2. SCHEMES

In this section we will define the notion of a scheme. First we define affine schemes: to any ring A we associate a topological space together with a sheaf of rings on it, called Spec A. This construction parallels the construction of affine varieties (I, §1) except that the points of Spec A correspond to all prime ideals of A, not just the maximal ideals. Then we define an arbitrary scheme to be something which locally looks like an affine scheme. This definition has no parallel in Chapter I. An important class of schemes is given by the construction of the scheme Proj S associated to any graded ring S. This construction parallels the construction of projective varieties in (I, §2). Finally, we will show that the varieties of Chapter I, after a slight modification, can be regarded as schemes. Thus the category of schemes is an enlargement of the category of varieties.

Now we will construct the space Spec A associated to a ring A. As a set, we define Spec A to be the set of all prime ideals of A. If $\mathfrak{a}$ is any ideal of A, we define the subset $V(\mathfrak{a}) \subseteq \operatorname{Spec} A$ to be the set of all prime ideals which contain $\mathfrak{a}$. (If $A = 0$ then Spec $A = \emptyset$)

LEMMA 2.1 (a) If $\mathfrak{a}$ and $\mathfrak{b}$ are two ideals of A, then $V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$.
(b) If $\{\mathfrak{a}_i\}$ is any set of ideals of A, then $V(\sum \mathfrak{a}_i) = \bigcap V(\mathfrak{a}_i)$
(c) If $\mathfrak{a}$ and $\mathfrak{b}$ are two ideals, $V(\mathfrak{a}) \subseteq V(\mathfrak{b})$ if and only if $\sqrt{\mathfrak{a}} \supseteq \sqrt{\mathfrak{b}}$.

PROOF (a) Certainly if $\mathfrak{p} \supseteq \mathfrak{a}$ or $\mathfrak{p} \supseteq \mathfrak{b}$ we have $\mathfrak{p} \supseteq \mathfrak{a}\mathfrak{b}$. Conversely if $\mathfrak{p} \supseteq \mathfrak{a}\mathfrak{b}$ and $\mathfrak{b} \not\subseteq \mathfrak{p}$, let $b \in \mathfrak{b} \setminus \mathfrak{p}$. Then $\forall a \in \mathfrak{a}$, $ab \in \mathfrak{p} \Rightarrow a \in \mathfrak{p}$, so $\mathfrak{a} \subseteq \mathfrak{p}$.
(b) $\mathfrak{p}$ contains each $\mathfrak{a}_i$ iff. it contains $\sum \mathfrak{a}_i$, simply because $\sum \mathfrak{a}_i$ is the smallest ideal containing all the $\mathfrak{a}_i$.
(c) The radical of $\mathfrak{a}$ (resp. $\mathfrak{b}$) is the intersection of all the prime ideals containing $\mathfrak{a}$ (resp. $\mathfrak{b}$). Hence if $V(\mathfrak{a}) \subseteq V(\mathfrak{b})$, every prime containing $\mathfrak{a}$ contains $\mathfrak{b}$, so $\sqrt{\mathfrak{b}} \subseteq \sqrt{\mathfrak{a}}$. The converse is clear.

Now we define a topology on Spec A by taking the subsets of the form $V(\mathfrak{a})$ to be the closed subsets. Note that $V(A) = \emptyset$; $V((0)) = \operatorname{Spec} A$; and the lemma shows that finite unions and arbitrary intersections of sets of the form $V(\mathfrak{a})$ are again of that form. Hence they do form the set of closed sets for a topology on Spec A.

Next we will define a sheaf of rings $\mathcal{O}$ on Spec A. For each prime ideal $\mathfrak{p} \subseteq A$, let $A_\mathfrak{p}$ be the localisation of A at $\mathfrak{p}$. For an open set $U \subseteq \operatorname{Spec} A$, we define $\mathcal{O}(U)$ to be the set of functions $s : U \longrightarrow \coprod_{\mathfrak{p} \in U} A_\mathfrak{p}$ (disjoint union of sets.) such that $s(\mathfrak{p}) \in A_\mathfrak{p}$ for each $\mathfrak{p}$, and such that s is locally a quotient of elements of A: to be precise, we require that for each $\mathfrak{p} \in U$, there is a neighborhood V of $\mathfrak{p}$, contained in U, and elements $a, f \in A$, such that for each $\mathfrak{q} \in V$, $f \notin \mathfrak{q}$, and $s(\mathfrak{q}) = a/f$ in $A_\mathfrak{q}$. (Note the similarity with the definition of the regular functions on a variety. The difference is that we consider functions into the various local rings, instead of to a field). (We define $\mathcal{O}(\emptyset) = 0$)(If $A = 0$ so Spec$A = 0$ then $\mathcal{O}(\emptyset) = 0$)

Now it is clear that sums and products of such functions are again such, and that the element 1 which gives 1 in each $A_\mathfrak{q}$ is an identity. Thus $\mathcal{O}(U)$ is a commutative ring with identity. If $V \subseteq U$ are two open subsets, the natural restriction map $\mathcal{O}(U) \longrightarrow \mathcal{O}(V)$ is a homomorphism of rings. It is then clear that $\mathcal{O}$ is a presheaf. Finally, it is clear from the local nature of the definition that $\mathcal{O}$ is a sheaf.

DEFINITION Let A be a ring. The <u>spectrum</u> of A is the pair consisting of the topological space Spec A together with the sheaf of rings $\mathcal{O}$ defined above.

Let us establish some basic properties of the sheaf $\mathcal{O}$ on Spec A. For any element $f \in A$, we denote by $D(f)$ the open complement of $V((f))$. Note that open sets of the form $D(f)$ form a base for the topology of Spec A. Indeed, if $V(\mathfrak{a})$ is a closed set, and $\mathfrak{p} \notin V(\mathfrak{a})$, then $\mathfrak{p} \not\supseteq \mathfrak{a}$, so there is an $f \in \mathfrak{a}$, $f \notin \mathfrak{p}$. Then $\mathfrak{p} \in D(f)$ and $D(f) \cap V(\mathfrak{a}) = \emptyset$.

NOTE If $A \neq 0$ and $\mathfrak{p} \in \operatorname{Spec} A$ then $A_\mathfrak{p} \neq 0$ since $1/1$ cannot be 0.

NOTE Just to be clear, by a <u>basis</u> of a space X we mean a nonempty collection $\{U_i\}_{i\in I}$ of open sets with the property that any open $U \subseteq X$ can be written as a union of elements of $\{U_i\}_{i \in I}$ (the empty union giving $\emptyset$). Equivalently given $x \in U \subseteq X$ (U open) $\exists i$ s.t. $x \in U_i \subseteq U$.

PROPOSITION 2.2 Let A be a ring, and $(\operatorname{Spec} A, \mathcal{O})$ its spectrum

(a) For any $\mathfrak{p} \in \operatorname{Spec} A$, the stalk $\mathcal{O}_{\mathfrak{p}}$ of the sheaf $\mathcal{O}$ is isomorphic to the local ring $A_{\mathfrak{p}}$

(b) For any element $f \in A$, the ring $\mathcal{O}(D(f))$ is isomorphic to the localised ring A_f.

(c) In particular, $\Gamma(\operatorname{Spec} A, \mathcal{O}) \cong A$.

⌈If $D(f) = \phi$ so f is nilpotent then trivially $A_f = 0$⌋

PROOF (a) First we define a homomorphism from $\mathcal{O}_{\mathfrak{p}}$ to $A_{\mathfrak{p}}$ by sending local sections s in a neighborhood of $\mathfrak{p}$ to its value $s(\mathfrak{p}) \in A_{\mathfrak{p}}$. This gives a well-defined homomorphism from $\mathcal{O}_{\mathfrak{p}}$ to $A_{\mathfrak{p}}$, call it φ. The map φ is surjective, because any element of $A_{\mathfrak{p}}$ can be represented as a quotient a/f, with $a, f \in A$, $f \notin \mathfrak{p}$. Then $D(f)$ will be an open neighborhood of $\mathfrak{p}$, and a/f defines a section of $\mathcal{O}$ over $D(f)$ whose value at $\mathfrak{p}$ is a/f. To show that φ is injective, let U be a neighborhood of $\mathfrak{p}$, and let $s, t \in \mathcal{O}(U)$ be elements having the same value at $\mathfrak{p}$. By shrinking U if necessary, we may assume that $s = a/f$, and $t = b/g$ on U, where $a, b, f, g \in A$, and $f, g \notin \mathfrak{p}$. Since a/f and b/g have the same image in $A_{\mathfrak{p}}$, it follows from the definition that there is $h \notin \mathfrak{p}$ s.t. $h(ga - fb) = 0$ in A. Therefore, $a/f = b/g$ in every local ring $A_{\mathfrak{q}}$ s.t. $f, g, h \notin \mathfrak{q}$. But the set of such $\mathfrak{q}$ is the open set $D(f) \cap D(g) \cap D(h)$, which contains $\mathfrak{p}$. Hence $s = t$ in a whole neighborhood of $\mathfrak{p}$, so they have the same germ at $\mathfrak{p}$. So φ is an isomorphism, which proves (a).

(b) and (c). Note that (c) is a special case of (b) when $f = 1$, and $D(f)$ is the whole space. So it is sufficient to prove (b). We define a homomorphism $\psi: A_f \longrightarrow \mathcal{O}(D(f))$ by sending a/f^n to the section $s \in \mathcal{O}(D(f))$ which assigns to each $\mathfrak{p}$ the image of a/f^n in $A_{\mathfrak{p}}$. First we show that ψ is injective. If

$$\psi(a/f^n) = \psi(b/f^m)$$

then for every $\mathfrak{p} \in D(f)$, a/f^n and b/f^m have the same image in $A_{\mathfrak{p}}$. Hence there is an element $h \notin \mathfrak{p}$ s.t.

$$h(f^m a - f^n b) = 0 \quad \text{in } A$$

Let $\mathfrak{a}$ be the annihilator of $f^m a - f^n b$. Then $h \in \mathfrak{a}$, and $h \notin \mathfrak{p}$, so $\mathfrak{a} \nsubseteq \mathfrak{p}$. This holds for any $\mathfrak{p} \in D(f)$, so we conclude that $V(\mathfrak{a}) \cap D(f) = \emptyset$. Therefore $f \in \sqrt{\mathfrak{a}}$, so some power $f^e \in \mathfrak{a}$, so

$$f^e(f^m a - f^n b) = 0$$

which shows that $a/f^n = b/f^m$ in A_f. Hence ψ is injective.

The hard part is to show that ψ is surjective. So let $s \in \mathcal{O}(D(f))$. Then by definition of $\mathcal{O}$, we can cover $D(f)$ with open sets V_i, on which s is represented by a quotient a_i/g_i, with $g_i \notin \mathfrak{p}$ for all $\mathfrak{p} \in V_i$, in other words, $V_i \subseteq D(g_i)$. Now the open sets of the form $D(h)$ form a base for the topology, so we may assume that $V_i = D(h_i)$ for some h_i. Since $D(h_i) \subseteq D(g_i)$, we have $V((h_i)) \supseteq V((g_i))$, hence $\sqrt{(h_i)} \subseteq \sqrt{(g_i)}$, and in particular $h_i^n \in (g_i)$ for some n. So $h_i^n = cg_i$, so $a_i/g_i = ca_i/h_i^n$. Replacing h_i by h_i^n (since $D(h_i) = D(h_i^n)$) and a_i by ca_i we may assume that $D(f)$ is covered by the open subsets $D(h_i)$, and that s is represented by a_i/h_i on $D(h_i)$. Next we observe that $D(h)$ can be covered by a finite number of the $D(h_i)$. Indeed, $D(f) \subseteq \bigcup D(h_i)$ iff. $V((f)) \supseteq \bigcap V((h_i)) = V(\sum (h_i))$. By (2.1c) again, this is equivalent to saying $f \in \sqrt{\sum (h_i)}$ or $f^n \in \sum (h_i)$ for some n. This means that f^n can be expressed as a finite sum $f^n = \sum b_i h_i$ $b_i \in A$. Hence a finite subset of h_i will do. So from now on we fix a finite set $h_1, \dots, h_r$ such that $D(f) \subseteq D(h_1) \cup \dots \cup D(h_r)$.

For the next step, note that on $D(h_i) \cap D(h_j) = D(h_i h_j)$ we have two elements of $A_{h_i h_j}$, namely a_i/h_i and a_j/h_j both of which represent s. Hence, according to the injectivity of ψ proved above, applied to $D(h_i h_j)$, we must have $a_i/h_i = a_j/h_j$ in $A_{h_i h_j}$. Hence for some n

$$(h_i h_j)^n (h_j a_i - h_i a_j) = 0$$

Since there are only finitely many indices involved, we may pick n so large that it works for all i, j at once. Rewrite this equation as

$$h_j^{n+1}(h_i^n a_i) - h_i^{n+1}(h_j^n a_j) = 0$$

Then replace each h_i by h_i^{n+1} and a_i by $h_i^n a_i$. The element s is still represented by a_i/h_i on $D(h_i)$, and furthermore, we have $h_j a_i = h_i a_j$ for all i,j. Now write $f^n = \sum b_i h_i$ as above, which is possible for some n since the $D(h_i)$ cover $D(f)$. Let $a = \sum b_i a_i$. Then for each j we have

$$h_j a = \textstyle\sum_i b_i a_i h_j = \sum_i b_i h_i a_j = f^n a_j$$

This says that $a/f^n = a_j/h_j$ on $D(h_j)$. So $\psi(a/f^n) = s$ everywhere, which shows that ψ is surjective, hence an isomorphism. $\square$ (As a Corollary if A is any ring and a,b have the same class in $A_\mathfrak{p}$ $\forall \mathfrak{p}$ then $a=b$ – since $\forall \mathfrak{p}\ \exists s \notin \mathfrak{p}\ s(a-b)=0 \therefore \operatorname{Ann}(a-b)$ cannot be proper)

To each nonzero ring we have now associated a spectrum $(\operatorname{Spec} A, \mathcal{O})$. We would like to say that this correspondence is functorial. For that we need a suitable category of spaces with sheaves of rings on them. The appropriate notion is the category of locally ringed spaces.

<u>DEFINITION</u> A <u>ringed space</u> is a pair $(X, \mathcal{O}_X)$ consisting of a topological space X (possibly empty) and a sheaf of rings $\mathcal{O}_X$ on X (as always O is a ring, and sheaf $\Rightarrow \mathcal{O}_X(\emptyset)=0$). A <u>morphism</u> of ringed spaces from $(X,\mathcal{O}_X)$ to $(Y,\mathcal{O}_Y)$ is a pair (f,ϕ) of a continuous map $f: X \longrightarrow Y$ and a morphism of sheaves of rings $\phi: \mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X$. (If $X=\emptyset$ $f_*\mathcal{O}_X = 0$) [0 is <u>not</u> a local ring]

The ringed space $(X,\mathcal{O}_X)$ is a <u>locally ringed space</u> if for all $P \in X$ the ring $\mathcal{O}_{X,P}$ is a local ring. A <u>morphism</u> of locally ringed spaces is a morphism (f,ϕ) of ringed spaces such that for all $P \in X$ the induced map of local rings $\phi_P: \mathcal{O}_{Y,f(P)} \longrightarrow \mathcal{O}_{X,P}$ is a <u>local homomorphism</u> of local rings. We explain this last condition: first of all, given a point $P \in X$ the morphism ϕ gives $\phi_V: \mathcal{O}_Y(V) \longrightarrow \mathcal{O}_X(f^{-1}V)$ for all neighborhoods V of P. Composed with $\mathcal{O}_X(f^{-1}V) \longrightarrow \mathcal{O}_{X,P}$ these maps induce

$$\phi_P: \mathcal{O}_{Y,f(P)} \longrightarrow \mathcal{O}_{X,P}$$
$$(V,s) \longmapsto (f^{-1}V, \phi_V(s))$$

[By convention any ringed space $(X,\mathcal{O}_X)$ with $X=\emptyset$ is locally ringed]

We require that this be a local homomorphism. If A, B are local rings with maximal ideals $\mathfrak{m}_A, \mathfrak{m}_B$ resp. a morphism of rings $\varphi: A \longrightarrow B$ is <u>local</u> if $\varphi^{-1}\mathfrak{m}_B = \mathfrak{m}_A$ (equiv. $a \in \mathfrak{m}_A \Rightarrow \varphi(a) \in \mathfrak{m}_B$).

Given two morphisms $(f,\phi):(X,\mathcal{O}_X) \longrightarrow (Y,\mathcal{O}_Y)$ and $(g,\psi):(Y,\mathcal{O}_Y) \longrightarrow (Z,\mathcal{O}_Z)$ of ringed spaces their composite is the pair $(gf, g_*(\phi)\psi): (X,\mathcal{O}_X) \longrightarrow (Z,\mathcal{O}_Z)$. That is, $gf: X \longrightarrow Z$ and for $V \subseteq Z$ the morphism of rings is

$$\mathcal{O}_Z(V) \xrightarrow{\psi_V} \mathcal{O}_Y(g^{-1}V) \xrightarrow{\phi_{g^{-1}V}} \mathcal{O}_X((gf)^{-1}V)$$
$$(V,s) \longmapsto ((gf)^{-1}V, \phi_{g^{-1}V}\psi_V(s))$$

[clearly $(\emptyset, 0)$ is initial for ringed and locally ringed spaces]

One checks that this definition makes ringed spaces and their morphisms into a category. An <u>isomorphism</u> of ringed spaces is a morphism (f,ϕ) for which f is a homeomorphism and ϕ is an isomorphism of sheaves of rings (i.e. ϕ_V bijective $\forall V$). $\phi: \mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X$. (if f is homeomorphic with inverse g and $\psi: f_*\mathcal{O}_X \longrightarrow \mathcal{O}_Y$ is inverse to ϕ then $(g, g_*\psi)$ is inverse to (f,ϕ)).

To show that the locally ringed spaces form a subcategory (<u>not</u> full!) of the ringed spaces, we need only show that if $(f,\phi):(X,\mathcal{O}_X) \longrightarrow (Y,\mathcal{O}_Y)$ and $(g,\psi):(Y,\mathcal{O}_Y) \longrightarrow (Z,\mathcal{O}_Z)$ are morphisms of <u>locally</u> ringed spaces then so is $(gf, g_*\phi\,\psi)$. But for a point $P \in X$ the induced morphism of local rings is

$$(g_*\phi\,\psi)_P: \mathcal{O}_{Z,gf(P)} \longrightarrow \mathcal{O}_{X,P}$$
$$(V,s) \longmapsto ((gf)^{-1}V, \phi_{g^{-1}V}\psi_V(s))$$

[Any morphism $(\emptyset,0) \longrightarrow$ is a local morphism. There are <u>no</u> morphisms $\longrightarrow (\emptyset,0)$ other than $1_{(\emptyset,0)}$]

Hence $(g_*\phi\,\psi)_P$ is the composite $\mathcal{O}_{Z,gf(P)} \xrightarrow{\psi_{f(P)}} \mathcal{O}_{Y,f(P)} \xrightarrow{\phi_P} \mathcal{O}_{X,P}$, and it suffices to show that if $A \xrightarrow{\Upsilon} B \xrightarrow{\epsilon} C$ are local morphisms of local rings then $(\epsilon\Upsilon)^{-1}\mathfrak{m}_C = \mathfrak{m}_A$. But $(\epsilon\Upsilon)^{-1}\mathfrak{m}_C = \Upsilon^{-1}(\epsilon^{-1}\mathfrak{m}_C) = \Upsilon^{-1}\mathfrak{m}_B = \mathfrak{m}_A$, as required. Hence the locally ringed spaces form a category. A morphism (f,ϕ) of locally ringed spaces is an isomorphism iff. f is a homeomorphism and $\phi: \mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X$ an isomorphism of sheaves of rings (i.e. iff iso in category of ringed spaces). Clearly if (f,ϕ) is an isomorphism of locally ringed spaces it is an iso. of ringed spaces. Conversely if (g,ψ) is an inverse for (f,ϕ) in the category of ringed spaces (so f is homeo and ϕ bijective) then for $P \in Y$

$$\psi_P: \mathcal{O}_{X,g(P)} \longrightarrow \mathcal{O}_{Y,P} \xrightarrow{\phi_P} \mathcal{O}_{X,g(P)} \qquad (\text{since } P = f(g(P)))$$
$$(V,s) \longmapsto (g^{-1}V, \psi_V(s)) \longmapsto (V,s)$$

So $\psi_P = \phi_P^{-1}$, hence ψ_P is a local morphism and (g,ψ) is a morphism of locally ringed spaces.

<u>NOTE</u> Obviously any ringed space isomorphic to a locally ringed space is locally ringed.

NOTE

We make a couple of simple observations:

(i) Any locally ringed space covered by schemes is a scheme. In fact the same is true of a <u>ringed</u> space.

(ii) If $\psi : X \longrightarrow Y$ is a morphism of schemes and $U \subseteq X$ is open, the restriction $\psi|_U : (U, \mathcal{O}_X|_U) \longrightarrow Y$ is a morphism of schemes (since the inclusion $(U, \mathcal{O}_X|_U) \longrightarrow X$ is).

(iii) The inclusion $(U, \mathcal{O}_X|_U) \longrightarrow X$, $\mathcal{O}_X(V) \longrightarrow \mathcal{O}_X(V \cap U)$ by restriction, is a morphism of schemes.

(iv) If $(f, \psi) : X \longrightarrow Y$ is a morphism of schemes and $f(X) \subseteq V$ where V is an open subset of Y, then (f, ψ) defines a morphism of schemes $X \longrightarrow (V, \mathcal{O}_Y|_V)$ in the obvious way, so that the diagram

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Y \\ & (V, \mathcal{O}_Y|_V) \arrow[ur] & \end{tikzcd} \qquad (1)$$

commutes. So in particular (f, ψ) can be recovered from $X \longrightarrow (V, \mathcal{O}_Y|_V)$. Moreover the morphism $X \longrightarrow V$ is <u>unique</u> making (1) commute

(v) The same as (iv) and (iii) for arbitrary ringed spaces. Ie if $(X, \mathcal{O}_X)$ ringed space there is an obvious morphism $(U, \mathcal{O}_X|_U) \longrightarrow (X, \mathcal{O}_X)$ and if $f : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces and $f(X) \subseteq V$ there is a unique morphism of ringed spaces $(X, \mathcal{O}_X) \longrightarrow (V, \mathcal{O}_Y|_V)$ making (1) commute.

PROPOSITION 2.3 (a) If A is a ring, then $(\mathrm{Spec}A, \mathcal{O})$ is a locally ringed space.

(b) If $\varphi: A \longrightarrow B$ is a homomorphism of rings, then φ induces a natural morphism of locally ringed spaces.

$$(f, \phi): (\mathrm{Spec}\, B, \mathcal{O}_B) \longrightarrow (\mathrm{Spec}\, A, \mathcal{O}_A)$$

(c) If A and B are rings, then any morphism of locally ringed spaces from $\mathrm{Spec}A$ to $\mathrm{Spec}B$ is induced by a homomorphism of rings $\varphi: A \longrightarrow B$ as in (b).

⌜If A or B are 0 then (b), (c) are trivial since there is only one morphism $(\emptyset, 0) \to (\mathrm{Spec}A, \mathcal{O})$⌟

PROOF (a) This follows from (2.2a).

(b) Given a morphism of rings $\varphi: A \longrightarrow B$ any prime ideal $\mathfrak{p} \in B$ induces a morphism of rings

$$\varphi_{\mathfrak{p}}: A_{\varphi^{-1}\mathfrak{p}} \longrightarrow B_{\mathfrak{p}}$$
$$a/s \longmapsto \varphi(a)/\varphi(s)$$

This is a local morphism since $\varphi_{\mathfrak{p}}^{-1}(\mathfrak{p}B_{\mathfrak{p}}) = \{a/s \mid \varphi(a) \in \mathfrak{p}\} = \{a/s \mid a \in \varphi^{-1}\mathfrak{p}\} = \varphi^{-1}\mathfrak{p}A_{\varphi^{-1}\mathfrak{p}}$. We define a map $f: \mathrm{Spec}B \longrightarrow \mathrm{Spec}A$ by $f(\mathfrak{p}) = \varphi^{-1}\mathfrak{p}$. If $V(\mathfrak{a})$ is a closed subset of $\mathrm{Spec}A$ then $f^{-1}V(\mathfrak{a}) = V(\mathfrak{b})$ where $\mathfrak{b}$ is the smallest ideal containing $\varphi(\mathfrak{a})$ – hence f is continuous.

