Call this morphism φ. Then φ is a dominant rational map $X \longrightarrow Y$.

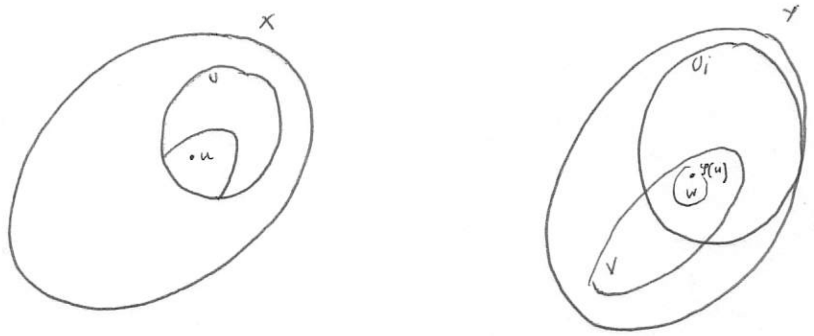
Suppose $Q \subseteq \mathbb{A}^n$ is affine s.t. $U_i \overset{\beta}{\cong} Q$ and so $\varphi : U \longrightarrow Y$ is defined by

$$\varphi(u) = \beta^{-1}(\theta(y_1)(u), \dots, \theta(y_n)(u)) \qquad u \in U \qquad (1)$$

This induces $\hat{\theta} : K(Y) \longrightarrow K(X)$ as follows

$$(V, f) = (V \cap U_i, f|_{U_i}) \longmapsto (\varphi^{-1}(V \cap U_i), f\varphi)$$

We claim $\hat{\theta}(V, f) = \theta(V, f)$.



Let $u \in \varphi^{-1}(V \cap U_i)$ so $\varphi(u) \in V \cap U_i$. Since $f|_{U_i \cap V} : U_i \cap V \longrightarrow k$ is regular and $U_i \cong Q$ there is an open set $\varphi(u) \in W \subseteq U_i \cap V$ and $g, h \in k[y_1, \dots, y_n]$ such that $\forall w \in W$

$$f(w) = \frac{g\beta(w)}{h\beta(w)}$$

$$= \frac{g(y_1(w), \dots, y_n(w))}{h(y_1(w), \dots, y_n(w))} \qquad y_i : U_i \longrightarrow k$$

If we confuse polynomials g, h with the induced regular function $U_i \longrightarrow k$, then $f = g/h$ on W. Now, $\hat{\theta}(V, f)$ can be represented by $(\varphi^{-1}(V \cap U_i), f\varphi)$ and for $u \in \varphi^{-1}W \subseteq \varphi^{-1}(V \cap U_i)$

$$f\varphi(u) = \frac{g(y_1\varphi(u), \dots, y_n\varphi(u))}{h(y_1\varphi(u), \dots, y_n\varphi(u))}$$

$$= \frac{g(\theta(y_1)(u), \dots, \theta(y_n)(u))}{h(\theta(y_1)(u), \dots, \theta(y_n)(u))} \qquad \text{by (1)}$$

$$= \frac{\theta(g)(u)}{\theta(h)(u)}$$

$$= \theta(g/h)(u) = \theta(f)(u)$$

Since $(V, f) = (W, f) = (W, g)(W, h^{-1})$ and θ is a morphism of rings, the remaining detail is routine. Hence $\theta(V, f) = \hat{\theta}(V, f)$, and so $\theta = \hat{\theta}$.

Note The assignment (ii) $\longrightarrow$ (i) is at this stage not known to be well-defined, since it involves an arbitrary choice of an affine open in Y. Suppose $U_i, U_j \subseteq Y$ are two such affine opens and let $\varphi : V_i \longrightarrow Y$, $\varphi' : V_j \longrightarrow Y$ be the end-products, where $V_i, V_j \subseteq X$. Let $\theta', \theta'' : K(Y) \longrightarrow K(X)$ be induced by φ, φ'. By the above we know $\theta' = \theta'' = \theta$. We must show that for $x \in V_i \cap V_j$ $\varphi(x) = \varphi'(x)$. Let $y_1, \dots, y_n : U_i \longrightarrow k$ be induced from the generators for $A(U_i)$. Then for $1 \le i \le n$ and $x \in \varphi^{-1}U_j \cap \varphi'^{-1}U_i$

$$y_i\varphi(x) = \theta(y_i)(x) = \theta(U_i, y_i)(x)$$

$$= \theta(U_i \cap U_j, y_i)(x) = \theta''(U_i \cap U_j, y_i)(x)$$

$$= y_i\varphi'(x)$$

Hence $\varphi(x) = \varphi'(x)$. Now apply 4.1 to see $\varphi = \varphi'$ on $V_i \cap V_j$.

We have shown (ii) $\mapsto$ (i) $\mapsto$ (ii) $= 1$. Now suppose $\varphi: U \longrightarrow Y$ is a dominant rational map and $\theta: K(Y) \longrightarrow K(X)$ is the induced morphism of k-algebras. Let $U_i \subseteq Y$ be affine and $\varphi': V \longrightarrow U_i$ dominant rational for $V \subseteq X$ open. For $x \in U \cap V$ we have by definition (if $\beta: U_i \cong$ affine variety $\subseteq \mathbb{A}^n$)

$$\begin{aligned} \varphi'(x) &= \beta^{-1}(\theta(y_1)(x), \dots, \theta(y_n)(x)) \\ &= \beta^{-1}(y_1\varphi(x), \dots, y_n\varphi(x)) \\ &= \beta^{-1}\beta\varphi(x) = \varphi(x) \end{aligned}$$

Hence as rational maps $\varphi' = \varphi$. This completes the proof of the Theorem. □

It is easily checked that $Y \mapsto K(Y)$ is a functor in the following:

<u>COROLLARY</u> There is an arrow reversing equivalence between the category of varieties and dominant rational maps with the category of all finitely generated field extensions of k.

<u>PROOF</u> The equivalence is defined by $Y \mapsto K(Y)$. For any variety Y, $K(Y)$ is isomorphic as a k-algebra to $K(U)$ for any open affine $U \subseteq Y$. By (3.2d) $K(U)$ is a finitely generated field extension of k. It remains to show that every f.g. field extension K/k is isomorphic to $K(Y)$ for some variety Y. Let $y_1, \dots, y_n \in K$ be a set of generators, and let B be the sub k-algebra of K generated by $y_1, \dots, y_n$. Then B is a quotient of the polynomial ring $A = k[x_1, \dots, x_n]$, so $B \cong A(Y)$ for some variety Y in $\mathbb{A}^n$, so $K \cong K(Y)$ and we are done. □

<u>NOTE</u> The above proof implies any variety is birational to an affine variety, which is kind of trivial anyway.

