5. NonSingular Varieties

The notion of nonsingular variety in algebraic geometry corresponds to the notion of manifold in topology. Over $\mathbb{C}$ for example, the nonsingular varieties are those which in the "usual" topology are complex manifolds. Accordingly, the most natural (and historically first) definition of nonsingularity uses the derivatives of the function defining the variety:

<u>DEFINITION</u> Let $Y \subseteq \mathbb{A}^n$ be an affine variety and let $f_1, \ldots, f_t \in A = k[x_1, \ldots, x_n]$ be a set of generators for the ideal of Y. Y is <u>nonsingular at a point</u> $P \in Y$ if the rank of the matrix

$$\| (\partial f_i / \partial x_j)(P) \|$$

is $n - r$, where r is the dimension of Y. Y is <u>nonsingular</u> if it is nonsingular at every point.
(By Cor 2.29 with $R = k$ rank = col-rank = row-rank = dimension image of $((\partial f_i/\partial x_j)(P))$) ⌜see Note overleaf for why $t \geq n-r$⌟

A few comments are in order. In the first place, the notion of partial derivative with respect to one of its variables makes sense over any field. One just applies the usual rules for differentiation. Thus no limiting process is needed. But funny things can happen in characteristic $p > 0$. For example if $f(x) = x^p$ then $df/dx = px^{p-1} = 0$ since $p = 0$ in k. In any case, if $f \in A$ is a polynomial, then for each i $\partial f / \partial x_i$ is a polynomial.

The matrix $\| (\partial f_i / \partial x_j)(P) \|$ is called the <u>Jacobian matrix</u> at P. One can show easily that this definition of nonsingularity is independent of the set of generators of Y chosen. (this follows from 5.1). One drawback of our definition is that apparently depends on the embedding of Y in affine space. However, it was shown by Zariski that nonsingularity could be described intrinsically in terms of the local rings. In our case the result is this:

<u>DEFINITION</u> Let A be a noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field $k = A/\mathfrak{m}$. A is a <u>regular local ring</u> if $\dim_k \mathfrak{m}/\mathfrak{m}^2 = \dim A$.

<u>THEOREM 5.1</u> Let $Y \subseteq \mathbb{A}^n$ be an affine variety. Let $P \in Y$ be a point. Then Y is nonsingular at P if and only if the local ring $\mathcal{O}_{P,Y}$ is a regular local ring.

<u>PROOF</u> We know from Theorem 3.4 (b) that $\mathcal{O}_{P,Y}$ is a Noetherian local ring of dimension $r = \dim Y$. Let P be the point $(a_1, \ldots, a_n)$ in $\mathbb{A}^n$ and let $\mathfrak{a}_P = (x_1 - a_1, \ldots, x_n - a_n)$. be the corresponding maximal ideal in $A = k[x_1, \ldots, x_n]$. We define a linear map $\theta : A \longrightarrow k^n$ by

$$\theta(f) = \left(\frac{\partial f}{\partial x_1}(P), \ldots, \frac{\partial f}{\partial x_n}(P) \right) \qquad (1)$$

It is clear that $\theta(x_i - a_i)$ for $i = 1, \ldots, n$ forms a basis of k^n, and that $\mathfrak{a}_P \cap \operatorname{Ker} \theta = \mathfrak{a}_P^2$ (see the Note overleaf) so θ induces an isomorphism of vector spaces $\theta' : \mathfrak{a}_P / \mathfrak{a}_P^2 \longrightarrow k^n$

Now let $\mathfrak{b}$ be the ideal of Y in A, and let $f_1, \ldots, f_t$ be a set of generators of $\mathfrak{b}$. The <u>rank</u> of the Jacobian J matrix $(\partial f_i / \partial x_j (P))$ is the maximum number of linearly independent rows, which is the dimension of the subspace of k^n generated by $\theta(f_1), \ldots, \theta(f_t)$. Since $\mathfrak{b} \subseteq \mathfrak{a}_P$ (by assumption $P \in Y$) if $f \in \mathfrak{b}$ then

$$\begin{aligned}
\theta(f) &= \theta(p_1 f_1 + \cdots + p_t f_t) \\
&= \textstyle\sum_i \theta(p_i f_i) \\
&= \textstyle\sum_i \left(\frac{\partial p_i}{\partial x_1}(P) f_i(P) + p_i(P) \frac{\partial f_i}{\partial x_1}(P), \ldots, \frac{\partial p_i}{\partial x_n}(P) f_i(P) + p_i(P) \frac{\partial f_i}{\partial x_n}(P) \right) \\
f_i(P) = 0 \ \therefore \quad &= \textstyle\sum_i p_i(P)\, \theta(f_i)
\end{aligned}$$

Clearly $\mathfrak{b}$ is a k-submodule of $\mathfrak{a}_P$, and we have just shown $\theta(\mathfrak{b})$ has dimension rank J. But

$$\theta(\mathfrak{b}) \cong \mathfrak{b} / \mathfrak{b} \cap \mathfrak{a}_P^2 \cong \mathfrak{b} + \mathfrak{a}_P^2 / \mathfrak{a}_P^2$$

On the other hand, the local ring $\mathcal{O}_P$ of P on Y is obtained from A by dividing by $\mathfrak{b}$ and localising at the maximal ideal $\mathfrak{a}_P$. If $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}_P$ then $\mathcal{O}_P/\mathfrak{m}$ is isomorphic as a ring to the quotient of $A/\mathfrak{b}$ by $\mathfrak{a}_P/\mathfrak{b}$, which is k (By the Note). Again by the note, $\mathfrak{m}/\mathfrak{m}^2$ is isomorphic as a k-vector space to the quotient $(\mathfrak{a}_P/\mathfrak{b})/(\mathfrak{a}_P/\mathfrak{b})^2$ which is $\dfrac{\mathfrak{a}_P/\mathfrak{b}}{\mathfrak{a}_P^2+\mathfrak{b}/\mathfrak{b}} \cong \mathfrak{a}_P/\mathfrak{a}_P^2+\mathfrak{b}$. So finally we have an isomorphism of k-modules:

$$\mathfrak{m}/\mathfrak{m}^2 \cong \mathfrak{a}_P/\mathfrak{a}_P^2+\mathfrak{b}$$

But $\dfrac{\mathfrak{a}_P/\mathfrak{a}_P^2}{\mathfrak{b}+\mathfrak{a}_P^2/\mathfrak{a}_P^2} \cong \mathfrak{a}_P/\mathfrak{a}_P^2+\mathfrak{b}$ so counting dimensions we have

$$\dim_k \mathfrak{m}/\mathfrak{m}^2 = n - \operatorname{rank} J$$

Now $\mathcal{O}_P$ is regular if and only if $\dim_k \mathfrak{m}/\mathfrak{m}^2 = r$, which is iff. $\operatorname{rank} J = n-r$, which says that P is a nonsingular point of Y. $\square$ (This even works when $r=n$ (i.e. $Y=\mathbb{A}^n$) or $r=0$ (Y a point))

Now that we know that the concept of nonsingularity is intrinsic, we can extend the definition:

<u>DEFINITION</u> Let Y be any variety. Y is <u>nonsingular</u> at a point $P\in Y$ if the local ring $\mathcal{O}_{P,Y}$ is a regular local ring. Y is <u>nonsingular</u> if it is nonsingular at every point. Y is <u>singular</u> if it is not nonsingular.
(By an earlier Note, $\dim \mathcal{O}_{P,Y} = \dim Y$ for any variety Y)

Our next objective is to show that most points of a variety are nonsingular. We need an algebraic preliminary.

<u>PROPOSITION 5.2A</u> If A is a Noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field k, then $\dim_k \mathfrak{m}/\mathfrak{m}^2 \geq \dim A$.

<u>PROOF</u> Atiyah-MacDonald [1, Cor 11.15 p121]. $\square$

<u>THEOREM 5.3</u> Let Y be a variety. Then the set $\operatorname{Sing} Y$ of singular points of Y is a proper closed subset of Y.

<u>PROOF</u> (See also II, 8.16). First we show $\operatorname{Sing} Y$ is a closed subset. It is sufficient to show for some open covering $Y = \bigcup Y_i$ of Y that $\operatorname{Sing} Y_i$ is closed for each i. Hence by (4.3) we may assume that Y is affine. By (5.2) and the proof of (5.1) we know that the rank of the Jacobian matrix is always $\leq n-r$. Hence the set of singular points is the set of all points of Y where the rank is $< n-r$. Let J' be the matrix

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_t}{\partial x_1} & \cdots & \frac{\partial f_t}{\partial x_n} \end{pmatrix} \in M_{t,n}(k[x_1,\dots,x_n])$$

Assume $Y \subseteq \mathbb{A}^n$ has dimension r

So that a point P of Y is nonsingular iff. $\operatorname{rank} J'(P) < n-r$. But by our "Reminder: Linear Algebra" Note this iff. every $(n-r)\times(n-r)$ submatrix of $\operatorname{rank} J'(P)$ has zero determinant. To take this determinant we can calculate the determinant of the corresponding submatrix of J' and then evaluate at P. If there are N $(n-r)\times(n-r)$ submatrices with determinants $F_1,\dots,F_N \in k[x_1,\dots,x_n]$ then

$$\operatorname{Sing} Y = Y \cap Z(F_1) \cap \cdots \cap Z(F_N) \subseteq \mathbb{A}^n$$

So $\operatorname{Sing} Y$ is closed.

