6. Nonsingular Curves

In considering the problem of classification of algebraic varieties, we can formulate several subproblems, based on the idea that a nonsingular projective variety is the best kind: (a) classify varieties up to birational equivalence; (b) within each birational equivalence class, find a nonsingular projective variety; (c) classify the nonsingular projective varieties in a given birational equivalence class.

In general, all three problems are very difficult. However, in the case of curves (varieties of dimension 1) the situation is much simpler. In this section we will answer problems (b) and (c) by showing that in each birational equivalence class, there is a unique nonsingular projective curve. We will also give an example to show that not all curves are birationally equivalent to each other (Ex 6.2). Thus for a given finitely generated extension field K of k of transcendence degree 1 (which we will call a <u>function field of dimension 1</u>) we can talk about <u>the</u> nonsingular projective curve C_K with function field K. We will see also that if K_1, K_2 are two function fields of dimension 1, then any k-homomorphism $K_2 \longrightarrow K_1$ is represented by a morphism of C_{K_1} to C_{K_2}.

We begin our study in an oblique manner by defining the notion of an "abstract nonsingular curve" associated with a given function field. It will not be clear a priori that this is a variety. However, we will see in retrospect that we have defined nothing new.

see Exs §5 A&M for def^N: is p.order $\leq$ s.t. if $a \leq b$ then $a+c \leq b+c$ $\forall c$

<u>DEFINITION</u> Let K be a field and G a totally ordered abelian group. A <u>valuation</u> of K with values in G is a map $v: K - \{0\} \longrightarrow G$ such that for all $x, y \in K$ with $x, y \neq 0$

$$(1) \quad v(xy) = v(x) + v(y)$$

$$(2) \quad \text{If } x+y \neq 0 \quad v(x+y) \geq \min\{v(x), v(y)\}$$

If v is a valuation, then the set $R = \{x \in K \mid v(x) \geq 0\} \cup \{0\}$ is a subring of K which we call the <u>valuation ring</u> of v. The subset $\mathfrak{m} = \{x \in K \mid v(x) > 0\} \cup \{0\}$ is an ideal in R and $(R, \mathfrak{m})$ is a local ring. A <u>valuation ring</u> is an integral domain which is the valuation ring of some valuation of its quotient field. By our A&M notes an integral domain is a valuation ring in this sense iff it is a subring of some field K s.t. $\forall\, 0 \neq x \in K$ $x \in A$ or $x^{-1} \in A$. In this case we say A is a <u>valuation ring of</u> K. If k is a subfield of K such that $v(x) = 0$ for all $x \in k - \{0\}$, then we say v is a <u>valuation</u> of K/k, and R is a <u>valuation ring</u> of K/k (Note that valuation rings are not in general Noetherian). (for R to be a valuation ring of K/k it is enough to have $k \subseteq R$, since v of a unit is 0)

<u>DEFINITION</u> If A, B are local rings contained in a field K, we say that B <u>dominates</u> A if $A \subseteq B$ and $\mathfrak{m}_B \cap A = \mathfrak{m}_A$ (equiv. $\mathfrak{m}_A \subseteq \mathfrak{m}_B$).

<u>THEOREM 6.1A</u> Let K be a field. A local ring R contained in K is a valuation ring of K if and only if it is a maximal element of the set of local rings contained in K, with respect to the relation of domination. Every local ring contained in K is dominated by some valuation ring of K.

<u>PROOF</u> Atiyah-MacDonald Exs Section 5. □

the <u>value group</u> is $v(K^*)$. So the zero valuation is <u>not</u> discrete.

<u>DEFINITION</u> A valuation v is <u>discrete</u> if its value group G is the integers. The corresponding valuation ring is called a discrete valuation ring. A discrete valuation ring is a Noetherian local domain of dimension 1.

<u>THEOREM 6.2A</u> Let A be a Noetherian local domain of dimension one, with maximal ideal $\mathfrak{m}$. Then the following conditions are equivalent:

(i) A is a discrete valuation ring (i.e. there is a discrete valuation on $Q(A)$ s.t. A is the val. ring) ⊛
(ii) A is integrally closed
(iii) A is a regular local ring
(iv) $\mathfrak{m}$ is principal

<u>PROOF</u> A&M Prop 9.2 p94.

Let v be a discrete valuation on K, A its valuation ring. Then A is a valuation ring of K, so K is isomorphic to the quotient field of A, so A is indeed a discrete valuation ring in the ⊛ sense.

<u>DEFINITION</u> A <u>Dedekind domain</u> is an integrally closed noetherian domain of dimension one.

Because integral closure is a local property, (A&M Prop 5.13 p63) every localisation of a Dedekind domain at a nonzero prime ideal is a DVR ($\dim A_{\mathfrak{p}} = \operatorname{ht}\mathfrak{p}$).

THEOREM 6.3A The integral closure of a Dedekind domain in a finite extension field of its quotient field is again a Dedekind domain.

PROOF Z&S 1, Vol.1 Th.19 p281 (see our Elements of Field Theory Notes).

We now turn to the case of a function field K of dimension 1 over k, where k is our fixed algebraically closed base field. We wish to establish a connection between non-singular curves with function field K and the set of discrete valuation rings of K/k. If P is a point on a nonsingular curve Y, then by (5.1) the local ring is a regular local ring of dimension one, and so by (6.2A) it is a discrete valuation ring. Its quotient field is the function field K of Y, and since $k \subseteq \mathcal{O}_P$, it is a valuation ring of K/k. ($k \subseteq \mathcal{O}_P \Rightarrow v(k) \geq 0$ where v is the discrete valuation determined by $\mathcal{O}_P \subseteq K$, but the elements of k are units, so $v(k) = 0$). Thus the local rings of Y define a subset of the set C_K of all discrete valuation rings of K/k. This motivates the definition of an abstract nonsingular curve below. But first we need a few preliminaries.

LEMMA 6.4 Let Y be a quasi-projective variety, let $P, Q \in Y$ and suppose that $\mathcal{O}_Q \subseteq \mathcal{O}_P$ as subrings of $K(Y)$. Then $P = Q$.