<u>COROLLARY 4.5</u> For any two varieties X, Y the following conditions are equivalent

(i) X and Y are birationally equivalent

(ii) there are two open subsets $U \subseteq X$ and $V \subseteq Y$ with U isomorphic to V (nonempty)

(iii) $K(X) \cong K(Y)$ as k-algebras

<u>PROOF</u> (i) $\Rightarrow$ (ii) Let $\varphi: X \longrightarrow Y$ and $\psi: Y \longrightarrow X$ be rational maps which are inverse to each other. Let φ be represented by (U, φ) and ψ by (V, ψ). Then $\psi\varphi$ is represented by $(\varphi^{-1}(V), \psi\varphi)$, and since $\psi\varphi = 1_X$ as a rational map, $\psi\varphi$ is the identity on $\varphi^{-1}V$. Similarly $\varphi\psi$ is the identity on $\psi^{-1}U$. We now take $\varphi^{-1}(\psi^{-1}(U))$ as our open set in X, and $\psi^{-1}(\varphi^{-1}(V))$ as our open set in Y. It follows from the construction that these two open sets are isomorphic (as varieties) via φ and ψ.

(ii) $\Rightarrow$ (iii) Follows from the definition of function field.

(iii) $\Rightarrow$ (i) Follows from the theorem. □

<u>NOTE</u> Birational Equivalence and Projective closure

Let $Y \subseteq \mathbb{A}^n$ be affine, $\varphi_0: \mathbb{A}^n \longrightarrow U_0 \subseteq \mathbb{P}^n$ and identify Y with the closed, irreducible subset $\varphi_0(Y)$ of U_0. Thus $Y = U_0 \cap Z$ for some $Z \subseteq \mathbb{P}^n$ closed, and thus it follows that $\overline{Y} \cap U_0 = Y$, so Y is an open nonempty subset of the projective variety $\overline{Y}$. Hence $K(\overline{Y}) \cong K(Y)$ as k-algebras, so <u>any affine variety is birational to its projective closure</u> (or use 4.5 ii to be even more trivial). Since clearly any quasi-affine variety is ... birational to an affine variety, we have

<u>Any variety is birational to a projective variety</u>

We will see in Prop 4.9 that this statement can be made a lot stronger.

<u>NOTE</u> Birational varieties have the same dimension

Either from (4.5 ii) since for any variety Y, $\dim Y = \dim \mathcal{O}_{P,Y} = \dim_{P,U}$ for any open neighborhood of P, or from (iii) since for any variety $\dim Y = \text{tr.deg.}\, K(Y)/k$.

As an illustration of the notion of birational correspondence, we will use some algebraic results on field extensions to show that every variety is birational to a hypersurface in projective space.

THEOREM 4.6A (Theorem of the Primitive Element) Let L be a finite separable field extension of a field K. Then there is an element $\alpha \in L$ with $L = K(\alpha)$. Furthermore, if $\beta_1, \dots, \beta_n$ is any set of generators of L over K, and if K is infinite, then α can be taken to be a linear combination $\alpha = c_1\beta_1 + \dots + c_n\beta_n$ for $c_i \in K$.

PROOF See p8 of our "Elements of Field Theory" notes. $\square$

DEFINITION A field extension K/k is separably generated if there is a transcendence base $\{x_i\}$ for K/k s.t. K is a separable algebraic extension of $k(\{x_i\})$. Such a transcendence base is called a separating transcendence basis.

THEOREM 4.7A If a field extension K/k is finitely generated and separably generated, then any set of generators contains a subset which is a separating transcendence base.

PROOF Combine the Theorem on p12 and the Note on p13 of our "Elements of Field Theory" notes. $\square$

THEOREM 4.8A If k is a perfect field (hence if char $k = 0$, or k is finite, or algebraically closed) any finitely generated field extension is separably generated.

PROOF p13 EFT Notes. $\square$

PROPOSITION 4.9 Any variety X of dimension r is birational to a hypersurface Y in $\mathbb{P}^{r+1}$.

PROOF For any variety the function field $K = K(X)$ is a finitely generated field extension of k with transcendence degree $\dim X$ over k (See Note following proof of Theorem 3.4). Hence by (4.8)A K is separably generated over k, so we can find a separating transcendence basis $a_1, \dots, a_r$ with K separable over $k(a_1, \dots, a_r)$. Since K/k is finitely generated, so is $K/k(a_1, \dots, a_r)$, whence the extension $K/k(a_1, \dots, a_r)$ is finite separable and so by (4.6A) admits a primitive element $y \in K$ with $K = k(a_1, \dots, a_r, y)$. Since $a_1, \dots, a_r$ are algebraically independent and $a_1, \dots, a_r, y$ are algebraically dependent, by Lemma 2 §13 2&5 (p11 of our EFT Notes) there is an irreducible polynomial

$$f(x_1, \dots, x_r, x_{r+1}) \in k[x_1, \dots, x_r, x_{r+1}]$$

which is primitive in $k[x_1, \dots, x_r][x_{r+1}]$ with $f(a_1, \dots, a_r, y) = 0$. Let Y' be the hypersurface $Y' = Z(f) \subseteq \mathbb{A}^{r+1}$. We claim that $K(Y') = K$. Since f is irreducible in $k[x_1, \dots, x_r][x_{r+1}]$, $f(a_1, \dots, a_r, x_{r+1})$ is irreducible in $k[a_1, \dots, a_r][x_{r+1}]$ (the a_i are alg. indept) and f is of positive x_{r+1}-degree since the a_i are alg. indept. Hence by Lemma 1 §13 2&5 (p11 EFT Notes) $f(a_1, \dots, a_r, x_{r+1})$ is irreducible in $k(a_1, \dots, a_r)[x_{r+1}]$. Hence it is the minimum polynomial (up to a unit) of y over $k(a_1, \dots, a_r)$, and it follows that we have k-algebra isomorphisms

$$\begin{aligned} K &\cong k(a_1, \dots, a_r, y) \\ &= k(a_1, \dots, a_r)[y] \\ &\cong k(a_1, \dots, a_r)[x_{r+1}] \,/\, (f(a_1, \dots, a_r, x_{r+1})) \end{aligned}$$

Since $f(x_1, \dots, x_r, x_{r+1})$ is primitive in $k[x_1, \dots, x_{r+1}][x_{r+1}]$, $f(a_1, \dots, a_r, x_{r+1})$ is primitive in $k[a_1, \dots, a_r][x_{r+1}]$, so a polynomial $g \in k[a_1, \dots, a_r][x_{r+1}]$ is a multiple of f in $k[a_1, \dots, a_r][x_{r+1}]$ iff it is a multiple of f in $k(a_1, \dots, a_r)[x_{r+1}]$ (again by Lemma 1 (iii)). Hence there is an injective homomorphism of k-algebras

$$\frac{k[a_1, \dots, a_r][x_{r+1}]}{(f)} \longrightarrow \frac{k(a_1, \dots, a_r)[x_{r+1}]}{(f)} \cong K \qquad (1)$$