To show that $\operatorname{Sing} Y$ is proper, we first apply (4.9) to get Y birational to a hypersurface Z in $\mathbb{P}^n$. That is, there are open nonempty subsets $U \subseteq Y$ and $V \subseteq Z$ which are isomorphic as varieties. Since $\operatorname{Sing} Y$ is closed and U is dense, $\operatorname{Sing} Y$ is proper iff. $\operatorname{Sing} Y \cap U$ is a proper subset of U, which is iff. $\operatorname{Sing} Z \cap V$ is a proper subset of V, which is iff. $\operatorname{Sing} Z$ is proper. So we reduce to the case of a hypersurface. If Y is a hypersurface in $\mathbb{P}^n$ then $\dim Y = n-1$ and $\operatorname{Sing} Y = \bigcup \operatorname{Sing} Y \cap Y_i$ (where $Y_i = Y \cap U_i$, U_i the canonical cover $\mathbb{P}^n$ with affines) $\operatorname{Sing} Y$ is proper iff. for $Y_i \neq \emptyset$ $\operatorname{Sing} Y \cap Y_i \subset Y_i$. But by Ex 2.6 $\dim Y_i = n-1$ so Y_i is a hypersurface in $\mathbb{A}^n$. So we reduce to the case of an affine hypersurface $Y = Z(f)$ for a single irreducible polynomial f.

Now $\operatorname{Sing} Y$ is the set of points $P\in Y$ such that $(\partial f/\partial x_i)(P) = 0$ for $i=1,\dots,n$. If $\operatorname{Sing} Y = Y$ (since $\dim Y = n-1$) then the polynomials $\partial f/\partial x_i$ are zero on Y, and hence $\partial f/\partial x_i \in I(Y)$ for each i. But $I(Y) = (f)$, and $\deg(\partial f/\partial x_i) \leq \deg f - 1$ (in the degree = largest n s.t. $f_n \neq 0$ sense) so $\partial f/\partial x_i = 0$ for each i.

In characteristic zero this is already impossible, because if x_i occurs in f then $\partial f/\partial x_i \neq 0$. So we must have char $k = p > 0$, and then the fact that $\partial f/\partial x_i = 0$ implies that f is actually a polynomial in x_i^p. This is true for each i, so by taking pth roots of the coefficients (possible since k is algebraically closed) we get a polynomial $g(x_1, \dots, x_n)$ such that $f = g^p$. But this contradicts the hypothesis that f is irreducible, so we conclude that $\operatorname{Sing} Y < Y$. $\square$

<u>EXAMPLES</u> If $Y = \mathbb{A}^n$ then $I(Y) = 0$ and the Jacobian for any point is zero, hence has rank $0 = n - n = n - r$, so $\mathbb{A}^n$ is nonsingular. Alternatively if $P \in \mathbb{A}^n$ then $\mathcal{O}_P = A_{\mathfrak{m}_P}$ so if $\mathfrak{m}$ is the maximal ideal in $\mathcal{O}_P$ $\mathfrak{m}/\mathfrak{m}^2 \cong \mathfrak{m}_P/\mathfrak{m}_P^2 \cong k^n$ (the case $\mathfrak{m}_P = (x_1, \dots, x_n)$ is obvious, and we can use an auto of A to get $\mathfrak{m}_P = \mathfrak{m}$) So $\dim_k \mathfrak{m}/\mathfrak{m}^2 = n = \dim Y$ so Y is nonsingular.

Let Y be any variety of dimension 0. Then Y is a point $Y = \{P\}$ and $\mathcal{O}_{P,Y} = k$ so $\mathfrak{m} = 0$ and clearly $\mathcal{O}_{P,Y}$ is a regular local ring. Hence Y is nonsingular.

Since $\mathbb{A}^n$ is nonsingular and $\mathbb{P}^n$ is covered by copies of $\mathbb{A}^n$, it follows that $\mathbb{P}^n$ is nonsingular. Hence in particular $\mathbb{P}^1$ is nonsingular, and by Ex(3.1c) every conic in $\mathbb{P}^2$ is nonsingular.

NOTE $A/\mathfrak{m} \cong A_\mathfrak{m}/\mathfrak{m}A_\mathfrak{m}$, and $\mathfrak{m}/\mathfrak{m}^2 \cong \mathfrak{m}A_\mathfrak{m}/\mathfrak{m}^2A_\mathfrak{m}$

The important fact is this: If A is any ring with maximal ideal $\mathfrak{m}$, and M is an A-module, then $M/\mathfrak{m}M$ is a module over the field $A/\mathfrak{m}$. Hence the action $(a+\mathfrak{m})(m+\mathfrak{m}M) = am+\mathfrak{m}M$ is <u>torsion-free</u>. That is, if $m \notin \mathfrak{m}M$ and $a \notin \mathfrak{m}$ then $am \notin \mathfrak{m}M$. Or put differently, if $a \notin \mathfrak{m}$ and $am \in \mathfrak{m}M$ then $m \in \mathfrak{m}M$. In particular this applies when $M = \mathfrak{m}$ to the $A/\mathfrak{m}$ module $\mathfrak{m}/\mathfrak{m}^2$. So we have shown

<u>LEMMA</u> If $m \in \mathfrak{m}$ and $s \notin \mathfrak{m}$ with $sm \in \mathfrak{m}^2$ then $m \in \mathfrak{m}^2$. Equivalently, the kernel of $\mathfrak{m} \longrightarrow \mathfrak{m}_\mathfrak{m}/\mathfrak{m}^2_\mathfrak{m}$ is $\mathfrak{m}^2$.

Let $\varphi: A \longrightarrow A_\mathfrak{m}$ be the canonical morphism of rings. If $a/1 = \varphi(a) \in \mathfrak{m}A_\mathfrak{m}$ then $sa \in \mathfrak{m}$ for some $s \notin \mathfrak{m}$. Since $\mathfrak{m}$ is prime, $a \in \mathfrak{m}$, so we have an injection $A/\mathfrak{m} \longrightarrow A_\mathfrak{m}/\mathfrak{m}A_\mathfrak{m}$. To see this is sujective, let $a \in A$ and $s \notin \mathfrak{m}$ be arbitrary. Let $t \notin \mathfrak{m}$ be s.t. $st + n = 1$ for some $n \in \mathfrak{m}$ (i.e. $t = s^{-1} \bmod \mathfrak{m}$). Then put $m = na = (1-st)a = a - sta$ and note that $m \in \mathfrak{m}$ and

$$\frac{a}{s} - \frac{m}{s} = \frac{sta}{s} = \frac{ta}{1}$$

Hence $A/\mathfrak{m} \longrightarrow A_\mathfrak{m}/\mathfrak{m}A_\mathfrak{m}$ is sujective and therefore an isomorphism of rings.

Next we claim there is an isomorphism of A-modules $\mathfrak{m}/\mathfrak{m}^2 \cong \mathfrak{m}A_\mathfrak{m}/\mathfrak{m}^2A_\mathfrak{m}$. We begin with the A-linear map $\varphi: \mathfrak{m} \longrightarrow \mathfrak{m}A_\mathfrak{m}$. If $m/1 = \varphi(m) \in \mathfrak{m}^2A_\mathfrak{m}$ then there is $s \notin \mathfrak{m}$ with $sm \in \mathfrak{m}^2$. By the Lemma, $m \in \mathfrak{m}^2$, so $\varphi^{-1}(\mathfrak{m}^2A_\mathfrak{m}) = \mathfrak{m}^2$. So $\mathfrak{m}/\mathfrak{m}^2 \longrightarrow \mathfrak{m}A_\mathfrak{m}/\mathfrak{m}^2A_\mathfrak{m}$ is injective. To see it is sujective use the above argument, but now $a \in \mathfrak{m}$ so $m \in \mathfrak{m}^2$. Hence the map is an isomorphism. If we make the identification

$$k \cong A/\mathfrak{m} \cong A_\mathfrak{m}/\mathfrak{m}A_\mathfrak{m}$$

Then $\mathfrak{m}/\mathfrak{m}^2$ and $\mathfrak{m}A_\mathfrak{m}/\mathfrak{m}^2A_\mathfrak{m}$ are isomorphic as vector spaces over k.

<u>NOTE</u> Definition of Nonsingularity makes sense

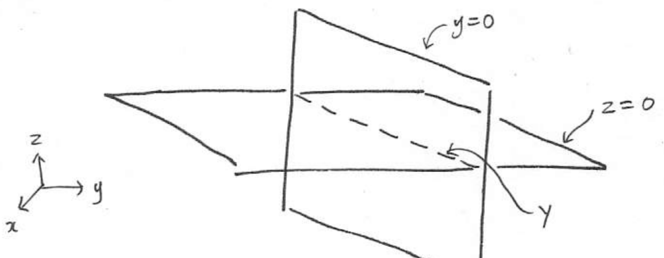
If $Y \subseteq \mathbb{A}^n$ is affine with $I(Y) = (f_1, \dots, f_t)$ then by Krull's generalised principal ideal theorem, $ht\, I(Y) \leq t$. Hence from $ht\, I(Y) + \dim Y = n$ we deduce that $t \geq n-r$. So the Jacobian matrix has dimensions exceeding $n-r$ on both sides, so having rank $n-r$ makes sense (i.e. we can't dismiss the possibility due to size constraints).