PROOF Embed Y in $\mathbb{P}^n$ for some n. Replacing Y by its closure, we may assume Y projective. After a suitable linear change of coordinates in $\mathbb{P}^n$ we assume that neither P nor Q is in the hyperplane H_0 defined by $x_0 = 0$. To be explicit, let $P = (a_0, \dots, a_n)$, $Q = (b_0, \dots, b_n)$. If already for some i $a_i \neq 0$ and $b_i \neq 0$ perform the linear change $x_0 \leftrightarrow x_i$. Otherwise we may assume $a_0 \neq 0$ and that $a_i = 0$, $b_i \neq 0$ for some $i > 0$. Then the linear change of coordinates determined by

$$\begin{array}{c} \\ \\ \end{array}\begin{pmatrix} 1 & 0 & & 1 & & 0 \\ \vdots & 1 & \cdots & 0 & \cdots & \vdots \\ \vdots & \vdots & & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & & 0 \\ & & \cdots & 0 & & 1 \end{pmatrix} = I_{n+1} + E_{1i}$$

has the desired effect. Note that $\mathcal{O}_Q \subseteq \mathcal{O}_P$ as subrings of $K(Y)$ means precisely that if (U, f) is a regular map with $Q \in U$ then there is a regular map (V, g) with $P \in V$ s.t. $f = g$ on $U \cap V$. This property is clearly preserved by automorphisms of $\mathbb{P}^n$, so we can safely reduce to $a_0 \neq 0$, $b_0 \neq 0$. Thus $P, Q \in Y_0$ which is affine, so we may assume that Y is an affine variety.

Let A be the affine ring of Y. Then there are maximal ideals $\mathfrak{m}, \mathfrak{n} \subseteq A$ s.t. $\mathcal{O}_P \cong A_{\mathfrak{m}}$ and $\mathcal{O}_Q \cong A_{\mathfrak{n}}$, that is, P and Q correspond to $\mathfrak{m}, \mathfrak{n}$ resp. under (3.2c). We claim that $\mathcal{O}_Q \subseteq \mathcal{O}_P$ implies $\mathfrak{m} \subseteq \mathfrak{n}$. Say $f \in \mathfrak{m}$ so $f(P) = 0$, and suppose $f(Q) \neq 0$ so $f \notin \mathfrak{n}$. Let $\varphi : Q(A(Y)) \longrightarrow K(Y)$ be the isomorphism of (3.2). Since $f \neq 0$ we have

$$(U, f) = \varphi(1/f) \quad \text{regular } U \longrightarrow k, \quad U = \{x \in Y \mid f(x) \neq 0\}$$

Since $Q \in U$, the fact that $\mathcal{O}_Q \subseteq \mathcal{O}_P$ in $K(Y)$ means that there is $g/h \in Q(A(Y))$ s.t. $h(P) \neq 0$ and $(U, f) = (D(h), g/h)$ in $K(Y)$. Since φ is injective this implies that $1/f = g/h$ in $Q(A(Y))$ so $fg = h$ in $A(Y)$. But this contradicts the fact that $h(P) \neq 0$. Hence we must have $f(Q) = 0$ and so $f \in \mathfrak{n}$, as required. Since $\mathfrak{m}, \mathfrak{n}$ are maximal $\mathfrak{m} \subseteq \mathfrak{n}$ implies $\mathfrak{m} = \mathfrak{n}$ so $P = Q$. □

As the proof shows, Lemma 6.4 holds for <u>any</u> variety (which is also obvious since every variety is isomorphic to a quasi-proj variety).

LEMMA 6.5 Let K be a function field of dimension 1 over k, and let $x \in K$. Then if

$$C_K = \{ R \mid R \text{ is the valuation ring of a discrete valuation } v \text{ of } K/k \}$$

The set $\{ R \in C_K \mid x \notin R \}$ is a finite set. (possibly empty)

PROOF

If R is a valuation ring then $x \notin R$ iff. $x^{-1} \in \mathfrak{m}_R$. So letting $y = x^{-1}$ we have to show that if $y \in K, y \neq 0$, then $\{ R \in C_K \mid y \in \mathfrak{m}_R \}$ is a finite set. If $y \in k$ there are no such R, so let us assume $y \notin k$.

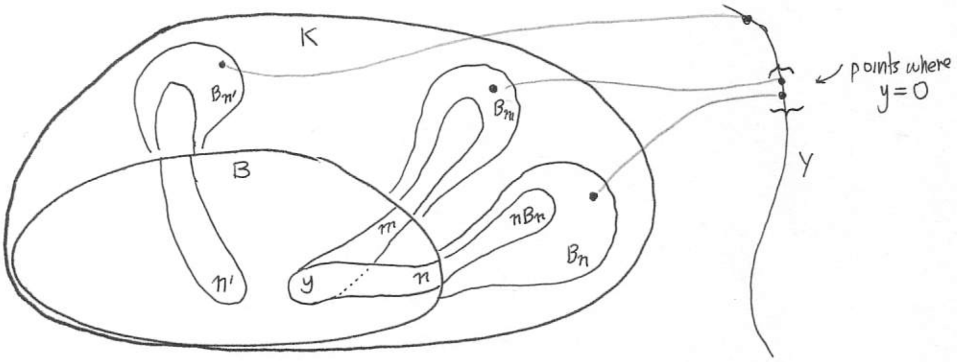
We consider the subring $k[y]$ of K generated by y. Since k is algebraically closed, y is transcendental over k, so $k[y]$ is a polynomial ring. Hence the extension $k(y)/k$ has tr. deg 1. If T were a transcendence basis for $K/k(y)$ then $T \cup \{y\}$ would be a transcendence basis for K/k – but then T must be empty since tr.deg.$K/k = 1$. Hence K is algebraic over $k(y)$ and since K/k is f.g. it follows that $K/k(y)$ is f.g. and algebraic – hence finite. Now let B be the integral closure of $k[y]$ in K. Then by (6.3A) B is a Dedekind domain. By (Corollary 1 to Theorem 7 §4 Ch.V Z&S) B is a finitely generated $k[y]$-module, and it is straightforward to check that B is thus a finitely-generated k-algebra.

NOTE: By Z&S Theorem 7 §Ch.V K is the quotient field of B, so if $\mathfrak{n}$ is any maximal ideal of B then $B_\mathfrak{n}$ is a discrete valuation ring with quotient field K. So the set C_K is nonempty – in fact, we can produce elements of C_K containing any given $y \in K - k$.