Since the first k-algebra is isomorphic to $k[x_1, \dots, x_r, x_{r+1}]/(f(x_1, \dots, x_r, x_{r+1}))$ we see immediately that $K(Y') \cong K$ as k-algebras. (Since the image of (1) in K contains $k, a_1, \dots, a_r$ and y). By Corollary 4.5, X is birational to Y'. By the note on the previous page, the hypersurface Y' is birational to its projective closure Y, which is a hypersurface in $\mathbb{P}^{r+1}$ by a note following Ex 2.9. This completes the proof. $\square$

Blowing Up (throughout $n \geq 2$)

As another example of a birational map, we will now construct the blowing up of a variety at a point. This important construction is the main tool in the resolution of singularities of an algebraic variety. First we will construct the blowing-up of $\mathbb{A}^n$ at the point $O = (0, \dots, 0)$. Consider the product $\mathbb{A}^n \times \mathbb{P}^{n-1}$, which is a quasi-projective variety (3.16). If $x_1, \dots, x_n$ are the affine coordinates of $\mathbb{A}^n$ and $y_1, \dots, y_n$ are the homogenous coordinates of $\mathbb{P}^{n-1}$ (note the unusual notation), then the closed subsets of $\mathbb{A}^n \times \mathbb{P}^{n-1}$ are defined by polynomials in the x_i, y_j which are homogenous with respect to the y_j. (put $\mathbb{A}^n \cong V_0 \subseteq \mathbb{P}^n$)

We now define the <u>blowing up of $\mathbb{A}^n$ at the point O</u> to be the closed subset X of $\mathbb{A}^n \times \mathbb{P}^{n-1}$ defined by the equations $\{x_i y_j = x_j y_i \mid i,j = 1, \dots, n\}$.

[see our Notes on how to look at $\mathbb{P}^n \times \mathbb{P}^{n-1}$ without using Segre. Closed subsets $\mathbb{P}^n \times \mathbb{P}^{n-1}$ are $Z(f)$ where $f \in k[x_0, \dots, x_n, y_1, \dots, y_n]$ is separately hom. in xs and ys ... i.e. $x_i y_j - x_j y_i$]

$$\begin{tikzcd}
X \arrow[r, hook] \arrow[dr, "\varphi"'] & \mathbb{A}^n \times \mathbb{P}^{n-1} \arrow[d] \\
& \mathbb{A}^n
\end{tikzcd}$$

We have a natural morphism $\varphi: X \longrightarrow \mathbb{A}^n$ obtained by restricting the projection map of $\mathbb{A}^n \times \mathbb{P}^{n-1}$ onto the first factor. We will now study the properties of X.

(1) If $P \in \mathbb{A}^n$, $P \neq O$ then $\varphi^{-1}P$ consists of a single point. In fact, φ gives an isomorphism of $X - \varphi^{-1}O$ onto $\mathbb{A}^n - O$. Indeed, let $P = (a_1, \dots, a_n)$, with some $a_i \neq 0$. Now if $P \times (y_1, \dots, y_n) \in \varphi^{-1}P$, then for each j $y_j = (a_j/a_i) y_i$, so $(y_1, \dots, y_n)$ is uniquely determined as a point in $\mathbb{P}^{n-1}$. In fact, setting $y_i = a_i$ we can take $(y_1, \dots, y_n) = (a_1, \dots, a_n)$. Thus $\varphi^{-1}P$ consists of a single point. Furthermore, for $P \in \mathbb{A}^n - O$, setting $\psi(P) = (a_1, \dots, a_n) \times (a_1, \dots, a_n)$ defines an inverse morphism to φ, showing $X - \varphi^{-1}(O)$ is isomorphic to $\mathbb{A}^n - O$. (as a topological space) (φ is clearly continuous bijective on $X - \varphi^{-1}O$ with $\mathbb{A}^n - O$, and $\psi: \mathbb{A}^n - O \longrightarrow X - \varphi^{-1}O$ is continuous since if $\theta: k[x_1, \dots, x_n, y_1, \dots, y_n] \longrightarrow k[x_1, \dots, x_n]$ is $x_i \mapsto x_i$ $y_i \mapsto x_i$ then $\psi^{-1}(Z(f) \cap X - \varphi^{-1}O) = \psi^{-1}(Z(f)) = Z\theta(f)$)

(2) $\varphi^{-1}O \cong \mathbb{P}^{n-1}$. Indeed $\varphi^{-1}O$ consists of all points $O \times Q$, with $Q = (y_1, \dots, y_n) \in \mathbb{P}^{n-1}$ subject to no restriction.

(3) The points of $\varphi^{-1}O$ are in 1-1 correspondence with the set of lines through O in $\mathbb{A}^n$. Indeed, a line L through O in $\mathbb{A}^n$ can be given by parametric equations

$$x_i = a_i t \qquad i = 1, \dots, n$$
$$a_i \in k \text{ not all zero}, \ t \in k$$

Now consider the line $L' = \varphi^{-1}(L - O)$ in $X - \varphi^{-1}O$. It is given parametrically by $x_i = a_i t$ $y_i = a_i t$ with $t \in \mathbb{A}^1 - O$. But the y_i are homogenous coordinates in $\mathbb{P}^{n-1}$, so we can equally well describe L' by the equations $x_i = a_i t$ $y_i = a_i$ for $t \in \mathbb{A}^1 - O$. Consider (assuming $a_k \neq 0$)

$$\begin{aligned} &y_i a_j - a_i y_j \qquad i,j = 1, \dots, n \\ &x_i a_j - a_i x_j \end{aligned} \qquad (2)$$

Let K denote the solutions in $\mathbb{A}^n \times \mathbb{P}^{n-1}$ to this family. Clearly $L' \subseteq K$, and $K \subseteq X$ since if $(b_1, \dots, b_n, c_1, \dots, c_n) \in K$

$$b_i c_j = \frac{1}{a_k} a_k b_i c_j = \frac{1}{a_k} b_k a_i c_j = \frac{1}{a_k} b_k c_i a_j$$
$$= \frac{1}{a_k} a_k b_j c_i = b_j c_i$$

Moreover, $K = L' \cup O \times (a_1, \dots, a_n)$ since (2) implies $x_i = a_i\, x_k / a_k$ and $y_i = a_i\, y_k / a_k$. Hence the closure of L' is $L' \cup O \times (a_1, \dots, a_n)$ in X.