Let $V \subseteq \mathrm{Spec}A$ be open and let $\Upsilon: V \longrightarrow \bigcup_{\mathfrak{q} \in V} A_{\mathfrak{q}}$ be regular. Composition with f gives a map $\Upsilon': f^{-1}V \longrightarrow \bigcup_{\mathfrak{p} \in f^{-1}V} A_{\varphi^{-1}\mathfrak{p}}$ and composition with the $\varphi_{\mathfrak{p}}$ gives $\Upsilon'': f^{-1}V \longrightarrow \bigcup_{\mathfrak{p} \in f^{-1}V} B_{\mathfrak{p}}$, defined by

$$\Upsilon''(\mathfrak{p}) = \varphi_{\mathfrak{p}}(\Upsilon'(\mathfrak{p})) = \varphi_{\mathfrak{p}}\Upsilon(\varphi^{-1}\mathfrak{p}) = \varphi_{\mathfrak{p}}\Upsilon(f(\mathfrak{p}))$$

We claim that Υ'' is regular. If $\mathfrak{p} \in f^{-1}V$ then $\varphi^{-1}\mathfrak{p} \in V$ so there is an open neighbourhood U of $\varphi^{-1}\mathfrak{p}$ in V and $a, s \in A$ s.t. $\forall \mathfrak{q} \in U$ $s \notin \mathfrak{q}$ and $\Upsilon(\mathfrak{q}) = a/s$. Then $\forall \mathfrak{p}' \in f^{-1}U$ we have

$$\Upsilon''(\mathfrak{p}') = \varphi_{\mathfrak{p}}\Upsilon(\varphi^{-1}\mathfrak{p}') = \varphi_{\mathfrak{p}}(a/s) = \varphi(a)/\varphi(s)$$

$\forall \mathfrak{p}' \in f^{-1}U$ we have $s \notin \varphi^{-1}\mathfrak{p}'$ and hence $\varphi(s) \notin \mathfrak{p}'$. So Υ'' is regular. This defines a map

$$\phi_V: \mathcal{O}_{\mathrm{Spec}A}(V) \longrightarrow \mathcal{O}_{\mathrm{Spec}B}(f^{-1}V)$$
$$\phi_V(\Upsilon)(\mathfrak{p}) = \varphi_{\mathfrak{p}}\Upsilon f(\mathfrak{p})$$

It is easily checked that ϕ_V is a morphism of rings (if $V = \emptyset$ or $f^{-1}V = \emptyset$ the obvious morphisms are used) and that ϕ is a natural transformation. There is a commutative diagram of rings for $\mathfrak{p} \in \mathrm{Spec}B$

$$\begin{array}{ccc} \mathcal{O}_{\mathrm{Spec}A, f(P)} & \xrightarrow{\phi_{\mathfrak{p}}} & \mathcal{O}_{\mathrm{Spec}B, \mathfrak{p}} \\ \wr\downarrow & & \downarrow\wr \\ A_{\varphi^{-1}\mathfrak{p}} & \xrightarrow[\varphi_{\mathfrak{p}}]{} & B_{\mathfrak{p}} \end{array}$$

Implying that ϕ is a morphism of locally ringed spaces.

(c) Conversely, suppose $(f, \phi): (\mathrm{Spec}B, \mathcal{O}_{\mathrm{Spec}B}) \longrightarrow (\mathrm{Spec}A, \mathcal{O}_{\mathrm{Spec}A})$ is a morphism of locally ringed spaces. There is an induced morphism of rings: $\varphi: A \longrightarrow B$

$$\begin{array}{ccc} \mathcal{O}(\mathrm{Spec}A) & \xrightarrow{\phi_{\mathrm{Spec}A}} & \mathcal{O}(\mathrm{Spec}B) \\ \wr\uparrow & & \uparrow\wr \\ A & \overset{\varphi}{\dashrightarrow} & B \end{array}$$

$\varphi(a) =$ the unique element $b \in B$ with $b = \phi_{\mathrm{Spec}A}(\dot{a})(\mathfrak{p})$ for all $\mathfrak{p} \in \mathrm{Spec}B$ where $\dot{a}(\mathfrak{q}) = a \in A_{\mathfrak{q}}$.

The inverse of the iso $\mathcal{O}_{\mathfrak{p}} \longrightarrow A_{\mathfrak{p}}$ of Prop 2.2 is $a/f \longmapsto (D(f), (\dot{a/f}))$ where $(\dot{a/f})(\mathfrak{q}) = a/f \in A_{\mathfrak{q}}$. So there is a diagram for any $\mathfrak{p} \in \mathrm{Spec}B$

$$
\begin{tikzcd}
& \mathcal{O}(\operatorname{Spec} A) \arrow[rr] \arrow[dl] & & \mathcal{O}(\operatorname{Spec} B) \arrow[dr] & \\
\mathcal{O}_{\operatorname{Spec} A, f(\mathfrak{p})} \arrow[rrrr, "\phi_{\mathfrak{p}}"] & & & & \mathcal{O}_{\operatorname{Spec} B, \mathfrak{p}} \\
& A \arrow[uu, "\wr"] \arrow[rr, "\varphi"] \arrow[dl, "\delta"'] & & B \arrow[uu, "\wr"'] \arrow[dr, "\epsilon"] & \\
A_{f(\mathfrak{p})} \arrow[uu, "m\ \wr"] \arrow[rrrr, dotted, "\gamma"'] & & & & B_{\mathfrak{p}} \arrow[uu, "\wr\ n"']
\end{tikzcd}
\qquad (1)
$$

The map γ is induced by $\phi_{\mathfrak{p}}$ and the isomorphisms. Since all other faces of this cube commute, $\epsilon\varphi = \gamma\delta$. Hence

$$
\begin{aligned}
f(\mathfrak{p}) = \delta^{-1}(f(\mathfrak{p})A_{f(\mathfrak{p})}) &= (\gamma\delta)^{-1}(\mathfrak{p}B_{\mathfrak{p}}) \qquad (\phi_{\mathfrak{p}} \text{ is local, hence } \gamma \text{ is}) \\
&= (\epsilon\varphi)^{-1}(\mathfrak{p}B_{\mathfrak{p}}) = \varphi^{-1}\mathfrak{p}
\end{aligned}
$$

So f coincides with the map $\operatorname{Spec} B \longrightarrow \operatorname{Spec} A$ induced by φ. Let the morphism $\phi': \mathcal{O}_{\operatorname{Spec} A} \longrightarrow f_* \mathcal{O}_{\operatorname{Spec} B}$ induced by φ. We claim that $\phi = \phi'$. Let $V \subseteq \operatorname{Spec} A$ be open and $\theta: V \longrightarrow \bigcup_{\mathfrak{q} \in V} A_{\mathfrak{q}}$ regular. We show that the maps $\phi'_V(\theta), \phi_V(\theta): f^{-1}V \longrightarrow \bigcup_{\mathfrak{p} \in f^{-1}V} B_{\mathfrak{p}}$ agree on every $\mathfrak{p} \in f^{-1}V$. Let such a $\mathfrak{p}$ be given, and find an open neighborhood U of $f(\mathfrak{p})$ and $a, s \in A$ s.t. $s \notin \mathfrak{q}$ for all $\mathfrak{q} \in U$ and $\theta(\mathfrak{q}) = a/s \in A_{\mathfrak{q}}$ $\forall \mathfrak{q} \in U$. Then since by (1) $\gamma = \varphi_{\mathfrak{p}}: A_{\varphi^{-1}\mathfrak{p}} \longrightarrow B_{\mathfrak{p}}$

$$
\begin{aligned}
\phi'_V(\theta)(\mathfrak{p}) &= \varphi_{\mathfrak{p}}(\theta(f(\mathfrak{p}))) \\
&= \varphi_{\mathfrak{p}}(a/s) \\
&= \varphi_{\mathfrak{p}} m^{-1}(D(s), (a/s)) \\
&= n^{-1}\phi_{\mathfrak{p}}(D(s), (a/s)) \\
&= n^{-1}(f^{-1}D(s), \phi_{D(s)}((a/s))) \\
&= \phi_{D(s)}((a/s))(\mathfrak{p}) \\
&= \phi_{D(s)}(\theta|_{D(s)})(\mathfrak{p}) = \phi_V(\theta)|_{D(s)}(\mathfrak{p}) = \phi_V(\theta)(\mathfrak{p})
\end{aligned}
$$

Hence $\phi = \phi'$, so (f, ϕ) is the morphism induced by φ. In particular given any morphism of locally ringed spaces $(\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B}) \longrightarrow (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ works by

$$
\begin{tikzcd}
\mathcal{O}_{\operatorname{Spec} A}(V) \arrow[r] & \mathcal{O}_{\operatorname{Spec} B}(f^{-1}V) & \\
\theta: V \longrightarrow \bigcup_{\mathfrak{q} \in V} A_{\mathfrak{q}} \arrow[r, mapsto] & V \arrow[r] & \bigcup_{\mathfrak{q} \in V} A_{\mathfrak{q}} \\
& f^{-1}V \arrow[u] \arrow[r] & \bigcup_{\mathfrak{p} \in f^{-1}V} B_{\mathfrak{p}}.
\end{tikzcd}
\qquad \square
$$

If Rng denotes the category of zero rings and LRng the category of locally ringed spaces, we have shown that $A \longmapsto (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ associates an object of LRng to every object in Rng', and that there is a bijection between ring morphisms $A \longrightarrow B$ and morphisms $(\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B}) \longrightarrow (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ in LRng. First we show that this defines a functor S: Rng' $\longrightarrow$ LRng, and then we show that S is actually injective on objects (i.e. if $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}) = (\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B})$ then $A = B$). (This is actually much harder than it appears, and I'm not sure it's true)

Let $\varphi: A \longrightarrow B$ and $\psi: B \longrightarrow C$ be ring morphisms, $(f, \varphi^{\#}): (\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B}) \longrightarrow (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ and $(g, \psi^{\#}): (\operatorname{Spec} C, \mathcal{O}_{\operatorname{Spec} C}) \longrightarrow (\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B})$ the associated morphisms of locally ringed spaces. Suppose (fg, γ) is associated to $\psi\varphi: A \longrightarrow C$. For $V \subseteq \operatorname{Spec} A$ and $\theta: V \longrightarrow \bigcup_{\mathfrak{p} \in V} A_{\mathfrak{p}}$ regular, we claim that

$$
\begin{aligned}
\gamma_V(\theta) &\in \mathcal{O}_{\operatorname{Spec} C}((fg)^{-1}V) \\
&= \psi^{\#}_{f^{-1}V}\varphi^{\#}_V(\theta)
\end{aligned}
$$

To test this, let $\mathfrak{q} \in (fg)^{-1}V$. Then

$$
\begin{aligned}
\gamma_V(\theta)(\mathfrak{q}) &= (\psi\varphi)_{\mathfrak{q}}\, \theta((fg)(\mathfrak{q})) \\
&= \psi_{\mathfrak{q}}\, \varphi_{g(\mathfrak{q})}(\theta(f(g(\mathfrak{q}))))
\end{aligned}
$$

(The cases where one of A, B, C is zero are easy)

$$
\begin{aligned}
\psi^{\#}_{f^{-1}V}(\varphi^{\#}_V(\theta))(\mathfrak{q}) &= \psi_{\mathfrak{q}}(\varphi^{\#}_V(\theta)(g(\mathfrak{q}))) \\
&= \psi_{\mathfrak{q}}\, \varphi_{g(\mathfrak{q})}(\theta(f(g(\mathfrak{q}))))
\end{aligned}
$$

as required. Since it is clear that S preserves identities, S is a functor, and by (2.3) it is fully faithful. (contravariant)

CAUTION 2.3.0 Statement (c) of the proposition would be false, if in the definition of a morphism of locally ringed spaces, we did not insist that the induced maps on stalks be local homomorphisms of local rings. (see 2.3.2 below)

DEFINITION An affine scheme is a locally ringed space $(X, \mathcal{O}_X)$ which is isomorphic (as a locally ringed space) to the spectrum of some ring. A scheme is a locally ringed space $(X, \mathcal{O}_X)$ in which every point has an open neighborhood U such that the topological space U, together with the restricted sheaf $\mathcal{O}_X|_U$, is an affine scheme.

We call X the underlying topological space of the scheme $(X, \mathcal{O}_X)$, and $\mathcal{O}_X$ its structure sheaf. By abuse of notation we will often write simply X for the scheme $(X, \mathcal{O}_X)$. If we wish to refer to the underlying topological space without its scheme structure, we write $sp(X)$, read "space of X". A morphism of schemes is a morphism as locally ringed spaces. An isomorphism is a morphism with a two sided inverse.

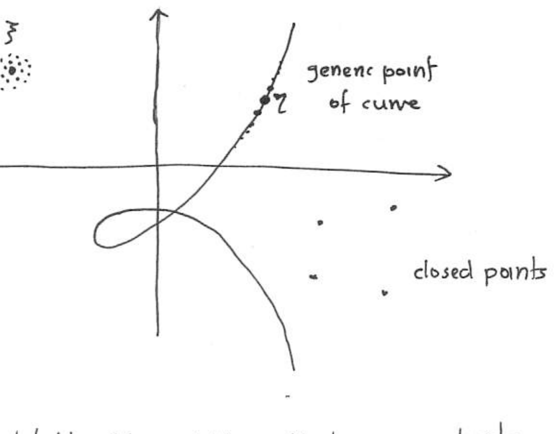
EXAMPLE 2.3.1 If k is a field, Spec$k$ is an affine scheme whose topological space consists of one point, and whose structure sheaf consists of the field k.

EXAMPLE 2.3.3 If k is a field, we define the affine line over k, $\mathbb{A}^1_k$, to be Spec$k[x]$. It has a point ξ, corresponding to the zero ideal, whose closure is the entire space. This is called a generic point. Notice that the other points, which correspond to maximal ideals in $k[x]$, are all closed points. They are in one-to-one correspondence with the nonconstant monic irreducible polynomials in x. In particular, if k is algebraically closed, the closed points of $\mathbb{A}^1_k$ are in one-to-one correspondence with the elements of k.

EXAMPLE 2.3.4 Let k be an algebraically closed field, and consider the affine plane over k, defined as

$$\mathbb{A}^2_k = \operatorname{Spec} k[x,y]$$

The closed points of $\mathbb{A}^2_k$ are in one-to-one correspondence with ordered pairs of elements of k. Furthermore, the set of all closed points of $\mathbb{A}^2_k$, with the induced topology, is homeomorphic to the variety called $\mathbb{A}^2$ in Chapter I. In addition to the closed points, there is a generic point ξ, corresponding to the zero ideal of $k[x,y]$, whose closure is the whole space. Also, for each irreducible polynomial $f(x,y)$ there is a point η, corresponding to $(f(x,y))$, whose closure is $V(f(x,y))$ and includes all the closed points (a,b) for which $f(a,b)=0$. We say that η is a generic point of the curve $f(x,y)=0$.



EXAMPLE 2.3.5 Let X_1 and X_2 be schemes, let $U_1 \subseteq X_1$ and $U_2 \subseteq X_2$ be open subsets, and let

$$\varphi : (U_1, \mathcal{O}_{X_1}|_{U_1}) \longrightarrow (U_2, \mathcal{O}_{X_2}|_{U_2})$$

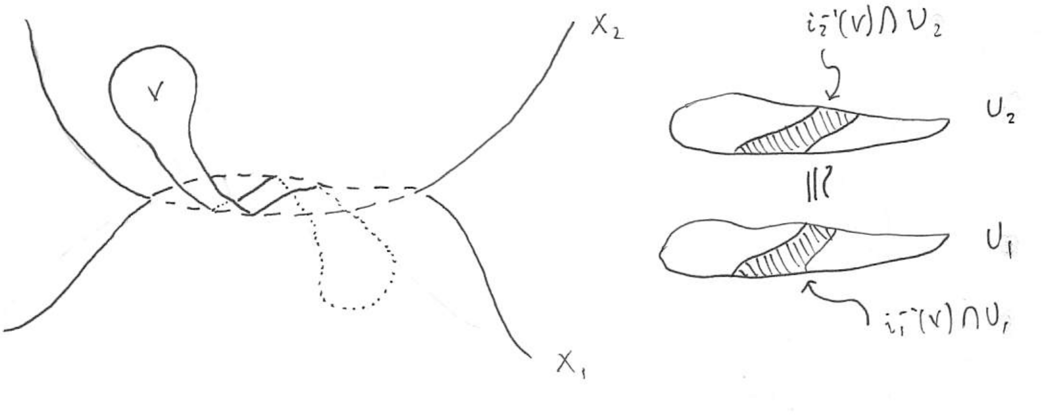
be an isomorphism of locally ringed spaces. Then we can define a scheme X, obtained by glueing X_1 and X_2 along U_1 and U_2 via the isomorphism φ. The topological space of X is the quotient of the disjoint union $X_1 \cup X_2$ by the equivalence relation generated by $x_1 \sim \varphi(x_1)$ for $x_1 \in U_1$, with the quotient topology. Explicitly, the open sets of $X_1 \cup X_2$ are disjoint unions $P \cup Q$ where P is open in X_1 and Q is open in X_2. The open sets of X are those sets $P \cup Q$ which involve for each $x_1 \in U_1$ either $x_1 \in P$ and $\varphi(x_1) \in Q$ or $x_1 \notin P$. In particular an open set avoiding U_1 (and U_2) remains open.

Hence there are maps $i_1 : X_1 \longrightarrow X$ and $i_2 : X_2 \longrightarrow X$ such that $V \subseteq X$ is open if and only if both $i_1^{-1}(V)$ is open in X_1 and $i_2^{-1}(V)$ is open in X_2. The structure sheaf $\mathcal{O}_X$ is defined as follows: for any open set $V \subseteq X$

$$\mathcal{O}_X(V) = \left\{ (s_1, s_2) \;\middle|\; s_1 \in \mathcal{O}_{X_1}(i_1^{-1}(V)) \text{ and } s_2 \in \mathcal{O}_{X_2}(i_2^{-1}(V)) \text{ and } \varphi(s_1|_{i_1^{-1}(V)\cap U_1}) = s_2|_{i_2^{-1}(V)\cap U_2} \right\}$$

It is not hard to verify that $\mathcal{O}_X(V)$ is a subring of $\mathcal{O}_{X_1}(i_1^{-1}(V)) \times \mathcal{O}_{X_2}(i_2^{-1}(V))$, and further that the image of the canonical restriction $\mathcal{O}_X(V) \longrightarrow \mathcal{O}_X(U)$ is contained in $\mathcal{O}_X(U)$. Suppose $V \subseteq X$ is open and is covered by open sets U_i, with sections $(s_i, t_i) \in \mathcal{O}_X(U_i)$ all agreeing on overlaps. Then by definition of the restriction, s_i is compatible on the cover $i_1^{-1}(U_i)$ of $i_1^{-1}(V)$, and likewise t_i. Hence produce $(s,t) \in \mathcal{O}_{X_1}(V) \times \mathcal{O}_{X_2}(V)$, s.t. $s|_{i_1^{-1}(V) \cap U_1}$ is the section on $i_1^{-1}(V) \cap U_1$ uniquely determined by the family $s_i|_{i_1^{-1}(U_i) \cap U_1}$, and hence $\varphi(s|_{i_1^{-1}(V)\cap U_1})$ is $t|_{i_2^{-1}(V) \cap U_2}$. Hence $\mathcal{O}_X$ is a sheaf.

Consider an open subset $V \subseteq X$ with $i_1^{-1}(V) = \phi$.



The open set V depicted has nonempty $i_1^{-1}(V)$ and $i_2^{-1}(V)$. Open sets with $i_1^{-1}(V) = \phi$ correspond to open sets $V \subseteq X_2$ not meeting U_2. It is obvious that for such V, $\mathcal{O}_X(V) = \mathcal{O}_{X_2}(V)$. For any $W \subseteq X$ contained in the overlap, $\mathcal{O}_X(W)$ can be identified canonically with $\mathcal{O}_{X_1}(W)$ or $\mathcal{O}_{X_2}(W)$. The "overlap" is obviously open in X, and for $x \in X$ the stalk $\mathcal{O}_{X,x}$ (x in overlap) is just $\mathcal{O}_{X_1,x} = \mathcal{O}_{X_2,x}$ which is local. Also note that X_1, X_2 form open subsets of X, so that for $x \in X_2 \setminus$ overlap, $\mathcal{O}_{X,x}$ is just $\mathcal{O}_{X_2,x}$ and likewise for $x \in X_1 \setminus$ overlap. Hence X is locally ringed. For an open set $V \subseteq X_2 \subseteq X$, intersecting U_2 or not, it is easy to see how we identify $\mathcal{O}_X(V)$ with $\mathcal{O}_{X_2}(V)$, and hence how $(V, \mathcal{O}_X|_V) \cong (V, \mathcal{O}_{X_2}|_V)$ as locally ringed spaces. Hence X is a scheme.

EXAMPLE 2.3.6 As an example of glueing, let k be a field, let $X_1 = X_2 = \mathbb{A}^1_k$, let $U_1 = U_2 = \mathbb{A}^1_k - \{P\}$, where P is the point corresponding to the maximal ideal (x), and let $\varphi: U_1 \longrightarrow U_2$ be the identity map. Let X be obtained by glueing X_1 and X_2 along U_1 and U_2 via φ. We get an "affine line with the point P doubled".

—— : ——

This is an example of a scheme which is not an affine scheme. It is also an example of a nonseperated scheme, as we will see later (4.0.1). The ring of global sections consists of pairs (f,g) where $f \in k[x]$ and $g \in k[x]$ and f, g agree on all of $\operatorname{Spec} k[x]$ except for (x), (possibly) — that is, f and g are identified in $k[x]_{\mathfrak{p}}$ for $\mathfrak{p} \neq (x)$ — in particular in $k[x]_{(0)} = k(x)$, and hence $f = g$. Hence the ring of global sections is $k[x]$, so that this scheme is certainly not affine.

NOTE Consider a field k. Two polynomials of $k[x]$ are equal iff. they have the same residue in each maximal ideal ($k[x]$ is Hilbert) and they are also equal iff. they agree in any localisation ($k[x]$ is a domain). When we give $\operatorname{Specm} k[x]$ (maximal ideals) the usual affine variety structure with sheaf $D(f) \longmapsto k[x]_f$, with stalk $k[x]_{\mathfrak{m}_a}$ at a point $\mathfrak{m}_a$, the sheaf $D(f) \longmapsto k[x]_f$ can be replaced by the sheaf of sections of germs, where sections on $D(f)$ are functions $D(f) \longrightarrow \coprod_{a \in D(f)} k[x]_{\mathfrak{m}_a}$ with the usual properties. Considering canonical maps $k[x]_{\mathfrak{m}_a} \longrightarrow k[x]_{\mathfrak{m}_a} / \mathfrak{m}_a k[x]_{\mathfrak{m}_a}$ which is just $k = k[x]/\mathfrak{m}_a$ (when k alg. closed) we recover the usual notion of f, g as functions. But as the functions $f: \mathfrak{m} \longmapsto f \bmod \mathfrak{m}$, $g: \mathfrak{m} \longmapsto g \bmod \mathfrak{m}$, f and g may agree at $\mathfrak{m}$ but still be distinct – hence the residues of f and g, identified in $k[x]_{\mathfrak{m}_a} / \mathfrak{m}_a k[x]_{\mathfrak{m}_a}$ must have their difference in $\mathfrak{m}_a k[x]_{\mathfrak{m}_a}$ as elements of $k[x]_{\mathfrak{m}_a}$.

That is, the two polynomials agree at $\mathfrak{m}$ iff. their residues $f/1, g/1$ in $k[x]_{\mathfrak{m}}$ are equal modulo $\mathfrak{m} k[x]_{\mathfrak{m}}$, or, equivalently, $f - g \in \mathfrak{m}$.

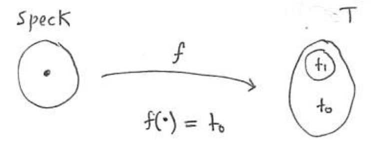
NOTE Any locally ringed space isomorphic to a scheme is a scheme (isomorphic as ringed spaces = iso as loc. ringed spaces) The zero scheme $(\phi, 0)$ is an affine scheme, and any ringed space locally isomorphic to a scheme is locally ringed and is a scheme.

EXAMPLE Let X be a scheme, $x \in X$ and assume $\mathfrak{m}_x$ is the only prime ideal in $\mathcal{O}_{X,x}$. Then there is a morphism of schemes $\operatorname{Spec} \mathcal{O}_{X,x} \longrightarrow X$ with $\mathfrak{m}_x \longmapsto x$ and $\mathcal{O}_X(U) \longrightarrow \mathcal{O}_{\operatorname{Spec}}(\operatorname{Spec} \mathcal{O}_{X,x})$ (for $x \in X$) given by $\mathcal{O}_X(U) \longrightarrow \mathcal{O}_{X,x} \cong \Gamma(\operatorname{Spec} \mathcal{O}_{X,x})$.