Let R be a discrete valuation ring of K/k and suppose $y \in \mathfrak{m}_R$. Then $k[y] \subseteq R$, and since R is integrally closed in K, we have $B \subseteq R$. Let $\mathfrak{n} = \mathfrak{m}_R \cap B$. Then $\mathfrak{n}$ is a prime ideal of B, and $y \in \mathfrak{n}$ so $\mathfrak{n} \neq 0$ and hence $\mathfrak{n}$ is maximal since $\dim B = 1$. If $s \in B - \mathfrak{n}$ then $s \notin \mathfrak{m}_R$ so s is a unit in R and so $s^{-1} \in R$. Hence $B_\mathfrak{n} \subseteq R$. Since $\mathfrak{m}_R \cap B_\mathfrak{n}$ is a proper prime ideal $\mathfrak{m}_R \cap B_\mathfrak{n} \subseteq \mathfrak{n}B_\mathfrak{n}$, but on the other hand $\mathfrak{m}_R \cap B_\mathfrak{n} \supseteq \mathfrak{m}_R \cap B = \mathfrak{n}$ so $\mathfrak{m}_R \cap B_\mathfrak{n} \supseteq \mathfrak{n}B_\mathfrak{n}$. Hence $B_\mathfrak{n}$, R are local rings in K and R dominates $B_\mathfrak{n}$. But B is a Dedekind domain and $\mathfrak{n} \neq 0$ so $B_\mathfrak{n}$ is a discrete valuation ring, and since K is the quotient field of B (see our Z&S notes Theorem 7 §4 (Ch.V)), K is the quotient field of $B_\mathfrak{n}$, so $B_\mathfrak{n}$ is a valuation ring of K (i.e. $\forall\, 0 \neq x \in K$ $x \in B \vee x^{-1} \in B$) and by (6.1A) we must have $R = B_\mathfrak{n}$.

Now B is the affine coordinate ring of some affine variety Y (1.4.6). Since B is a Dedekind domain, Y has dimension 1 and is nonsingular. To say that $y \in \mathfrak{n}$ says that y, as a regular function on Y, vanishes at the point of Y corresponding to $\mathfrak{n}$. But $y \neq 0$, so it vanishes only at a finite set of points; these are in (1-1) correspondence with the maximal ideals of B by (3.2), and $R = B_\mathfrak{n}$ is determined by the maximal ideal $\mathfrak{n}$. (Since points where $y = 0$ closed subset of Y, which has dimension 1, so since $y \neq 0$ in B by Ex1.10 dimension is 0, so finite set of points). Hence we conclude that $y \in \mathfrak{m}_R$ for only finitely many $R \in C_K$, as required. □

This is a nice result. In pictures: each $y \neq 0$ in K with $y \notin k$ determines a Dedekind domain $B \subseteq K$ with quotient field K s.t. $k[y] \subseteq B \subseteq K$. Let $\mathfrak{n}$ be a maximal ideal of B. Then $B_\mathfrak{n}$ is a discrete valuation ring of K/k and $B_\mathfrak{n}$ is the <u>unique</u> element of C_K dominating $(B, \mathfrak{n})$ – that is, s.t. $B \subseteq B_\mathfrak{n}$ and $\mathfrak{n}B_\mathfrak{n} \cap B = \mathfrak{n}$. The maximal ideals with $y \in \mathfrak{n}$ are in bijection with the points of Y satisfying $y = 0$, are in bijection with the elements of C_K containing y.



<u>COROLLARY 6.6</u> Any discrete valuation ring of K/k is isomorphic to the local ring of a point on some nonsingular affine curve, which has function field K.

<u>PROOF</u>

Let v be a discrete valuation of K/k with valuation ring R. Then $k \subset R \subset K$ since R cannot be a field. Let $0 \neq y \in \mathfrak{m}_R$ (so $y \notin k$ since $k \subseteq R$). Then as we saw in the proof of (6.5) R is isomorphic as a k-algebra to $B_\mathfrak{n}$ (where $\mathfrak{n}$ is a maximal ideal of B) which is isomorphic as a k-algebra to the local ring of the nonsingular affine curve Y at the point corresponding to $\mathfrak{n}$. Moreover, the function field of Y is isomorphic as a k-algebra to the quotient field of B, which is isomorphic as a k-algebra to K:

R isomorphic to $\mathcal{O}_{P,Y}$ as k-algebras
K isomorphic to $K(Y)$ as k-algebras. □

We now come to the definition of an abstract nonsingular curve. Let K be a function field of dimension 1 over k (i.e. a finitely generated extension field of transcendence degree 1). Let C_K be the set of all discrete valuation rings of K/k. We will sometimes call the elements of C_K points, and write $P \in C_K$, where P stands for the valuation ring R_P. As noted in the proof of (6.5) the set C_K is nonempty. In fact, C_K is infinite: from (6.5) and (6.6) we know there is a nonsingular affine curve Y s.t. $K(Y)$ is isomorphic as a k-algebra to K. The set Y is infinite since it has dimension 1 and is irreducible, and for each $P \in Y$ the k-algebra $\mathcal{O}_{P,Y}$ is a subring of $K(Y)$ containing k, and moreover $\mathcal{O}_{P,Y}$ is a discrete valuation ring. Since $K(Y)$ is the quotient field of any such $\mathcal{O}_{P,Y}$ (see our §3 Notes), and since by (6.4) all the $\mathcal{O}_{P,Y}$ are distinct subrings of $K(Y)$, there is an infinite number of discrete valuation rings of K/k. Hence C_K is infinite. Note that another equivalent definition of C_K is

$$C_K = \{ R \mid R \text{ is an integrally closed nothenan local subring of } K \text{ of dim } 1 \text{ containing } k, \text{ such that } K \text{ is the smallest subfield of } K \text{ containing } R \}$$