NOTE

One may well ask: how do you know L' isn't already closed? Suppose $f(x_1, \dots, x_n, y_1, \dots, y_n)$ is zero on L' but not on $O \times (a_1, \dots, a_n)$. That is, $\forall t \neq 0$

$$f(a_1 t, \dots, a_n t, a_1, \dots, a_n) = 0$$

But then considering t as a variable, this is a polynomial in $k[t]$ with way too many roots. Hence as a polynomial in t, $f(t) = 0$. In particular $f(0, \dots, 0, a_1, \dots, a_n) = 0$.

Now $\overline{L'}$ meets $\varphi^{-1}O$ at $Q = (a_1, \dots, a_n) \in \mathbb{P}^{n-1}$, so we see that sending L to Q gives a 1-1 correspondence between lines through O in $\mathbb{A}^n$ and points of $\varphi^{-1}O$.

(4) X is irreducible. Indeed, X is the union of $X - \varphi^{-1}O$ and $\varphi^{-1}O$. The first piece is isomorphic to $\mathbb{A}^n - O$, hence irreducible. On the other hand, we have just seen that every point of $\varphi^{-1}O$ is in the closure of some subset (the line L') of $X - \varphi^{-1}O$. Hence $X - \varphi^{-1}O$ is dense in X, and X is irreducible.

i.e. Y = aff. variety

DEFINITION If Y is a closed subvariety of $\mathbb{A}^n$ passing through O, we define the blowing-up of Y at the point O to be $\tilde{Y} = (\varphi^{-1}(Y - O))^{-}$, where $^{-}$ denotes closure and $\varphi : X \longrightarrow \mathbb{A}^n$ is the blowing up of $\mathbb{A}^n$ at the point O described above. We denote also by $\varphi : \tilde{Y} \longrightarrow Y$ the morphism obtained by restricting $\varphi : X \longrightarrow \mathbb{A}^n$ to $\tilde{Y}$.

Note that $X \subseteq \mathbb{A}^n \times \mathbb{P}^{n-1}$ is closed and irreducible, hence is a subvariety of the quasi-projective variety $\mathbb{A}^n \times \mathbb{P}^{n-1}$. (see Ex. 3.10) It follows that X is also quasi-projective (as a subspace of $\mathbb{P}^N$, $N = n(n-1) + n + n - 1$). Also by Ex 3.10 the restriction to X of $\mathbb{A}^n \times \mathbb{P}^{n-1} \longrightarrow \mathbb{A}^n$ is a morphism of varieties.

Now let $P \in \mathbb{A}^n$ be any point $P = (c_1, \dots, c_n)$ and define the blowing up of $\mathbb{A}^n$ at P to be the closed subset of $\mathbb{A}^n \times \mathbb{P}^{n-1}$ defined by the equations $\{ (x_i - c_i) y_j = (x_j - c_j) y_i \mid i, j = 1, \dots, n \}$. Once again we have the continuous map

$$\varphi : X \longrightarrow \mathbb{A}^n$$

and

(1) If $Q \in \mathbb{A}^n$, $Q \neq P$ then $\varphi^{-1}Q$ consists of the single point (if $Q = (a_1, \dots, a_n)$)

$$\varphi^{-1}Q = \{ (a_1, \dots, a_n) \times (a_1 - c_1, \dots, a_n - c_n) \}$$

The map $\psi : \mathbb{A}^n - P \longrightarrow X - \varphi^{-1}P$ defined by $(a_1, \dots, a_n) \longmapsto (a_1, \dots, a_n) \times (a_1 - c_1, \dots, a_n - c_n)$ is again continuous and so $\mathbb{A}^n - P$ is homeomorphic to $X - \varphi^{-1}P$.

(2) Clearly $\varphi^{-1}P \cong \mathbb{P}^{n-1}$, this time the elements are $P \times Q$ for any $Q \in \mathbb{P}^{n-1}$. So $E = \varphi^{-1}P$ is a closed, irreducible subset of X.

(3) A line L through P in $\mathbb{A}^n$ can be given by parametric equations

$$x_i = a_i t + c_i \qquad i = 1, \dots, n \quad t \in \mathbb{A}^1$$

where not all the a_i are zero. Now consider $L' = \varphi^{-1}(L - P)$ in $X - \varphi^{-1}P$. It is given parametrically by

$$x_i = a_i t + c_i \qquad y_i = a_i \qquad t \neq 0$$

Now consider the polynomials

$$\begin{matrix} y_i a_j - a_i y_j \\ (x_i - c_i) a_j - (x_j - c_j) a_i \end{matrix} \qquad i, j = 1, \dots, n$$

Let $K \subseteq \mathbb{A}^n \times \mathbb{P}^n$ be the closed subset carved out by these equations. Again it is checked that $K \subseteq X$, $L' \subseteq K$ and finally that $K = \overline{L'}$, and

$$K = L' \cup P \times (a_1, \dots, a_n)$$

So sending L to the point $P \times (a_1, \dots, a_n)$ gives a 1-1 correspondence between lines through P and points of $\varphi^{-1}P$.

(4) It is now easily seen that X is irreducible.

Hence X becomes a quasi-projective variety and $\varphi: X \longrightarrow \mathbb{A}^n$ a morphism. The topology on X is easily derived from the topology of $\mathbb{A}^n \times \mathbb{P}^{n-1}$. We now want the variety structure. First, for $\mathbb{A}^n \times \mathbb{P}^{n-1}$

$\boxed{\mathbb{A}^n \times \mathbb{P}^{n-1}}$ Let $S \subseteq \mathbb{A}^n \times \mathbb{P}^{n-1}$ a map $f: S \longrightarrow k$ is regular iff. $\forall s \in S$ there is $U \subseteq S$ open s.t. $x \in U$, and $g, h \in k[x_0, x_1, \dots, x_n, y_1, \dots, y_n]$ seperately homogenous in the x_i and ys (both g, h having the same x-deg, y-deg) s.t. $h \neq 0$ on U and for all $(a_1, \dots, a_n, b_1, \dots, b_n) \in U$

$$f(a_1, \dots, a_n, b_1, \dots, b_n) = \frac{g(1, a_1, \dots, a_n, b_1, \dots, b_n)}{h(1, a_1, \dots, a_n, b_1, \dots, b_n)}$$

This is clearly equivalent to there being $g, h \in k[x_1, \dots, x_n, y_1, \dots, y_n]$ homogenous in the ys of the same degree s.t. $h \neq 0$ on U and $f(u) = g(u)/h(u)$ $\forall u \in U$.