EXAMPLES Let $Y = \mathbb{A}^n$ be affine space. Then $\mathcal{O}_{0,Y} = (k[x,y])_{(x,y)}$ where $0 \in Y$ is the origin. So in the Noetherian local domain $\mathcal{O}_{0,Y}$ $\mathfrak{m}/\mathfrak{m}^2 \cong (x,y)/(x,y)^2$ (by the above). But the latter is in bijection (actually k-module iso) with k^n via $ax+by \longleftrightarrow (a,b)$. So we think of $\mathfrak{m}/\mathfrak{m}^2$ as <u>tangent vectors at O</u>. The dimension $\dim_k \mathfrak{m}/\mathfrak{m}^2$ counts the dimension of the tangent space of Y at P (where $\mathfrak{m}$ is the maximal ideal in $\mathcal{O}_{P,Y}$).

Now let $Y \subseteq \mathbb{A}^n$ be affine. From the proof of Theorem 5.1 we know that for $P \in Y$

$$\operatorname{rank} J = n - \dim_k \mathfrak{m}/\mathfrak{m}^2$$

where $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}_{P,Y}$, and $J = \left(\partial f_i/\partial x_j(P)\right)$ for $I(Y) = (f_1, \dots, f_t)$. For example let $Y = Z(y,z) \subseteq \mathbb{A}^3$. Then Y is the intersection of the hyperplanes $z=0$ and $y=0$, that is, the x-axis:



Then $J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ so $\operatorname{rank} J = 2$ and $\dim_k \mathfrak{m}/\mathfrak{m}^2 = n-2 = 1$ at any $P \in Y$. That is, every point on Y only has one tangent.

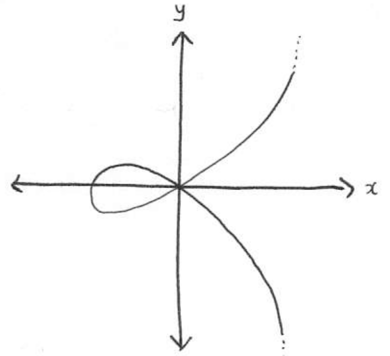
At any time $\dim \mathcal{O}_{P,Y} = \dim Y$ and $\dim_k \mathfrak{m}/\mathfrak{m}^2 \geqslant \dim \mathcal{O}_{P,Y}$. When these two are equal, the point is nonsingular (i.e. if $\dim \mathcal{O}_{P,Y} = r$ then the intuition is that near P, Y looks like $\mathbb{A}^r$ – thus should have r L.I. tangents. If this isn't so, if it has <u>more</u> than r tangents, P is a screwy point).

We consider some examples of curves in $\mathbb{A}^2$, which by Prop 1.13 must be $Z(f)$ for some single, nonconstant, irreducible $f \in k[x,y]$. For $P \in \mathbb{A}^2$ let $\mathfrak{a}_P$ denote the maximal ideal of P. Then by S.1, for $P \in Z(f)$ we have

$$\dim_k \mathfrak{m}/\mathfrak{m}^2 = \dim_k \frac{\mathfrak{a}_P}{\mathfrak{a}_P^2 + (f)}$$

Of course $\dim_k \mathfrak{a}_P/\mathfrak{a}_P^2 = 2$ (one way is to $k[x,y] \cong k[x,y]$ with $x - a_1 \longleftrightarrow x$, $y - a_2 \longleftrightarrow y$) so if $(f) \subseteq \mathfrak{a}_P^2$ then $\dim_k \mathfrak{m}/\mathfrak{m}^2 = 2$. But $\dim Y = 1$, so if $(f) \not\subseteq \mathfrak{a}_P^2$ then P is a nonsingular point of f. Of course, this is obvious since $J = (\partial f/\partial x, \partial f/\partial y)$ and $f \in \mathfrak{a}_P^2$ iff. $f(P) = 0$ and $\partial f/\partial x(P) = \partial f/\partial y(P) = 0$ (see our Note). So $P \in Y$ is singular iff. $f \in \mathfrak{a}_P^2$. In this case $\dim_k \mathfrak{m}/\mathfrak{m}^2 = 2$ ($\mathfrak{m} \subseteq \mathcal{O}_{P,Y}$).

$\boxed{y^2 - x^2(x+1)}$

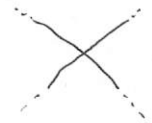


This was proved to be irreducible in our Section 1 Note on Irreducibility. We have $f = y^2 - x^3 - x^2$ and

$$\frac{\partial f}{\partial x} = -3x^2 - 2x \qquad \frac{\partial f}{\partial y} = 2y$$

see ⊛

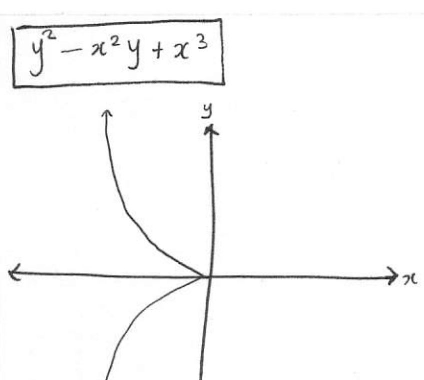
So $P \in Y$ is nonsingular, $P = (a,b)$ iff. $b = 0$ and $a = 0$ (or $a = -2/3$, but $(-2/3, 0) \notin Y$). So the origin is the only singular point on Y. Since $f \in (x,y)^2$ we see that $\dim_k \mathfrak{m}/\mathfrak{m}^2 = 2$, which matches with the two lines at the origin:



N.B. All the pictures are in $\mathbb{R}^2$, so we may gain solutions over $\mathbb{C}^2$ or whatever field k. But the calculations of singular points are valid in k^2

⊛ This is true for char $k = 0$. If char $k = 2$ the only singular point is O, and similarly if char $k = 3$ and above.

$\boxed{y^2 - x^2 y + x^3}$



Test for irreducibility (see our Notes at the end of §1):

$$\begin{aligned}
0 &= G_0 H_0 \qquad (\text{wlog } G_0 = 0) \\
0 &= H_0 G_x \\
0 &= H_0 G_y \qquad \Big\} \Rightarrow H_0 = 0 \\
0 &= G_x H_x \\
0 &= H_y G_x + H_x G_y \\
1 &= G_y H_y \\
1 &= G_x H_{x^2} \\
0 &= G_y H_{y^2} \\
0 &= G_x H_{y^2} + G_y H_{xy} \\
-1 &= G_x H_{xy} + G_y H_{x^2}
\end{aligned}$$

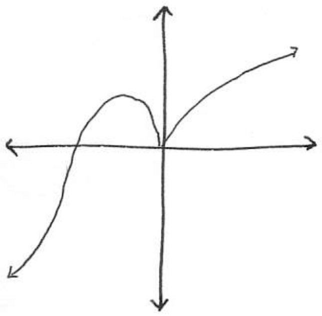
Case (A) $G_x = 0$. Then $H_x G_y = 0$ and $G_y \neq 0$ since $G \neq 0$, so $H_x = 0$. (well immediately $1 = G_x H_{x^2} \Rightarrow$ contradiction)

(B) $G_x \neq 0$, so $H_x = 0$. Hence $0 = H_y G_x$ so $H_y = 0$, contradicting $1 = G_y H_y$.

So $f = y^2 - x^2 y + x^3$ is irreducible, and $\partial f/\partial x = -2xy + 3x^2$, $\partial f/\partial y = 2y - x^2$. The only common ⊛ solutions are $(0,0)$ and $(3, 4.5)$, which is not on Y. So the origin is the only singular point on Y. Although it seems like perhaps there should be <u>one</u> tangent at O (x-axis?) what we see is that there is a "horizontal tangent" for each piece ... ie — and /. (For any curve in $\mathbb{A}^2$ either a point is nonsingular or it's singular and $\dim_k \mathfrak{m}/\mathfrak{m}^2 = 2$).

⊛ So in char 0, O is the only singular point. Similar calculations for 2, 3 show O is the only singular point. If char > 3 get $(0,0)$ and $(3, 9/2)$ as possibilities, but $(3, 9/2) \in Y$ iff. char $= 3$. So in any characteristic O is the only singular point.

$\boxed{y^3 - x^2(x+1)}$



Irreducibility test: $f = y^3 - x^3 - x^2$

$$\begin{aligned}
0 &= G_0 H_0 && (\text{wlog } G_0 = 0)\\
0 &= H_0 G_x && (\text{so } H_0 = 0)\\
0 &= H_0 G_y\\
-1 &= G_x H_x\\
0 &= H_y G_x + H_x G_y\\
0 &= G_y H_y\\
-1 &= G_x H_{x^2}\\
1 &= G_y H_{y^2}\\
0 &= G_x H_{y^2} + G_y H_{xy}\\
0 &= G_x H_{xy} + G_y H_{x^2}
\end{aligned}$$

Since $-1 = G_x H_x$, $G_x \neq 0$ and $H_x \neq 0$.
Also $G_y \neq 0$ ($1 = G_y H_{y^2}$) so $H_y = 0$ ($0 = G_y H_y$).