We make C_K into a topological space by taking the closed sets to be the finite subsets and the whole space. If $U \subseteq C_K$ is an open subset of C_K, we define the ring of regular functions on U to be

$$\mathcal{O}(U) = \bigcap_{P \in U} R_P \qquad (\mathcal{O}(\emptyset) = 0)$$

If U is nonempty every $R_P, P \in U$ is a subring of K with $k \subseteq R \subseteq K$. Hence $\mathcal{O}(U)$ is a subring of K containing k. If U, V are two open subsets of C_K with $U \subseteq V$ then clearly $\mathcal{O}(V)$ is a subring of $\mathcal{O}(U)$, so there is a canonical morphism of rings $\mathcal{O}(V) \longrightarrow \mathcal{O}(U)$. Hence $\mathcal{O}$ is a presheaf of rings. From (6.6) we know that for $R \in C_K$ in the map $R \longrightarrow R/m_R$ every element is the image of some element of k. That is, the residue field of R is k, in the obvious way. So if $f \in \mathcal{O}(U)$ with $U \neq \emptyset$ then f defines a function from $U \longrightarrow k$ by taking $f(P)$ to be the residue of f modulo the maximal ideal of R_P. If two elements $f, g \in \mathcal{O}(U)$ define the same function, then $f - g \in m_P$ for infinitely many $P \in C_K$, so by (6.5) and its proof $f = g$. Thus we can identify elements of $\mathcal{O}(U)$ with functions from $U \longrightarrow k$. Note that by the proof of (6.5) every $f \in K$ belongs to some $R \in C_K$. That is, every element of K belongs to some $\mathcal{O}(U)$ (since $\{R \in C_K \mid f \notin R\}$ is finite). Notice that C_K is an irreducible noethenan topological space in which points are closed (this follows immediately from the definition of C_K's topology). We can define the function field $K(C_K)$ as in §3: the elements of $K(C_K)$ are pairs (U, f) where U is a nonempty open set and $f \in \mathcal{O}(U)$. We define an equivalence relation by $(U,f) \sim (V,g)$ if $f = g$ (as functions) on $U \cap V$. But $U \cap V$ is an infinite subset of C_K, so again by (6.5) $f = g$ as elements of K, so $\sim$ is an equivalence relation. The ring operations on $K(C_K)$ coincide with the operations in K, and since every $f \in K$ belongs to some $\mathcal{O}(U)$ it is clear that the function field of C_K is just K. So in summary:

- C_K is an infinite, noethenan topological space in which points are closed
- C_K is irreducible and has dimension 1
- $\mathcal{O}$ is a sheaf of k-algebras (the sheaf condition is easily checked)

So that C_K looks a lot like a nonsingular curve!

<u>DEFINITION</u> An <u>abstract nonsingular curve</u> is an nonempty open subset $U \subseteq C_K$ where K is a function field of dimension 1 over k, with the induced topology, and the induced notion of regular functions on its open subsets. So U is infinite, irreducible, noethenan, has closed points and the structure sheaf is a sheaf of k-algebras.

Note that it is not clear a priori that such an abstract curve is a variety. So we will enlarge the category of varieties by adjoining the abstract curves.

<u>DEFINITION</u> A <u>morphism</u> $\varphi: X \longrightarrow Y$ between abstract nonsingular curves or varieties is a continuous mapping such that for every open set $V \subseteq Y$ and every regular function $f: V \longrightarrow k$, $f\varphi$ is regular on $\varphi^{-1}V$.

where, as defined above, if $V \subseteq C_K$ is open $f: V \longrightarrow k$ is <u>regular</u> if there is $g \in \mathcal{O}(V)$ s.t. $\forall P \in V$ $f(P) =$ residue of g modulo m_P. It is easily checked that this is a category. Now that we have apparently enlarged our category, our task will be to show that every nonsingular quasi-projective curve is isomorphic to an abstract nonsingular curve, and conversely. In particular, we will show that C_K itself is isomorphic to a nonsingular projective curve.

PROPOSITION 6.7 Every nonsingular quasi-projective curve Y is isomorphic to an abstract nonsingular curve.

PROOF Let K be the function field of Y. Then each local ring of a point $P \in Y$ gives a subring of K containing k, and K is the quotient field of each $\mathcal{O}_P$. Since Y is a nonsingular curve, each $\mathcal{O}_P$ is a discrete valuation ring of K/k. Furthermore, by (6.4) distinct points give rise to distinct subrings of K. So let $U \subseteq C_K$ be the set of all local rings of Y, and let $\varphi : Y \longrightarrow U$ be the bijective map defined by $\varphi(P) = \mathcal{O}_P$.

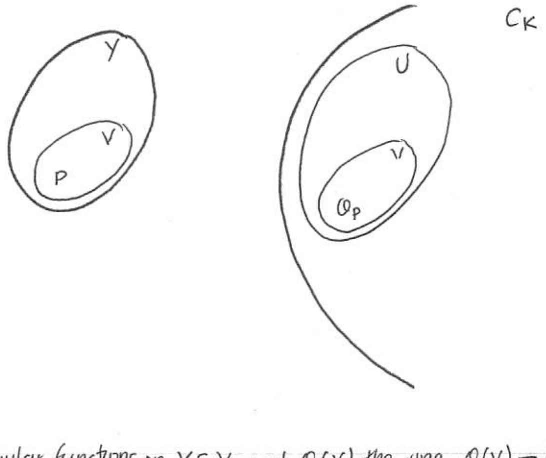
We know U is infinite since Y is, but we need to show its complement is finite.

First, we need to show that U is an open subset of C_K. Because open sets are complements of finite sets, it is sufficient to show that U contains a nonempty open set. Thus by (4.3), we may assume Y is affine, with affine ring A. Then A is a finitely-generated k-algebra and by (3.2) K is the quotient field of A, and U is the set of localisations of A at its maximal ideals (at least $Q(A)$ is k-iso to K, so this all works). Since these local rings are all discrete valuation rings of K/k, U consists in fact of all discrete valuation rings of K containing A: for if B is such a ring with maximal ideal $\mathfrak{m}$, then if $\mathfrak{m} \cap A = 0 \Rightarrow$ nonzero elts. A units in $B \Rightarrow B = K$ a contradiction. Hence $\mathfrak{n} = \mathfrak{m} \cap A$ is a maximal ideal of A (it is prime $\neq 0$ and $\dim A = 1$). As in the proof of Lemma 6.5 it follows that B dominates $A_{\mathfrak{n}}$ whence $B = A_{\mathfrak{n}}$ by maximality of valuation rings (6.1A).

Now let $x_1, \dots, x_n$ be a set of generators of A over k. Then $A \subseteq R_P$ iff. $x_1, \dots, x_n \in R_P$. Thus $U = \bigcap U_i$ where $U_i = \{P \in C_K \mid x_i \in R_P\}$. But by (6.5) $\{P \in C_K \mid x_i \notin R_P\}$ is a finite set (possibly $\emptyset$). Therefore each U_i and hence U is open.