NOTE Let A be a ring, $S \subseteq A$ multiplicatively closed $\varphi: A \longrightarrow S^{-1}A$ canonical, $\Phi: \operatorname{Spec} S^{-1}A \longrightarrow \operatorname{Spec} A$ induced by φ. Then by Ex 2.4 Φ is a monomorphism, since φ is an epimorphism of rings. Φ gives a bijection of $\operatorname{Spec} S^{-1}A$ with the subset $\{\mathfrak{p} \mid \mathfrak{p} \cap S = \emptyset\}$ of $\operatorname{Spec} A$. We claim this is a homeomorphism, since any ideal of $S^{-1}A$ is $S^{-1}\mathfrak{a}$ and

$$\Phi(V(S^{-1}\mathfrak{a})) = \{\mathfrak{p} \mid S^{-1}\mathfrak{p} \supseteq S^{-1}\mathfrak{a}\} = V(\mathfrak{a}) \cap \operatorname{Im} \Phi$$

EXAMPLE 2.3.2 If R is a discrete valuation ring, then $T = \operatorname{Spec} R$ is an affine scheme whose topological space consists of two points: 0 and the maximal ideal $\mathfrak{m}$. The point $t_0 = \mathfrak{m}$ is closed with local ring R (since $R - \mathfrak{m} = \text{units}$, $R_\mathfrak{m} \cong R$), the other point $t_1 = 0$ is open and dense (open since $T - t_1 = t_0$ closed) with local ring equal to K, the quotient field of R. The inclusion map $R \longrightarrow K$ corresponds to the morphism $\operatorname{Spec} K \longrightarrow T$ which sends the unique point of $\operatorname{Spec} K$ to t_1. There is another morphism of ringed spaces $\operatorname{Spec} K \longrightarrow T$ which sends the unique point of $\operatorname{Spec} K$ to t_0.



Opens are $\{t_1\}$, T

To define $\phi : \mathcal{O}_T \longrightarrow f_* \operatorname{Spec} K$ we need morphisms $\mathcal{O}_T(T) \to \mathcal{O}_{\operatorname{Spec} K}(\cdot)$ and $\mathcal{O}_T(t_1) \to 0$. So all that is required is a morphism of rings $\mathcal{O}_T(T) \longrightarrow K$. But a regular map $f : \{t_0, t_1\} \longrightarrow R \cup K$ can be mapped to $f(t_0) \in R \subseteq K$. This is not a morphism of locally ringed spaces because the induced map on stalks is $R \cong \mathcal{O}_{T,t_0} \longrightarrow \mathcal{O}_{\operatorname{Spec} K, \cdot} \cong K$ $r \mapsto r \in K$. The inverse image of $0 \subseteq K$ is not $\mathfrak{m}$. Hence (f, ϕ) is not induced by any ring morphism $R \longrightarrow K$.

NOTE Let A be a ring, $\mathfrak{p} \in \operatorname{Spec} A$. The maximal ideal of $\mathcal{O}_{\operatorname{Spec} A, \mathfrak{p}}$ are those regular functions f with $f(\mathfrak{p}) \in \mathfrak{p} A_\mathfrak{p}$. We can consider regular functions to have values in fields using $A_\mathfrak{p} \longrightarrow k(\mathfrak{p}) = A_\mathfrak{p} / \mathfrak{p} A_\mathfrak{p}$. Then the maximal ideal of $\mathcal{O}_{\operatorname{Spec} A, \mathfrak{p}}$ are those regular f's which "vanish" at $\mathfrak{p}$ in this sense.

Next we will define an important class of schemes, constructed from graded rings, which are analogous to projective varieties. Let S be a graded ring, $S = \bigoplus_{d \geq 0} S_d$. We denote by S_+ the ideal $\bigoplus_{d > 0} S_d$. We define the set $\operatorname{Proj} S$ to be the set of all homogenous prime ideals $\mathfrak{p}$ which do not contain S_+. If $\mathfrak{a}$ is a homogenous ideal of S, we define the subset

$$V(\mathfrak{a}) = \{ \mathfrak{p} \in \operatorname{Proj} S \mid \mathfrak{a} \subseteq \mathfrak{p} \}$$

Note always $1 \in S_0$ see A & M notes

LEMMA 2.4 (a) If $\mathfrak{a}$ and $\mathfrak{b}$ are homogenous ideals in S, then $V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$

(b) If $\{\mathfrak{a}_i\}$ is any family of homogenous ideals in S, then $V(\sum \mathfrak{a}_i) = \cap V(\mathfrak{a}_i)$

PROOF Clearly $\mathfrak{p} \supseteq \mathfrak{a}\mathfrak{b}$ if $\mathfrak{p} \supseteq \mathfrak{a}$ or $\mathfrak{p} \supseteq \mathfrak{b}$, and if $\mathfrak{p} \supseteq \mathfrak{a}\mathfrak{b}$ and say $\mathfrak{b} \not\subseteq \mathfrak{p}$ then there is a $b \in \mathfrak{b}$ s.t. $b \notin \mathfrak{p}$. Then $ab \in \mathfrak{p}$ $\forall a \in \mathfrak{a}$ implies $\mathfrak{a} \subseteq \mathfrak{p}$. (b) $\mathfrak{p}$ contains $\sum \mathfrak{a}_i$ iff. it contains each $\mathfrak{a}_i$, simply because $\sum \mathfrak{a}_i$ is the smallest ideal containing all the $\mathfrak{a}_i$. $\square$

Because of the Lemma we can define a topology on $\operatorname{Proj} S$ by taking the closed subsets to be the subsets of the form $V(\mathfrak{a})$, for homogenous ideals $\mathfrak{a}$. Clearly $V(0) = \operatorname{Proj} S$ and $V(S) = \phi$ (also $V(S_+) = \phi$). Next we will define a sheaf of rings $\mathcal{O}$ on $\operatorname{Proj} S$. For each $\mathfrak{p} \in \operatorname{Proj} S$, we consider the ring $S_{(\mathfrak{p})}$ of elements of degree zero in the localised ring $T^{-1}S$, where T is the multiplicative system consisting of all homogenous elements of S which are not in $\mathfrak{p}$. Since S is not a domain, necessarily, it is possible for $f/g = f'/g$ in $T^{-1}S$ with f homogenous and f' non-homogenous. So to form the subring $S_{(\mathfrak{p})}$ we take the equivalence class of (f, g) in $T^{-1}S$ for $g \in T$ and $f \in S_e$ where $g \in S_e$. Two such pairs are equal in $S_{(\mathfrak{p})}$ iff. $\exists\, G \in T$ s.t. $G(fg' - gf') = 0$ (so we could define $S_{(\mathfrak{p})}$ without reference to $T^{-1}S$). The ring $S_{(\mathfrak{p})}$ is local with maximal ideal consisting of f/g with $f \in \mathfrak{p}$. (Again $S_{(\mathfrak{p})} \neq 0$ since $1/1 \neq 0$) (It is easy to check if $f/g = f'/g$ and $f \in \mathfrak{p}$ then $f' \in \mathfrak{p}$).

For any open subset $U \subseteq \operatorname{Proj} S$, we define $\mathcal{O}(U)$ to be the set of functions $s : U \longrightarrow \coprod_{\mathfrak{p} \in U} S_{(\mathfrak{p})}$ such that for each $\mathfrak{p} \in U$, $s(\mathfrak{p}) \in S_{(\mathfrak{p})}$, and such that s is locally a quotient of elements of s: for each $\mathfrak{p} \in U$, there exists a neighborhood V of $\mathfrak{p}$ in U, and homogenous elements $a, f \in S$ of the same degree, s.t. $\forall \mathfrak{q} \in V$ $f \notin \mathfrak{q}$ and $s(\mathfrak{q}) = a/f$ in $S_{(\mathfrak{q})}$. ($\mathcal{O}(\phi) = 0$). Now it is clear that $\mathcal{O}$ is a presheaf of rings, with the natural restrictions, and it is also clear from the local nature of the definition that $\mathcal{O}$ is a sheaf. If $S = 0$ then $\operatorname{Proj} S = \phi$ and $\mathcal{O} = 0$.

PROPOSITION 2.5 Let S be a graded ring.

(a) For any $\mathfrak{p} \in \operatorname{Proj} S$, the stalk $\mathcal{O}_\mathfrak{p}$ is isomorphic to the local ring $S_{(\mathfrak{p})}$

(b) For any homogenous $f \in S_+$, let $D_+(f) = \{ \mathfrak{p} \in \operatorname{Proj} S \mid f \notin \mathfrak{p} \}$. Then $D_+(f)$ is open in $\operatorname{Proj} S$. Furthermore these open sets cover $\operatorname{Proj} S$, and for each such open set, we have an isomorphism of locally ringed spaces

$$(D_+(f), \mathcal{O}|_{D_+(f)}) \cong \operatorname{Spec} S_{(f)}$$

where $S_{(f)}$ is the subring of elements of degree zero in S_f.

(c) $\operatorname{Proj} S$ is a scheme.

PROOF Note first that (a) says that Proj S is a locally ringed space, and (b) tells us it is covered by open affine schemes, so (c) is a consequence of (a) and (b).

(a) Same technique as (2.2a): First we define a morphism of rings $\mathcal{O}_{\mathfrak{p}} \xrightarrow{\varphi} S_{(\mathfrak{p})}$ by sending any $(V,s) \longmapsto s(\mathfrak{p})$. The map is surjective, since if $a/f \in S_{(\mathfrak{p})}$ with $f \notin \mathfrak{p}$ and a homogenous then (f) is a homogenous ideal, so $D(f) = \operatorname{Proj} S - V((f))$ is open and contains $\mathfrak{p}$. The map $\mathfrak{q} \longmapsto a/f \in S_{(\mathfrak{q})}$ on $D(f)$ is regular and maps to $a/f \in S_{(\mathfrak{p})}$ on the stalks. To show that φ is injective let $(V,s), (U,t) \in \mathcal{O}_{\mathfrak{p}}$ be given. Let V', U' be open neighborhoods of $\mathfrak{p}$ and $a, f, b, g \in S$ s.t. $s|_{V'} = a/f$ and $t|_{U'} = b/g$. It suffices to show that if $\varphi((V,s)) = \varphi((U,t))$ then s,t agree on a neighborhood of $\mathfrak{p}$.
Then $s(\mathfrak{p}) = t(\mathfrak{p})$ implies that $a/f = b/g$ in $S_{(\mathfrak{p})}$. Hence there is $w \notin \mathfrak{p}$ homogenous with $w(ga - fb) = 0$ in S. Therefore $a/f = b/g$ in every local ring $S_{(\mathfrak{q})}$ s.t. $f, g, w \notin \mathfrak{q}$. But the set of such $\mathfrak{q}$ is the open set $D(f) \cap D(g) \cap D(w)$, which contains $\mathfrak{p}$. Hence $s = t$ in a neighborhood of $\mathfrak{p}$, as required.

(b) First note that $D_+(f) = \operatorname{Proj} S - V((f))$ is open. Since the elements of Proj$S$ are those homogenous prime ideals $\mathfrak{p}$ of S which do not contain all of S_+, it follows that the open sets $D_+(f)$ for homogenous $f \in S_+$ cover Proj S.
Now fix a homogenous $f \in S_+$. We will define an isomorphism $(\varphi, \varphi^\#)$ of locally ringed spaces from $D_+(f)$ to Spec $S_{(f)}$. If $f = 0$ of course $(D_+(f), \mathcal{O}|_{D_+(f)}) = (\emptyset, 0) = \operatorname{Spec} 0 = \operatorname{Spec} S_{(f)}$ so the result is trivial. Indeed $D_+(f) = \emptyset$ iff. f is nilpotent (f hom. and in S_+)(see following note) so $D_+(f) \neq \emptyset$ and f not nilpotent can be assumed (so $S_f \neq 0$ also). Then $S_{(f)}$ is the subring of S_f consisting of the equivalence classes of fractions a/f^n where $a \in S_n$ $(n \geq 0)$. Since f is not nilpotent, $S_{(f)} \neq 0$. (If Proj$S = \emptyset$ then it is clearly affine, so assume Proj$S \neq \emptyset$ in which case Proj$S$ is covered by $D_+(f) \neq \emptyset$).
For any homogenous ideal $\mathfrak{a} \subseteq S$ let $\varphi(\mathfrak{a}) = (\mathfrak{a}S_f) \cap S_{(f)}$ (using the canonical $S \longrightarrow S_f$). This is an ideal of $S_{(f)}$ and is improper iff. $\mathfrak{a} \cap \{1, f, f^2, \ldots\} \neq \emptyset$. Hence for $\mathfrak{p} \in D_+(f)$ $\varphi(\mathfrak{p})$ is a prime ideal of $S_{(f)}$, hence an element of Spec $S_{(f)}$. This defines $\varphi: D_+(f) \longrightarrow \operatorname{Spec} S_{(f)}$. See the notes on following pages for the proof that this is a homeomorphism.

Next we define an isomorphism $\mathcal{O}_{\operatorname{Spec} S_{(f)}} \xrightarrow{\phi} \varphi_* \mathcal{O}|_{D_+(f)}$ of ringed spaces. First we show that for all $\mathfrak{p} \in D_+(f)$ there is a natural isomorphism $\psi_{\mathfrak{p}}: (S_{(f)})_{\varphi(\mathfrak{p})} \longrightarrow S_{(\mathfrak{p})}$. The natural ring morphism $S_f \longrightarrow S_{\mathfrak{p}}$ induces a homomorphism of the subrings $S_{(f)} \longrightarrow S_{(\mathfrak{p})}$ as in the following diagram: (say $f \in S_d$ $d > 0$)

$$
\begin{tikzcd}
S \arrow[r] \arrow[d] & S_f \arrow[dl] \\
S_{\mathfrak{p}} & S_{(f)} \arrow[u, hook] \arrow[d] \arrow[dl, "\varphi"'] \\
S_{(\mathfrak{p})} \arrow[u, hook] & (S_{(f)})_{\varphi(\mathfrak{p})} \arrow[l, dashed, "\psi"]
\end{tikzcd}
\qquad
\begin{aligned}
&\varphi(a/f^n) \quad a \in S_{nd} \\
&= a/f^n
\end{aligned}
$$

Here $\varphi(\mathfrak{p}) = \mathfrak{p}S_f \cap S_{(f)}$ consists of those classes $a/f^n \in S_{(f)}$ with $a \in \mathfrak{p}$. Hence φ maps elements not in $\varphi(\mathfrak{p})$ to units, inducing ψ, which is defined by

$$\psi(a/f^n, b/f^m) = \frac{af^m}{bf^n}$$

A tuple $(a/f^n, b/f^m)$ is zero in $(S_{(f)})_{\varphi(\mathfrak{p})}$ iff. $\exists q \notin \mathfrak{p}$ and $k \geq 0$ s.t. $f^k qa = 0$. It is thus clear that ψ is injective. To see that it is surjective, let $a/q \in S_{(\mathfrak{p})}$ be given, with $q \notin \mathfrak{p}$. Then $a/q = aq^{d-1}/q^d$ so we may assume $q \in S_{nd}$ and hence $a \in S_{nd}$ for some $n \geq 0$. Then

$$\psi(a/f^n, q/f^n) = a/q$$

So ψ is surjective, as required. If we are considering multiple primes, we denote this ψ by $\psi_{\mathfrak{p}}$.

Let $V \subseteq \operatorname{Spec} S_{(f)}$ be nonempty and open and $g: V \longrightarrow \bigcup_{\mathfrak{p} \in \varphi^{-1}V} (S_{(f)})_{\varphi(\mathfrak{p})}$ regular. Define a map $\phi_V(g): \varphi^{-1}V \longrightarrow \bigcup_{\mathfrak{p} \in \varphi^{-1}V} S_{(\mathfrak{p})}$ by

$$\phi_V(g)(\mathfrak{p}) = \psi_{\mathfrak{p}}(g(\varphi(\mathfrak{p})))$$

To see that $\phi_V(g)$ is regular let $\mathfrak{p} \in \varphi^{-1}V$ be given, find $\varphi(\mathfrak{p}) \in W \subseteq V$ and $a/f^n, b/f^m \in S_{(f)}$ with $b/f^m \notin \varphi(\mathfrak{q})$ for all $\mathfrak{q} \in \varphi^{-1}W$ and $g(\varphi(\mathfrak{q})) = (a/f^n, b/f^m)$ $\forall \mathfrak{q} \in \varphi^{-1}W$. Then $\forall \mathfrak{q} \in \varphi^{-1}W$

$$\phi_V(g)(\mathfrak{q}) = af^m / bf^n$$

so $\phi_V(g)$ is regular. The morphism of rings ϕ_V is clearly injective.

To see that ϕ_V is also surjective, let $h : \varphi^{-1}V \longrightarrow \bigcup_{\mathfrak{p} \in \varphi^{-1}V} S_{(\mathfrak{p})}$ be regular and define $H : V \longrightarrow \bigcup_{\mathfrak{p} \in \varphi^{-1}V} (S_{(f)})_{\varphi(\mathfrak{p})}$ by

$$H(\varphi(\mathfrak{p})) = \psi_{\mathfrak{p}}^{-1}(h(\mathfrak{p}))$$

This is regular since if $\mathfrak{p} \in V$ there is $\mathfrak{p} \in W \subseteq V$ and $a, q \in S$ with $q \notin \mathfrak{q}$ $\forall \mathfrak{q} \in W$ and $h(\mathfrak{q}) = a/q$ $\forall \mathfrak{q} \in W$. Then as above we may assume $a, q \in S_{nd}$ for some $n \geqslant 0$, so $(a/f^n, q/f^n) \in (S_{(f)})_{\varphi(\mathfrak{q})}$ $\forall \mathfrak{q} \in W$ and so $\forall \varphi(\mathfrak{q}) \in \varphi W$

$$H(\varphi(\mathfrak{q})) = (a/f^n, q/f^n)$$

So H is regular. Clearly $\phi_V(H) = h$, so ϕ is an isomorphism, as required. (ϕ is clearly natural). So $D_+(f)$ and $\operatorname{Spec} S_{(f)}$ are isomorphic as ringed spaces, hence as locally ringed spaces, completing the proof. $\square$

<u>LEMMA</u> The open sets $D_+(f)$ for homogenous $f \in S_+$ form a basis for the topology of $\operatorname{Proj} S$, and $D_+(f) \cap D_+(g) = D_+(fg)$.

<u>PROOF</u> Let $U = X - V(\mathfrak{a})$ be open, with $\mathfrak{a}$ homogenous. If $\mathfrak{p} \in U$ then $\mathfrak{p} \not\supseteq \mathfrak{a}$. If some homogenous $f \in S_+ \cap \mathfrak{a}$ exists with $f \notin \mathfrak{p}$, then $\mathfrak{p} \in D_+(f) \subseteq U$ and we are done. So suppose $\mathfrak{p} \not\supseteq \mathfrak{a}$ but $\mathfrak{p} \supseteq S_+ \cap \mathfrak{a}$. Let $a \in \mathfrak{a} \cap S_0$ be some element with $a \notin \mathfrak{p}$. Then for all homogenous $g \in S_+$ we have $ag \in \mathfrak{a} \cap S_+ \subseteq \mathfrak{p} \Longrightarrow g \in \mathfrak{p}$. But then $S_+ \subseteq \mathfrak{p}$, which is a contradiction. $\square$

<u>NOTE</u> Let S be a graded ring, $S = \operatorname{Proj} S$, $\mathfrak{p} \in \operatorname{Proj} S$. Then $\overline{\{\mathfrak{p}\}} = V(\mathfrak{p})$.

LEMMA Let $S = \bigoplus_{d\geq 0} S_d$ be a graded ring, and f a homogenous element of S_+. Then f is nilpotent iff. f is contained in every prime $\mathfrak{p} \in \operatorname{Proj} S$.

PROOF Clearly if f is nilpotent it belongs to all the homogenous primes of $\operatorname{Proj} S$. Conversely suppose f is not nilpotent but belongs to all $\mathfrak{p} \in \operatorname{Proj} S$. Then $T = \{1, f, f^2, \dots\}$ is a multiplicatively closed set not containing 0. Let Z be the set of all homogenous ideals in S not meeting T. It is not hard to check that T has upper bounds for all its chains — use Zorn to produce a maximal homogenous ideal $\mathfrak{p}$. We claim $\mathfrak{p}$ is prime. It suffices to show that if a, b are homogenous then $ab \in \mathfrak{p} \Rightarrow a \in \mathfrak{p}$ or $b \in \mathfrak{p}$. Suppose $ab \in \mathfrak{p}$ but $a \notin \mathfrak{p}, b \notin \mathfrak{p}$. Then $(a) + \mathfrak{p}$ and $(b) + \mathfrak{p}$ are homogenous ideals properly containing $\mathfrak{p}$, hence must intersect T. Say

$$t_1 = am + p \qquad\qquad t_1, t_2 \in T$$
$$t_2 = bn + q$$

Then $t_1 t_2 \in T$, but $t_1 t_2 = ambn + amq + pbn + pq \in \mathfrak{p}$ since $ab \in \mathfrak{p}$. This contradiction shows that $\mathfrak{p}$ is prime. Since $f \in S_+$ and $f \notin \mathfrak{p}$ it follows that $\mathfrak{p} \in \operatorname{Proj} S$, which is a contradiction since f belongs to all primes of $\operatorname{Proj} S$. Hence f must have been nilpotent. $\square$

Hence for f homogenous of degree ≥ 1, $D_+(f) = \emptyset$ iff. f is nilpotent iff. $S_f = 0$ iff. $S_{(f)} = 0$.

LEMMA Let $S = \bigoplus_{d \geq 0} S_d$ be a graded ring, $f \in S_1$ not nilpotent. Then there is a bijection betweeen homogenous prime ideals of S not containing f and prime ideals of $S_{(f)}$, given by

$$\begin{array}{ccc} D_+(f) & & \operatorname{Spec} S_{(f)} \\ \mathfrak{p} & \longmapsto & (\mathfrak{p} S_f) \cap S_{(f)} \\ \varphi^{-1}(\mathfrak{q} S_f) & \longleftarrow & \mathfrak{q} \end{array} \qquad \varphi : S \longrightarrow S_f$$

PROOF Note that $\mathfrak{p} \in D_+(f)$ iff. $\mathfrak{p}$ is homogenous, prime, does not contain f and does not contain S_+. But $f \in S_+$, so $D_+(f) = $ hom-prime $\not\ni f$. Let $\mathfrak{p} \in D_+(f)$. Then $\mathfrak{p} S_f$ is a prime ideal of S_f. Let $\{a_i\}_{i \in I}$ be a set of homogenous generators for $\mathfrak{p}$, and say $a_i \in S_{n_i}$, $n_i \geq 0$ for $i \in I$. The ideal $(\mathfrak{p} S_f) \cap S_{(f)}$ is clearly a prime ideal of $S_{(f)}$ and we claim it is generated in $S_{(f)}$ by a_i / f^{n_i}. Suppose $b \in S_m$ and $b/f^m \in (\mathfrak{p} S_f) \cap S_{(f)}$, so $b/f^m = a/f^n$ for some $a \in \mathfrak{p}$ and $n \geq 0$. Then for some $j \geq 0$

$$f^j (b f^n - a f^m) = 0$$
$$b f^{j+n} \in \mathfrak{p} \Rightarrow b f^{j+n} = \sum a_i r_i \qquad r_i \in S$$

Hence

$$\begin{aligned} b/f^m &= \sum_i \frac{a_i r_i}{f^{j+n+m}} \\ &= \sum_i \frac{a_i}{f^{n_i}} \frac{r_i}{f^{j+n+m-n_i}} \end{aligned}$$

As required. Now let $\mathfrak{q} \subseteq S_{(f)}$ be prime. For $d \geq 0$ let $\mathfrak{q}_d = \{ a \in S_d \mid a/f^d \in \mathfrak{q} \}$. Then $\mathfrak{p} = \bigoplus_{d \geq 0} \mathfrak{q}_d$ is an abelian group. We claim it is an ideal in S. If $a = a_0 + \cdots + a_d \in \mathfrak{p}$ then for $1 \leq i \leq d$ $a_i \in \mathfrak{q}_i$ so $a_i / f^i \in \mathfrak{q}$. If $b \in S$, say $b = b_0 + \cdots + b_e$, then for $x \geq 0$

$$(ab)_x = \sum_{i+j=x} a_i b_j$$

But for $i + j = x$ $a_i b_j / f^x = a_i/f^i \cdot b_j / f^{x-i} \in \mathfrak{q}$, so $(ab)_x \in \mathfrak{q}_x$. Thus $\mathfrak{p}$ is a homogenous ideal, which is clearly proper, since if $1 \in \mathfrak{p}$ then $1/1 \in \mathfrak{q}$, which is a contradiction. To see that $\mathfrak{p}$ is prime, let $a \in S_d$ and $b \in S_e$ be homogenous and suppose $ab \in \mathfrak{p}$. Then $ab \in \mathfrak{q}_{d+e}$, so $ab/f^{d+e} \in \mathfrak{q}$. Hence $a/f^d \cdot b/f^e \in \mathfrak{q}$. Since $\mathfrak{q}$ is prime either a/f^d or b/f^e is in $\mathfrak{q}$, so $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$. Since $\mathfrak{q}$ is proper it also follows that $\mathfrak{p} \in D_+(f)$.