Hence we can show that $X - \varphi^{-1}P \cong \mathbb{A}^n - P$ as varieties: we know φ restricts to a morphism $X - \varphi^{-1}P \longrightarrow \mathbb{A}^n - P$. It only remains to show that the inverse $\psi: \mathbb{A}^n - P \longrightarrow X - \varphi^{-1}P$ is a morphism. Let $U \subseteq X - \varphi^{-1}P$ be open and $f: U \longrightarrow k$ regular. Then we find for $x \in \psi^{-1}U$ an open neighborhood V of $\psi(x)$ and $g, h \in k[x_1, \dots, x_n, y_1, \dots, y_n]$ homogenous in the y s.t. $h \neq 0$ on V and $f(v) = g(v)/h(v)$ $\forall v \in V$. For any $(a_1, \dots, a_n) \in \psi^{-1}V$ we have

$$\begin{aligned} f(\psi(a_1, \dots, a_n)) &= f(a_1, \dots, a_n, a_1 - c_1, \dots, a_n - c_n) \\ &= \frac{g(a_1, \dots, a_n, a_1 - c_1, \dots, a_n - c_n)}{h(a_1, \dots, a_n, a_1 - c_1, \dots, a_n - c_n)} \\ &= \frac{K(g)(a_1, \dots, a_n)}{K(h)(a_1, \dots, a_n)} \end{aligned}$$

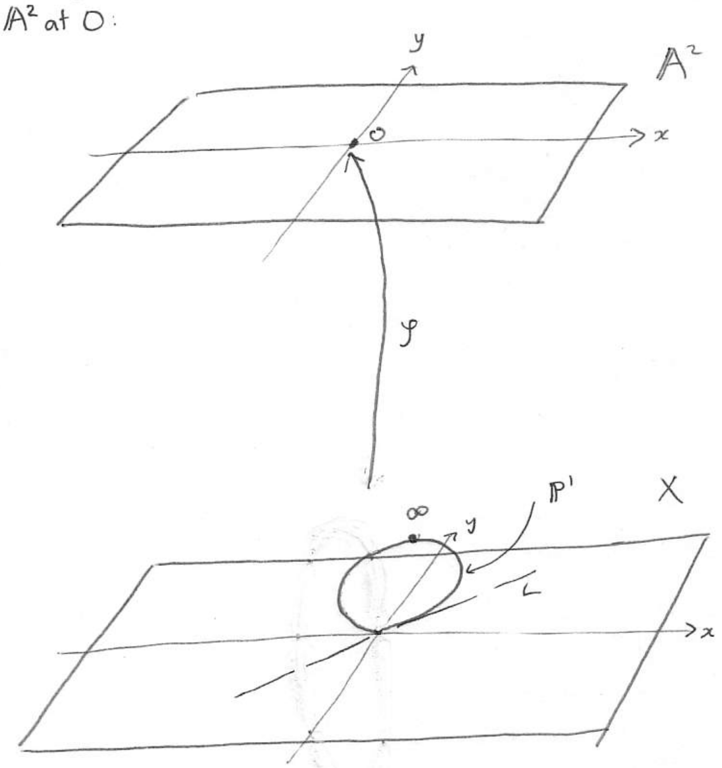
Where $K: k[x_1, \dots, x_n, y_1, \dots, y_n] \longrightarrow k[x_1, \dots, x_n]$ is $x_i \mapsto x_i$ $y_i \mapsto x_i - c_i$. Clearly $K(h) \neq 0$ on $\psi^{-1}V$, so $f\psi$ is regular, as required. Hence $X - \varphi^{-1}P \cong \mathbb{A}^n - P$.

Now let $Y \subseteq \mathbb{A}^n$ be closed and irreducible. Then if $P \in Y$, $Y - P \subseteq Y$ is an open subset, which is thus irreducible. Let $\varphi: X \longrightarrow \mathbb{A}^n$ be the blowup of $\mathbb{A}^n$ at P. Then $X - \varphi^{-1}P \cong \mathbb{A}^n - P$, so $\varphi^{-1}(Y - P)$ is an irreducible subset of X. Hence the closure $\varphi^{-1}(Y-P)^-$ is a closed, irreducible subset of X. Hence $\tilde{Y}$ is an irreducible locally closed subset of X, a quasi-projective variety. Hence $\tilde{Y}$ becomes a quasi-projective variety. Note that $\varphi^{-1}Y$ is closed and contains $\varphi^{-1}(Y-P)$. So at most $\varphi^{-1}(Y-P)^-$ adds points of $\mathbb{P}^{n-1}$ to $\varphi^{-1}(Y-P)$. Hence $\varphi(\tilde{Y}) \subseteq Y$. Thus φ restricts to a morphism

$$\varphi: \tilde{Y} \longrightarrow Y$$

The isomorphism $X - \varphi^{-1}P \cong \mathbb{A}^n - P$ restricts to an isomorphism of $\tilde{Y} - \varphi^{-1}P$ with $Y - P$. That is, φ is a birational morphism of $\tilde{Y}$ to Y. (A birational <u>morphism</u> is a birational map defined on the whole variety). Note that this definition apparently depends on the embedding of Y in $\mathbb{A}^n$, but in fact, we will see later that blowing-up is intrinsic (II, 7.5.1).

The effect of blowing up a point of Y is to "pull apart" Y near O according to the different directions of lines through O.
We blow up $\mathbb{A}^2$ at O:



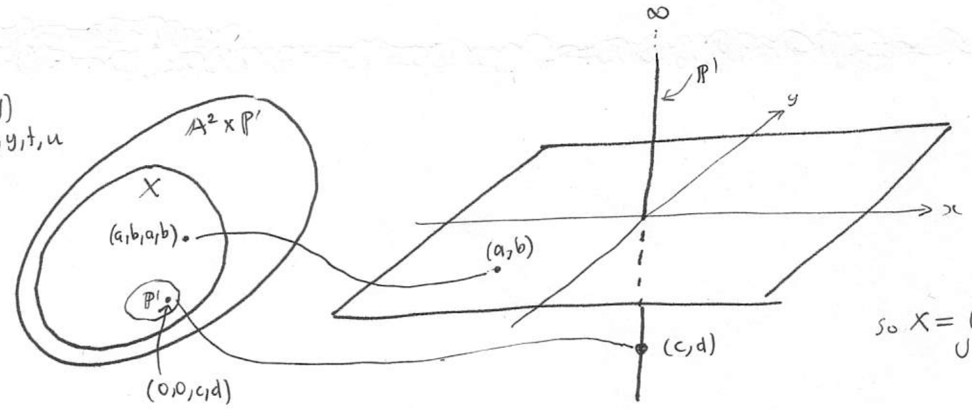
⌜NOTE: The topology of X is defined by: ϕ, X open and $Z \subseteq X$ closed iff. Z is the intersection of sets $Z(f)$ where $f(x,y,t,u)$ is homogeneous in t,u (e.g. $xu+yt$) and

$$Z(f) = \{(a,b) \in X - \mathbb{P}^1 \mid f(a,b,a,b)=0\} \cup \{(c,d) \in \mathbb{P}^1 \mid f(0,0,c,d)=0\}$$

This is derived from the topology on $\mathbb{A}^n \times \mathbb{P}^{n-1}$ ⌟

We attach an infinite loop $\mathbb{P}^1$ at O. A line L passing through O has $O \in L$ replaced by its slope. So the x-axis is unchanged, y-axis becomes $(y\text{-axis} - O) \cup \infty$, etc. The above identification works as:

$X = Z(xu - ty)$
with coordinates x, y, t, u



(1)

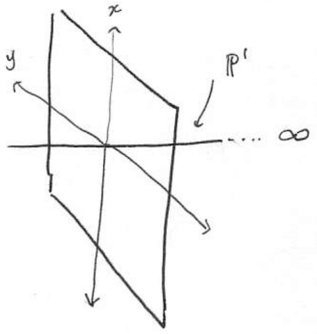
So $X = (\mathbb{A}^2 - O) \cup \mathbb{P}^1$.