Now back to $Y =$ quasi-proj.

So we have shown that the U defined above is an abstract nonsingular curve. To show that φ is a homeomorphism, we need only note that Y is a variety of dimension 1. Clearly finite sets in Y are closed, and if $Z \subseteq Y$ is closed either $Z = Y$ or $\dim Z < \dim Y = 1$ (Ex 1.10) so $\dim Z = 0$ and Z is a finite set. So the topology on any curve is the finite closed set topology, so φ is a homeomorphism. To show that φ is an isomorphism of varieties (or of abstract nonsingular curves more properly) we need only check that the regular functions on any open set are the same. Let $V \subseteq Y$ be open



Denote by $\mathcal{O}_Y(V)$ the regular functions on $V \subseteq Y$ and $\mathcal{O}(V)$ the ring $\mathcal{O}(V) = \bigcap_{P \in V} \mathcal{O}_P$. Recall that $\mathcal{O}_Y(Y)$, $\mathcal{O}_{P,Y}$ any $P \in Y$ can be considered as subrings of $K(Y)$. Under this identification, $\mathcal{O}_Y(V) = \bigcap_{P \in V} \mathcal{O}_{P,Y}$ ($\subseteq$ is clear) since if $(U, f) \in K(Y)$ is in each $\mathcal{O}_{P,Y}$ – say $(U, f) = (U_P, f_P)$ $P \in U_P$ so that $f = f_P$ on $U \cap U_P$. We can paste the f_P to make $(V, g) \in \mathcal{O}_Y(V)$ s.t. $g(P) = f_P(P)$ $\forall P \in V$. Then if $P \in U \cap V$, $g(P) = f_P(P) = f(P)$ since $P \in U \cap U_P$, so $(U, f) = (V, g)$ in $K(Y)$ and hence $(U, f) \in \mathcal{O}_Y(V)$. So the rings $\mathcal{O}_Y(V)$ and $\mathcal{O}(V)$ coincide. Moreover $f \in \mathcal{O}(V)$ is interpreted as the map $P \mapsto$ residue of f in $\mathcal{O}_P/\mathfrak{m}_P = f(P) \in k$. Hence φ is an isomorphism. $\square$

NOTE (6.7) says every nonsingular curve Y (i.e. variety Y nonsingular $\dim Y = 1$) is isomorphic to an open subset of $C_{K(Y)}$.
(i.e. the proof never uses quasi-proj)

Now we need a result about extensions of morphisms from curves to projective varieties, which is interesting in its own right.

PROPOSITION 6.8 Let X be an abstract nonsingular curve, let $P \in X$, let Y be projective variety, and let $\varphi : X - P \longrightarrow Y$ be a morphism. Then there exists a unique morphism $\bar{\varphi} : X \longrightarrow Y$ extending φ.

PROOF Embed Y as a closed subset of $\mathbb{P}^n$ for some n. Then φ is a morphism $X - P \longrightarrow \mathbb{P}^n$. Suppose we could extend this uniquely to $\bar{\varphi} : X \longrightarrow \mathbb{P}^n$. Since X is irreducible, $\bar{\varphi}(X) = \varphi(X - P) \cup \bar{\varphi}(P)$ is irreducible – but $\bar{\varphi}(P)$ is a point, hence closed, and if $\bar{\varphi}(P) \notin Y$ then $\varphi(X - P) = \bar{\varphi}(X) \cap Y$ is also closed in $\bar{\varphi}(X)$ – but this leads to a contradiction, so we must have had $\bar{\varphi}(P) \in Y$. Hence $\bar{\varphi}$ gives a morphism $X \longrightarrow Y$ extending φ, and the extension is clearly unique. Thus we reduce to the case $Y = \mathbb{P}^n$.

[For this we must know that if $U \xrightarrow{\phi} X$ is a morphism where U is an abstract nonsingular curve, and if $Y \subseteq X$ is a subset with the induced ringed space structure, if $\operatorname{Im} \phi \subseteq Y$ then $U \longrightarrow Y$ is a morphism of ringed spaces – use sheaf condition on U and usual arg. see §3 Exercises Note]

Let $\mathbb{P}^n$ have homogenous coordinates $x_0,\dots,x_n$ and let U be the open set where $x_0,\dots,x_n$ are all nonzero. By using induction on n, we may assume that $\varphi(X-P)\cap U\neq\phi$. Because if $\varphi(X-P)\cap U=\phi$, then $\varphi(X-P)\subseteq\mathbb{P}^n-U$. But $\mathbb{P}^n-U$ is the union of the hyperplanes $H_i=Z(x_i)$. Since $\varphi(X-P)$ is irreducible, it must be contained in H_i for some i. Now $H_i\cong\mathbb{P}^{n-1}$, so the result follows by a quick check of the details and induction. Of course, we need to know the result for $n=1$: but then $\mathbb{P}^1=U\cup\{(0,1),(1,0)\}$ so $\varphi(X-P)$ consists of a single point, if $\varphi(X-P)\cap U=\phi$. So in that case there is a trivial unique extension $\bar{\varphi}$ mapping P to this same point.

So, if we prove the result for all n in the case where $\varphi(X-P)\cap U\neq\phi$ then a) it will be true for $n=1$ in either case and so b) proceeding by induction it will be true for any n, whether $\varphi(X-P)\cap U=\phi$ or not.

So assume $\varphi(X-P)\cap U\neq\phi$. For each i,j x_i/x_j is a regular function on U. Pulling it back by φ, we obtain a regular function f_{ij} on a nonempty open subset of X, which we view as a rational function on X, i.e. an element $f_{ij}\in K$, where is the function field of X. Let v be the valuation of K associated with the valuation ring R_P. Let $r_i=v(f_{i0})$ $i=0,1,\dots,n$ $r_i\in\mathbb{Z}$. Then since $x_i/x_j=(x_i/x_0)/(x_j/x_0)$ we have $f_{ij}=f_{i0}f_{j0}^{-1}$ so

$$v(f_{ij})=r_i-r_j\qquad i,j=0,\dots,n$$

[Note $f_{ii}=1\in K$ for all i]