But then $0 = H_x G_y$, a contradiction.

So $f = y^3 - x^3 - x^2$ is irreducible, and $\partial f/\partial x = -3x^2 - 2x$ $\partial f/\partial y = 3y^2$

In characteristic 0, the only solutions are $(0,0)$ and $(-2/3, 0)$, but $(-2/3, 0) \notin Y$ in char 0. If Char $= 2$, $-3x^2 - 2x = x^2$ and $3y^2 = y^2$ so $(0,0)$ is the only solution. In char 3, $-2x = 0$ and any $(0, a)$ $a \in k$ is a solution, but $(0,a) \in Y \Rightarrow a^3 = 0 \Rightarrow a = 0$, so $(0,0)$ is again the only singular point. In char > 3, we get $(0,0)$ and $(-2/3, 0)$, which is on Y iff. $-2/3 + 1 = 0$, so iff. $1 = 0$, a contradiction. So in any char, 0 is the only singular pt.

$\boxed{y^2 - x^2(x-a), \quad a \text{ varies}}$

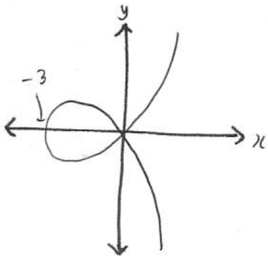
Irreducibility: $f = y^2 - x^3 + ax^2$. Using the normal technique assume $G_0 = 0$, then

$$\begin{aligned}
1 &= G_y H_y\\
0 &= G_y H_{y^2}\\
0 &= G_x H_{y^2} + G_y H_{xy}\\
0 &= G_x H_{xy} + G_y H_{x^2}
\end{aligned}$$

So $H_{y^2} = 0 \Rightarrow H_{xy} = 0 \Rightarrow H_{x^2} = 0$, but this contradicts $G_x H_{x^2} = -1$. So f is irreducible.

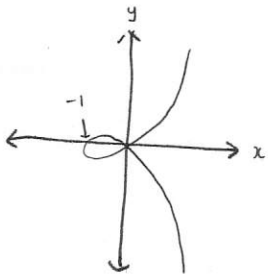
$$\frac{\partial f}{\partial x} = -3x^2 + 2ax \qquad \frac{\partial f}{\partial y} = 2y$$

$\boxed{a = -3}$

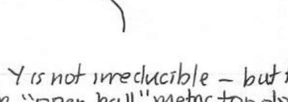


In char 0, solutions are $(0,0)$ and $(2a/3, 0)$. The second point belongs to Y iff. $a = 0$. So the only solution is $(0,0)$, provided $(0,0) \in Y$, which is always true. One checks in higher characteristics that 0 is the only singular point.

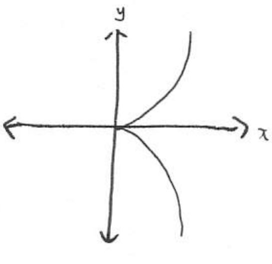
$\boxed{a = -1}$



NOTE For $y = 0$ the points on Y are $(0,0)$ and $(0, a)$ for all a. For positive a we have a solution at the origin and then another "piece" moving off to the right



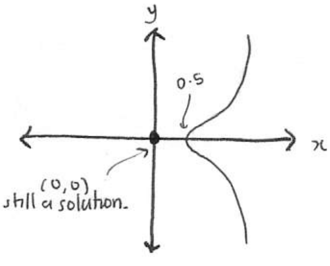
$\boxed{a = 0}$



It may seem like Y is not irreducible – but this because our intuition is in the "open ball" metric topology on $\mathbb{C}^2 / \mathbb{R}^2$ etc. Not in the Zariski topology. Since f is irreducible, you cannot write the piece as a closed set. In fact, in the topological space Y, this piece is open! $(Y - (0,0))$.

Although the dimension of the tangent space at $(0,0)$ in Y is of dimension 2, this is difficult to visualise?

$\boxed{a = 0.5}$



$a \in k$

$\boxed{y^2 - x^2(x-a) = 1}$

Irreducibility: $f = y^2 - x^3 + ax^2 - 1$

$1 = G_0H_0$
$0 = G_0H_x + H_0G_x$
(3) $0 = G_0H_y + H_0G_y$
$a = G_0H_{x2} + G_xH_x$
$0 = G_0H_{xy} + H_yG_x + H_xG_y$
(6) $1 = G_0H_{y2} + G_yH_y$
(7) $-1 = G_xH_{x2}$
$0 = G_yH_{y2}$
(9) $0 = G_xH_{y2} + G_yH_{xy}$
(10) $0 = G_xH_{xy} + G_yH_{x2}$

As always, pictures in $\mathbb{R}^2$ so be careful

Use $0 = G_yH_{y2}$

Case (A) $G_y = 0$, then (3) $\Rightarrow G_0H_y = 0$, so $H_y = 0$. By (10) $H_{xy} = 0$ ($G_x \neq 0$ since otherwise $G = G_0$ and we are done)
So by (9) $0 = G_xH_{y2}$, so $H_{y2} = 0$, contradicting (6)

Case (B) $G_y \neq 0$, $H_{y2} = 0$. Then from (6) $1 = G_yH_y$, (9) $0 = G_yH_{xy}$, so $H_{xy} = 0$ $\therefore$ (10) $0 = G_yH_{x2} \Rightarrow H_{x2} = 0$, contradicting (7)

Hence f is irreducible.

$\boxed{\text{Intercepts}}$ If $y = 0$, solving $x^3 - ax^2 + 1 = 0$. Three solutions in k (our pictures are in $\mathbb{R}$, so may be less). If $x = 0$ $y^2 = 1$ so $y = \pm 1$.

$\boxed{\text{Singular Pts}}$ $\partial f/\partial x = -3x^2 + 2ax$ $\quad \partial f/\partial y = 2y$

$\boxed{\text{Char 2}}$ Solutions to $x^2 = 0$, so $(0,b)$, if $(0,b) \in Y$. This is iff. $b^2 = 1$. Since char $k = 2$ the only singular point is $(0,1)$.

$\boxed{\text{Char 3}}$ Solutions to $2ax = 0$, $2y = 0$. If $a \neq 0$, only $(0,0)$ which is not $\in Y$. If $a = 0$, have $(b,0)$ provided $(b,0) \in Y$ which is iff. $b^3 - ab^2 + 1 = 0$, so iff $b^3 + 1 = 0$. But in char 3 $(b+1)^3 = b^3 + 1 = 0$, so $b = -1$.

$\boxed{a \neq 0}$ Y is nonsingular

$\boxed{a = 0}$ singular pt. $(-1,0)$

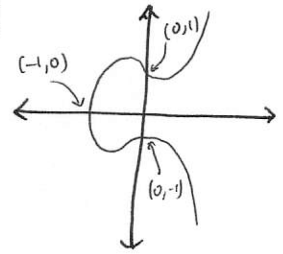
$\boxed{\text{Char 0}}$ Solutions to $-3x^2 + 2ax = x(2a - 3x) = 0$ and $2y = 0$, so $(0,0)$ and $(2a/3, 0)$ provided these points are on Y. But $(0,0) \notin Y$, and $(2a/3, 0) \in Y$ iff. $8a^3 - 12a^2 + 27 = 0$. So

$\boxed{8a^3 - 12a^2 + 27 \neq 0}$ Y is nonsingular

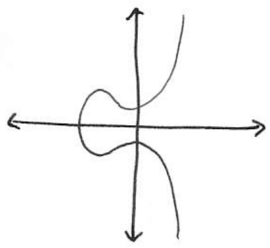
$\boxed{8a^3 - 12a^2 + 27 = 0}$ Y has a singular point $(\frac{2a}{3}, 0)$

$\boxed{\text{Char } p > 3}$ Solution possibility is $(2a/3, 0)$, provided $8a^3 - 12a^2 + 27 = 0$

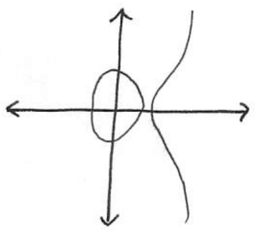
$\boxed{a = 0}$



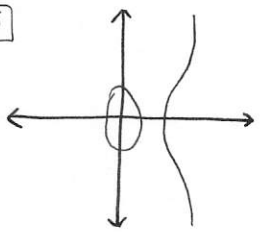
$\boxed{a = 1}$



$\boxed{a = 1.9}$

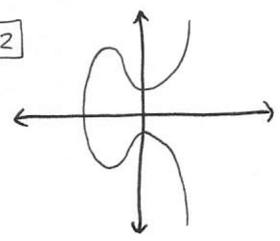


$\boxed{a = 2.5}$



As $a \to \infty$ the ellipse $\to$ and flattens and moves off $\longrightarrow$

$\boxed{a = -2}$



as $a \to -\infty$, the bulge enlarges.