Let $\mathfrak{q} S_f$ be the extension of $\mathfrak{q}$ to S_f. Then $\mathfrak{q} S_f \subseteq \mathfrak{p} S_f$ since if $a/f^n \in \mathfrak{q}$ then $a \in \mathfrak{q}_n \subseteq \mathfrak{p}$ so $a/f^n \in \mathfrak{p} S_f$. Conversely any ideal in S containing $\mathfrak{q}$ must contain $\mathfrak{p} S_f$, since if $a \in \mathfrak{q}_d$ $d \geq 0$ then $a/f^d \in \mathfrak{q}$ and so $a/1$ is in the ideal. Hence $\mathfrak{p} S_f \subseteq \mathfrak{q} S_f$, so $\mathfrak{p} S_f = \mathfrak{q} S_f$. Hence $\varphi^{-1}(\mathfrak{q} S_f) = \varphi^{-1}(\mathfrak{p} S_f) = \mathfrak{p}$.

To show that the association is bijective, we first show that $\varphi^{-1}(\mathfrak{q}S_f)$ gives $\mathfrak{q} = (\varphi^{-1}(\mathfrak{q}S_f)S_f)\cap S_{(f)}$. Since $\varphi^{-1}(\mathfrak{q}S_f)$ is prime avoiding f it suffices to show $\mathfrak{q}S_f \cap S_{(f)} = \mathfrak{q}$, or $\mathfrak{p}S_f \cap S_{(f)} = \mathfrak{q}$. The inclusion $\supseteq$ is clear. But if $a \in S_n$ and $a/f^n \in \mathfrak{p}S_f$ then $a \in \mathfrak{p} \Rightarrow a/f^n \in \mathfrak{q}$, so $\subseteq$ is clear as well.

Let $\mathfrak{p} \in D_+(f)$ be given and put $\mathfrak{q} = (\mathfrak{p}S_f)\cap S_{(f)}$. Let $\mathfrak{p}'$ be the homogenous prime ideal $\bigoplus_{d\geq 0} \mathfrak{q}_d$. Since $\mathfrak{p}, \mathfrak{p}'$ are both homogenous to show that $\mathfrak{p} = \mathfrak{p}'$ it suffices to show that they have the same homogenous elements. But if $a \in S_d$, $d \geq 0$, then $a \in \mathfrak{p}' \iff a \in \mathfrak{q}_d \iff a/f^d \in \mathfrak{q} \iff a \in \mathfrak{p}$. This completes the proof. □

not nilpotent

There is a similar result when f has any positive degree $d > 0$. But to prove this result we first introduce the graded ring

$$S^{(d)} = \bigoplus_{n \geq 0} S_{nd}$$

which has vanishing graded pieces in degrees not divisible by d. Since $(S^{(d)})_f \cong S^{(d)}[X]/(1 - Xf)$, by assigning X the degree $-d$ (see the next page) the ring $(S^{(d)})_f$ becomes $\mathbb{Z}$-graded, where s/f^n is assigned the degree $\deg(s) - nd$ for $s \in S^{(d)}$ homogenous. (so vanishing grading for degrees not divisible by d. If $m = nd$ then the graded piece of degree m of $(S^{(d)})_f$ (m may be < 0) are sums $\sum s_i/f^{n_i}$ where $\deg(s_i) - n_i d = m$)

<u>PROPOSITION</u> Let $S = \bigoplus_{d \geq 0} S_d$ be a graded ring, f any non-nilpotent homogenous element of degree ≥ 1. Then the map $\varphi : D_+(f) \longrightarrow \operatorname{Spec} S_{(f)}$ given by

$$\varphi(\mathfrak{p}) = (\mathfrak{p}S_f) \cap S_{(f)}$$

is a homeomorphism.

<u>PROOF</u> For any homogenous ideal $\mathfrak{a} \subseteq S$ we define $\varphi(\mathfrak{a}) = (\mathfrak{a}S_f)\cap S_{(f)}$. For any $\mathfrak{p} \in D_+(f)$ we claim that $\varphi(\mathfrak{a}) \subseteq \varphi(\mathfrak{p}) \iff \mathfrak{a} \subseteq \mathfrak{p}$. Once this is proved, it will follow that φ is at least injective. The $\Leftarrow$ implication is obvious, and for the converse it suffices to prove that if $a \in \mathfrak{a}$ is a homogenous element then $a \in \mathfrak{p}$. Let $n = \deg a \geq 0$ and let $d = \deg f > 0$. It follows that

$$\frac{a^d}{f^n} \in \mathfrak{a}S_f \cap S_{(f)} = \varphi(\mathfrak{a}) \subseteq \varphi(\mathfrak{p}) = \mathfrak{p}S_f \cap S_{(f)}$$

so there exists homogenous $x \in \mathfrak{p}$ such that $a^d/f^n = x/f^m$ in S_f with $md = \deg(x)$. Thus, for some $e \geq 0$

$$f^e(f^m a^d - x f^n) = 0$$

Since $d > 0$ and $f \notin \mathfrak{p}$ we see that $a \in \mathfrak{p}$ as required. This shows that φ is injective.

To show that φ is surjective we begin by showing that every prime ideal $\mathfrak{q}$ of $(S^{(d)})_{(f)}$ determines a homogenous prime ideal $\mathfrak{p}$ of S not containing f. Let $\mathfrak{q}$ be given, and make $S^{(d)}$ into a graded ring and $(S^{(d)})_f$ into a $\mathbb{Z}$-graded ring as above. Let $\mathfrak{q}'$ be the homogenous ideal in $(S^{(d)})_f$ generated by the elements $x/f^e f^r$ where $x \in S_{de}$ and $x/f^e \in \mathfrak{q}$, and $r \in \mathbb{Z}$ (i.e. $\mathfrak{q}'$ is the ideal generated, which is clearly homogenous). Since $\mathfrak{q}' \supseteq \mathfrak{q}$ it is clear that $\mathfrak{q}' = \mathfrak{q}(S^{(d)})_f$.

We claim that $\mathfrak{q}'$ actually consists of sums $\sum_i x_i/f^{e_i} f^{r_i}$ with $x_i \in S_{de_i}$ and $r_i \in \mathbb{Z}$. To see this, let $a/f^n \in (S^{(d)})_f$ be given, with $a \in S_{dk}$. Then

$$\left(\sum_i x_i/f^{e_i} f^{r_i}\right) a/f^n = \sum_i \frac{x_i a}{f^{e_i + k}} f^{r_i + k - n}$$

Moreover if $x/f^e f^r$ and $y/f^k f^r$ are two generators of the same degree, their sum is $(x/f^e + y/f^k)f^r$, which is of the same form. Hence a homogenous element of $\mathfrak{q}'$ must take the form $x/f^e f^r$ for some x, e and r. From this observation it follows immediately that $\mathfrak{q}' \cap (S^{(d)})_{(f)} = \mathfrak{q}$.

Let $\mathfrak{p}'$ the the contraction of $\mathfrak{q}(S^{(d)})_f$ under $S^{(d)} \longrightarrow (S^{(d)})_f$. Then $\mathfrak{p}'(S^{(d)})_f \cap (S^{(d)})_{(f)} = \mathfrak{q}$. Since $\mathfrak{q}'$ is a homogenous proper ideal ($\mathfrak{q}'$ contracts to $\mathfrak{q}$) it follows that $\mathfrak{p}'$ is also proper and homogenous (and does not contain f). We claim that $\mathfrak{p}'$ is prime. Let $a, a' \in S^{(d)}$ be homogenous with degrees dn, dn' and assume $aa' \in \mathfrak{p}'$. Thus $aa'/1 \in \mathfrak{q}'$, so $aa'/1 = (x/f^e)f^r$ for some $r \in \mathbb{Z}$, $x \in S_{ed}$, $x/f^e \in \mathfrak{q}$. Comparing degrees we find that $dr = dn + dn'$, so $n + n' = r$. Hence $aa'/f^r = (a/f^n)(a'/f^{n'}) \in (S^{(d)})_f$ is a product of terms with deg. 0. However,

$$(a/f^n)(a'/f^{n'}) = x/f^e \in \mathfrak{q}$$

Since $\mathfrak{q}$ is prime in $(S^{(d)})_{(f)}$ it follows that one of a/f^n, $a'/f^{n'}$ belongs to $\mathfrak{q}$. Hence $a \in \mathfrak{p}'$ or $a' \in \mathfrak{p}'$, as required. Hence any prime ideal in $(S^{(d)})_{(f)}$ has the form $\mathfrak{p}'(S^{(d)})_f \cap (S^{(d)})_{(f)}$ for some homogenous prime ideal $\mathfrak{p}'$ of $S^{(d)}$, which does not contain f.

Let $\mathfrak{p}'$ be a homogenous prime ideal of $S^{(d)}$ not containing f and let $\mathfrak{p}$ be the homogenous prime ideal generated by all homogenous $a \in S$ with $a^d \in \mathfrak{p}' \subseteq S^{(d)}$. Then $\mathfrak{p}$ does not contain f: hence $\mathfrak{p} \in D_+(f)$. Moreover $S^{(d)}$ is a graded subring of S and it is not hard to see that $\mathfrak{p} \cap S^{(d)} = \mathfrak{p}'$. Hence any prime ideal in $(S^{(d)})_{(f)}$ can be produced via

$$\begin{array}{ccccc} \mathfrak{p} \in D_+(f) & \longmapsto & \mathfrak{p} \cap S^{(d)} \subseteq S^{(d)} & \longmapsto & (\mathfrak{p} \cap S^{(d)})(S^{(d)})_f \\ \mathfrak{p} \subseteq S & & & & \downarrow \\ & & & & (\mathfrak{p} \cap S^{(d)})(S^{(d)})_f \cap (S^{(d)})_{(f)} \end{array}$$

The inclusion $S^{(d)} \longrightarrow S$ induces $(S^{(d)})_f \longrightarrow S_f$ which restricts to an isomorphism $\phi : (S^{(d)})_{(f)} \longrightarrow S_{(f)}$ that identifies $(\mathfrak{p} \cap S^{(d)})(S^{(d)})_f \cap (S^{(d)})_{(f)}$ and $\mathfrak{p} S_f \cap S_{(f)}$, proving that φ is surjective.

A closed subset of $D_+(f)$ is $D_+(f) \cap V(\mathfrak{a})$ for some homogenous $\mathfrak{a} \subseteq S$. But then φ identifies $D_+(f) \cap V(\mathfrak{a})$ with $V(\varphi(\mathfrak{a}))$, so φ is a homeomorphism. $\square$

<u>LEMMA</u> Let S be a graded ring, $\mathfrak{a}$ a homogenous ideal. If $\mathfrak{a}$ is proper, there is a homogenous prime ideal $\mathfrak{p}$ with $\mathfrak{p} \supseteq \mathfrak{a}$.

<u>PROOF</u> Let Z be the partially ordered set of all proper homogenous ideals containing $\mathfrak{a}$. Then upper bounds for chains are given by unions, so we can apply Zorn's Lemma to produce a maximal element $\mathfrak{p}$. We claim $\mathfrak{p}$ is prime. For this it suffices to show that if $a, b \in S$ are homogenous and $ab \in \mathfrak{p}$ then $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$. Suppose that $a \notin \mathfrak{p}$, $b \notin \mathfrak{p}$. Then $(a)+\mathfrak{p}$, $(b)+\mathfrak{p}$ are homogenous ideals containing $\mathfrak{p}$, hence must be improper. Say $ar + p = 1$, $bs + q = 1$. Then $abrs + arq + bsp + pq = 1$, which implies $1 \in \mathfrak{p}$ since $ab \in \mathfrak{p}$. This contradiction shows that $ab \in \mathfrak{p} \Rightarrow a \in \mathfrak{p}$ or $b \in \mathfrak{p}$. $\square$

NOTE Polynomial gradings

Let $S = \bigoplus_{d \geq 0} S_d$ be a nonzero graded ring. Let $e \in \mathbb{Z}$ be any nonzero integer. We define a $\mathbb{Z}$-grading on $S[X]$ as follows:

$$(S[X])_d = \{ \sum s_i X^{n_i} \mid \text{for each } i \ s_i \text{ is homogenous and } \deg(s_i) + n_i \cdot e = d \}$$

So X has degree e and homogenous elements of S keep their degrees ($n_i = 0$). Each $(S[X])_d$ is clearly an abelian group, and by writing coefficients as sums of homogenous elements it is readily seen that $\sum_{d \in \mathbb{Z}} (S[X])_d = S[X]$. It remains to show the sum is direct.

We need to show that $S[X]_d \cap (S[X]_{d_1} + \cdots + S[X]_{d_m}) = 0$ for distinct $d, d_1, \ldots, d_m$. Suppose

$$\sum s_i X^{n_i} = \sum s_{1,i} X^{n_{1,i}} + \cdots + \sum s_{m,i} X^{n_{m,i}}$$

with $\deg(s_i) + n_i e = d$ and $\deg(s_{j,i}) + n_{j,i} e = d_j$ for $1 \leq j \leq m$. But then

$$\sum_k \Big(\sum_{\deg s_i = d - ke} s_i \Big) X^k$$
$$= \sum_k \Big(\sum_{\deg s_{1,i} = d_1 - ke} s_i \Big) X^k + \cdots + \sum_k \Big(\sum_{\deg s_{m,i} = d_m - ke} s_{m,i} \Big) X^k$$

so $\forall k \geq 0$ $\sum_{\deg s_i = d - ke} s_i = \sum_{\deg s_{1,i} = d_1 - ke} s_{1,i} + \cdots + \sum_{\deg s_{m,i} = d_m - ke} s_{m,i}$. Since S is graded and $e \neq 0$ it follows that $\forall k \geq 0$ $\sum_{\deg s_i = d - ke} s_i = 0$, so $\sum s_i X^{n_i} = 0$, as required.

NOTE S_f is $\mathbb{Z}$-graded

Let $S = \bigoplus_{d \geq 0} S_d$ be a nonzero graded ring, $f \in S_e$ homogenous of degree $e > 0$. There is an isomorphism of rings

$$\varphi : S[X]/(1 - Xf) \longrightarrow S_f$$

Thus S_f is a $\mathbb{Z}$-graded ring, since $(1 - Xf)$ is a homogenous ideal. If T is any $\mathbb{Z}$-graded ring $\mathfrak{a} \subseteq T$ a homogenous ideal, $T/\mathfrak{a}$ is $\mathbb{Z}$-graded via $(T/\mathfrak{a})_d = T_d + \mathfrak{a}$ $(d \in \mathbb{Z})$. Hence the grading on S_f is defined by $(d \in \mathbb{Z})$

$$(S_f)_d = \varphi(S[X]_d + (1 - Xf))$$
$$= \{ \textstyle\sum_i s_i / f^{n_i} \mid s_i \text{ homogenous and } \deg(s_i) - n_i e = d \}$$
$$= \{ s / f^n \mid s \text{ homogenous and } \deg(s) - ne = d \}$$

where put the degree of $X = -e$ in the above.

EXAMPLE 2.5.1 If A is a ring, we define projective n-space over A to be the scheme $(n \geq 0)$

$$\mathbb{P}^n_A = \operatorname{Proj} A[x_0, \dots, x_n]$$

In particular, if A is algebraically closed and a field, then $\mathbb{P}^n_A$ is a scheme whose subspace of closed points is naturally homeomorphic to the variety called projective n-space. $(n \geq 1)$

To summarise: $\mathbb{A}^n_k = \operatorname{Spec} k[x_1, \dots, x_n]$ for an algebraically closed field k has closed points in bijection with k^n and these points with the subspace topology are homeomorphic to the variety called affine n-space, since a subset of k^n is closed iff. it is $Z(\mathfrak{a})$ for some $\mathfrak{a} \subseteq k[x_1, \dots, x_n]$.

A subset of projective n-space ✓ the variety is closed iff. it is $Z(T)$ for a set T of homogenous elements, but $Z(T) = Z((T))$ so iff. it is $Z(\mathfrak{a})$ for a homogenous ideal $\mathfrak{a} \subseteq k[x_0, \dots, x_n]$. Here $Z(\mathfrak{a})$ denotes $Z(T')$ where T' is the set of homogenous elements in $\mathfrak{a}$. There is a bijective correspondence between closed sets in projective n-space and homogenous radical ideals in $S = k[x_0, \dots, x_n]$ other than S_+, identifying proj. varieties and homogenous primes. Since $S_+ = (x_0, \dots, x_n)$ is maximal, $\operatorname{Proj} S$ consists simply of all homogenous ideals other than S_+. For a point $P = (a_0, \dots, a_n)$, $I(P) = (a_i x_0 - a_0 x_i, \dots, a_i x_n - a_n x_i)$ $(a_i \neq 0)$. By def^N $I(P)$ is the ideal generated by all homogenous elements vanishing on P, so for $f \in S$ homogenous $f \in I(P)$ iff. $f(P) = 0$. So for a homogenous ideal $\mathfrak{a}$, let T be the set of all homogenous elements of $\mathfrak{a}$

$$\begin{aligned} P \in Z(\mathfrak{a}) &\iff \forall f \in T \ f(P) = 0 \\ &\iff \forall f \in T \ f \in I(P) \\ &\iff \mathfrak{a} \subseteq I(P) \\ &\iff I(P) \in V(\mathfrak{a}) \end{aligned}$$

It is easy to see that the closed points of $\operatorname{Proj} S$ are the $I(P)$, for points P of the variety $\mathbb{P}^n$, and the above shows that these closed points form a subspace homeomorphic to $\mathbb{P}^n$ the variety.

NOTE If $A = 0$ then $\mathbb{A}^n_A = \mathbb{P}^n_A = 0 \ \forall n$.

Next we will show that the notion of scheme does in fact generalise the notion of variety. It is not quite true that a variety is a scheme (although any variety is a locally ringed space). As we have already seen above, the underlying topological space of a scheme such as $\mathbb{A}^1_k$ or $\mathbb{A}^2_k$ has more points than the corresponding variety. However, we will show that there is a natural way of adding generic points (Ex 2.9) for every irreducible subset of a variety so that the variety becomes a scheme.

Let S be a fixed scheme. A scheme over S is a scheme X together with a morphism $X \longrightarrow S$. If X and Y are schemes over S a morphism of X to Y as schemes over S, (also called an S-morphism) is a morphism $f: X \longrightarrow Y$ making

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

commute. We denote by $\underline{\mathrm{Sch}}(S) = \underline{\mathrm{Sch}}/S$ the category of schemes over S. If A is a ring, then by abuse of notation we write $\underline{\mathrm{Sch}}(A)$ for $\underline{\mathrm{Sch}}/\operatorname{Spec} A$.

PROPOSITION 2.6 Let k be an algebraically closed field. There is a natural fully faithful functor

$$t : \underline{\mathrm{Var}}(k) \longrightarrow \underline{\mathrm{Sch}}(k)$$

from the category of varieties over k to the category of schemes over k. For any variety V, its topological space is isomorphic to the set of closed points of $sp(t(V))$, and its sheaf of regular functions is obtained by restricting the structure sheaf of $t(V)$ via this homeomorphism.

PROOF To begin with, let X be any topological space, and let $t(X)$ be the set of all (nonempty) irreducible closed subsets of X. If $X = \emptyset$ then $t(X) = \emptyset$ but if $P \in X$ then $\overline{\{P\}} \in t(X)$ so $t(X)$ is nonempty. If Y is a closed subset of X then $t(Y) \subseteq t(X)$. Furthermore, $t(Y_1 \cup Y_2) = t(Y_1) \cup t(Y_2)$ and $t(\bigcap Y_i) = \bigcap t(Y_i)$. So we can define a topology on $t(X)$ by taking as closed sets the subsets of the form $t(Y)$, where $Y \subseteq X$ is closed (possibly $\emptyset$). If $f: X_1 \longrightarrow X_2$ is a continuous map, define $t(f): t(X_1) \longrightarrow t(X_2)$ by

$$t(f)(Q) = \overline{f(Q)}$$

Then $t(f)$ is continuous since $t(f)^{-1}(t(Y)) = t(f^{-1}Y)$ for $Y \subseteq X_2$ closed. Furthermore t is a functor, since if X_1, X_2, X_3 are nonempty and

$$\begin{tikzcd} X_1 \arrow[r, "g"] & X_2 \arrow[r, "f"] & X_3 \\ t(X_1) \arrow[r, "t(g)"] \arrow[rr, bend right, "t(fg)"'] & t(X_2) \arrow[r, "t(f)"] & t(X_3) \end{tikzcd}$$

Then for $Q \in t(X_1)$ we have $t(fg)(Q) = \overline{f(g(Q))}$ and $t(f)t(g)(Q) = \overline{f(\overline{g(Q)})}$. Clearly $\overline{f(g(Q))} \subseteq \overline{f(\overline{g(Q)})}$. To show the reverse inclusion, it suffices to show that any open set U meeting $f(\overline{g(Q)})$ also meets $f(g(Q))$. If U meets $f(\overline{g(Q)})$ then $f^{-1}U \cap \overline{g(Q)}$ is nonempty, hence $f^{-1}U \cap g(Q)$ is nonempty (since otherwise $(f^{-1}U)^c \cap \overline{g(Q)}$ would be a closed set containing $g(Q)$ properly contained in $\overline{g(Q)}$), so finally U meets $f(g(Q))$ as required. Hence $t(fg) = t(f)t(g)$, and t is a functor.

We define a continuous map $\alpha : X \longrightarrow t(X)$ by $\alpha(P) = \overline{\{P\}}$. An open subset of $t(X)$ is $t(Y)^c$ for some closed $Y \subseteq X$, but $\alpha^{-1}(t(Y)^c) = Y^c$, so α sets up a bijection between the open sets of X and $t(X)$ via $U \longleftrightarrow \alpha^{-1}U$. This is a bijection since $t(Y) \subseteq t(Z) \iff Y \subseteq Z$. In particular if $U \subseteq t(X)$ is nonempty $\exists P \in X$ s.t. $\alpha(P) \in U$.

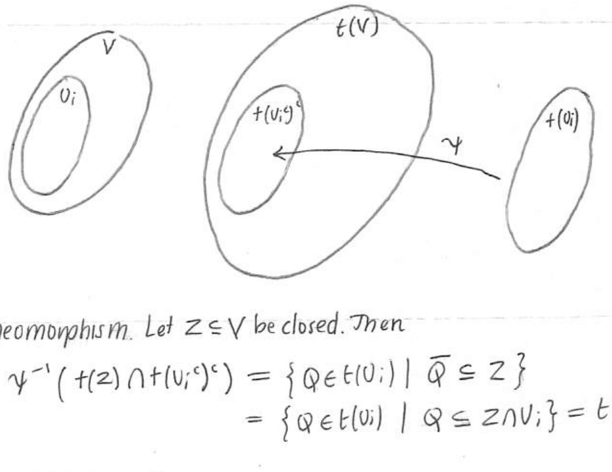
Now let k be an algebraically closed field. Let V be a variety over k, and let $\mathcal{O}_V$ be its sheaf of regular functions (1.0.1). We will show that $(t(V), \alpha_* \mathcal{O}_V)$ is a scheme over k. Since the lattice of open subsets is isomorphic to the lattice of open subsets of V, $\alpha_* \mathcal{O}_V$ and $\mathcal{O}_V$ are essentially the same.

To show that $(t(V), \alpha_* \mathcal{O}_V)$ is a scheme it suffices to show it is covered by open subsets which are schemes (this also shows $(t(V), \alpha_* \mathcal{O}_V)$ is locally ringed). But any variety is covered by open affine varieties (I, 4.3). Suppose we can show that $(t(Y), \alpha_* \mathcal{O}_Y)$ is affine when Y is affine. Since V is quasi-compact, we can write $V = U_1 \cup \cdots \cup U_n$ with U_i affine opens. Then $\phi = U_1^c \cap \cdots \cap U_n^c$ so $\phi = t(\phi) = t(U_1^c) \cap \cdots \cap t(U_n^c)$ and hence $t(V)$ is covered by $t(U_1^c)^c, \ldots, t(U_n^c)^c$. We will show that $(t(U_i^c)^c, (\alpha_* \mathcal{O}_V)|_{t(U_i^c)^c}) \cong (t(U_i), \alpha_{i*} \mathcal{O}_{U_i})$ where $\alpha_i : U_i \longrightarrow t(U_i)$ is canonical. By assumption this second ringed space is affine, showing that $(t(V), \alpha_* \mathcal{O}_V)$ is a scheme.