There are some other useful pictures we can draw, but first consider the cover $\mathbb{A}^2 \times \mathbb{P}^1 = U_0 \cup U_1$ where $U_i = \mathbb{A}^2 \times \mathbb{P}^1 - Z(y_i)$. By earlier notes (in section 3 exercises) each U_i is isomorphic to $\mathbb{A}^3$, via

$$\varphi : \mathbb{A}^3 \longrightarrow U_0$$
$$(a,b,c) \longmapsto (a,b,1,c)$$
$$\phi : U_0 \longrightarrow \mathbb{A}^3$$
$$(a,b,c,d) \longmapsto (a,b,d/c)$$

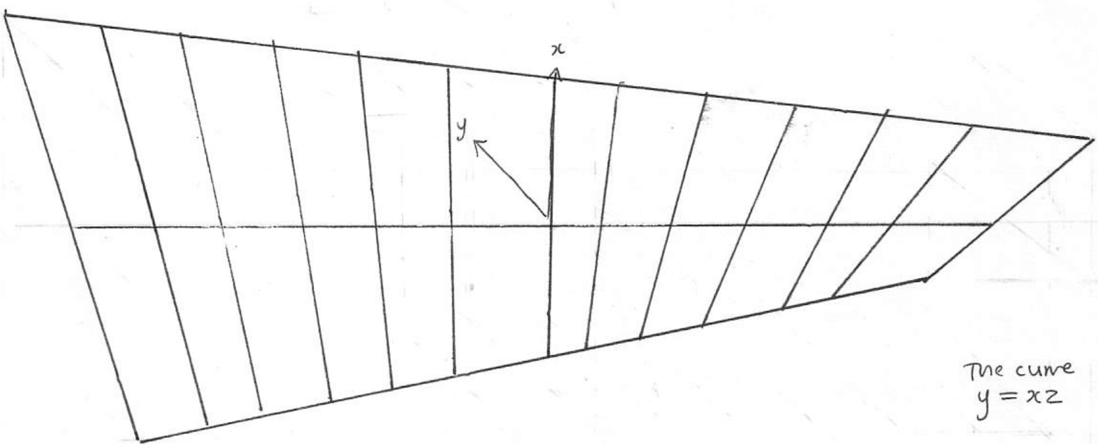
Now under this isomorphic the closed irreducible subset $X \cap U_0 \subseteq U_0$ corresponds to an affine variety in $\mathbb{A}^3$, consisting of points (a,b,c) with $b = ac$. That is, $X \cap U_0$ is the affine variety $Z(y - xz) \subseteq \mathbb{A}^3$. The only points of X not in U_0 are those on the y-axis (i.e. $(0,b) \sim (0,b,0,b)$) and the point $(0,1) \in \mathbb{P}^1$ (i.e. ∞).

Visualising X as the plane $\cup\ \mathbb{P}^1$ as in (1), the isomorphism $\varphi : Z(y-xz) \longrightarrow X \cap U_0$ is $(a, ac, c) \longmapsto (a, ac, 1, c)$
(i.e. the picture on p29 of Hartshorne)



Or graph $x(u,v) = u$, $y(u,v) = uv$, $z(u,v) = v$ with u range $-0.5, 0.5$ and v range $-4, 4$.

We divide the points of $Z(y-xz)$ into two groups: those of the form $(0,0,c)$ (i.e. the z-axis) and $(a, b, b/a)$ with $a \neq 0$ (i.e. those points not lying over the y-axis). The former points are mapped to points $(1,c)$ of $\mathbb{P}^1$ whereas the latter cover $\mathbb{A}^2$ minus the y-axis.



The curve
$y = xz$

Similarly $X \cap U_1$ is isomorphic to $Z(x-yz)$ via $(ab, a, b) \mapsto (ab, a, b, 1)$, or looking at $X \cap U_1$ as $\mathbb{A}^2 \cup \mathbb{P}^1$ minus the x-axis and $\infty = (1,0) \in \mathbb{P}^1$, the z-axis corresponds to the points of $\mathbb{P}^1$ other than $(1,0)$, and the points of $\mathbb{A}^2$ off the x-axis are in bijection with all other points of $Z(x-yz)$.

Consider the curve $y^2 = x^2(x+1) \subseteq \mathbb{A}^2$, say $Y = Z(y^2 - x^2(x+1))$. The closed set $\varphi^{-1}Y \subseteq X$ is precisely $Y \cup \mathbb{P}^1$, so $\varphi^{-1}Y = Z(xu - ty) \cap Z(y^2 - x^2(x+1)) \subseteq \mathbb{A}^2 \times \mathbb{P}^1$. Note that

$$\varphi^{-1}Y \cap U_0 = \{(a,b,a,b) \mid a \neq 0 \text{ and } b^2 = a^2(a+1)\}$$
$$\cup \{(0,0,c,d) \mid c \neq 0\} = \varphi^{-1}Y - (0,0,0,1).$$

Under the isomorphism $\mathbb{A}^3 \longrightarrow U_0$, $\varphi^{-1}Y - (0,0,0,1)$ corresponds to

$$\{(a,b,c) \mid b = ac \text{ and } b^2 = a^2(a+1)\}$$
$$= z\text{-axis} \cup \{(a,b,c) \mid b = ac \text{ and } c^2 = a+1\}$$
$$= Z(x,y) \cup Z(x - yz) \cap Z(z^2 - (x+1)) \qquad (2)$$

If we visualise $Z(x-yz)$ lying over $X - y\text{-axis} \cup \infty$ as in the above pictures, $\varphi^{-1}Y - (0,0,0,1)$ lifts to the curve depicted on p29 of Hartshorne, since $\varphi^{-1}Y - (0,0,0,1)$ corresponds to z-axis $\cup\ Q$ where $Q = Z(x-yz) \cap Z(z^2 - (x+1))$.