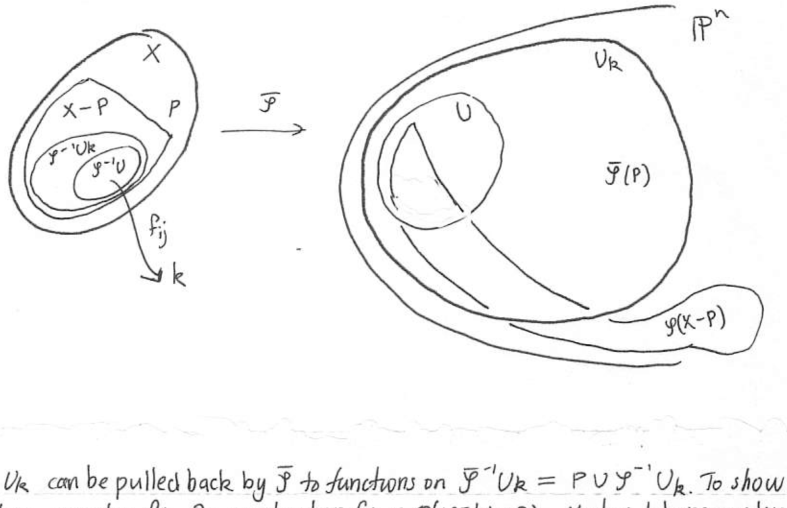
Choose k such that r_k is minimal among $r_0,\dots,r_n$ (these may be ≤ 0). Then $v(f_{ik})\geq 0$ for all i hence $f_{0k},\dots,f_{nk}\in R_P$. Now define $\bar{\varphi}:X\longrightarrow\mathbb{P}^n$ by

$$\bar{\varphi}(P)=(f_{0k}(P),\dots,f_{nk}(P))$$
$$\bar{\varphi}(Q)=\varphi(Q)\qquad Q\neq P$$

Note that $v(f_{kk})=0$ so $f_{kk}(P)\neq 0$ and $\bar{\varphi}(P)$ is well-defined. I claim that $\bar{\varphi}$ is a morphism of X to $\mathbb{P}^n$ which extends φ, and that $\bar{\varphi}$ is unique. Uniqueness follows from (4.1). To show that $\bar{\varphi}$ is a morphism, we first show that $\bar{\varphi}$ is continuous; if $V\subseteq\mathbb{P}^n$ is open and $\bar{\varphi}(P)\notin V$ then $\bar{\varphi}^{-1}(V)=\varphi^{-1}V$ which is open in X. If $\bar{\varphi}(P)\in V$ then $\bar{\varphi}^{-1}(V)=\varphi^{-1}V\cup P$. If $\varphi^{-1}V$ is nonempty then it has finite complement in C_K, so $\varphi^{-1}V\cup P$ will also have finite complement and is thus open. If $\varphi^{-1}V=\phi$ then in particular $\varphi(X-P)\subseteq V^c$, which is a closed set in $\mathbb{P}^n$. Hence $\varphi(X)$, which is an irreducible subspace of $\mathbb{P}^n$, is the union $\varphi(X-P)\cup\bar{\varphi}(P)$ of two closed subsets, and so $P\in\varphi(X-P)$ — contradicting the assumption $\bar{\varphi}(P)\in V$. Hence $\varphi^{-1}V$ is nonempty and $\bar{\varphi}$ is continuous.

To check that $\bar{\varphi}$ is a morphism, it will be sufficient to show that regular functions in a neighborhood of $\bar{\varphi}(P)$ pull back to regular functions on X. To start with, let $U_k\subseteq\mathbb{P}^n$ be the open set $x_k\neq 0$. Then $\bar{\varphi}(P)\in U_k$, and U_k is affine with affine coordinate ring

$$k[x_0/x_k,\dots,x_{k-1}/x_k,x_{k+1}/x_k,\dots,x_n/x_k]$$



The regular functions x_i/x_k on U_k can be pulled back by $\bar{\varphi}$ to functions on $\bar{\varphi}^{-1}U_k=P\cup\varphi^{-1}U_k$. To show that these functions are regular: say x_i/x_k, consider f_{ik}. By construction $f_{ik}\in\mathcal{O}(\varphi^{-1}U\cup P)$ — that is, taking residues of f_{ik} gives a regular map $\varphi^{-1}U\cup P\longrightarrow k$. But $x_i/x_k\,\bar{\varphi}(P)=f_{ik}(P)/f_{kk}(P)=f_{ik}(P)$ ($f_{kk}(P)=1$ since $f_{kk}=1$). So $x_i/x_k\,\bar{\varphi}:\bar{\varphi}^{-1}U_k\to k$ agrees with a regular map on $\varphi^{-1}U\cup P$. But $x_i/x_k:U_k\longrightarrow k$ is regular, so $x_i/x_k\,\varphi:\varphi^{-1}U_k\to k$ is regular — so there is $g\in K$ with the right residues — but then this implies $g=f_{ik}$, so $x_i/x_k\,\bar{\varphi}$ is regular on all of $\bar{\varphi}^{-1}U_k$.

Now let V be any open neighborhood of $\bar{\varphi}(P)$ in $\mathbb{P}^n$, $f:V\longrightarrow k$ regular. As noted above, $\varphi(X-P)\cap V$ is nonempty, so $\bar{\varphi}^{-1}V=\varphi^{-1}V\cup P$. Since f is regular there is an open set W with $\bar{\varphi}(P)\in W\subseteq V\cap U_k$ and polynomials $f(x_1/x_k,\dots,x_n/x_k)$ $g(x_1/x_k,\dots,x_n/x_k)$ with $g\neq 0$ on W s.t. $f=g/h$ on W (using $U_k\cong\mathbb{A}^n$). Clearly $\bar{\varphi}^{-1}V=\varphi^{-1}V\cup\bar{\varphi}^{-1}W$. We already know that $f\bar{\varphi}$ is regular on $\varphi^{-1}V$, so it suffices to show $f\bar{\varphi}$ is regular on $\bar{\varphi}^{-1}W$. But the $x_i/x_k\,\bar{\varphi}$ are regular on $\bar{\varphi}^{-1}U_k$, so this is straightforward to check. Hence $\bar{\varphi}$ is a morphism. $\square$

THEOREM 6.9 Let K be a function field of dimension 1 over k. Then the abstract nonsingular curve C_K defined above is isomorphic to a nonsingular projective curve.