We define $\psi : t(U_i) \longrightarrow t(U_i^c)^c$ by $\psi(Q) = \overline{Q}$. To be explicit

$$\begin{aligned} t(U_i^c)^c &= \{Q \subseteq V \mid Q \text{ closed, irreducible and } Q \nsubseteq U_i^c\} \\ &= \{Q \subseteq V \mid Q \text{ closed, irreducible and } Q \cap U_i \neq \phi\} \end{aligned}$$

If $Q \subseteq U_i$ is closed and irreducible (in the subspace topology on $U_i \subseteq V$) then the closure of Q in V belongs to $t(U_i^c)^c$. Next we claim that for $Q \in t(U_i)$, $\overline{Q} \cap U_i = Q$. One inclusion is clear — in the other direction, note that since Q is closed in U_i we have $Q = U_i \cap Z$ for some $Z \subseteq V$ closed. Thus $\overline{Q} \subseteq Z$ and consequently $\overline{Q} \cap U_i \subseteq Z \cap U_i = Q$. Hence ψ is injective. Let $Q \subseteq V$ be closed and irreducible, and suppose $Q \cap U_i \neq \phi$. Then $\overline{Q \cap U_i} \subseteq Q$ is a closed subset of Q containing $Q \cap U_i$, which is open in the subspace topology on Q. Since Q is irreducible, $Q = \overline{Q \cap U_i}$. But the induced topology on $Q \cap U_i \subseteq U_i$ is the induced topology of $Q \cap U_i \subseteq Q$ — and in Q, $Q \cap U_i$ is a nonempty open subset of an irreducible space. Hence by (1.1.3) $Q \cap U_i \in t(U_i)$ and so ψ is a bijection.



Next we show that ψ is a homeomorphism. Let $Z \subseteq V$ be closed. Then

$$\begin{aligned} \psi^{-1}(t(Z) \cap t(U_i^c)^c) &= \{Q \in t(U_i) \mid \overline{Q} \subseteq Z\} \\ &= \{Q \in t(U_i) \mid Q \subseteq Z \cap U_i\} = t(Z \cap U_i) \end{aligned}$$

If $M \subseteq U_i$ is closed then

$$\Psi(t(M)) = \{ \overline{Q} \mid Q \in t(M) \} = t(\overline{M}) \cap t(U_i^c)^c$$

Since for $Q \in t(U_i)$, $\overline{Q} \subseteq \overline{M}$ iff. $Q \subseteq M$ $(\overline{M} \cap U_i = M)$. Hence Ψ is a homeomorphism. Next we define an iso $\phi : (\alpha_* \mathcal{O}_V)|_{t(U_i^c)^c} \longrightarrow \Psi_*(\alpha_{i*} \mathcal{O}_{U_i})$. For an open subset $t(Z)^c \subseteq t(U_i^c)^c$ (hence $Z^c \subseteq U_i$) we have

$$\begin{aligned} \phi : (\alpha_* \mathcal{O}_V)(t(Z)^c) &= \mathcal{O}_V(Z^c) = \mathcal{O}_{U_i}(Z^c) \\ &= (\alpha_{i*} \mathcal{O}_{U_i})(t(Z \cap U_i)^c) \\ &= (\alpha_{i*} \mathcal{O}_{U_i})(\Psi^{-1}(t(Z)^c)) \end{aligned}$$

which is clearly an isomorphism. Hence as claimed, $(t(U_i^c)^c, (\alpha_* \mathcal{O}_V)|_{t(U_i^c)^c}) \cong (t(U_i), \alpha_{i*} \mathcal{O}_{U_i})$. To complete the proof that $(t(V), \alpha_* \mathcal{O}_V)$ is a scheme, it suffices to show that if Y is affine (i.e. a variety isomorphic to an affine variety in $\mathbb{A}^n$ some n) then $(t(Y), \alpha_* \mathcal{O}_Y)$ is an affine scheme.

If a variety Y is affine, say it is isomorphic to $Y' \subseteq \mathbb{A}^n$, then it is readily checked that $(t(Y), \alpha_* \mathcal{O}_Y) \cong (t(Y'), \alpha_* \mathcal{O}_{Y'})$ as ringed spaces. We may thus reduce to the case where Y is actually an affine variety, with affine coordinate ring A, and let $(X, \mathcal{O}_X) = \operatorname{Spec} A$. We define a morphism of locally ringed spaces

$$\beta : (Y, \mathcal{O}_Y) \longrightarrow (X, \mathcal{O}_X)$$

as follows. For each point $P \in Y$, let $\beta(P) = \mathfrak{m}_P$, the ideal of A consisting of all regular functions vanishing at P. Then by (I, 3.2b), β is a bijection of Y onto the set of closed points of X. It is easy to see that β is a homeomorphism onto its image. Now for any open set $U \subseteq X$, we will define a homomorphism of rings $\mathcal{O}_X(U) \longrightarrow \beta_*(\mathcal{O}_Y)(U) = \mathcal{O}_Y(\beta^{-1}U)$. Given a section $s \in \mathcal{O}_X(U)$ we define a regular map $\beta(s) : \beta^{-1}U \longrightarrow k$ as follows

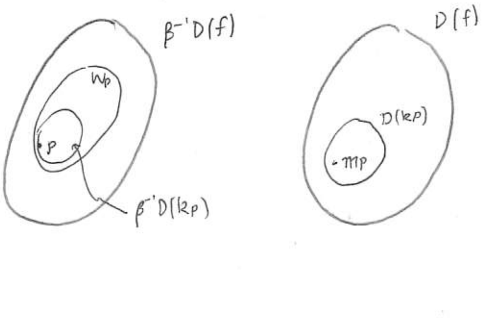
$$\beta(s)(P) = \mu_P(s(\mathfrak{m}_P)) \qquad \mu_P : A_{\mathfrak{m}_P} \longrightarrow A_{\mathfrak{m}_P}/\mathfrak{m}_P \cong k$$
$$(\cong A/\mathfrak{m}_P)$$

Then $\beta(s)$ is regular since $\mu_P(a/f) = a(P)/f(P)$. We claim that this map $\mathcal{O}_X(U) \longrightarrow \beta_*(\mathcal{O}_Y)(U)$ is an isomorphism of rings. The important fact is that since A is a quotient of $k[x_1, \dots, x_n]$ by a prime ideal, A is a Hilbert ring – in particular, the radical of an ideal is the intersection of all the <u>maximal</u> ideals containing it. So for example if $U \subseteq X$ is nonempty open, say $U = V(\mathfrak{a})^c$, then some closed point of X is in U. Otherwise $\mathfrak{a}$ is contained in every max. ideal, hence is 0 (A domain). But then $U = V(\mathfrak{a})^c$ would be empty – a contradiction.

Let us show that $\mathcal{O}_X(U) \longrightarrow \mathcal{O}_Y(\beta^{-1}U)$ is injective. Let $U \neq \emptyset$ (so also $\beta^{-1}U \neq \emptyset$ by the above) and suppose $s \in \mathcal{O}_X(U)$ is s.t. $\beta(s) = 0$. That is, $\forall P \in \beta^{-1}U$ $s(\mathfrak{m}_P) \in \mathfrak{m}_P A_{\mathfrak{m}_P}$. We need to show that $s(\mathfrak{p}) = 0$ $\forall \mathfrak{p} \in U$. Let $\mathfrak{p} \in U$ be given, and find $\mathfrak{p} \in W \subseteq U$ with W open, and $a, f \in A$ with $f \notin \mathfrak{q}$ $\forall \mathfrak{q} \in W$ and $s(\mathfrak{q}) = a/f \in A_{\mathfrak{q}}$ $\forall \mathfrak{q} \in W$. Any open set in $\operatorname{Spec} A$ is covered by opens of the form $D(h), h \in A$. So let $\mathfrak{p} \in D(h) \subseteq W$. Let $\mathfrak{m} \subseteq A$ be a maximal ideal with $h \notin \mathfrak{m}$. Then $a/f = s(\mathfrak{m}) \in \mathfrak{m} A_{\mathfrak{m}}$. Hence $a \in \mathfrak{m}$. It follows that ah belongs to every maximal ideal of A, and thus $ah = 0$. Since $D(h) \neq \emptyset$, we see that $a = 0$, and consequently $s(\mathfrak{p}) = 0$. Since $\mathfrak{p}$ was arbitrary, we conclude that $s = 0$ and the map is injective. (For 0 cases, $\mathcal{O}_X(U) = 0 \Longleftrightarrow U = \emptyset$)

Let a regular map $g : \beta^{-1}D(f) \longrightarrow k$ be given.

We first show that the map is surjective in the case where $U = D(f)$ for some $f \in A$. For each $P \in \beta^{-1}D(f)$ let $W_P \subseteq \beta^{-1}D(f)$ be an open neighborhood of P and $a_P, h_P \in A$ s.t. $\forall Q \in W_P$ $g(Q) = a_P(Q)/h_P(Q)$ and $h_P(Q) \neq 0$. Since β is a homeomorphism onto its image there is $k_P \in A$ s.t. $P \in \beta^{-1}D(k_P) \subseteq W_P$.



The open sets $\beta^{-1}D(k_P)$ cover $\beta^{-1}D(f)$, so the open sets $D(k_P)$, $P \in \beta^{-1}D(f)$ cover $D(f)$, since if $U \subseteq V$ are open sets in Spec$A$ with U containing all the closed points in V, then $U = V$, since if $U = V(\mathfrak{a})^c$, $V = V(\mathfrak{b})^c$ with $\mathfrak{a}, \mathfrak{b}$ radical, then $\mathfrak{a} \subseteq \mathfrak{b}$ by assumption and $\mathfrak{b} \subseteq \mathfrak{a}$ since $\mathfrak{b}$ is contained in every maximal ideal containing $\mathfrak{a}$, and A is Hilbert.

For each P define $g_P : D(k_P) \longrightarrow \bigcup_{\mathfrak{p} \in D(k_P)} A_{\mathfrak{p}}$ by $g_P(\mathfrak{p}) = a_P/h_P$. (By replacing $D(k_P)$ by $D(k_P h_P) = D(k_P) \cap D(h_P)$ we may assume $h_P \notin \mathfrak{p}$ $\forall \mathfrak{p} \in D(k_P)$ - i.e. $k_P' = k_P h_P$). Then clearly $g_P \in \mathcal{O}_X(D(k_P))$. For distinct $P, Q \in \beta^{-1}D(f)$ the maps g_P, g_Q agree on $D(k_P) \cap D(k_Q)$ since for $t = g_P|_{D(k_P)\cap D(k_Q)} - g_Q|_{D(k_P)\cap D(k_Q)} \in \mathcal{O}_X(D(k_P) \cap D(k_Q))$ we have

$$\begin{aligned} \beta(t)(Z) &= \mu_Z(g_P(\mathfrak{m}_Z) - g_Q(\mathfrak{m}_Z)) \qquad\qquad Z \in \beta^{-1}(D(k_P)\cap D(k_Q)) \\ &= \mu_Z(a_P/h_P - a_Q/h_Q) \qquad\qquad\qquad = \beta^{-1}D(k_P) \cap \beta^{-1}D(k_Q) \\ &= a_P(Z)/h_P(Z) - a_Q(Z)/h_Q(Z) \\ &= g(Z) - g(Z) = 0 \end{aligned}$$

So by the injectivity proved earlier, $g_P|_{D(k_P)\cap D(k_Q)} = g_Q|_{D(k_P)\cap D(k_Q)}$. Since $\mathcal{O}_X$ is a sheaf there is a unique $g' \in \mathcal{O}_X(D(f))$ with $g'|_{D(k_P)} = g_P$ $\forall P \in \beta^{-1}D(f)$. Hence for $P \in \beta^{-1}D(f)$

$$\begin{aligned} \beta(g')(P) &= \mu_P(g'(\mathfrak{m}_P)) \\ &= \mu_P(g_P(\mathfrak{m}_P)) \\ &= \mu_P(a_P/h_P) = a_P(P)/h_P(P) = g(P) \end{aligned}$$

So $\mathcal{O}_X(D(f)) \longrightarrow \mathcal{O}_Y(\beta^{-1}D(f))$ is surjective, as claimed. If $U \subseteq X$ is an arbitrary open set, then cover U with $D(f_i)$, say $U = \bigcup_{i\in I} D(f_i)$. Then $\beta^{-1}U = \bigcup_i \beta^{-1}D(f_i)$ and if $g \in \mathcal{O}_Y(\beta^{-1}U)$ is regular, let $g_i \in \mathcal{O}_X(D(f_i))$ be s.t. $\beta(g_i) = g|_{\beta^{-1}D(f_i)}$. Since $\mathcal{O}_X(D(f_i)\cap D(f_j)) \longrightarrow \mathcal{O}_Y(\beta^{-1}D(f_i) \cap \beta^{-1}D(f_j))$ is injective, the g_i are a matching family, hence have a unique amalgamation $g' \in \mathcal{O}_X(U)$. Clearly $\beta(g') = g$, so $\mathcal{O}_X(U) \longrightarrow \mathcal{O}_Y(\beta^{-1}U)$ is surjective, as required.

Hence β gives an isomorphism of locally ringed spaces $\mathcal{O}_X \longrightarrow \beta_* \mathcal{O}_Y$. Since the prime ideals of A are in 1-1 correspondence with the irreducible closed subsets of Y, there is a map $\gamma : X \longrightarrow t(Y)$, $\mathfrak{p} \longmapsto Z(\mathfrak{p})$ which is easily checked to be a homeomorphism. It is actually an isomorphism of ringed spaces $(X, \mathcal{O}_X) \cong (t(Y), \alpha_* \mathcal{O}_Y)$, as is easily checked. Hence $(t(Y), \alpha_* \mathcal{O}_Y)$ is in fact an affine scheme, as claimed. This completes the proof that $(t(V), \alpha_* \mathcal{O}_V)$ is a scheme for any variety V.

To give a morphism of $(t(V), \alpha_* \mathcal{O}_V)$ to Spec$k$, we need only give a morphism of rings $k \longrightarrow \Gamma(t(V), \alpha_* \mathcal{O}_V) = \mathcal{O}_V(V)$. We send $\lambda \in k$ to the constant function λ on V. Thus $t(V)$ becomes a scheme over Spec$k$.

NOTE The assignment $t : \underline{Var(k)} \longrightarrow \underline{Sch}(k)$ is actually injective on objects. If V, W are varieties and $(t(V), \alpha_* \mathcal{O}_V) = (t(W), \beta_* \mathcal{O}_W)$ where $\alpha : V \longrightarrow t(V)$ and $\beta : W \longrightarrow t(W)$ are canonical, then $P \in V \Rightarrow \{P\} \in t(V) = t(W)$ so $P \in W$. Hence $V \subseteq W$ and similarly $W \subseteq V$. If $U \subseteq V$ is open then $t(V - U) \subseteq t(V)$ is closed, hence is $t(Q) \subseteq t(W) = t(V)$ for $Q \subseteq W$ closed. It follows that $V - U = Q$ and so V is also open in W. Hence $V = W$ as spaces. It is then easy to see that $\mathcal{O}_V = \mathcal{O}_W$, so V and W are the same variety.

Next we define t on morphisms. Let $f : V \longrightarrow W$ be a morphism of varieties. Then $t(f) : t(V) \longrightarrow t(W)$ is continuous. For an open subset $t(Z)^c \subseteq t(W)$, define $(\beta_* \mathcal{O}_W)(t(Z)^c) = \mathcal{O}_W(Z^c) \longrightarrow \mathcal{O}_V(f^{-1}Z^c) = (\alpha_* \mathcal{O}_V)(t(f)^{-1} t(Z)^c)$ in the obvious way, giving a morphism of schemes over Spec$k$ $(t(V), \alpha_* \mathcal{O}_V) \longrightarrow (t(W), \beta_* \mathcal{O}_W)$. It is easily checked that with this definition $t : \underline{Var(k)} \longrightarrow \underline{Sch}(k)$ is a functor. We show in Ex 2.15 that t is fully faithful. In particular, this will imply that $t(V)$ is isomorphic to $t(W)$ iff. V is isomorphic to W.

It is clear from the construction that $\alpha : V \longrightarrow t(V)$ induces a homeomorphism from V onto the set of closed points of $t(V)$, with the induced topology. □

Since points of V are closed, $\alpha(P) = \{P\}$. By def^n the sheaf on $t(V)$ is $\alpha_* \mathcal{O}_V$, so sections on V = reg. maps on $V \cap$ closed pts.

NOTE If X is a scheme and k a field, a morphism of schemes $X \longrightarrow$ Spec$k$ corresponds precisely to a morphism of rings $k \longrightarrow \Gamma(X)$ (see Ex 2.4 for generalisation).

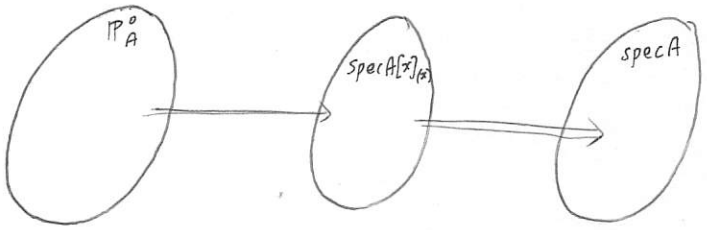
NOTE Let A be a ring. A scheme over A consists of $X \longrightarrow$ Spec$A$ which is a ring morphism $A \longrightarrow \mathcal{O}_X(X)$. A morphism $(f, \phi) : X \longrightarrow Y$ of schemes is a morphism of schemes over A iff. the following diagram commutes

$$\begin{tikzcd} \mathcal{O}_Y(Y) \arrow[rr, "\phi_Y"] & & \mathcal{O}_X(X) \\ & A \arrow[ul] \arrow[ur] & \end{tikzcd}$$

That is, ϕ_Y is a morphism of A-algebras.

NOTE $\mathbb{P}^0_A \cong \operatorname{Spec} A$

Let A be a ring, $\mathbb{P}^0_A = \operatorname{Proj} A[x]$. We claim that $\mathbb{P}^0_A \cong \operatorname{Spec} A$. Of course $D_+(x) = \mathbb{P}^0_A$ and hence if $X = \operatorname{Proj} A[x] = \mathbb{P}^0_A$, $\mathcal{O}_X|_{D_+(x)} \cong \operatorname{Spec} A[x]_{(x)}$. Now the ring morphism $A \xrightarrow{\alpha} A[x]_{(x)}$ $a \mapsto a/1$ is an isomorphism of rings, so $\mathbb{P}^0_A \cong \operatorname{Spec} A[x]_{(x)} \cong \operatorname{Spec} A$.



Clearly if $\mathfrak{p} \in \mathbb{P}^0_A$ is a homogenous prime, $\mathfrak{p} \cap A = \alpha^{-1}(\mathfrak{p} A[x]_x \cap A[x]_{(x)})$. Checking the definitions of the composite isomorphisms, we see that there is an isomorphism of schemes

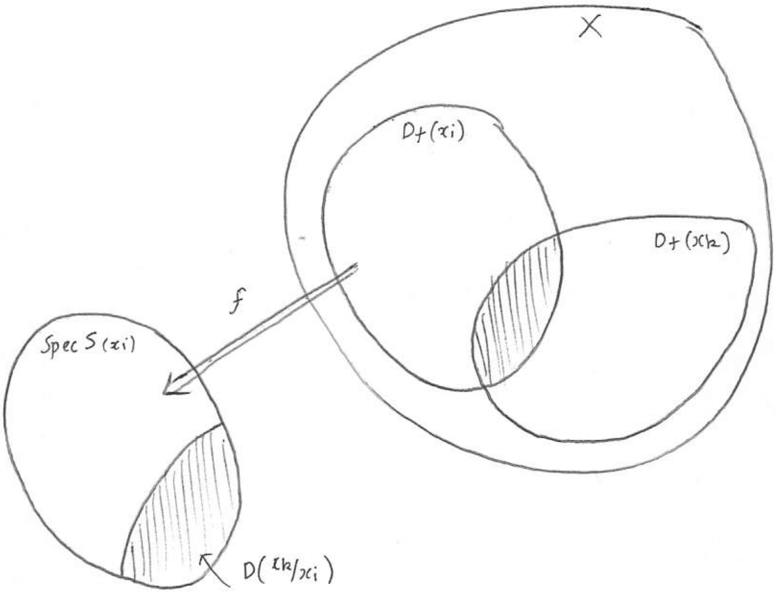
$$f : \mathbb{P}^0_A \longrightarrow \operatorname{spec} A$$
$$f(\mathfrak{p}) = \mathfrak{p} \cap A$$
$$f^{\#}_V(s)(\mathfrak{p}) = \gamma_{\mathfrak{p}}(s(\mathfrak{p} \cap A))$$

$$\gamma_{\mathfrak{p}} : A_{\mathfrak{p} \cap A} \longrightarrow S_{(\mathfrak{p})}$$
$$\gamma_{\mathfrak{p}}(a/s) = a/s$$

The morphism on global sections is $a \mapsto a/1$, so in fact f is the canonical morphism $\mathbb{P}^n_A \longrightarrow \operatorname{Spec} A$ (see later notes).

NOTE $A[x_0/x_i,\dots,x_n/x_i]_{x_k/x_i} \cong A[x_0,\dots,x_n]_{(x_i x_k)}$ $\qquad i\neq k$

Let A be a ring, $n \geqslant 1$. If $A=0$ the above isomorphism is trivial, so assume $A\neq 0$. Let $i\neq k$. As we know, $A[x_0,\dots,x_n]_{(x_i)} \cong A[x_0/x_i,\dots,x_n/x_i]$, where the latter is a polynomial ring in $n-1$ variables with funny labels on the variables (makes writing down explicit maps more transparent). Let $X = \mathrm{Proj}\,A[x_0,\dots,x_n]$. Then $D_+(x_i)$ is isomorphic to $\mathrm{Spec}\,A[x_0,\dots,x_n]_{(x_i)}$ via $\mathfrak{p} \longmapsto \mathfrak{p}S_{x_i} \cap S_{(x_i)}$ ($S = A[x_0,\dots,x_n]$).



We claim that f identifies $D(x_k/x_i)$ and $D_+(x_ix_k) = D_+(x_i)\cap D_+(x_k)$, since

$$
\begin{aligned}
f^{-1}D(x_k/x_i) &= \{\mathfrak{p}\in D_+(x_i) \mid x_k/x_i \notin \mathfrak{p}S_{x_i}\cap S_{(x_i)}\} \\
&= \{\mathfrak{p}\in D_+(x_i) \mid x_k \notin \mathfrak{p}\} \\
&= D_+(x_i)\cap D_+(x_k)
\end{aligned}
$$

Of course $D_+(x_ix_k) \cong \mathrm{Spec}\,A[x_0,\dots,x_n]_{(x_ix_k)}$ and $D(x_k/x_i) \cong \mathrm{Spec}\,(S_{(x_i)})_{x_k/x_i} \cong \mathrm{Spec}\,A[x_0/x_i,\dots,x_n/x_i]_{x_k/x_i}$. Putting these together we see that the morphism of A-algebras

$$
\begin{aligned}
A[x_0/x_i,\dots,x_n/x_i] &\longrightarrow A[x_0,\dots,x_n]_{(x_ix_k)} \\
x_j/x_i &\longmapsto x_j\cdot x_k/x_ix_k
\end{aligned}
$$

Maps x_k/x_i to a unit x_kx_k/x_ix_k (inverse x_i^2/x_ix_k) and the induced morphism of rings (A-algebras)

$$K:\ A[x_0/x_i,\dots,x_n/x_i]_{x_k/x_i} \longrightarrow A[x_0,\dots,x_n]_{(x_ix_k)}$$

is an isomorphism. By definition the following diagram commutes: (trivially for $D_+(x_i)=\phi$)

$$
\begin{tikzcd}
\mathrm{Spec}\,A[x_0/x_i,\dots,x_n/x_i] & D_+(x_i) \arrow[l, Leftarrow] \\
\mathrm{Spec}\,A[x_0,\dots,x_n]_{(x_ix_k)} \arrow[u] & D_+(x_ix_k) \arrow[l, Leftarrow] \arrow[u]
\end{tikzcd}
$$

Still assuming $i \neq k$, the isomorphisms

$$A[x_0/x_i, \dots, x_n/x_i]_{x_k/x_i} \cong A[x_0, \dots, x_n]_{(x_i x_k)} \cong A[x_0/x_k, \dots, x_n/x_k]_{x_i/x_k}$$

Map $x_k/x_i \longmapsto x_k x_k / x_i x_k = (x_i x_i / x_i x_k)^{-1} \longmapsto 1/(x_i/x_k)$ and similarly going right to left x_i/x_k is identified with $1/(x_k/x_i)$. Suppose R is an A-algebra and $b_0, \dots, b_n \in R$ with b_i, b_k units. Define

$$\phi : A[x_0/x_i, \dots, x_n/x_i] \longrightarrow R \qquad \phi(x_j/x_i) = b_j b_i^{-1}$$
$$\varphi : A[x_0/x_k, \dots, x_n/x_k] \longrightarrow R \qquad \varphi(x_j/x_k) = b_j b_k^{-1}$$

Then ϕ maps x_k/x_i to a unit and φ maps x_i/x_k to a unit. We claim that the inner square in the following diagram commutes:

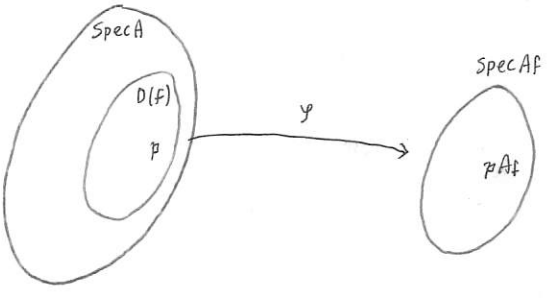
$$\begin{tikzcd}
& & R & & \\
A[x_0/x_i, \dots, x_n/x_i] \arrow[urr, bend left, "\phi"] \arrow[r, "\alpha"'] & A[x_0/x_i, \dots, x_n/x_i]_{x_k/x_i} \arrow[ur] & & A[x_0/x_k, \dots, x_n/x_k]_{x_k/x_i} \arrow[ul] & A[x_0/x_k, \dots, x_n/x_k] \arrow[l] \arrow[ull, bend right, "\varphi"'] \\
& & A[x_0, \dots, x_n]_{(x_i x_k)} \arrow[ul, Rightarrow, "\beta"] \arrow[ur, Rightarrow] & &
\end{tikzcd} \tag{1}$$

But α is an epimorphism, so after turning β around it suffices to check commutativity of the two maps $A[\text{—}]_{x_k/x_i} \longrightarrow R$ on x_j/x_i $j \neq i$. For $j \neq k$ $x_j/x_i/1$ becomes $x_j x_k / x_k x_i$ in $A[x_0, \dots, x_n]_{(x_i x_k)}$ and then $(x_j/x_k)/(x_i/x_k)$, then $\varphi(x_j/x_k)\varphi(x_i/x_k)^{-1} = b_j b_k^{-1}(b_i b_k^{-1})^{-1} = b_j b_i^{-1} = \phi(x_j/x_i)$, as required. And x_k/x_i maps to $1/(x_i/x_k)$ whence to $\varphi(x_i/x_k)^{-1} = b_k b_i^{-1} = \phi(x_k/x_i)$, completing the proof.