The above motivates the following trick: let $Y = Z(y^2 - x^2(x+1)) \subseteq \mathbb{A}^2$. The blowup $\tilde{Y}$ of Y at O is $\varphi^{-1}(Y-O)^-$. The closed set $Z(y^2 - x^2(x+1)) \subseteq \mathbb{A}^2 + \mathbb{P}^1$ intersects with X to give $\varphi^{-1}(Y-O) \cup \mathbb{P}^1$, so $\tilde{Y} \subseteq \varphi^{-1}(Y-O) \cup \mathbb{P}^1$. Here is the trick: on $X - \mathbb{P}^1$ which is in bijection with $\mathbb{A}^2 - O$ via $(a,b,a,b) \leftrightarrow (a,b)$ $y^2 - x^2(x+1)$ has the same solutions as the homogenous polynomial $u^2 - t^2(x+1)$. But

$$Z(u^2 - t^2(x+1)) \cap X = \varphi^{-1}(Y-O) \cup \{(1,-1),(1,1)\}$$

Note that $\varphi^{-1}Y = \mathbb{P}^1 \cup \tilde{Y}$ is the decomposition of $\varphi^{-1}Y$ into its irreducible components, where $\tilde{Y} = \varphi^{-1}(Y-O)^-$. Hence, identifying U_0 and $\mathbb{A}^3$, $\varphi^{-1}Y \cap U_0 = \mathbb{P}^1 \cap U_0 \cup \tilde{Y} \cap U_0$ is the decomposition of the closed set $\varphi^{-1}Y \cap U_0 \subseteq \mathbb{A}^3$ into its irreducible components. By uniqueness and (2), which implies

$$\begin{aligned} \varphi^{-1}Y \cap U_0 &= Z(x,y) \cup Z(x-yz) \cap Z(z^2-(x+1)) \\ &= \mathbb{P}^1 \cap U_0 \cup \quad \text{"} \end{aligned}$$

We must have $\tilde{Y} \cap U_0 = Z(x-yz) \cap Z(z^2-(x+1))$, which implies $\tilde{Y} = \varphi^{-1}(Y-O) \cup \{(1,-1),(1,1)\}$. So we adjoin to Y the slopes of lines through O which are tangents — i.e. seperate out the singular point O according to the tangent slopes.

EXERCISES I.4

Q4.1 Let f,g be regular functions on open subsets of a variety X, $f: U \to k$ $g: V \to k$ and suppose $f = g$ on $U \cap V$. We claim the function $F: U \cup V \to k$ defined to be f on U and g on V is regular. But this is obvious since we can paste reg. functions. Now let f be a <u>rational</u> function on X. That is, f is an equivalence class $\{(U_i, f_i)\}_{i \in I}$ of regular maps. Let $W = \bigcup_i U_i$. By def^N for $i,j \in I$ $f_i = f_j$ on $U_i \cap U_j$. By def^N of sheaf there is a regular map $F: W \to k$ s.t. $F|_{U_i} = f_i$. Then (W, F) belongs to f, and W is clearly the largest open set on which f is represented. We say f is <u>defined</u> on W.

Q4.2 Now let $\varphi: X \to Y$ be a rational map, given by an equivalence class $\{(U_i, \varphi_i)\}_{i \in I}$ $\varphi_i: U_i \to Y$, where $\varphi_i = \varphi_j$ on $U_i \cap U_j$. The union $W = \bigcup_i U_i$ is open in X and $\phi: W \to Y$ defined by $\phi|_{U_i} = \varphi_i$ is a well-defined morphism.

Q4.3 (a) Let f be the rational function on $\mathbb{P}^2$ given by $f = x_1/x_0$. Clearly if $U_0 = \mathbb{P}^2 - V(x_0)$ then f is defined on U_0 by $(a,b,c) \mapsto b/a$

(b) Think of this as a rational map $\mathbb{P}^2 \to \mathbb{A}^1$ and compose with $\mathbb{A}^1 = U_0 \hookrightarrow \mathbb{P}^1$. This map is a rational map $\mathbb{P}^2 \to \mathbb{P}^1$ defined on U_0 by

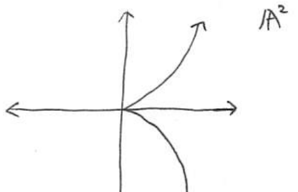
$$\varphi: (a,b,c) \mapsto (1, \tfrac{b}{a})$$

Let φ' be defined on $\mathbb{P}^2 - (0,0,1)$ by $(a,b,c) \mapsto (a,b)$. This is easily checked to be a rational map $\varphi': \mathbb{P}^2 \to \mathbb{P}^1$. Moreover $\varphi'|_{U_0} = \varphi$. Hence φ is defined on $\mathbb{P}^2 - (0,0,1)$.

Q4.4 A variety Y is <u>rational</u> if it is birationally equivalent to $\mathbb{P}^n$ for some n. Since $K(\mathbb{P}^n) = k(x_0, \dots, x_n)$ by (4.5) Y is rational iff. $K(Y) \cong k(x_0, \dots, x_n)$ as k-algebras for some n, which is iff. $K(Y)$ is a pure transcendental extension of k (since $K(Y)/k$ is f.g. $\therefore$ any transcendence basis is finite).

(a) By Ex 3.1 c) any conic in $\mathbb{P}^2$ is isomorphic to $\mathbb{P}^1$, hence rational.

(b) Note that $\mathbb{A}^n$ is birational to $\mathbb{P}^n$, so any variety birational to $\mathbb{A}^n$ is also rational. Consider the cuspidal cubic $y^2 = x^3$:



The usual test (see end §1) shows $y^2 - x^3$ is irreducible, so $Y = Z(y^2 - x^3)$ is an affine variety. Define $\phi: \mathbb{A}^1 \to Y$ by $\phi(t) = (t^2, t^3)$. We saw in Ex 3.2 that this is a bijective morphism (but not an isomorphism). We already checked in Ex 3.2 that Y is birational to $\mathbb{A}^1$ and hence rational.

(c) Let Y be the nodal cubic curve $y^2 z - x^2(x+z)$ in $\mathbb{P}^2$ (coordinates x,y,z). Let $P = (0,0,1)$ and $H = Z(z)$ be a hyperplane. Let $\varphi: \mathbb{P}^2 - P \to \mathbb{P}^1$ be the projection of Ex 3.14. The map is defined by $\varphi(a,b,c) = (a,b)$. This restricts to a morphism $\varphi': U \to \mathbb{P}^1$ where U is the open subset $Y - P$ of Y. Define $\psi: \mathbb{P}^1 - \{(1,1),(1,-1)\} \to$ by

$$\psi(a,b) = ((b^2 - a^2)a,\ (b^2 - a^2)b,\ a^3)$$

The same formula defines a map $\mathbb{P}^1 \to \mathbb{P}^2$ which is a morphism by earlier Notes (§ Note: Morphisms $\mathbb{P}^n \to \mathbb{P}^m$). Hence ψ is a morphism. So $(U, \varphi'): Y \to \mathbb{P}^1$ and $(\mathbb{P}^1 - \{(1,1),(1,-1)\}, \psi): \mathbb{P}^1 \to Y$ are rational maps, and they are clearly inverse to each other. Hence φ gives a birational map $Y \to \mathbb{P}^1$ and Y is rational.