NOTE Characterising powers of $\mathfrak{m}_P = (x_1-a_1,\dots,x_n-a_n)$

LEMMA Let k be a field, $P=(a_1,\dots,a_n)\in k^n$. Then $\mathfrak{m}_P = (x_1-a_1,\dots,x_n-a_n)$ is a maximal ideal in $k[x_1,\dots,x_n]$, and

$$f\in \mathfrak{m}_P \iff f(P)=0$$

$$f\in \mathfrak{m}_P^2 \iff \frac{\partial f}{\partial x_1}(P) = \frac{\partial f}{\partial x_2}(P) = \dots = \frac{\partial f}{\partial x_n}(P) = 0 \quad \text{and } f(P)=0$$

$$\vdots$$

$$f\in \mathfrak{m}_P^t \iff \frac{\partial^\alpha f}{\partial x^\alpha}(P)=0 \qquad [\alpha]<t$$

Where for $\alpha\in\mathbb{N}^n$ $[\alpha]=\sum \alpha_i$ and $\partial^\alpha f/\partial x^\alpha = \partial f/\partial x_1^{\alpha_1}\dots\partial x_n^{\alpha_n}$. (if $\alpha=(0,\dots,0)$ $\frac{\partial^\alpha f}{\partial x^\alpha}=f$)

This is well-defined, i.e. $\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}f = \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i}f$ as our other notes in p9 EFT notes p.derivatives polys show.

PROOF The statement for $t=1$ is proved in our Ueno notes. If $f\in\mathfrak{m}_P$ and

$$f = p_1(x_1-a_1)+\dots+p_n(x_n-a_n) \qquad (*)$$

Then it is easily checked that $\partial f/\partial x_i(P)=p_i(P)$. Hence if $\partial f/\partial x_1(P)=\dots=\partial f/\partial x_n(P)=f(P)=0$ then $f\in\mathfrak{m}_P^2$. The converse is easily checked. Hence $t=2$ also holds. We proceed by induction. Suppose the result holds for $t-1$. If $f\in\mathfrak{m}_P^t$ then $f\in\mathfrak{m}_P^{t-1}\mathfrak{m}_P$ so

$$f = \sum m_i n_i \qquad m_i\in\mathfrak{m}_P^{t-1} \quad n_i\in\mathfrak{m}_P$$

Let $[\alpha]<t$. Since $f\in\mathfrak{m}_P^t\subseteq\mathfrak{m}_P^{t-1}\subseteq\dots\subseteq\mathfrak{m}$ we can assume $[\alpha]=t-1$

$$\frac{\partial^\alpha f}{\partial x^\alpha}(P) = \sum \frac{\partial^\alpha}{\partial x^\alpha}(m_i n_i)(P)$$

$$= 0$$

Since $\frac{\partial^\alpha m_i n_i}{\partial x^\alpha}$ is a linear combination of terms $\partial^\beta m_i/\partial x^\beta \ \partial^\gamma n_i/\partial x^\gamma$ $\beta+\gamma=\alpha$ and these are zero on P if $\beta\neq\alpha$ and otherwise $\partial^\gamma n_i/\partial x^\gamma = n_i\in\mathfrak{m}_P$. Hence we have shown $\Rightarrow$.

For the converse we know by the inductive hypothesis that $f\in\mathfrak{m}_P^{t-1}$, so

$$f = \sum_{\substack{\beta\\ [\beta]=t-1}} p_\beta (x_1-a_1)^{\beta_1}\dots(x_n-a_n)^{\beta_n}$$

Then by the following Lemma for all β $p_\beta(P)=0$, so $p_\beta\in\mathfrak{m}_P$ and consequently $f\in\mathfrak{m}_P$. $\square$

LEMMA If $f=\sum_{\beta,[\beta]=k} p_\beta(x_1-a_1)^{\beta_1}\dots(x_n-a_n)^{\beta_n}$ then for any α with $[\alpha]=k$,

$$\frac{\partial^\alpha f}{\partial x^\alpha}(P) = p_\alpha(P)\,\alpha_1!\dots\alpha_n!$$

PROOF The only way for a sum $\sum p_\beta(\dots)$ to have nonzero terms in $\partial^\alpha/\partial x^\alpha$ is when we sink all the ∂/∂'s into (x_i-a_i) terms and not the ps. So we end up with

$$\frac{\partial^\alpha f}{\partial x^\alpha}(P) = \alpha_1!\dots\alpha_n!\,p_\alpha(P).\ \square$$

Reminder: Linear Algebra

Daniel Murfet

September 18, 2004

Let k be a field, let $M_{m,n}(k)$ denote the set of all $m \times n$ matrices with entries in k. The *column rank* of a matrix $A \in M_{m,n}(k)$ is the maximum number of linearly independent columns. The *row rank* is the maximum number of linearly independent rows. The matrix A corresponds to a linear transformation $\varphi : k^n \longrightarrow k^m$ and there is an exact sequence

$$0 \longrightarrow Ker\varphi \longrightarrow k^n \longrightarrow Im\varphi \longrightarrow 0$$

The image of φ in k^m is generated by the columns of A, hence the columns contain a basis for the image, so $dim(Im\varphi)$ is equal to the column rank of A.

For $1 \leq i \leq m$ the $m \times m$ matrix E_{ij} has a single nonzero entry 1 at position (i,j). If $\rho : \{1,\dots,n\} \longrightarrow \{1,\dots,n\}$ is any permutation then E_ρ is defined by putting row j of E_ρ equal to row $\rho(j)$ of the identity matrix. Any such matrix is clearly invertible. If $i \neq j$ then we define the matrix $A_{ij} = I + E_{ij}$. This matrix has inverse $I - E_{ij}$. If ℓ is a nonzero field element, we define for $1 \leq j \leq n$ the matrix $I_{\ell,j}$ which is the identity with the 1 at position (j,j) replaced by k. Clearly $I_{\ell,j}$ is invertible.

Suppose B is an $m \times m$ matrix which can be written as a product of matrices of the form E_ρ, A_{ij} and $I_{\ell,j}$. Then B is invertible and corresponds to an isomorphism $\psi : k^m \longrightarrow k^m$. The column rank of the product BA is the dimension of the image of $\psi\varphi$, which is the dimension of the image of φ, which is the column rank of A. The row rank of A is the dimension of the subspace of k^n spanned by the rows. If we interchange the order of the rows (the matrices E_ρ) or multiply rows by nonzero scalars (the matrices $I_{\ell,j}$) or even add rows to other rows (the matrices A_{ij}) the subspace spanned remains unchanged. So the row rank of A is equal to the row rank of B.

But by applying the three types of matrices, we can produce a matrix B whose nonzero rows all begin with 1s in such a way that these 1s are the only nonzero elements in their column. So clearly the row rank of B is the number of nonzero rows. But the columns corresponding to the leading 1s of nonzero rows clearly span the column space, and are linearly independent, so the number of these columns (which is the number of nonzero rows, i.e. the row rank) is also the column rank. Hence we have shown

Lemma 1. *If A is any $m \times n$ matrix over a field, then the column rank of A equals the row rank of A. We call this common value the* rank *of A.*

Theorem 2. *Let A be a square $m \times m$ matrix over a field. Then A is invertible if and only if $detA \neq 0$.*

Proof. Theorem 2.17 of Adkins & Weintraub. □

Corollary 3. *Let A be a square $m \times m$ matrix over a field. Then $detA = 0$ if and only if $rankA < m$.*

Proof. Since A corresponds to a morphism $\varphi : k^m \longrightarrow k^m$ by a dimension counting argument A is invertible if and only if φ is surjective, so if and only if the rank of A is m. So using the Theorem, $\det A = 0$ if and only if $\operatorname{rank} A < m$. □

Proposition 4. *Let A be an $m \times n$ matrix over the field k, and let k be a positive integer with $k \leq m$ and $k \leq n$. Then $rankA < k$ if and only if every $k \times k$ submatrix over A has zero determinant.*

Proof. By a *submatrix* we mean a matrix obtained from A by omitting rows and columns, which are not necessarily adjacent. Suppose $\mathrm{rank} A < k$ and consider the k rows contributing to a specific submatrix. These rows are linearly dependent, so the rank of the submatrix is $< k$ and hence by the Corollary the determinant of the submatrix is zero.

Conversely suppose all the $k \times k$ submatrices have zero determinant, but that $\mathrm{rank} A \geq k$. Then there are k linearly independent rows. Throwing away the other rows, we have a $k \times n$ matrix with rank k. Thus there are k linearly independent columns. Throwing away the other columns, we have produced a $k \times k$ submatrix of A with rank k, which thus has nonzero determinant. This contradiction shows that $\mathrm{rank} A < k$. $\square$

Completion

For the local analysis of singularities we will now describe the technique of completion. Let A be a local ring with maximal ideal $\mathfrak{m}$. The powers of $\mathfrak{m}$ define a topology on A, called the <u>$\mathfrak{m}$-adic topology</u>. By completing with respect to this topology, one defines the <u>completion</u> of A, denoted $\hat{A}$. Alternatively one can define $\hat{A}$ as the inverse limit $\varprojlim A/\mathfrak{m}^n$.