PROOF The idea of the proof is this: we first cover $C = C_K$ with open subsets U_i which are isomorphic to nonsingular affine curves (by Ex 1.10 this implies $\dim C = 1$). Let Y_i be the projective closure of this affine curve. Then we use (6.8) to define a morphism $\varphi_i : C \longrightarrow Y_i$. Next we consider the product mapping $\varphi : C \longrightarrow \prod Y_i$, and let Y be the closure of the image of C. Then Y is a projective curve, and we show that φ is an isomorphism of C onto Y.

To begin with, let $P \in C$ be any point. Then by (6.6) there is a nonsingular affine curve V and a point $Q \in V$ with $R_P \cong \mathcal{O}_Q$ as k-algebras. It follows that $K(V) \cong K$ as k-algebras, hence $C_K \cong C_{K(V)}$ as abstract nonsingular curves. By (6.7) V is isomorphic to an open subset of $C_{K(V)}$, hence to an open subset of $C_{K(V)}$ (containing P). Thus we have shown that every point $P \in C$ has an open neighborhood which is isomorphic to an affine variety (an affine nonsingular curve, in fact). Hence $\dim C = 1$.

Since C is quasi-compact, we can cover it with a finite number of open subsets U_i, each of which is isomorphic to an affine nonsingular curve V_i. Embed $V_i \subseteq \mathbb{A}^{n_i}$, think of $\mathbb{A}^{n_i}$ as an open subset of $\mathbb{P}^{n_i}$, and let Y_i be the closure of V_i in $\mathbb{P}^{n_i}$. Then Y_i is a projective curve, and we have a morphism $\varphi_i : U_i \longrightarrow Y_i$ which is an isomorphism of U_i onto its image (which is an open subset)

By (6.8) applied to the finite set of points $C - U_i$, we can find a morphism $\overline{\varphi}_i : C \longrightarrow Y_i$ extending φ_i. Let $\prod Y_i$ be the product of the projective varieties Y_i (Ex 3.16). Then $\prod Y_i$ is also a projective variety - it is a product in the category of varieties. But C is <u>not</u> a variety, so we need to exercise some caution. Fix j and consider the morphisms $\overline{\varphi}_i|_{U_j} : U_j \longrightarrow Y_i$. Using the fact that $U_j \cong V_j$ and the categorical product $\prod Y_i$ the map

$$\begin{aligned} U_j &\longrightarrow \prod Y_i \\ u &\longmapsto (\overline{\varphi}_i(u))_i \end{aligned}$$

is a morphism. But this is the restriction to U_j of the map $\varphi : C \longrightarrow \prod Y_i$, $c \longmapsto (\overline{\varphi}_i(c))_i$, so φ is a morphism. Let Y be the closure of the image of φ. Since C is irreducible, Y is a projective variety, and $\varphi : C \longrightarrow Y$ is a morphism whose image is dense in Y.

We must show that Y is nonsingular, has dimension 1, and that φ is an isomorphism. For every point $P \in C$ we have $P \in U_i$ for some i. There is a commutative diagram

$$\begin{tikzcd} C \arrow[r, "\varphi"] & Y \arrow[d, "\pi"] \\ U_i \arrow[u, hook] \arrow[r, "\varphi_i"] & Y_i \end{tikzcd}$$

It is easy to see that there is a morphism of k-algebras $\mathcal{O}_{\varphi(P),Y} \longrightarrow \mathcal{O}_P$ ($\mathcal{O}_P \subseteq K$). The usual argument (Ex 3.3.c) still works to show that this map is injective since $\operatorname{Im}\varphi$ is dense. Similarly there is an injective morphism of rings $\mathcal{O}_{\varphi_i(P),Y_i} \longrightarrow \mathcal{O}_P$ induced by $\varphi_i : U_i \longrightarrow Y_i$. It is not hard to check that

$$\begin{tikzcd} \mathcal{O}_{\varphi(P),Y} \arrow[dr] & \\ & \mathcal{O}_P \\ \mathcal{O}_{\varphi_i(P),Y_i} \arrow[uu] \arrow[ur] & \end{tikzcd}$$

is a commutative diagram of k-algebras. The bottom morphism is also surjective, since any $f \in \mathcal{O}_P$ becomes a regular map on a neighborhood of P (by 6.5) and φ_i is an isomorphism onto its image. It follows that the top morphism is an isomorphism of k-algebras. So for any $P \in C$ the morphism $\varphi_P^* : \mathcal{O}_{\varphi(P),Y} \longrightarrow \mathcal{O}_P$ is an isomorphism. Since every $\mathcal{O}_P$ is a discrete valuation ring, it follows that Y is a nonsingular projective curve, provided we can show that φ is surjective. The fact that φ_P^* is an isomorphism of k-algebras also implies that $K(Y)$ and K are isomorphic as k-algebras. To see this in a more useful way, note that since $\varphi : C \longrightarrow Y$ has dense image we can define a morphism of k-algebras $\phi : K(Y) \longrightarrow K$ as in §4. Considering $\mathcal{O}_{\varphi(P),Y}$ as a subring of $K(Y)$, ϕ extends φ_P^* (for all P). Since ϕ is trivially injective and also surjective because the φ_P^* are, ϕ is an isomorphism. (any $y \in K$ is contained in some $\mathcal{O}_P$)

To see that φ is surjective let $Q \in Y$. Then the local subring $\mathcal{O}_{Q,Y} \subseteq K(Y)$ is contained in some discrete valuation ring of $K(Y)/k$ (take for example a localisation of the integral closure of $\mathcal{O}_{Q,Y}$ at a maximal ideal. Integral extensions preserve dimension, and $\dim \mathcal{O}_{Q,Y} = \dim Y = 1$, To see that φ is surjective, let $Q \in Y$. Then the local subring $\mathcal{O}_{Q,Y} \subseteq K(Y)$ is contained in some discrete valuation ring of $K(Y)/k$ (see the next Lemma), say R. Identifying $K(Y)$ and K via ϕ, R corresponds to R_P for some $P \in C$ and it follows that $R = \mathcal{O}_{\varphi(P),Y}$ since ϕ identifies $\mathcal{O}_{\varphi(P),Y}$ with R_P. By (6.4) we must have $Q = \varphi(P)$. This shows that φ is surjective. Again using the fact that under ϕ $\mathcal{O}_{\varphi(P),Y} \longleftrightarrow R_P$ we see that φ is injective as well. Hence φ is a bijective morphism of C to Y (and as observed earlier this means Y is a nonsingular projective curve). Since the topology on any curve is the finite-complement topology, φ is trivially a homeomorphism. For every $P \in C$ φ_P^* is an isomorphism, so the argument of Ex 3.3b can be easily adapted to show that φ is an isomorphism. $\square$

<u>LEMMA</u> Let Y be any curve, $P \in Y$ a point of Y. Then the local subring $\mathcal{O}_{P,Y} \subseteq K(Y)$ is contained in some discrete valuation ring of $K(Y)/k$.