$\boxed{Q4.5}$ Let Q be the quadric surface $Q: xy = zw$ in $\mathbb{P}^3$ (coordinates w, x, y, z). By Ex 2.15 Q is the image of the segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^3$. By definition then Q is the product variety $\mathbb{P}^1 \times \mathbb{P}^1$ (Ex 3.16), and we have shown that the product $\mathbb{P}^1 \times \mathbb{P}^1$ is covered by open subsets isomorphic to $\mathbb{A}^2$. Hence Q is birational to $\mathbb{A}^2$ which is birational to $\mathbb{P}^2$. But by Ex 2.15 Q contains curves which do not intersect: hence Q cannot be isomorphic to $\mathbb{P}^2$ by Ex 3.7a).

$\boxed{Q4.6}$ Plane Cremona Transformations A birational map $\mathbb{P}^2 \longrightarrow \mathbb{P}^2$ is called a plane Cremona transformation. We give an example, called a quadratic transformation.

Q4.7 Let X and Y be two varieties and suppose $P\in X$ and $Q\in Y$ are such that the local rings $\mathcal{O}_{P,X}$ and $\mathcal{O}_{Q,Y}$ are isomorphic as k-algebras. Since $\dim X = \dim \mathcal{O}_{P,X} = \dim \mathcal{O}_{P,Y} = \dim Y$, by Prop 4.3 there are open subsets $P\in U\subseteq X$ and $Q\in V\subseteq Y$ with U, V isomorphic to affine varieties so we may reduce to the case of $X\subseteq \mathbb{A}^n$ and $Y\subseteq \mathbb{A}^m$ affine.

Let $\phi : \mathcal{O}_{P,X} \longrightarrow \mathcal{O}_{Q,Y}$ and $\psi : \mathcal{O}_{Q,Y} \longrightarrow \mathcal{O}_{P,X}$ form an isomorphism of k-algebras. We write the coordinate rings of and respectively as $k[x_1,\dots,x_n]$ and $k[y_1,\dots,y_m]$, and let $f_i, g_i \in k[y_1,\dots,y_m]$ and $t_j, s_j \in k[x_1,\dots,x_n]$ be such that

$$\phi(X, x_i) = (W_i, f_i/g_i) \qquad Q\in W_i \subseteq Y$$
$$\psi(Y, y_i) = (M_i, t_i/s_i) \qquad P\in M_i \subseteq X$$

Notice that since ϕ,ψ are morphisms of k-algebras, if $F(x_1,\dots,x_n)\in k[x_1,\dots,x_n]$ then

$$\begin{aligned}\phi(X, F(x_1,\dots,x_n)) &= \phi(F((X,x_1),\dots,(X,x_n)))\\ &= F(\phi(X,x_1),\dots,\phi(X,x_n))\\ &= F((W_1, f_1/g_1),\dots,(W_n, f_n/g_n))\end{aligned}$$

And similarly for ψ. The fact that $\psi\phi = 1$, $\phi\psi = 1$ means that

$$\begin{aligned}(X,x_i) = \psi\phi(X,x_i) &= \psi(W_i, f_i/g_i)\\ &= \psi(Y, f_i(y_1,\dots,y_m))\,\psi((Y, g_i(y_1,\dots,y_m))^{-1})\\ &= f_i(\psi(Y,y_1),\dots,\psi(Y,y_m))\; g_i(\psi(Y,y_1),\dots,\psi(Y,y_m))^{-1}\end{aligned}$$

So for $1\le i\le n$ there is an open set $P_i \subseteq X$ s.t. $\forall (a_1,\dots,a_n)\in P_i$

$$a_i = \frac{f_i(\psi(Y,y_1)(a_1,\dots,a_n),\dots,\psi(Y,y_m)(a_1,\dots,a_n))}{g_i(\psi(Y,y_1)(a_1,\dots,a_n),\dots,\psi(Y,y_m)(a_1,\dots,a_n))} \qquad (1)$$

similarly for $1\le j\le m$ there is $Q_j\subseteq Y$ s.t. $\forall (b_1,\dots,b_m)\in Q_j$

$$b_j = \frac{t_j(\phi(X,x_1)(b_1,\dots,b_m),\dots,\phi(X,x_n)(b_1,\dots,b_m))}{s_j(\phi(X,x_1)(b_1,\dots,b_m),\dots,\phi(X,x_n)(b_1,\dots,b_m))} \qquad (2)$$

Let $M\subseteq X$ be the open set $M_1\cap\dots\cap M_m\cap P_1\cap\dots\cap P_n$ and let $W\subseteq Y$ be $W_1\cap\dots\cap W_n\cap Q_1\cap\dots\cap Q_m$, and define $f: M\longrightarrow \mathbb{A}^m$ and $g: W\longrightarrow \mathbb{A}^n$ by

[Nonempty since X, Y irred.]

$$f(a_1,\dots,a_n) = (\psi(Y,y_1)(a_1,\dots,a_n),\dots,\psi(Y,y_m)(a_1,\dots,a_n))$$
$$g(b_1,\dots,b_m) = (\phi(X,x_1)(b_1,\dots,b_m),\dots,\phi(X,x_n)(b_1,\dots,b_m))$$

By Lemma 3.6 both f and g are morphisms. Since ϕ and ψ are isomorphisms, ϕ identifies the nonunits in $\mathcal{O}_{P,X}$ with the nonunits in $\mathcal{O}_{Q,Y}$, hence identifies the maximal ideals (which consist of regular maps with $f(P)=0$ resp. $g(Q)=0$). Since $(Y, y_i - Q_i)\in \mathfrak{m}_Q$ we have

$$(M_i, t_i/s_i - Q_i) = \psi(Y, y_i - Q_i)\in \mathfrak{m}_P$$
$$\therefore\ t_i(P)/s_i(P) = Q_i.$$

It follows that $f(P)=Q$ and similarly $g(Q)=P$. Hence $M' = f^{-1}W$ and $W' = g^{-1}M$ are nonempty open neighborhoods of P, Q resp. Equations (1), (2) imply that $gf = 1$ on M' and $fg = 1$ on W'. In particular $f(M')\subseteq W'$ and $g(W')\subseteq M'$. Restricting gives morphisms $f: M'\longrightarrow W'$ and $g: W'\longrightarrow M'$ with $f(P)=Q$ and $g(Q)=P$, $fg=1$ and $gf=1$, as required.