The significance of completion in algebraic geometry is that by passing to the completion $\hat{\mathcal{O}}_P$ of the local ring of a point P on a variety X, one can study the very local behaviour of X near P. We have seen (Ex 4.7) that if points $P \in X$ and $Q \in Y$ have isomorphic local rings, then already P and Q have isomorphic neighborhoods, so in particular X and Y are birational. Thus the ordinary local ring $\mathcal{O}_P$ contains information about almost all of X. However the completion $\hat{\mathcal{O}}_P$, as we will see, carries much more local information, closer to our intuition of what "local" means in topology or differential geometry. We will recall some of the algebraic properties of completion and then give some examples.

<u>THEOREM 5.4A</u> Let A be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and let $\hat{A}$ be its completion

(a) $\hat{A}$ is a local ring, with maximal ideal $\hat{\mathfrak{m}} = \mathfrak{m}\hat{A}$, and there is a natural injective homomorphism $A \longrightarrow \hat{A}$.

(b) If M is a finitely-generated A-module, its completion $\hat{M}$ with respect to its $\mathfrak{m}$-adic topology is isomorphic to $M \otimes_A \hat{A}$.

(c) $\dim A = \dim \hat{A}$

(d) A is regular iff. $\hat{A}$ is regular

<u>PROOF</u> (a) Prop 10.16 of A&M. Note that $\hat{\mathfrak{m}}$ consists of Cauchy sequences (a_r) with $a_r \in \mathfrak{m}$.

(b) Prop 10.13 of A&M

(c) Prop 5.4.3 p71 our notes.

(d) Prop 11.24 p89 our notes. $\square$

<u>THEOREM 5.5A</u> (Cohen Structure Theorem) If A is a complete regular local ring of dimension n containing some field, then $A \cong k[[x_1, \dots, x_n]]$, the ring of formal power series over the residue field k of A.

<u>PROOF</u> If A is a complete regular local ring of dimension n, then A is equicharacteristic (p95 Regular Local Ring Notes) hence is either a field ($\dim = 0$) or is of dimension $n \geqslant 1$ and by p98 Reg. Local Ring notes $A \cong k[[x_1, \dots, x_n]]$ where $k = A/\mathfrak{m}$. $\square$

<u>DEFINITION</u> Let X, Y be varieties. We say two points $P \in X$ and $Q \in Y$ are <u>analytically isomorphic</u> if there is an isomorphism $\hat{\mathcal{O}}_P \cong \hat{\mathcal{O}}_Q$ of k-algebras.

<u>EXAMPLE 5.6.1</u> If $P \in X$ and $Q \in Y$ are analytically isomorphic, then $\dim X = \dim Y$. This follows from (5.4A) and the fact that any local ring of a point on a variety has the same dimension as the variety. (Ex.3.12)

<u>EXAMPLE 5.6.2</u> If $P \in X$ and $Q \in Y$ are nonsingular points on varieties of the same dimension, then $n = \dim \mathcal{O}_{P,X} = \dim \mathcal{O}_{Q,Y}$. Both $\hat{\mathcal{O}}_{P,Y}$ and $\hat{\mathcal{O}}_{Q,Y}$ are complete regular local rings, which contain $\mathcal{O}_{P,X}$ and $\mathcal{O}_{Q,Y}$ as subrings and hence the field k. By (5.5A) either $n = 0$ and $k = \mathcal{O}_{P,Y} = \hat{\mathcal{O}}_{P,Y}$ and $k = \mathcal{O}_{Q,Y} = \hat{\mathcal{O}}_{Q,Y}$ or $n > 0$ and both $\hat{\mathcal{O}}_{P,Y}$ and $\hat{\mathcal{O}}_{Q,Y}$ are formal power series rings in n variables over their residue fields, but $\hat{\mathcal{O}}_{P,Y}/\hat{\mathfrak{m}} = \mathcal{O}_{P,Y}/\mathfrak{m} = k$, so P and Q are analytically isomorphic. This example is the algebraic analogue of the fact that any two manifolds (topological, differentiable, or complex) of the same dimension are locally isomorphic.

NOTE For any ring A, $A/\mathfrak{m}^k \cong A_\mathfrak{m}/\mathfrak{m}^k A_\mathfrak{m}$ $k \geq 1$

First see our earlier note where we showed for any ring A with maximal ideal $\mathfrak{m}$ that

$$A/\mathfrak{m} \cong A_\mathfrak{m}/\mathfrak{m}A_\mathfrak{m} \quad \text{and} \quad \mathfrak{m}/\mathfrak{m}^2 \cong \mathfrak{m}A_\mathfrak{m}/\mathfrak{m}^2 A_\mathfrak{m}$$

We used the fact that if $s \notin \mathfrak{m}$ and $m \in \mathfrak{m}$, then $sm \in \mathfrak{m}^2 \Rightarrow m \in \mathfrak{m}^2$. This generalises:

LEMMA If $s \notin \mathfrak{m}$ and $a \in A$, then $sa \in \mathfrak{m}^k \Rightarrow a \in \mathfrak{m}^k$ $\forall k \geq 1$

PROOF By induction. It is trivially true for $k=1$. If $k>1$ assume it is true for $k-1$. Then $sa \in \mathfrak{m}^k \subseteq \mathfrak{m}^{k-1}$ implies $a \in \mathfrak{m}^{k-1}$. But $\mathfrak{m}^{k-1}$ is an A-module which becomes a module $\mathfrak{m}^{k-1}/\mathfrak{m}^k$ over the field $A/\mathfrak{m}$. Since this module is torsion free, $sa \in \mathfrak{m}^k \Rightarrow a \in \mathfrak{m}^k$ as required.

LEMMA $\mathfrak{m}^k/\mathfrak{m}^{k+1} \cong \mathfrak{m}^k A_\mathfrak{m}/\mathfrak{m}^{k+1}A_\mathfrak{m}$ $k \geq 1$ (as $A/\mathfrak{m}$-modules)

PROOF The canonical map $\mathfrak{m}^k \longrightarrow \mathfrak{m}^k A_\mathfrak{m}/\mathfrak{m}^{k+1}A_\mathfrak{m}$ of A-modules has kernel $\mathfrak{m}^{k+1}$ by the previous Lemma, so the $A/\mathfrak{m}$-linear map $\mathfrak{m}^k/\mathfrak{m}^{k+1} \longrightarrow \mathfrak{m}^k A_\mathfrak{m}/\mathfrak{m}^{k+1}A_\mathfrak{m}$ is injective. To see it is surjective, let $a \in \mathfrak{m}^k$ and $s \notin \mathfrak{m}$, $n \in \mathfrak{m}$ $t \notin \mathfrak{m}$ s.t. $st+n=1$ and put $m = na \in \mathfrak{m}^{k+1}$. Then

$$\frac{a}{s} - \frac{m}{s} \in \operatorname{Im}(\mathfrak{m}^k/\mathfrak{m}^{k+1} \longrightarrow \mathfrak{m}^k A_\mathfrak{m}/\mathfrak{m}^{k+1}A_\mathfrak{m})$$

as required. $\square$

LEMMA For any ring A with maximal ideal $\mathfrak{m}$, $A/\mathfrak{m}^k \cong A_\mathfrak{m}/\mathfrak{m}^k A_\mathfrak{m}$ as rings $\forall k \geq 1$.

PROOF By the first Lemma the canonical ring homomorphism $A/\mathfrak{m}^k \longrightarrow A_\mathfrak{m}/\mathfrak{m}^k A_\mathfrak{m}$ is injective, $\forall k \geq 1$. By induction we show that $A \longrightarrow A_\mathfrak{m}/\mathfrak{m}^k A_\mathfrak{m}$ is surjective. We checked the $k=1$ case earlier, so say $k>1$ and assume the map is surjective for $k-1$. Let a/s be given, $a \in A$ and $s \notin \mathfrak{m}$, and let $m \in \mathfrak{m}^{k-1}$, $t \notin \mathfrak{m}$ and $a' \in A$ be s.t.

$$a/s - m/t = \frac{a'}{1}$$

By the previous Lemma, $\mathfrak{m}^{k-1}/\mathfrak{m}^k \cong \mathfrak{m}^{k-1}A_\mathfrak{m}/\mathfrak{m}^k A_\mathfrak{m}$, so that there is $n \in \mathfrak{m}^{k-1}$ and $e \in \mathfrak{m}^k$, $q \notin \mathfrak{m}$ s.t. $m/t - e/q = n/1$. But then

$$a'/1 = a/s - (n/1 + e/q) \Rightarrow \frac{(a'+n)}{1} = a/s - e/q$$

So $A \longrightarrow A_\mathfrak{m}/\mathfrak{m}^k A_\mathfrak{m}$ is surjective, as required.

COROLLARY If A is any ring with maximal ideal $\mathfrak{m}$ then $\hat{A} \cong \widehat{A_\mathfrak{m}}$ as rings, where the completions are $\mathfrak{m}$-adic and $\mathfrak{m}A_\mathfrak{m}$-adic respectively.