<u>PROOF</u> By covering Y with affine open sets we may reduce to the case where $Y \subseteq \mathbb{A}^n$ is an affine curve. Say $Y = Z(\mathfrak{p})$ for the prime ideal $\mathfrak{p} \subseteq k[x_1, \ldots, x_n]$. Let K be the quotient field of the domain $A(Y) = k[x_1, \ldots, x_n]/\mathfrak{p}$. Then by (3.9A) with $L = K$ it follows that the integral closure R of $A(Y)$ in K is a f.g. k-domain (in particular, R is Noetherian). Let S be the multiplicatively closed set $A(Y) - \mathfrak{m}_P$. By Proposition 5.12 of A&M, R_S is the integral closure of $A(Y)_{\mathfrak{m}_P}$ in K (since K_S may be identified with K). Since there is an isomorphism of K with $K(Y)$ identifying $\mathcal{O}_{P,Y}$ with $A(Y)_{\mathfrak{m}_P}$ it follows that the integral closure V of $\mathcal{O}_{P,Y}$ in $K(Y)$ is noetherian (since R, hence R_S, is). But integral extensions preserve dimension, so V is an integrally closed noetherian domain of dimension 1 (since $K(Y)$ is the quotient field of $\mathcal{O}_{P,Y}$, hence of V), that is, V is Dedekind. Let $\mathfrak{m}$ be a maximal ideal of V. Then $V_{\mathfrak{m}}$ is a discrete valuation ring of $K(Y)/k$ containing $\mathcal{O}_{P,Y}$. $\square$

COROLLARY 6.10 Every abstract nonsingular curve is isomorphic to a quasi-projective curve. Every nonsingular quasi-projective curve is isomorphic to an open subset of a nonsingular projective curve.

COROLLARY 6.11 Every curve is birationally equivalent to a nonsingular projective curve

PROOF Let Y be a curve (i.e. a variety of dimension 1). By (5.3) the singular points of Y form a proper closed subset. Let $U \subseteq Y$ be the nonempty open subset of Y consisting of all the nonsingular points. Then U is a nonsingular curve, which by (6.10) is isomorphic to an open subset of a nonsingular projective curve Z. Hence Y is birational to Z. □

Combining (6.7) and (6.8) gives

PROPOSITION Let X be a nonsingular curve, $P \in X$ and let Y be a projective variety, $\varphi: X - P \longrightarrow Y$ a morphism. Then there exists a unique morphism $\bar{\varphi}: X \longrightarrow Y$ extending φ.

COROLLARY 6.12 There is a diagram of equivalences of categories

$$\begin{tikzcd} ① \arrow[rr, "\sim"] & & ② \arrow[dl, "d"] \\ & ③ \arrow[ul, "d"] & \end{tikzcd}$$

(we are not claiming this diagram commutes)

where d denotes a dual equivalence and:

① Nonsingular projective curves, and dominant morphisms
② Curves and dominant rational maps
③ Function fields of dimension 1 over k and k-homomorphisms

PROOF The functor ① ⟶ ② is easy to define: map curves to themselves and $\psi: X \longrightarrow Y$ to (X, ψ). By (6.11) every object in ② is isomorphic in ② to an image of ①. Moreover ① ⟶ ② is clearly injective on morphism sets. To see it is surjective let $(U, \varphi): X \longrightarrow Y$ be a dominant rational map between nonsingular projective curves. Since $U \subseteq X$ is nonempty open it must have finite complement, so we can use the previous proposition to extend $\varphi: U \longrightarrow Y$ to $\psi: X \longrightarrow Y$. Since $\operatorname{Im}\varphi$ is dense, ψ is a dominant morphism, and $(X, \psi) = (U, \varphi)$ as rational maps, so ① ⟶ ② is indeed an equivalence.

The functor ② ⟶ ③ is defined by $Y \longmapsto K(Y)$ and on morphisms by (4.4) (we have already checked this is a functor which is contravariant and bijective on morphism sets). Let K be a function field of dimension 1 over k. Then C_K is nonempty and by (6.6) we can produce an affine curve Y with $K(Y) \cong K$ as k-algebras, so ② ⟶ ③ is an arrow-reversing equivalence.

Define the functor ③ ⟶ ① as follows: let a function field K of dimension 1 over k be given. Then $K \longmapsto F(K)$ where $F(K)$ is the nonsingular projective curve constructed in (6.9) (Note certain choices were made in this construction, so ③ ⟶ ① will not be unique in this sense. The only choice is the choice of the affine open cover – and for each K there are a set of options: we may use NBG's global axiom of choice) If $K_2 \longrightarrow K_1$ is a homomorphism then $K_2 \cong K(F(K_2))$ and $K_1 \cong K(F(K_1))$ as k-algebras so there is a homomorphism $K(F(K_2)) \longrightarrow K(F(K_1))$ and hence by (4.4) a dominant rational map $F(K_1) \longrightarrow F(K_2)$ represented by say $\varphi: U \longrightarrow F(K_2)$, where $U \subseteq F(K_1)$ is an open subset. Again by the previous Proposition φ can be extended to $\bar{\varphi}: F(K_1) \longrightarrow F(K_2)$ which is a dominant morphism. Using the fact that ① ⟶ ② and ② ⟶ ③ are functors it is not hard to check that ③ ⟶ ① is a functor. Note that the fact that ① ⟶ ② is an equivalence means that nonsingular projective curves are birational iff. they are isomorphic as varieties. To see ③ ⟶ ① is an equivalence, let a nonsingular projective curve Y be given. By (6.7) Y is isomorphic to an open subset of $C_{K(Y)}$ and hence birational to $F(K(Y))$. Hence Y is isomorphic as a variety (hence also in ①) to $F(K(Y))$ as required. □

NOTE An important point in the proof: nonsingular projective curves are birational iff. they are isomorphic. So a birational equivalence class of curves contains precisely one nonsingular projective curve (up to isomorphism).