<u>NOTE</u> Alternatively, if we are given an A-algebra R and $f_{ij} \in R$ for $0 \leq i, j \leq n$ with $f_{ii} = 1$ $\forall i$ and $f_{ij} f_{jk} = f_{ik}$, then for any fixed $i \neq k$ we define ϕ, φ by $x_j/x_i \mapsto f_{ji}$, $x_j/x_k \mapsto f_{jk}$ and induce all the morphisms in (1). It is not hard to check the central square still commutes.

EXERCISES (II, §2)

$\boxed{Q2.1}$ Let A be a ring, $X = \operatorname{Spec} A$, let $f \in A$ and $D(f) \subseteq X$ be the open complement of $V(f)$. Then $(D(f), \mathcal{O}_X|_{D(f)})$ is isomorphic to $\operatorname{Spec} A_f$. Let $\varphi : D(f) \longrightarrow \operatorname{Spec} A_f$ be $\varphi(\mathfrak{p}) = \mathfrak{p}A_f$. (Note that $D(f) = \emptyset$ iff. f nilpotent iff. $A_f = 0$ so this case is trivial and we may assume $D(f) \neq \emptyset$). Then φ is a bijection. For any prime $\mathfrak{p} \in D(f)$ and ideal $\mathfrak{a} \subseteq A$ it is easily checked that $\mathfrak{a}A_f \subseteq \mathfrak{p}A_f \iff \mathfrak{a} \subseteq \mathfrak{p}$, so φ identifies $V(\mathfrak{a}A_f) \subseteq \operatorname{Spec} A_f$ and $V(\mathfrak{a}) \cap D(f) \subseteq D(f)$. Since any ideal in A_f is of the form $\mathfrak{a}A_f$, this shows that φ is a homeomorphism.



We define $\phi : \mathcal{O}_{\operatorname{Spec} A_f} \longrightarrow \varphi_* \mathcal{O}_X$ as follows. First notice that for $\mathfrak{p} \in D(f)$ the various canonical ring morphisms induce an isomorphism $\kappa_{\mathfrak{p}} : (A_f)_{\mathfrak{p}A_f} \longrightarrow A_{\mathfrak{p}}$

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A_f \arrow[d] \arrow[dl] \\ A_{\mathfrak{p}} & (A_f)_{\mathfrak{p}A_f} \arrow[l, dotted, "\kappa_{\mathfrak{p}}"] \end{tikzcd}$$

$$\kappa_{\mathfrak{p}}\left(a/f^n, q/f^m\right) = \frac{af^m}{qf^n} \qquad q \notin \mathfrak{p}$$

$$\kappa_{\mathfrak{p}}^{-1}(a/q) = (a/1, q/1)$$

Let $U \subseteq \operatorname{Spec} A_f$ be open and $s : U \longrightarrow \bigcup_{\mathfrak{p} \in \varphi^{-1}U} (A_f)_{\mathfrak{p}A_f}$. Define $\phi_U(s) : \varphi^{-1}U \longrightarrow \bigcup_{\mathfrak{p} \in \varphi^{-1}U} A_{\mathfrak{p}}$ by

$$\phi_U(s)(\mathfrak{p}) = \kappa_{\mathfrak{p}}(s(\mathfrak{p}A_f))$$

It is clear that $\phi_U(s)$ is regular and that ϕ is an isomorphism. This morphism φ is the inverse to the morphism $\operatorname{Spec} A_f \longrightarrow \operatorname{Spec} A$ induced by $A \longrightarrow A_f$ on $D(f)$.

$\boxed{Q2.2}$ Let $(X, \mathcal{O}_X)$ be a scheme and $U \subseteq X$ any open subset. If $U = \emptyset$ then $(U, \mathcal{O}_X|_U)$ is trivially a scheme. Assume $U \neq \emptyset$. First of all, for $P \in U$ $(\mathcal{O}_X|_U)_P \cong \mathcal{O}_{X,P}$ so $(U, \mathcal{O}_X|_U)$ is a locally ringed space. To see that $(U, \mathcal{O}_X|_U)$ is a scheme, let $P \in U$ and let $V \subseteq X$ be open s.t. $(V, \mathcal{O}_X|_V)$ is affine. Then $U \cap V$ is an open subset of V. If $(V, \mathcal{O}_X|_V) \cong \operatorname{Spec} A$ then there is $f \in A$ s.t. $P \in D(f) \subseteq U \cap V$. Then

$$\begin{aligned} (D(f), (\mathcal{O}_X|_U)|_{D(f)}) &= (D(f), \mathcal{O}_X|_{D(f)}) \\ &= (D(f), (\mathcal{O}_X|_V)|_{D(f)}) \\ &\cong (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}|_{D(f)}) \\ &\cong \operatorname{Spec} A_f \end{aligned}$$

Using Ex 2.1. Hence $(U, \mathcal{O}_X|_U)$ is a scheme, called the <u>induced scheme structure</u> on U, and we refer to $(U, \mathcal{O}_X|_U)$ as an <u>open subscheme</u> of X.

$\boxed{Q2.3}$ <u>Reduced Schemes</u> A scheme $(X, \mathcal{O}_X)$ is <u>reduced</u> if for every open set $U \subseteq X$ the ring $\mathcal{O}_X(U)$ has no nilpotent elements (i.e. the nilradical is 0).

(a) We claim that $(X, \mathcal{O}_X)$ is reduced iff. $\mathcal{O}_{X,P}$ has no nilpotent elements $\forall P \in X$. Suppose $(X, \mathcal{O}_X)$ is reduced and $0 = (U, s^n) = (U, s)^n$ for some $(U, s) \in \mathcal{O}_{X,P}$ $P \in U$. Then there is $P \in V \subseteq U$ open with $s^n|_V = 0$. But $(s|_V)^n = s^n|_V = 0$, so $s|_V \in \mathcal{O}_X(V)$ is nilpotent. Hence $s|_V = 0$ and consequently $(U, s) = 0$ in $\mathcal{O}_{X,P}$, as required. Conversely, suppose $\mathcal{O}_{X,P}$ is reduced $\forall P \in X$ and let $s \in \mathcal{O}_X(U)$ be nilpotent, say $s^n = 0$. Then $\forall P \in U$ (U, s) is nilpotent in $\mathcal{O}_{X,P}$ so $s = 0$ in a neighborhood of P. Hence $s = 0$ on all of U, as required.

(b) Let $(X, \mathcal{O}_X)$ be a scheme. Let $(\mathcal{O}_X)_{red}$ be the sheaf associated to the presheaf $U \longmapsto \mathcal{O}_X(U)_{red}$, for any ring A, A_{red} denotes the quotient of A by its nilradical. Let P be the presheaf $P(U) = \mathcal{O}_X(U)/\mathfrak{N} = \mathcal{O}_X(U)_{red}$. Then restriction $\mathcal{O}_X(U) \longrightarrow \mathcal{O}_X(V)$ for $V \subseteq U$ induces $\mathcal{O}_X(U)_{red} \longrightarrow \mathcal{O}_X(V)_{red}$ and with this defn P is clearly a presheaf of rings.

For $x \in X$ the stalk P_x is the colimit of the direct system of rings $\mathcal{O}_X(U)/\mathfrak{N}$, $x \in U$. But for $x \in U$ there is a morphism of rings $\mathcal{O}_X(U)/\mathfrak{N} \longrightarrow \mathcal{O}_{X,x}/\mathfrak{N}$ compatible with this direct system. Hence a morphism of rings

$$\theta_x : P_x \longrightarrow \mathcal{O}_{X,x}/\mathfrak{N}$$
$$(U, s+\mathfrak{N}) \longmapsto (U,s)+\mathfrak{N}$$

We claim that θ_x is an isomorphism $\forall x \in X$. If $\theta_x(U, s+\mathfrak{N}) = 0$, so $(U,s) \in \mathfrak{N}$, then $0 = (U,s)^n = (U, s^n)$ in $\mathcal{O}_{X,x}$ for some $n \geq 1$. Hence $(s|_V)^n = 0$ in $\mathcal{O}_X(V)$ for some $V \subseteq U$. Thus $s|_V \in \mathfrak{N} \subseteq \mathcal{O}_X(V)$ so $(U, s+\mathfrak{N}) = (V, (s+\mathfrak{N})|_V)$ $= (V, s|_V + \mathfrak{N}) = 0$, so θ_x is injective. It is trivially surjective, hence an isomorphism.

For any $x \in X$, $(\mathcal{O}_X)_{red,x} \cong P_x \cong \mathcal{O}_{X,x}/\mathfrak{N}$. Since $\mathcal{O}_X$ is locally ringed, so is $(\mathcal{O}_X)_{red}$. We claim that $(X, (\mathcal{O}_X)_{red})$ is a scheme. This is trivial in the case $(\emptyset, 0) = (X, \mathcal{O}_X)$ since then $(\mathcal{O}_X)_{red} \cong \mathcal{O}_X$, so assume $X \neq \emptyset$. We begin by showing that for any ring A, $(\operatorname{Spec} A)_{red} \cong \operatorname{Spec}(A_{red})$. This is trivial for $A = 0$, so assume $A \neq 0$ has nilradical $\mathfrak{n} \subseteq A$ and define

$$f : (\operatorname{Spec} A, \mathcal{O}) \longrightarrow (\operatorname{Spec}(A_{red}), \mathcal{O}_{red}) \qquad \mathcal{O} = (\mathcal{O}_{\operatorname{Spec} A})_{red}$$
$$f(\mathfrak{p}) = \mathfrak{p} + \mathfrak{n} \qquad \mathcal{O}_{red} = \mathcal{O}_{\operatorname{Spec}(A_{red})}$$

The map f is well-defined and bijective since $\mathfrak{n}$ is contained in every prime ideal. Moreover f is clearly a homeomorphism of spaces. To define a morphism between the structure sheaves we use an isomorphism of rings $K_{\mathfrak{p}} : (A/\mathfrak{n})_{\mathfrak{p}} \longrightarrow A_{\mathfrak{p}}/\mathfrak{n}A_{\mathfrak{p}}$

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A_{\mathfrak{p}} \arrow[d] \\ A/\mathfrak{n} \arrow[r] \arrow[d] & A_{\mathfrak{p}}/\mathfrak{n}A_{\mathfrak{p}} \\ (A/\mathfrak{n})_{\mathfrak{p}} \arrow[ur, "K_{\mathfrak{p}}"'] & \end{tikzcd}$$

$$q \notin \mathfrak{p}$$
$$K_{\mathfrak{p}}(a+\mathfrak{n}, q+\mathfrak{n}) = a/q + \mathfrak{n}A_{\mathfrak{p}}$$

<u>NOTE</u> $A_{red} = A/\mathfrak{n}$

Let P be the following presheaf on $\operatorname{Spec} A$: $P(U) = \mathcal{O}_{\operatorname{Spec} A}(U)/\mathfrak{N} = (\mathcal{O}_{\operatorname{Spec} A}(U))_{red}$. Then $\mathcal{O}$ is the sheafification of P. There is an isomorphism (using the fact that localisation preserves radicals)

$$P_{\mathfrak{p}} \cong (\mathcal{O}_{\operatorname{Spec} A, \mathfrak{p}})_{red} \cong (A_{\mathfrak{p}})_{red} = A_{\mathfrak{p}}/\mathfrak{n}A_{\mathfrak{p}} \cong (A/\mathfrak{n})_{\mathfrak{p}}$$
$$(U, s+\mathfrak{N}) \quad (U,s)+\mathfrak{N} \quad s(\mathfrak{p}) + \mathfrak{n}A_{\mathfrak{p}} \quad K_{\mathfrak{p}}^{-1}(s(\mathfrak{p}) + \mathfrak{n}A_{\mathfrak{p}})$$

Denote this isomorphism by $\eta_{\mathfrak{p}} : P_{\mathfrak{p}} \longrightarrow (A/\mathfrak{n})_{\mathfrak{p}}$. Let $U \subseteq \operatorname{Spec} A$ be open and $s : U \longrightarrow \bigcup_{\mathfrak{p} \in U} P_{\mathfrak{p}}$ an element of $\mathcal{O}(U)$. Let $\phi_U(s) \in \mathcal{O}_{red}(U)$ be $\phi_U(s) : f(U) \longrightarrow \bigcup_{\mathfrak{p} \in U} (A/\mathfrak{n})_{\mathfrak{p}}$, $\phi_U(s)(\mathfrak{p}+\mathfrak{n}) = \eta_{\mathfrak{p}}(s(\mathfrak{p}))$. We need to show that $\phi_U(s)$ is regular: if $\mathfrak{p} \in U$ there is $\mathfrak{p} \in V \subseteq U$ and $t+\mathfrak{N} \in P(V)$ $(= \mathcal{O}_{\operatorname{Spec} A}(V)/\mathfrak{N})$ s.t. $\forall q \in V$ $s(q) = \operatorname{germ}_q(t+\mathfrak{N})$. Since $t \in \mathcal{O}_{\operatorname{Spec} A}(V)$ there is $a, f \in A$ and $\mathfrak{p} \in W \subseteq V$ s.t. $f \notin q$ $\forall q \in W$ and $t(q) = a/f \in A_q$ $\forall q \in W$. Then for all $q + \mathfrak{n} \in f(W)$ we have

$$\begin{aligned} \phi_U(s)(q+\mathfrak{n}) &= \eta_q(s(q)) \\ &= \eta_q(\operatorname{germ}_q(t+\mathfrak{N})) \\ &= K_q^{-1}(t(q) + \mathfrak{n}A_q) \\ &= K_q^{-1}(a/f + \mathfrak{n}A_q) = (a+\mathfrak{n}, f+\mathfrak{n}) \end{aligned}$$

So $\phi_U(s)$ is regular. (i.e. is in $\mathcal{O}_{red}(U)$).

Conversely let $f(U) \subseteq \operatorname{Spec}(A_{red})$ be open and $s : f(U) \longrightarrow \bigcup_{\mathfrak{p} \in U} (A/\mathfrak{n})_{\mathfrak{p}}$ regular. Define $\psi_U(s) : U \longrightarrow \bigcup_{\mathfrak{p} \in U} P_{\mathfrak{p}}$ by

$$\psi_U(s)(\mathfrak{p}) = \eta_{\mathfrak{p}}^{-1}(s(\mathfrak{p}+\mathfrak{n}))$$

To see that $\psi_U(s)$ belongs to $\mathcal{O}(U)$, let $\mathfrak{p} \in U$ and find $\mathfrak{p} \in V \subseteq U$ s.t. there are $a+\mathfrak{n}, f+\mathfrak{n} \in A/\mathfrak{n}$ with $f \notin q$ $\forall q \in V$ and s.t. $\forall q \in V$ $s(q+\mathfrak{n}) = (a+\mathfrak{n}, f+\mathfrak{n})$. Then $(a+\mathfrak{n}, f+\mathfrak{n}) \in (A/\mathfrak{n})_{\mathfrak{p}}$ corresponds to $a/f + \mathfrak{n}A_{\mathfrak{p}} \in (A_{\mathfrak{p}})_{red} \cong (\mathcal{O}_{\operatorname{Spec} A, \mathfrak{p}})_{red}$. Hence there is an open neighborhood W of $\mathfrak{p}$ in $\operatorname{Spec} A$ and $t \in \mathcal{O}_{\operatorname{Spec} A}(W)$ s.t. $a/f + \mathfrak{n}A_{\mathfrak{p}}$ corresponds to $(W, t) + \mathfrak{N}$ (so $t(\mathfrak{p}) = a/f$). We may as well assume $W \subseteq V$. Then $\forall q \in W$ $\psi_U(s)(q) = \eta_q^{-1}(s(q+\mathfrak{n})) = \eta_q^{-1}((a+\mathfrak{n}, f+\mathfrak{n})) = (W, t+\mathfrak{N}) = \operatorname{germ}_q(t+\mathfrak{N})$. Hence $\psi_U(s) \in \mathcal{O}(U)$. Clearly ϕ_U, ψ_U are inverse, so finally we conclude that $(\operatorname{Spec} A)_{red} \cong \operatorname{Spec}(A_{red})$ as ringed spaces.

Returning to the general case, where $(X, \mathcal{O}_X)$ is a scheme and $(X, (\mathcal{O}_X)_{red})$ a locally ringed space, we can now show that $(X, (\mathcal{O}_X)_{red})$ is a scheme. Let $x \in X$ be given and find $x \in V \subseteq X$ s.t. $(U, \mathcal{O}_X|_U) \cong \operatorname{Spec} A$ is affine. Then it is easily checked that $(U, (\mathcal{O}_X)_{red}|_U) \cong (U, (\mathcal{O}_X|_U)_{red})$ and $(U, (\mathcal{O}_X|_U)_{red}) \cong (\operatorname{Spec} A)_{red} \cong \operatorname{Spec} A_{red}$. Hence $(\mathcal{O}_X)_{red}|_U$ is affine, as required. We denote $(X, (\mathcal{O}_X)_{red})$ by X_{red}.

There is a morphism of schemes $\phi : X_{red} \longrightarrow X$ which is the identity on the underlying spaces and for $U \subseteq X$

$$\phi_U : \mathcal{O}_X(U) \longrightarrow (\mathcal{O}_X)_{red}(U)$$
$$\phi_U(s) : U \longrightarrow \textstyle\bigcup_{x \in U} P_x$$
$$\phi_U(s)(x) = (U, s + \mathfrak{N})$$

This is readily seen to be a morphism of ringed spaces. For $x \in X$ the induced morphism on stalks is

$$\phi_x : \mathcal{O}_{X,x} \longrightarrow (\mathcal{O}_X)_{red,x} \cong P_x \cong \mathcal{O}_{X,x}/\mathfrak{N}$$
$$(U, s) \longrightarrow (U, \phi_U(s)) \longrightarrow (U, s + \mathfrak{N}) \longrightarrow (U, s) + \mathfrak{N}$$

which makes it clear ϕ is a morphism of schemes. It is clear that X_{red} is a reduced scheme (since $(\mathcal{O}_X)_{red,x} \cong \mathcal{O}_{X,x}/\mathfrak{N}$).

(c) The reduced schemes form a full subcategory of the category of all schemes, call it <u>RedSch</u>. We claim that $X \longmapsto X_{red}$ defines a <u>functor</u> $r : \underline{Sch} \longrightarrow \underline{RedSch}$. Let $(f, \phi) : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ be a morphism of schemes. We construct a morphism $(f, \phi') : (X, (\mathcal{O}_X)_{red}) \longrightarrow (Y, (\mathcal{O}_Y)_{red})$ unique making

$$\begin{tikzcd}
(X, (\mathcal{O}_X)_{red}) \arrow[r, "{(f, \phi')}"] \arrow[d] & (Y, (\mathcal{O}_Y)_{red}) \arrow[d] \\
(X, \mathcal{O}_X) \arrow[r, "{(f, \phi)}"'] & (Y, \mathcal{O}_Y)
\end{tikzcd} \qquad (0)$$

commute. Clearly the map of spaces underlying $X_{red} \longrightarrow Y_{red}$ must be f for this diagram to commute. For $x \in X$ the local morphism $\phi_x : \mathcal{O}_{Y, f(x)} \longrightarrow \mathcal{O}_{X,x}$ induces

$$\begin{tikzcd}
\mathcal{O}_{Y,f(x)}/\mathfrak{N} \arrow[r] \arrow[d, phantom, "\wr\!\!\parallel"] & \mathcal{O}_{X,x}/\mathfrak{N} \arrow[d, phantom, "\wr\!\!\parallel"] \\
P_{Y, f(x)} \arrow[r, "\eta_x"'] & P_{X,x}
\end{tikzcd}$$

$$P_Y(V) = \mathcal{O}_Y(V)/\mathfrak{N}$$
$$P_X(U) = \mathcal{O}_X(U)/\mathfrak{N}$$
$$\eta_x(V, t + \mathfrak{N}) = (f^{-1}V, \phi_V(t) + \mathfrak{N})$$

Thus η_x is a local morphism of local rings. For $U \subseteq Y$ we are requiring the following diagram to commute

$$\begin{tikzcd}
(\mathcal{O}_X)_{red}(f^{-1}U) & & (\mathcal{O}_Y)_{red}(V) \arrow[ll, "\phi'_U"'] \\
\boxed{x \mapsto (f^{-1}U, \phi_U(t) + \mathfrak{N})} & & \boxed{x \mapsto (U, t + \mathfrak{N})} \\
(\mathcal{O}_X)(f^{-1}U) \arrow[uu] \arrow[u, dotted, phantom] & & \mathcal{O}_Y(V) \arrow[ll] \arrow[uu] \\
\boxed{\phi_U(t)} \arrow[u, dotted, shift left=8] & & \boxed{t} \arrow[ll, dotted] \arrow[uu, dotted, shift right=8]
\end{tikzcd} \qquad (1)$$

which makes it clear that ϕ' is unique, if it exists. We define $\phi'_U : (\mathcal{O}_Y)_{red}(U) \longrightarrow (\mathcal{O}_X)_{red}(f^{-1}U)$ by

$$\phi'_U(s)(x) = \eta_x(s(f(x)))$$

It is easily checked that $\phi' : (\mathcal{O}_Y)_{red} \longrightarrow f_*(\mathcal{O}_X)_{red}$ is a morphism of ringed spaces. It is a morphism of locally ringed spaces since the following diagram commutes:

$$\begin{tikzcd}
(\mathcal{O}_Y)_{red,f(x)} \arrow[r, "\phi'_x"] \arrow[d, phantom, "\cong" rotate=90] & (\mathcal{O}_X)_{red,x} \arrow[d, phantom, "\cong" rotate=90] \\
P_{Y,f(x)} \arrow[r, "\eta_x"'] & P_{X,x}
\end{tikzcd}$$

The morphism (f, ϕ') makes (o) commute, which defines the functor $r : \underline{Sch} \longrightarrow \underline{RedSch}$ on morphisms (r is a functor using the uniqueness property of ϕ'). Moreover for every scheme X there is a morphism $\eta_X : X_{red} \longrightarrow X$ which gives a natural transformation $\eta : ir \longrightarrow 1$ ($i : \underline{RedSch} \longrightarrow \underline{Sch}$ being the inclusion).

To actually answer the exercise, suppose $X \in \underline{RedSch}$ and let $(f,\phi) : X \longrightarrow Y$ be a morphism of schemes. The morphism $r(f,\phi) : X_{red} \longrightarrow Y_{red}$ is unique making the diagram

$$\begin{tikzcd}
X_{red} \arrow[r, "r(f{,}\phi)"] \arrow[d, "\iota"'] & Y_{red} \arrow[d, "\iota"] \\
X \arrow[r, "(f{,}\phi)"'] & Y
\end{tikzcd}$$

commute. Hence (f,ϕ) factors uniquely through $Y_{red} \longrightarrow Y$, since if X is reduced, $X_{red} \longrightarrow X$ is an isomorphism. These considerations show that $i \dashv r$, so the inclusion of reduced schemes has a right adjoint.