PROOF Using the fact that $\hat{A} = \varprojlim A/\mathfrak{m}^n$, $\widehat{A_\mathfrak{m}} = \varprojlim A_\mathfrak{m}/\mathfrak{m}^n A_\mathfrak{m}$ this is trivial. The isomorphism is

$$\phi : \hat{A} \longrightarrow \widehat{A_\mathfrak{m}}$$
$$(a_n) \longmapsto (a_n/1). \square$$

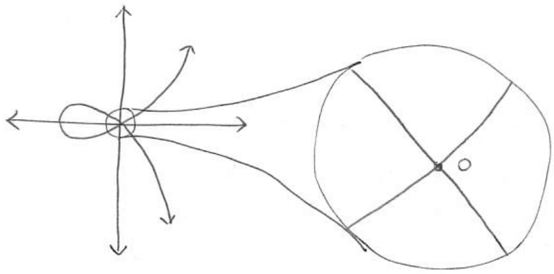
Intuitively, $\hat{A}$ is a really local view of the point $\mathfrak{m}$, whereas $A_\mathfrak{m}$ is a local view, so the corollary says you can go straight to the really local view or do it in two steps.

NOTE For any Noetherian ring A, we have $\widehat{(a)} = (a)$ in $\hat{A}$ (completion w.r.t. any ideal)

By Prop 10.15 A & M, if $\mathfrak{b} = (a)$ then $\hat{\mathfrak{b}} = \{(a_n) \mid a_n \in \mathfrak{b}\}$ is the smallest ideal in $\hat{A}$ containing the image of (a) under $A \longrightarrow \hat{A}$, but this is $(a) \subseteq \hat{A}$.

NOTE Let $\mathfrak{b}$ be an ideal in a Noetherian ring A, and let $\hat{}$ denote $\mathfrak{a}$-adic completion. Then if $\mathfrak{b} \subseteq \mathfrak{a}$, so $(A/\mathfrak{b})$ has its $\mathfrak{a}$-filtration as an A-module coincide with the filtration induced by the ideal $\mathfrak{a}/\mathfrak{b}$, then $\widehat{(A/\mathfrak{b})} \cong \hat{A}/\hat{\mathfrak{b}}$ as rings (see 10.12 A & M) via $(a_n) + \hat{\mathfrak{b}} \mapsto (a_n + \mathfrak{b})$

<u>EXAMPLE 5.6.3</u> Let X be the plane nodal cubic curve given by the equation $y^2 = x^2(x+1)$.



Let Y be the algebraic set in $\mathbb{A}^2$ defined by the equation $xy = 0$. We will show that the point $O = (0,0)$ on X is analytically isomorphic to the point O on Y. (Since we haven't yet developed the general theory of local rings of points on reducible algebraic sets, we use an ad hoc definition

$$\mathcal{O}_{O,Y} = \left(k[x,y]/(xy) \right)_{(x,y)}$$

By the Notes on the previous page, $\hat{\mathcal{O}}_{O,Y} \cong k[[x,y]]/(xy)$ (one checks that under the isomorphism $\widehat{k[x,y]} \cong k[[x,y]]$ the constant sequence $(xy, xy, \dots) \longleftrightarrow xy$) (see p42 A&M) (Note that $\hat{\mathcal{O}}_{O,Y} \cong k[[x,y]]/(xy)$ as <u>k-algebras</u>). This example corresponds to the fact that near O, X looks like two lines crossing.

To prove this result, we consider the completion $\hat{\mathcal{O}}_{O,X}$:

$$\begin{aligned} \hat{\mathcal{O}}_{O,X} &= \widehat{\left(k[x,y]/(y^2-x^2-x^3) \right)_{(x,y)}} \\ &\cong \widehat{k[x,y]}/(y^2-x^2-x^3) \cong k[[x,y]]/(y^2-x^2-x^3) \end{aligned}$$

The key point is that the leading form of the equation, namely $y^2 - x^2$, factors into two distinct factors $(y+x)$ and $(y-x)$ (we assume char $k \neq 2$). We claim there are formal power series

$$\begin{aligned} g &= y + x + g_2 + g_3 + \cdots \\ h &= y - x + h_2 + h_3 + \cdots \end{aligned}$$

in $k[[x,y]]$ where g_i, h_i are homogenous of degree i, such that $y^2 - x^2 - x^3 = gh$. We construct g and h step by step. To determine g_2 and h_2 we need to have

$$(y-x)g_2 + (y+x)h_2 = -x^3$$

See p44 A&M notes for some background

This is possible, because $y-x$ and $y+x$ generate the maximal ideal of $k[[x,y]]$. To determine g_3 and h_3 we need

$$(y-x)g_3 + (y+x)h_3 = -g_2 h_2$$

which is again possible, and so on.

Thus $\hat{\mathcal{O}}_{O,X} \cong k[[x,y]]/(gh)$. Since g and h begin with linearly independent linear terms there is an automorphism of $k[[x,y]]$ sending g and h to x and y respectively. (see next page). This shows that $\hat{\mathcal{O}}_{O,X} \cong k[[x,y]]/(xy)$ as required.

Note in this example that $\mathcal{O}_{O,X}$ is an integral domain, but its completion is not.

Automorphisms of Power Series Rings

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Let k be a field. We have seen earlier that if $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_2(k)$ is an invertible 2×2 matrix over k then $\varphi : k[x, y] \longrightarrow k[x, y]$ defined by

$$\varphi(x) = ax + by, \quad \varphi(y) = cx + dy$$

is an automorphism of k-algebras, and this extends to polynomial rings over any number of variables. We wish to establish an analgous result for power series rings.

From our Analytic Independence notes (p.85 of our A&M notes) we know that if $a_1, \ldots, a_n \in k[[x_1, \ldots, x_n]]$ are power series with no constant term (i.e. not units) then there is a unique continuous morphism of k-algebras $\varphi : k[[x_1, \ldots, x_n]] \longrightarrow k[[x_1, \ldots, x_n]]$ with $\varphi(x_i) = a_i$ for $1 \le i \le n$. If $f(x_1, \ldots, x_n)$ is a power series then we denote $\varphi(f)$ by $f(a_1, \ldots, a_n)$.

Let $\mathfrak{m} = (x)$ be the unique maximal ideal in $k[[x]]$ and consider the power series $u(x) \in k[[x]]$ defined by

$$u = x + x^2 + x^3 + x^4 + \ldots + x^n + \ldots$$

Put $g(x) = x^2 + x^3 + \ldots$ so $u \in \mathfrak{m}, g \in \mathfrak{m}^2$ and $u = x + g(x)$. Notice that $x = u - g$ and $g = xu$ so that $x = (1 - x)u$. We can use this fact to gradually replace the xs in $g(x)$ with us until we have $x = h(u)$ for some power series h:

$$\begin{aligned} x &= u - ux \\ &= u - u(u - ux) \\ &= u - u^2 + u^2 x = u - u^2 + u^2(u - ux) \\ &= u - u^2 + u^3 - u^3 x \end{aligned}$$

This suggests that $x = u - u^2 + u^3 - u^4 + \ldots + (-1)^{n+1} u^n + \ldots$ and one can check directly that this is the case. If we let $h(x)$ be the power series $x - x^2 + x^3 - x^4 + \ldots$ then we have $x = h(u)$ as required.

A similar technique works in the general case:

Proposition 1. *Let $u(x) = u_1 x + u_2 x^2 + \ldots$ be a power series with $u_1 \neq 0$. Then there is a power series $h(x) \in k[[x]]$ with $x = h(u)$.*

Proof. First we prove the following claim by induction: For each $n \ge 1$ there is a polynomial $h_n(x) \in k[x]$ of degree $\le n$ and a power series $b_n(x) \in \mathfrak{m}^{n+1}$ with $x = h_n(u) + b_n(x)$. Since

$$x = \frac{1}{u_1}(u - g)$$

where $g(x) = u_2 x^2 + u_3 x^3 + \ldots$ this is trivial for $n = 1$. Suppose it is true for $n \ge 1$ and let $x = h_n(u) + b_n(x)$. We can then write

$$\begin{aligned} x &= h_n(u) + b_{n,1} x^{n+1} + b_{n,2} x^{n+2} + \ldots \\ &= h_n(u) + b_{n,1} \left(\frac{1}{u_1} u - \frac{1}{u_1} g(x) \right)^{n+1} + b_{n,2} x^{n+2} + \ldots \\ &= h_n(u) + b_{n,1} \left(\frac{1}{u_1} \right)^{n+1} u^{n+1} + b_{n+1}(x) \\ &= h_{n+1}(u) + b_{n+1}(x) \end{aligned}$$

where $b_{n+1}(x)$ belongs to $\mathfrak{m}^{n+2}$ since $g \in \mathfrak{m}^2$ and $u \in \mathfrak{m}$. Notice that the sequence $h_1(x), h_2(x), \ldots$ is a Cauchy sequence in $k[[x]]$ and hence converges to some power series $h(x) \in k[[x]]$. For each $n \geq 1$ write $h(x) = h_n(x) + h_{>n}(x)$ and note that

$$h(u) = h_n(u) + h_{>n}(u) = x - b_n(x) + h_{>n}(u)$$

But $-b_n(x) + h_{>n}(u) \in \mathfrak{m}^{n+1}$ since $u \in \mathfrak{m}$ and $b_n \in \mathfrak{m}^{n+1}$. Since n was arbitrary and the limits of Cauchy sequences in $k[[x]]$ are unique, this shows that $h(u) = x$ as required. □

Note that in the above construction $h(x) = \frac{1}{u_1}x + \ldots$.