$\boxed{Q2.4}$ Let A be a ring and let $(X, \mathcal{O}_X)$ be a scheme. Given a morphism $f : X \longrightarrow \operatorname{Spec} A$, $f^\# : \mathcal{O}_{\operatorname{Spec} A} \longrightarrow f_* \mathcal{O}_X$ there is an induced ring morphism $A \longrightarrow \mathcal{O}_X(X)$.

$$\alpha : \operatorname{Hom}_{\underline{Sch}}(X, \operatorname{Spec} A) \longrightarrow \operatorname{Hom}_{\underline{Rng}}(A, \mathcal{O}_X(X))$$

We claim that α is bijective and natural. First we show that α is bijective by defining $\beta : \operatorname{Hom}_{\underline{Rng}}(A, \mathcal{O}_X(X)) \longrightarrow \operatorname{Hom}_{\underline{Sch}}(X, \operatorname{Spec} A)$. Note that if $(X, \mathcal{O}_X)$ is the zero scheme $(\emptyset, 0)$ or $A = 0$ α is trivially bijective. So we may assume $(X, \mathcal{O}_X) \neq (\emptyset, 0)$ and $A \neq 0$. Note that since $\mathcal{O}_X$ is a scheme and $X \neq \emptyset$ it follows that $\mathcal{O}_X(X) \neq 0$.

For $b \in \mathcal{O}_X(X)$ let $D(b) = \{x \in X \mid (X, b) \text{ is a unit in } \mathcal{O}_{X,x}\}$. We claim that $D(b) \subseteq X$ is open. Suppose $D(b) \neq 0$ and $x \in D(b)$, so there is an open neighborhood V of x and $s \in \mathcal{O}_X(V)$ with $(X, b)(V, s) = 1$ in $\mathcal{O}_{X,x}$. That is, there is $x \in U \subseteq X$ s.t. $b|_U s|_U = 1$ in $\mathcal{O}_X(U)$. So $b|_U$ is a unit in $\mathcal{O}_X(U)$, and consequently in $\mathcal{O}_{X,y}$ $\forall y \in U$. Hence $U \subseteq D(b)$ and it follows that $D(b)$ is open.

Let a morphism of rings $\theta : A \longrightarrow \mathcal{O}_X(X)$ be given. For an ideal $\mathfrak{b} \in A$ let $\mathfrak{b}^e$ denote the ideal in $\mathcal{O}_X(X)$ generated by $\theta(\mathfrak{b})$. For $x \in X$ let $\varphi_x : \mathcal{O}_X(X) \longrightarrow \mathcal{O}_{X,x}$ be canonical and $\mathfrak{m}_x \subseteq \mathcal{O}_{X,x}$ the maximal ideal. We define a map $f : X \longrightarrow \operatorname{Spec} A$ by

$$f(x) = (\varphi_x \theta)^{-1}(\mathfrak{m}_x)$$

To see that this is continuous, let $V(\mathfrak{b})$ be a closed set in $\operatorname{Spec} A$. Then

$$\begin{aligned} f^{-1}V(\mathfrak{b}) &= \{x \in X \mid \mathfrak{b} \subseteq (\varphi_x\theta)^{-1}(\mathfrak{m}_x)\} \\ &= \{x \in X \mid \mathfrak{b}^e \subseteq \varphi_x^{-1}\mathfrak{m}_x\} \\ &= \bigcap_{b \in \mathfrak{b}^e} D(b)^c \end{aligned}$$

The last step follows from the fact that for $b \in \mathcal{O}_X(X)$, $x \in D(b)^c \Leftrightarrow b$ not a unit in $\mathcal{O}_{X,x} \Leftrightarrow b \in \varphi_x^{-1}\mathfrak{m}_x$. Hence f is continuous. Next we define a morphism $\phi : \mathcal{O}_{\operatorname{Spec} A} \longrightarrow f_* \mathcal{O}_X$. Let $U \subseteq \operatorname{Spec} A$ be open. For $x \in f^{-1}U$ there is a morphism of rings $K_x : A_{f(x)} \longrightarrow \mathcal{O}_{X,x}$ induced by $\theta : A \longrightarrow \mathcal{O}_X(X) \longrightarrow \mathcal{O}_{X,x}$ (since $f(x) = \theta^{-1}(\varphi_x^{-1}\mathfrak{m}_x)$). The elements of A not in $f(x)$ are precisely those mapped to units in $\mathcal{O}_{X,x}$. For $a \in A$ and $s \notin f(x)$ we have

$$K_x(a/s) = (X, \theta(a))(X, \theta(s))^{-1} \in \mathcal{O}_{X,x}$$

Given $t \in \mathcal{O}_{\operatorname{spec}A}(U)$, so $t: U \longrightarrow \bigcup_{p\in U} A_p$ we define $t': f^{-1}U \longrightarrow \bigcup_{x\in f^{-1}U} \mathcal{O}_{X,x}$ by

$$t'(x) = K_x(t(f(x)))$$

The map t' is regular since for $x \in f^{-1}U$ there is an open neighborhood V of $f(x)$ in U and $a, s \in A$ with $a \notin q$ $\forall q \in V$ with $t(q) = a/s \in A_q$ $\forall q \in V$. Hence $(X, \theta(s)) \in \mathcal{O}_{X,x}$ is a unit – say $m \in \mathcal{O}_X(W)$ with $(X, \theta(s))(W, m) = 1$ in $\mathcal{O}_{X,x}$. Then $\theta(a)|_W m \in \mathcal{O}_X(W)$ and $\forall y \in W$ (we may assume $W \subseteq f^{-1}V$)

$$\begin{aligned} t'(y) &= K_x(t(f(y))) \\ &= K_x(a/s) = (W, \theta(a)|_W m) \end{aligned}$$

Showing t' is regular. Since $\mathcal{O}_X$ is actually a sheaf, there is unique $\phi_U(t) \in \mathcal{O}_X(f^{-1}U)$ with $(f^{-1}U, \phi_U(t)) = t'(x)$ in $\mathcal{O}_{X,x}$ $\forall x \in U$. Thus defined it is straightforward to check that $\phi: \mathcal{O}_{\operatorname{spec}A} \longrightarrow f_*\mathcal{O}_X$ is a morphism of sheaves of rings. Since K_x is trivially a local morphism of local rings and the diagram

$$\begin{tikzcd}
\mathcal{O}_{\operatorname{spec}A, f(x)} \arrow[r, "\phi_x"] \arrow[d, phantom, "\cong" rotate=90] & \mathcal{O}_{X,x} \\
A_{f(x)} \arrow[ur, "K_x"'] &
\end{tikzcd}$$

commutes. It follows that $(f, \phi): (X, \mathcal{O}_X) \longrightarrow (\operatorname{Spec}A, \mathcal{O}_{\operatorname{spec}A})$ is a morphism of schemes. This defines the map β.

Next we check that α, β are inverse to each other. Let $(f, \psi): X \longrightarrow \operatorname{Spec}A$ be given and let $\theta: A \longrightarrow \mathcal{O}_X(X)$ be $\theta(a) = \psi_X(\dot{a})$. This morphism θ induces $K_x: A_{f(x)} \longrightarrow \mathcal{O}_{X,x}$ via $A \longrightarrow \mathcal{O}_X(X) \longrightarrow \mathcal{O}_{X,x}$ $\forall x \in X$. K_x is unique making the upper face commute in

$$\begin{tikzcd}
& A \arrow[dl, "\theta"'] \arrow[dr, "\cong"] \arrow[dd] & \\
\mathcal{O}_X(X) \arrow[dd, "\rho_x"'] & & \mathcal{O}_{\operatorname{spec}A}(\operatorname{Spec}A) \arrow[ll] \arrow[dd] \\
& A_{f(x)} \arrow[dl, "K_x"'] \arrow[dr, "\cong"] & \\
\mathcal{O}_{X,x} & & \mathcal{O}_{\operatorname{spec}A, f(x)} \arrow[ll, "\psi_x"]
\end{tikzcd}$$

It follows that $K_x = A_{f(x)} \Rightarrow \mathcal{O}_{\operatorname{spec}A, f(x)} \xrightarrow{\psi_x} \mathcal{O}_{X,x}$. Let $(f', \phi): X \longrightarrow \operatorname{Spec}A$ be $\beta(\theta) = \beta\alpha(f, \psi)$. Then for $x \in X$

$$\begin{aligned} f'(x) &= (\rho_x\theta)^{-1}(m_x) \\ &= \theta^{-1}(\rho_x^{-1} m_x) \\ &= \{a \in A \mid \psi_X(\dot{a}) \in \rho_x^{-1} m_x\} \\ &= \{a \in A \mid \rho_x(\psi_X(\dot{a})) \in m_x\} \\ &= \{a \in A \mid \psi_x((\operatorname{Spec}A, \dot{a})) \in m_x\} \\ &= \{a \in A \mid (\operatorname{Spec}A, \dot{a}) \in m\} \qquad m \in \mathcal{O}_{\operatorname{spec}A, f(x)} \text{ since } \psi_x \text{ local.} \\ &= \{a \in A \mid a/1 \in f(x)\} = f(x) \end{aligned}$$

So $f = f'$. To show that $\phi = \psi$ let $U \subseteq \operatorname{Spec}A$ be open, and let $t \in \mathcal{O}_{\operatorname{spec}A}(U)$. It suffices to show $\forall x \in f^{-1}U$ that $(f^{-1}U, \psi_U(t)) = (f^{-1}U, \phi_U(t))$ in $\mathcal{O}_{X,x}$. But $(f^{-1}U, \phi_U(t)) = t'(x) = K_x(t(f(x))) = \psi_x(U, t) = (f^{-1}U, \psi_U(t))$, as required. Hence $\beta\alpha = 1$.

If $\theta: A \longrightarrow \mathcal{O}_X(X)$ and $(f, \phi) = \beta(\theta)$ then $\alpha(\beta(\theta)): A \longrightarrow \mathcal{O}_X(X)$ is $\phi_{\operatorname{spec}A}: \mathcal{O}_{\operatorname{spec}A}(\operatorname{Spec}A) \longrightarrow \mathcal{O}_X(X)$. But for $a \in A$, $(X, \phi_{\operatorname{spec}A}(\dot{a})) = t'(x) = K_x(\dot{a}(f(x))) = K_x(a/1) = (X, \theta(a))$ in $\mathcal{O}_{X,x}$ $\forall x \in X$. Hence $\alpha(\beta(\theta))(a) = \theta(a)$, so $\alpha\beta = 1$ also. Hence α is a bijection. It only remains to check that α is natural in X and A.

If $(g, \psi): X \longrightarrow Y$ is any morphism of schemes, the following diagram commutes:

$$\begin{tikzcd}
\boxed{(f,\phi)(g,\psi) = (fg,(f_*\psi)\phi)} \arrow[rrr, dashed] & & & \boxed{\begin{array}{l} A \longrightarrow \mathcal{O}_X(X) \\ a \longmapsto ((f_*\psi)\phi)_{\operatorname{Spec} A}(\dot{a}) \end{array}} \\
& \operatorname{Hom}_{\underline{Sch}}(X, \operatorname{Spec} A) \arrow[r] & \operatorname{Hom}_{\underline{Rng}}(A, \mathcal{O}_X(X)) & \boxed{\begin{array}{l} A \longrightarrow \mathcal{O}_X(X) \\ a \longmapsto \psi_X(\phi_{\operatorname{Spec} A}(\dot{a})) \end{array}})) \\
& \operatorname{Hom}_{\underline{Sch}}(Y, \operatorname{Spec} A) \arrow[r] \arrow[u] & \operatorname{Hom}_{\underline{Rng}}(A, \mathcal{O}_Y(Y)) \arrow[u] & \\
\boxed{(f,\phi)} \arrow[uuu, dashed] \arrow[rrr, dashed] & & & \boxed{\begin{array}{l} \theta : A \longrightarrow \mathcal{O}_Y(Y) \\ a \longmapsto \phi_{\operatorname{Spec} A}(\dot{a}) \end{array}} \arrow[uu, dashed]
\end{tikzcd}$$

This commutes for any ring A, so α is natural in X. Similarly if $\eta : A \longrightarrow B$ is a morphism of rings and $\operatorname{Spec}\eta : \operatorname{Spec} B \longrightarrow \operatorname{Spec} A$ the corresponding morphism of schemes, the following diagram commutes

$$\begin{tikzcd}
\boxed{\operatorname{Spec}\eta\,(f,\phi)} \arrow[rrr, dashed] & & & \boxed{a \longmapsto \phi_{\operatorname{Spec} B}(\dot{\eta(a)})} \\
& \operatorname{Hom}_{\underline{Sch}}(X, \operatorname{Spec} A) \arrow[r] & \operatorname{Hom}_{\underline{Rng}}(A, \mathcal{O}_X(X)) & \\
& \operatorname{Hom}_{\underline{Sch}}(X, \operatorname{Spec} B) \arrow[r] \arrow[u] & \operatorname{Hom}_{\underline{Rng}}(B, \mathcal{O}_X(X)) \arrow[u] & \\
\boxed{(f,\phi)} \arrow[uuu, dashed] \arrow[rrr, dashed] & & & \boxed{b \longmapsto \phi_{\operatorname{Spec} B}(\dot{b})} \arrow[uuu, dashed]
\end{tikzcd}$$

since $(\operatorname{Spec}\eta)_{\operatorname{Spec} A}(\dot{a}) = \dot{\eta(a)}$. Hence α is natural in A. Consider the two covariant functors:

$$T : \underline{Sch} \longrightarrow \underline{Rng}^{op} \qquad\qquad T \dashv \operatorname{Spec}.$$
$$\operatorname{Spec} : \underline{Rng}^{op} \longrightarrow \underline{Sch}$$

The above discussion shows that there is a natural bijection $\operatorname{Hom}_{\underline{Sch}}(X, \operatorname{Spec}(A)) \cong \operatorname{Hom}_{\underline{Rng}^{op}}(T(X), A)$ for any ring A and scheme X. Hence T is left adjoint to Spec. It follows that the contravariant functors have the following properties

$\boxed{\operatorname{Spec} : \underline{Rng} \longrightarrow \underline{Sch}}$ Maps colimits in $\underline{Rng}$ to limits in $\underline{Sch}$
Maps epimorphisms of $\underline{Rng}$ to monomorphisms of $\underline{Sch}$

$\boxed{T : \underline{Sch} \longrightarrow \underline{Rng}}$ Maps colimits in $\underline{Sch}$ to limits in $\underline{Rng}$
Maps epimorphisms in $\underline{Sch}$ to monomorphisms in $\underline{Rng}$

In particular if R, S are rings their coproduct in $\underline{Rng}$ is $R \longrightarrow R \otimes_{\mathbb{Z}} S \longleftarrow S$, so in $\underline{Sch}$ the following diagram is a product:

$$\begin{tikzcd}
\operatorname{Spec} R & & \operatorname{Spec} S \\
& \operatorname{Spec}(R \otimes_{\mathbb{Z}} S) \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

The fact that T maps epis to monos means that if $f : X \longrightarrow Y$ is an epimorphism of schemes, then $T(Y) \longrightarrow T(X)$ is injective. The unit of this adjunction is $\eta : 1 \longrightarrow \operatorname{Spec} T$ given for a scheme X by

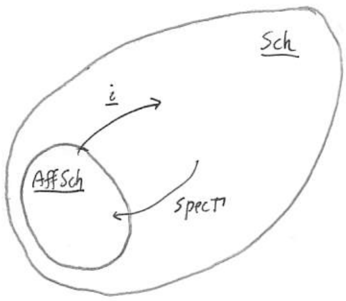
$$\eta_X : X \longrightarrow \operatorname{Spec} T(X)$$

This is $\beta(1_{T(X)})$, so $\eta_X = (f,\phi)$ where $f(x) = \varphi_x^{-1}\mathfrak{m}_x$ ($\varphi_x : \mathcal{O}_X(X) \longrightarrow \mathcal{O}_{X,x}$ canonical) and if $\kappa_x : T(X)_{f(x)} \longrightarrow \mathcal{O}_{X,x}$ is defined by $\kappa_x(a/s) = (X,a)(X,s)^{-1}$ then for $U \subseteq \operatorname{Spec} T(X)$ and $t \in \mathcal{O}_{\operatorname{Spec} T(X)}(U)$ the element $\phi_U(t) \in \mathcal{O}_X(f^{-1}U)$ is the unique section with $(f^{-1}U, \phi_U(t)) = \kappa_x(t(f(x)))\ \forall x \in f^{-1}U$.

Since $\Gamma \dashv \operatorname{Spec}$, if S is a ring and $X \longrightarrow \operatorname{Spec} S$ a morphism of schemes there is a unique morphism of rings $S \xrightarrow{\gamma} \Gamma(X)$ making the following diagram commute

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & \operatorname{Spec}\Gamma(X) \arrow[d, "\operatorname{Spec}\gamma"] \\ & \operatorname{Spec} S \end{tikzcd}$$

Of course γ is just $S \cong \mathcal{O}_{\operatorname{Spec} S}(\operatorname{Spec} S) \longrightarrow X$. Using Prop (2.3) and denoting by <u>AffSch</u> the category of affine schemes (a full subcategory of <u>Sch</u> consisting of $\operatorname{Spec} S$ $\forall S \in$ <u>Rng</u> - i.e. the actual spectrums, not any old scheme iso. to a spectrum) it is clear that $\operatorname{Spec}\Gamma :$ <u>Sch</u> $\longrightarrow$ <u>AffSch</u> is left adjoint to the inclusion <u>AffSch</u> $\longrightarrow$ <u>Sch</u>



$\operatorname{Spec}\Gamma \dashv i$

If we extend <u>AffSch</u> to mean all affine schemes (so <u>AffSch</u> is now replete), then $\operatorname{Spec}\Gamma$ is still left adjoint to the inclusion. So <u>AffSch</u> is a reflective subcategory of <u>Sch</u>.

$\boxed{Q2.5}$ The primes of $\mathbb{Z}$ are the principal ideals (p), p prime (>0). So $\operatorname{Spec}\mathbb{Z} = \{(2),(3),(5),(7),\dots\}$. The closed sets are $V((a)) = \{(p) \mid (a) \subseteq (p)\} = \{(p) \mid p \mid a\}$. Since $V((a)) = V((-a))$, the closed sets are in bijection with $\operatorname{Spec}\mathbb{Z}$, ϕ together with the set of distinct prime divisors of $a > 1$ in $\mathbb{Z}$. For example $V((6)) = \{(2),(3)\}$. By Q2.4 we have for any scheme X

$$\operatorname{Hom}_{\underline{Sch}}(X, \operatorname{Spec}\mathbb{Z}) \cong \operatorname{Hom}_{\underline{Rng}}(\mathbb{Z}, \Gamma(X)) = \{*\}$$

Since $\mathbb{Z}$ is the initial ring. Hence $\operatorname{Spec}\mathbb{Z}$ is a final object in <u>Sch</u>.

$\boxed{Q2.6}$ The spectrum of the zero ring is $(\phi, 0)$, which is clearly an initial object in <u>Sch</u>.

$\boxed{Q2.7}$ Let X be a scheme. For $x \in X$ let m_x be the maximal ideal of $\mathcal{O}_{X,x}$. Let $k(x)$ be the residue field $\mathcal{O}_{X,x}/m_x$. We claim that for any field K, morphisms $\operatorname{Spec} K \longrightarrow X$ are characterised by giving $x \in X$ and and a ring morphism $k(x) \longrightarrow K$ (any such morphism is injective).

Let $\mathcal{S} : \operatorname{Spec} K \longrightarrow X$ be a morphism. Since $\operatorname{Spec} K = \{(0)\}$ the image of $\mathcal{S}$ is a point $x \in X$. The morphism $\mathcal{S}_x : \mathcal{O}_{X,x} \longrightarrow \mathcal{O}_{\operatorname{Spec} K,(0)} = K$ is local, so $\operatorname{Ker}\mathcal{S}_x = \mathcal{S}_x^{-1}(0) = m_x$. Hence $\mathcal{S}_x : k(x) = \mathcal{O}_{X,x}/m_x \longrightarrow K$ is injective.

Conversely, let $x \in X$ and injective morphism $\phi : k(x) \longrightarrow K$ be given. Let $(f, \Psi) : \operatorname{Spec} K \longrightarrow X$ be defined by

$$f : \operatorname{Spec} K \longrightarrow X \qquad \text{Trivially continuous}$$
$$f(*) = x$$

For $U \subseteq X$ open there are two cases: $x \notin U$, in which case $f^{-1}U = \phi$ and $\Psi_U : \mathcal{O}_X(U) \longrightarrow 0$ is the zero map, and $x \in U$, in which case Ψ_U is the composite $\mathcal{O}_X(U) \longrightarrow \mathcal{O}_{X,x} \longrightarrow k(x) \xrightarrow{\phi} K = \mathcal{O}_{\operatorname{Spec} K}(f^{-1}U)$. It is easily checked that this defines a morphism of schemes - for $x \in X$, Ψ_x is $\mathcal{O}_{X,x} \longrightarrow k(x) \longrightarrow K$ which has kernel m_x since ϕ is injective. Moreover, the above correspondences are inverse, completing the proof.

$\boxed{Q2.8}$ Let X be a scheme. For any point $x \in X$, we define the <u>Zariski tangent space</u> T_x to X at x to be the dual of the $k(x)$-vector space $\mathfrak{m}_x / \mathfrak{m}_x^2$. Now assume that X is a scheme over a field k, and let $k[\varepsilon]/\varepsilon^2$ be <u>the ring of dual numbers</u> over k. We claim that to give a k-morphism of $\operatorname{Spec} k[\varepsilon]/\varepsilon^2$ to X is equivalent to giving a point $x \in X$, <u>rational over k</u> (i.e. such that $k(x) = k$) and an element of T_x.

<u>NOTE</u> If X is a scheme over k via $\eta : X \longrightarrow \operatorname{Spec} k$ and if $\theta : k \longrightarrow \mathcal{O}_X(X)$ is $\eta_{\operatorname{Spec} k}$, then every local ring $\mathcal{O}_{X,x}$ becomes a k-algebra via $k \cdot (U,s) = (X, \theta(k))(U,s) = (U, \theta(k)|_U s)$. Thus $k(x)$ is a k-algebra and x <u>is rational over k</u> if $k(x)$ is equal to its subfield k.

Let $(f,\phi) : \operatorname{Spec} k[\varepsilon]/\varepsilon^2 \longrightarrow X$ be a morphism of schemes over k (where $k[\varepsilon]/\varepsilon^2$ is a k-algebra canonically). Let $x = f((\varepsilon))$, and note that $\phi_x : \mathcal{O}_{X,x} \longrightarrow k[\varepsilon]/\varepsilon^2$ is a morphism of k-algebras since $\phi_X : \mathcal{O}_X(X) \longrightarrow k[\varepsilon]/\varepsilon^2$ is by assumption. The fact that $\phi_x^{-1}((\varepsilon^2)) = \mathfrak{m}_x$ means that there is an injective morphism of k-algebras $k(x) \longrightarrow (k[\varepsilon]/\varepsilon^2)/(\varepsilon) = k$. It follows that $k(x) = k$ and x is rational.

Since $k(x) = k$, $\mathfrak{m}_x/\mathfrak{m}_x^2$ naturally becomes a k-vector space and ϕ_x induces a k-linear map $\mathfrak{m}_x/\mathfrak{m}_x^2 \longrightarrow (\varepsilon)/(\varepsilon^2) = k$, which gives a $k(x)$-linear map $\mathfrak{m}_x/\mathfrak{m}_x^2 \longrightarrow k(x)$ – i.e., an element of T_x. ?

$\boxed{Q2.9}$ If X is a topological space and $Z \subseteq X$ irreducible and closed, a <u>generic point</u> for Z is a point ζ such that $Z = \overline{\{\zeta\}}$. Let X be a scheme. We claim that every (nonempty) irreducible closed subset has a unique generic point. Let $Z \subseteq X$ be such a subset and $X = \bigcup_i V_i$ an affine open cover of X. Say $V_i \cap Z \neq \emptyset$. Then if $V_i \cong \operatorname{Spec} B$, $V_i \cap Z$ corresponds to an irreducible closed subset of $\operatorname{Spec} B$, which has a unique generic point $\mathfrak{p}$ ($V(\mathfrak{p})$ has uniq. gen. point $\mathfrak{p}$). The closure of $\mathfrak{p}$ in X must be a closed set P lying between $V_i \cap Z$ and Z. But as a nonempty open subset of Z, $V_i \cap Z$ is dense, so $\mathfrak{p}$ is a generic point of Z.