Corollary 2. *Let $u(x) = u_1 x + u_2 x^2 + \ldots$ be a power series with $u_1 \neq 0$. Then the morphism of k-algebras $\varphi : k[[x]] \longrightarrow k[[x]]$ defined by $x \mapsto u$ is an automorphism.*

Proof. Let $h(x) \in k[[x]]$ be such that $h(u) = x$, that is, $\varphi(h) = x$. Let $\phi : k[[x]] \longrightarrow k[[x]]$ be defined by $x \mapsto h$. Then $\varphi\phi$ is a continuous morphism of k-algebras with $\varphi\phi(x) = x$, so by uniqueness $\varphi\phi = 1$. Since $h(x) = \frac{1}{u_1}x + \ldots$ we can apply the same argument to produce a continuous morphism of k-algebras $\psi : k[[x]] \longrightarrow k[[x]]$ with $\phi\psi = 1$. An elementary calculation shows that $\varphi = \psi$ and so φ is an automorphism. □

In particular any power series in one variable with no constant term and a nonzero linear term is analytically independent. We now extend this result to more than one variable. Consider the power series ring $k[[x_1, \ldots, x_n]]$ in n variables with maximal ideal $\mathfrak{m} = (x_1, \ldots, x_n)$. We have shown elsewhere that the power $\mathfrak{m}^k$ consist of those power series whose only nonzero terms involve monomials of order k or greater. For any $g(x_1, \ldots, x_n) \in k[[x_1, \ldots, x_n]]$ we write

$$g_i = \sum_{\alpha, |\alpha| = i} g(\alpha) x_1^{\alpha_1} \ldots x_n^{\alpha_n}$$

It is easily seen that g is the sum of the series $g_0 + g_1 + \ldots$. We call g_1 the *linear term* of g. Recall that is a power series is a unit iff. it has a nonzero constant term.

Proposition 3. *Let $u_1, \ldots, u_n \in k[[x_1, \ldots, x_n]]$ be nonunit power series whose linear terms are linearly independent. Then there are nonunit power series $h_1, \ldots, h_n$ whose linear terms are linearly independent with $x_i = h_i(u_1, \ldots, u_n)$ for $1 \leq i \leq n$.*

Proof. Suppose we write

$$u_i(x_1, \ldots, x_n) = u_{i,1} x_1 + \ldots u_{i,n} x_n + G_i(x_1, \ldots, x_n)$$

where $G_i \in \mathfrak{m}^2$, for $1 \leq i \leq n$. Since the linear terms are linearly independent they span k^n, so for $1 \leq i \leq n$ there are elements $\lambda_{i,1}, \ldots, \lambda_{i,n}$ with

$$\lambda_{i,1} u_1 + \ldots + \lambda_{i,n} u_n = x_i + \lambda_{i,1} G_1 + \ldots + \lambda_{i,n} G_n$$

That is,

$$x_i = \lambda_{i,1}(u_1 - G_1) + \ldots + \lambda_{i,n}(u_n - G_n) \tag{1}$$

Let us make some comments before proceeding with the proof. If $f(x_1, \ldots, x_n) \in k[x_1, \ldots, x_n]$ is any homogenous polynomial of degree $k \geq 1$ then we have

$$\begin{aligned} f(x_1, \ldots, x_n) &= f\left(\sum_j \lambda_{1,j}(u_j - G_j), \ldots, \sum_j \lambda_{n,j}(u_j - G_j) \right) \\ &= \sum_{\alpha, |\alpha| = k} f(\alpha) \prod_{i=1}^{n} (\lambda_{i,1} u_1 + \ldots + \lambda_{i,n} u_n - \lambda_{i,1} G_1 - \ldots - \lambda_{i,n} G_n)^{\alpha_i} \end{aligned}$$

Since the u_i belong to $\mathfrak{m}$ and the G_i all belong to $\mathfrak{m}^2$ we can expand this and write

$$\begin{aligned} f(x_1,\dots,x_n) &= f\left(\sum_j \lambda_{1,j}u_j,\dots,\sum_j \lambda_{n,j}u_j\right) + b(x_1,\dots,x_n) \\ &= H(u_1,\dots,u_n) + B(x_1,\dots,x_n) \end{aligned}$$

where $H(x_1,\dots,x_n) \in k[x_1,\dots,x_n]$ is a homogenous polynomial of degree k and B is a power series belonging to $\mathfrak{m}^{k+1}$.

Next we produce for each i a power series h_i with $h_i(u_1,\dots,u_n) = x_i$. Let $1 \le i \le n$ be fixed. By induction we show that for each $m \ge 1$ there is a polynomial $H_m(x_1,\dots,x_n) \in k[x_1,\dots,x_n]$ with degree $\le m$ (i.e. the highest degree monomial occurring in H_m has degree $\le m$) and a power series $b_m \in k[[x_1,\dots,x_n]]$ with $b_m \in \mathfrak{m}^{m+1}$ such that

$$x_i = H_m(u_1,\dots,u_n) + b_m$$

To see the claim is true for $m = 1$ we set $H_1 = \lambda_{i,1}x_1 + \ldots + \lambda_{i,n}x_n$ and $b_m = -\lambda_{i,1}G_1 - \ldots - \lambda_{i,n}G_n$ and use Equation 1. Suppose the claim is true for m and let $x_i = H_m(u_1,\dots,u_n) + b_m$. Denote by $b_{m,j}$ the homogenous part of b_m of degree j defined earlier. Using Equation 1 and the preceeding comment

$$\begin{aligned} b_m(x_1,\dots,x_n) &= b_{m,m+1}(x_1,\dots,x_n) + \sum_{j=m+2}^{\infty} b_{m,j} \\ &= H(u_1,\dots,u_n) + B(x_1,\dots,x_n) + \sum_{j=m+2}^{\infty} b_{m,j} \end{aligned}$$

Where $H(x_1,\dots,x_n) \in k[x_1,\dots,x_n]$ is a homogenous polynomial of degree $m+1$ and B is a power series belonging to $\mathfrak{m}^{m+2}$. Putting $H_{m+1} = H + H_m$ and $b_{m+1} = B + \sum_{j=m+2}^{\infty} b_{m,j}$ we have $x_i = H_{m+1}(u_1,\dots,u_n) + b_{m+1}$ as required.

Notice that in the above $H_{m+1} - H_m = H$, so at each stage we add to H_m a homogenous polynomial of degree $m+1$. Hence the sequence $H_1, H_2, \dots$ is Cauchy and converges to some power series $h_i(x_1,\dots,x_n) \in k[[x_1,\dots,x_n]]$. For each $m \ge 1$ write $h_i = H_m + H_{>m}$ and note that

$$\begin{aligned} h_i(u_1,\dots,u_n) &= H_m(u_1,\dots,u_n) + H_{>m}(u_1,\dots,u_n) \\ &= x_i - b_m(x_1,\dots,x_n) + H_{>m}(u_1,\dots,u_n) \end{aligned}$$

But $-b_m(x_1,\dots,x_n) + H_{>m}(u_1,\dots,u_n) \in \mathfrak{m}^{m+1}$ since the u_i belong to $\mathfrak{m}$ and $b_m \in \mathfrak{m}^{m+1}$. Since m was arbitrary this shows that $h_i(u_1,\dots,u_n) = x_i$, as required. By construction h_i has no constant term, and the linear term is $\lambda_{i,1}x_1 + \ldots + \lambda_{i,n}x_n$. So the power series $h_1,\dots,h_n$ satisfy all the conditions of the Proposition, since the coefficients of the linear terms form the matrix inverse to the matrix formed from the linear coefficients of the u_i, which are linearly independent by assumption. $\square$

Theorem 4. *Let $u_1,\dots,u_n \in k[[x_1,\dots,x_n]]$ be nonunit power series whose linear terms are linearly independent. Then the morphism of k-algebras*

$$\begin{aligned} \varphi : k[[x_1,\dots,x_n]] &\longrightarrow k[[x_1,\dots,x_n]] \\ x_i &\mapsto u_i \end{aligned}$$

is an automorphism.

Proof. Let $h_1,\dots,h_n$ be the power series produced by the Proposition with $h_i(u_1,\dots,u_n) = x_i$, that is, $\varphi(h_i) = x_i$. Let $\phi : k[[x_1,\dots,x_n]] \longrightarrow k[[x_1,\dots,x_n]]$ be defined by $x_i \mapsto h_i$. Then $\varphi\phi$ is a continuous morphism of k-algebras with $\varphi\phi(x_i) = x_i$ for all i. Hence by uniqueness $\varphi\phi = 1$. Since the h_i are also a family of nonunit power series whose linear terms are linearly independent, the same argument produces a continuous morphism of k-algebras ψ with $\phi\psi = 1$. We see immediately that $\varphi = \psi$ and hence φ is an automorphism. $\square$