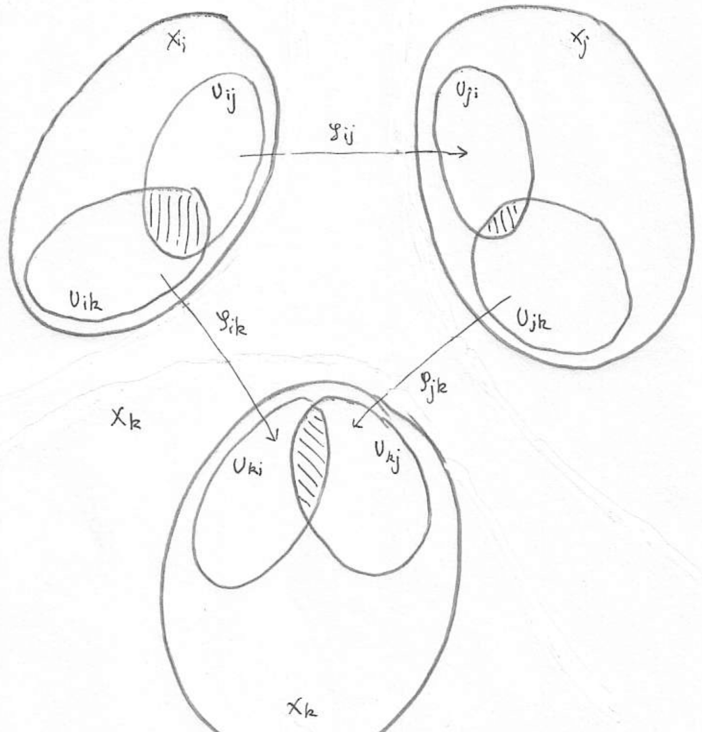
is another generic point of Z

Suppose $y \in Z \setminus V_i$ and let U_j be one of the open covers which contains y. Since Z is irreducible the intersection $U_i \cap U_j \cap Z$ must be nonempty. But then $U_j^c \cap U_i$ will be a closed subset of U_i, and if $\mathfrak{p} \notin U_j$ this would be a closed subset strictly contained in the closure of $\mathfrak{p}$ – a contradiction. Hence $\mathfrak{p} \in U_j$. But then in $\operatorname{Spec} B_j$ the points corresponding to $\mathfrak{p}$ and y are both generic points for $U_j \cap Z$. Hence $\mathfrak{p} = y$ and the generic point of Z is unique.

Q2.12 <u>Glueing Lemma</u> Let $\{X_i\}$ be a nonempty family of schemes (possibly infinite). Suppose for each $i \neq j$ we are given an open subset $U_{ij} \subseteq X_i$ (possibly empty) and let it have the induced scheme structure. Suppose we are also given for each $i \neq j$ an isomorphism of schemes $\varphi_{ij} : U_{ij} \longrightarrow U_{ji}$ such that

(1) For each $i \neq j$, $\varphi_{ij} = \varphi_{ji}^{-1}$

(2) For each i, j, k $\varphi_{ij}(U_{ij} \cap U_{ik}) = U_{jk} \cap U_{ji}$ and $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ on $U_{ij} \cap U_{ik}$.



We define a scheme X which "glues" the X_i along the U_{ij}. First let $X' = \coprod_i X_i = \{(i,x) \mid x \in X_i\}$ be the disjoint union topologised in the normal way: that is, if $\psi_i' : X_i \longrightarrow X'$ is $x \mapsto (i,x)$ then $V \subseteq X'$ is open iff. $\psi_i'^{-1}(V)$ is open $\forall i$. So iff. V is the union over I of an open set in each X_i. The following defines an equivalence relation on X':

$$(i,x) \sim (i,x) \qquad \forall i \in I, x \in X_i$$

$$(i,x) \sim (j, \varphi_{ij}(x)) \qquad \forall i \neq j \text{ and } x \in U_{ij}$$

(These couples <u>are</u> an equiv. relation: no need to generate). Let $X = X'/\sim$ with $\theta : X' \longrightarrow X$ canonical (and continuous by def^N of the quotient topology on X. So $W \subseteq X$ is open iff. $\theta^{-1}W$ is open which is iff. $\psi_i(\theta^{-1}W)$ is open $\forall i$. So letting $\psi_i : X_i \longrightarrow X$ be the continuous map $\theta\psi_i'$, a subset $W \subseteq X$ is open iff $\psi_i^{-1}W$ is open $\forall i \in I$. For $W \subseteq X$ open we define a ring $\mathcal{O}_X(W)$ as follows:

$$\mathcal{O}_X(W) = \Big\{ (s_i) \in \prod_{i \in I} \mathcal{O}_{X_i}(\psi_i^{-1}W) \;\Big|\; \forall i \neq j \;\; \varphi_{ij}^{\#}\big|_{\psi_j^{-1}W \cap U_{ji}} \big(s_j|_{\psi_j^{-1}W \cap U_{ji}}\big) = s_i|_{\psi_i^{-1}W \cap U_{ij}} \Big\}$$

Note that $\varphi_{ij}^{-1}(\psi_j^{-1}W \cap U_{ji}) = (\psi_j \varphi_{ij})^{-1} W \cap U_{ij} = \psi_i^{-1} W \cap U_{ij}$ since on U_{ij} it is easily checked that $\psi_j \varphi_{ij} = \psi_i$. So sections of $\mathcal{O}_X$ on W are sections over $\psi_i^{-1}W$ which are "glueable" together. It is immediate that $\mathcal{O}_X(W)$ is a subring of $\prod_i \mathcal{O}_{X_i}(\psi_i^{-1}W)$ and that the restriction $\prod_i \mathcal{O}_{X_i}(\psi_i^{-1}W) \longrightarrow \prod_i \mathcal{O}_{X_i}(\psi_i^{-1}Q)$ maps $\mathcal{O}_X(W) \longrightarrow \mathcal{O}_X(Q)$ for any $Q \subseteq W$. Hence $\mathcal{O}_X$ is a presheaf of rings (note $\mathcal{O}_X(\emptyset) = \prod_i 0 = 0$). Using the fact that the $\mathcal{O}_{X_i}$ are all sheaves we see that $\mathcal{O}_X$ too is a sheaf, We claim that $\forall i \in I$ the map ψ_i is a homeomorphism onto its image. If $V \subseteq X_i$ is open then $\psi_i(V)$ is open iff $\forall j \neq i$ $\psi_j^{-1}(\psi_i(V))$ is open in X_j. But $\psi_j^{-1}(\psi_i(V)) = \{x \in X_j \mid (j,x) \in \psi_i(V)\} = \varphi_{ji}^{-1} V$, which is open, as required. In particular $\psi_j^{-1}(\psi_i(V)) \subseteq U_{ji}$.

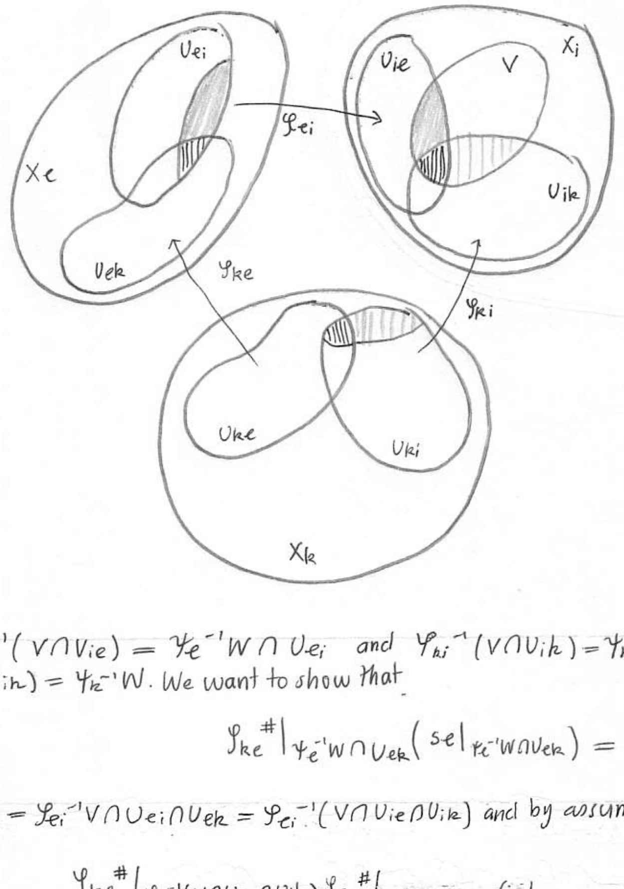
Let $W \subseteq \psi_i(X_i)$ be open, say $W = \psi_i(V)$. Then there is a canonical morphism of rings

$$\begin{aligned} \mathcal{O}_X(W) &\longrightarrow \mathcal{O}_{X_i}(V) \\ (s_j)_{j\in I} &\longmapsto s_i \end{aligned} \qquad (2)$$

which is clearly natural with respect to open inclusions $W' \subseteq W \subseteq \psi_i(X_i)$. These maps are injective since if $(s_j),(t_j) \in \mathcal{O}_X(W)$ and $s_i = t_i$ then for $j \neq i$ $s_j, t_j \in \mathcal{O}_{X_j}(\psi_j^{-1}W) = \mathcal{O}_{X_j}(\psi_j^{-1}\psi_i V) = \mathcal{O}_{X_j}(\varphi_{ji}^{-1}V)$. But by def^N of $\mathcal{O}_X(W)$

$$\begin{aligned} s_j &= s_j|_{\psi_j^{-1}W \cap U_{ji}} \\ &= \varphi_{ji}^{\#}|_{\psi_i^{-1}W\cap U_{ij}}(s_i|_{\psi_i^{-1}W \cap U_{ij}}) \\ &= \varphi_{ji}^{\#}|_{\psi_i^{-1}W\cap U_{ij}}(t_i|_{\psi_i^{-1}W\cap U_{ij}}) = t_j \end{aligned}$$

The map in (2) is also surjective, since if $s \in \mathcal{O}_{X_i}(V)$ define $s_j = \varphi_{ji}^{\#}|_{V\cap U_{ij}}(s|_{V\cap U_{ij}})$ $\forall j \in I, j \neq i$. Put $s_i = s$. Then for $e, k \in I$ $e \neq k$ and $e \neq i$, $k \neq i$ we have the following setup:



Note that $\varphi_{ei}^{-1}(V \cap U_{ie}) = \psi_e^{-1}W \cap U_{ei}$ and $\varphi_{ki}^{-1}(V \cap U_{ik}) = \psi_k^{-1}W \cap U_{ki}$. So $\varphi_{ei}^{-1}(V\cap U_{ie}) = \psi_e^{-1}W$ and $\varphi_{ki}^{-1}(V \cap U_{ik}) = \psi_k^{-1}W$. We want to show that

$$\varphi_{ke}^{\#}|_{\psi_e^{-1}W\cap U_{ek}}(s_e|_{\psi_e^{-1}W\cap U_{ek}}) = s_k|_{\psi_k^{-1}W\cap U_{ke}}$$

But $\psi_e^{-1}W \cap U_{ek} = \varphi_{ei}^{-1}V \cap U_{ei} \cap U_{ek} = \varphi_{ei}^{-1}(V \cap U_{ie} \cap U_{ik})$ and by assumption (2)

$$\begin{aligned} &\varphi_{ke}^{\#}|_{\varphi_{ei}^{-1}(V\cap U_{ie}\cap U_{ik})}\varphi_{ei}^{\#}|_{V\cap U_{ie}\cap U_{ik}}(s|_{V\cap U_{ie}\cap U_{ik}}) \\ &= \varphi_{ik}^{\#}|_{V\cap U_{ie}\cap U_{ik}}(s|_{V\cap U_{ie}\cap U_{ik}}) = s_k|_{\psi_k^{-1}W\cap U_{ke}} \end{aligned}$$

as required. For the case where $e = i$ or $k = i$ we use the fact that $\varphi_{ie} = \varphi_{ei}^{-1}$. Clearly (s_j) is mapped to s via (2), which is thus an isomorphism of ringed spaces.

Since X is covered by open sets $\psi_i(X_i)$ which are schemes, X is itself a scheme, and the $\psi_i : X_i \longrightarrow X$ are morphisms of schemes. (in fact they are open immersions). It is easy to check that $\psi_i(U_{ij}) = \psi_i(X_i) \cap \psi_j(X_j)$ and that $\psi_i = \psi_j \circ \varphi_{ij}$ on U_{ij}. An interesting special case is when the family X_i is arbitrary but the U_{ij} are all empty.

In this case $X = \coprod X_i$ and for $W \subseteq X$ open $\mathcal{O}_X(W) = \prod_{i\in I} \mathcal{O}_{X_i}(\varphi_i^{-1}W)$ where $\varphi_i : X_i \longrightarrow X$ is canonical. Then the scheme X is called the <u>disjoint union</u> of the X_i.

$\boxed{Q2.13}$ A topological space is <u>quasi-compact</u> if every open cover has a finite subcover.

(a) A space X is noetherian iff. every open subset is quasi-compact. The implication $\Rightarrow$ is trivial since by Ex 1.7b) of Ch. I if X is noetherian it is quasi-compact, and any subspace of X is noetherian, hence quasi-compact. For $\Leftarrow$ suppose every open subset is quasi-compact and let $U_1 \subseteq U_2 \subseteq \cdots$ be an ascending chain of open subsets of X. Put $V = \bigcup_i U_i$. Since V is quasi-compact, $V = U_N$ for some N. Hence $U_1 \subseteq \cdots \subseteq U_N = U_{N+1} = \cdots$ so the chain terminates, so by Ex Ch. I 1.7a) X is noetherian.

(b) Let $X = \operatorname{Spec} A$ be affine. As usual, for $f \in A$ $D(f) = \{\mathfrak{p} \in \operatorname{Spec} A \mid f \notin \mathfrak{p}\} \subseteq sp(X)$. Let $sp(X) = \bigcup_{i \in I} V_i$. Since we can write each V_i as a union of $D(f_{ij})$ $j \in I_i$, it suffices to show that any cover of the form $sp(X) = \bigcup_{k \in K} D(f_k)$ has a finite subcover. But

$$\operatorname{spec} A = \bigcup_k D(f_k)$$

$$\text{iff. } \forall \mathfrak{p} \in \operatorname{Spec} A \text{ for some } f_k,\ \mathfrak{p} \not\ni f_k.$$

$$\text{iff. } (f_k)_{k \in K} \text{ is improper}$$

But if $(f_k)_{k \in K} = A$, there are a finite number of the f_k, say $f_{i_1}, \ldots, f_{i_n}$ with $a_1 f_{i_1} + \cdots + a_n f_{i_n} = 1$. Hence $(f_{i_1}, \ldots, f_{i_n}) = A$ and this gives a finite subcover. Hence $sp(X)$ is quasi-compact, but not in general noetherian: for example, take the polynomial ring $k[x_1, x_2, x_3, \ldots]$ in a countably infinite set of variables. It is not noetherian (see A&M notes on Artin rings) and $(x_1) \subset (x_1, x_2) \subset (x_1, x_2, x_3) \ldots$ is a strictly ascending chain of primes, giving a strictly decreasing chain of closed subsets $V(x_1) \supset V(x_1, x_2) \supset V(x_1, x_2, x_3) \supset \cdots$ of $\operatorname{Spec} k[x_1, x_2, x_3, \ldots]$.

We say that a scheme X is <u>quasi-compact</u> if $sp(X)$ is.

(c) Let A be a noetherian ring. We claim that $\operatorname{Spec} A$ is a noetherian topological space. If $A = 0$ this is trivial, so assume $A \neq 0$. Let $Z_1 \supseteq Z_2 \supseteq Z_3 \supseteq \cdots$ be a descending chain of closed subsets. Say $Z_i = V(\mathfrak{a}_i)$, where we may assume $\mathfrak{a}_i$ is radical. Then $V(\mathfrak{a}_1) \supseteq V(\mathfrak{a}_2) \supseteq \cdots$ implies $\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \cdots$. Since A is noetherian, for some j, $\mathfrak{a}_j = \mathfrak{a}_{j+1} = \cdots$ hence $Z_j = Z_{j+1} = \cdots$ as required.

(d)

$\boxed{Q2.14}$ (a) Let S be a graded ring. $\operatorname{Proj} S = \emptyset$ iff. $\forall f \in S_+$ $D_+(f) = \emptyset$ iff. $\forall f \in S_+$ f is nilpotent.

(b) Let $\varphi : S \longrightarrow T$ be a graded homomorphism of graded rings (preserving degrees). Let

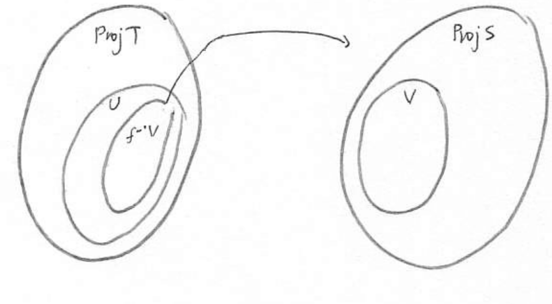
$$U = \{ \mathfrak{p} \in \operatorname{Proj} T \mid \mathfrak{p} \not\supseteq \varphi(S_+) \}$$

If $\mathfrak{p} \in U$ and say $f \in S_+$ with $\varphi(f) \notin \mathfrak{p}$ then $\varphi(f) \in T_+$ and $\mathfrak{p} \in D_+(\varphi(f)) \subseteq U$, hence U is open.
Next we define a morphism of schemes $f : U \longrightarrow \operatorname{Proj} S$. Firstly define

$$f(\mathfrak{p}) = \varphi^{-1}\mathfrak{p}$$

This is well-defined since if $\mathfrak{p} \in U$ then $\varphi^{-1}\mathfrak{p}$ is a homogenous prime not containing S_+. If $\mathfrak{a} \subseteq S$ is homogenous then the ideal in T generated by $\varphi(\mathfrak{a})$ is homogenous and $f^{-1}V(\mathfrak{a}) = V((\varphi(\mathfrak{a}))) \cap U$ so f is continuous.

Next we define a morphism of sheaves of rings $\mathcal{O}_{\operatorname{Proj} S} \longrightarrow f_*(\mathcal{O}_{\operatorname{Proj} T}|_U)$.



For $\mathfrak{p} \in U$ there is a morphism of rings $\varphi_{\mathfrak{p}} : S_{(\varphi^{-1}\mathfrak{p})} \longrightarrow T_{(\mathfrak{p})}$ defined by $\varphi_{\mathfrak{p}}(a/s) = \varphi(a)/\varphi(s)$.
This morphism is local since if $\varphi_{\mathfrak{p}}(a/s) \in \mathfrak{m}_{\mathfrak{p}}$ then $\varphi(a) \in \mathfrak{p}$ so $a \in \varphi^{-1}\mathfrak{p}$. Define

$$f^\#_V : \mathcal{O}_{\operatorname{Proj} S}(V) \longrightarrow \mathcal{O}_{\operatorname{Proj} T}(f^{-1}V)$$

$$f^\#_V(s)(\mathfrak{p}) = \varphi_{\mathfrak{p}}(s(\varphi^{-1}\mathfrak{p}))$$

One easily checks this makes $(f, f^\#) : U \longrightarrow \operatorname{Proj} S$ into a morphism of schemes, since $\varphi_{\mathfrak{p}}$ is local, and the following commutes

$$\begin{tikzcd}
\mathcal{O}_{\operatorname{Proj} S, \varphi^{-1}\mathfrak{p}} \arrow[rr] \arrow[d] & & \mathcal{O}_{U,\mathfrak{p}} \Rightarrow \mathcal{O}_{\operatorname{Proj} T, \mathfrak{p}} \arrow[d, Rightarrow] \\
S_{(\varphi^{-1}\mathfrak{p})} \arrow[rr] & & T_{(\mathfrak{p})}
\end{tikzcd}$$

?

(c) Suppose that $\varphi_d : S_d \longrightarrow T_d$ is an isomorphism for all $d \geq d_0$. Then $U = \operatorname{Proj} T$ since if $\mathfrak{p} \in \operatorname{Proj} T$ and $\mathfrak{p} \supseteq \varphi(S_+)$ then for sufficiently large N, $\mathfrak{p} \supseteq T_n$ $\forall n \geq N$. But taking for example $f \in T_1$, we have $f^N \in T_N$ so $f \in \mathfrak{p}$. Hence $\mathfrak{p} \supseteq T_+$, contradicting $\mathfrak{p} \in \operatorname{Proj} T$. Hence $U = \operatorname{Proj} T$. To prove that $f : \operatorname{Proj} T \longrightarrow \operatorname{Proj} S$ iso we need:

<u>LEMMA</u> Let S be a graded ring, $\mathfrak{p}, \mathfrak{q} \in \operatorname{Proj} S$. If $\exists d_0 \geq 0$ s.t. $\mathfrak{p}_d \subseteq \mathfrak{q}_d$ $\forall d \geq d_0$ then $\mathfrak{p} \subseteq \mathfrak{q}$.
In particular if $\mathfrak{p}, \mathfrak{q} \in \operatorname{Proj} S$ agree in all sufficiently high degrees, they are equal.

<u>PROOF</u> By induction on d_0 (or recursion, more precisely). If $d_0 = 0$ we are done since for any homogenous ideal $\mathfrak{a}$, $\mathfrak{a} = \bigoplus_{d \geq 0} \mathfrak{a}_d$ ($\mathfrak{a}_d = \mathfrak{a} \cap S_d$). For the recursive step we assume $d_0 > 0$ and $\forall d \geq d_0$ $\mathfrak{p}_d \subseteq \mathfrak{q}_d$ and we show $\mathfrak{p}_{d_0 - 1} \subseteq \mathfrak{q}_{d_0 - 1}$. Since $\mathfrak{q} \not\supseteq S_+$ there is some $e > 0$ and $f \in S_e$ with $f \notin \mathfrak{q}$.
If $a \in \mathfrak{p}_{d_0 - 1}$ then $af \in \mathfrak{p}_{d_0 - 1 + e} \subseteq \mathfrak{q}_{d_0 - 1 + e}$. Since $\mathfrak{q}$ is prime, $a \in \mathfrak{q}$, as required. $\square$

We now prove that $f : \mathrm{Proj}\, T \longrightarrow \mathrm{Proj}\, S$ is an isomorphism, under the following assumptions

$\varphi : S \longrightarrow T$ graded

- $\exists d_0 \geq 0$ s.t. $\forall d \geq d_0$ $\varphi_d : S_d \longrightarrow T_d$ is iso.
- T_0 is integral over $\varphi(S_0)$.

In particular if φ_0 is surjective. We could probably remove the second hypothesis by "homogenising" the Going Up Theorem. First we show that f is a homeomorphism:

?

$\boxed{\text{Injective}}$ Suppose $\mathfrak{p}, \mathfrak{q} \in \mathrm{Proj}\, T$ and $\varphi^{-1}\mathfrak{p} = \varphi^{-1}\mathfrak{q}$. Then $\forall d \geq d_0$ we have $\mathfrak{p}_d = \mathfrak{q}_d$ and hence $\mathfrak{p} = \mathfrak{q}$.

$\boxed{\text{Surjective}}$ Let $\mathfrak{p} \in \mathrm{Proj}\, S$ be given. Let C be the integral closure of $\varphi(S)$ in T. Then $C = T$ since C contains T_0 and T_+: $T_0 \subseteq C$ by assumption and if $f \in T_+$ is homogenous then $f^n \in T_d$ for some $d \geq d_0$ for sufficiently large n, so $f^n - \varphi(s) = 0$, $s \in S$, and hence f is integral over $\varphi(S)$. So T is integral over $\varphi(S)$. By the going up Theorem there exists a prime ideal $\mathfrak{q}$ of T with $\varphi^{-1}\mathfrak{q} = \mathfrak{p}$, provided $T \neq 0$. But if $T = 0$ then $S_d = 0$ $\forall d \geq d_0$ and it follows that $\mathrm{Proj}\, S = \emptyset$ (if $f \in S_e$ $e > 0$ $f^n = 0$ some n). So the result is trivial in this case, and we can assume $T \neq 0$. ?

$\boxed{Q2.15}$ (a) Let V be a variety over the algebraically closed field k. We claim that a point P is closed ($P \in t(V)$) iff. the residue field of $(t(V), \alpha_* \mathcal{O}_V)$ at P coincides with its subfield k. As in the proof of Prop 2.6 we can cover V with open affines U_i, so $t(V)$ is covered by open affine sets $t(U_i^c)^c$ with

$$(t(U_i^c)^c, (\alpha_* \mathcal{O}_V)|_{t(U_i^c)^c}) \cong (t(U_i), \alpha_{i*} \mathcal{O}_{U_i})$$
$$\cong \operatorname{Spec} A(Y_i)$$

where Y_i affine varieties with $Y_i \cong U_i$. Thus we reduce to the case where V is an affine variety. One checks that $(t(U_i), \alpha_{i*} \mathcal{O}_{U_i}) \cong \operatorname{Spec} A(Y_i)$ induces an isomorphi of local rings which is an isomorphism of k-algebras. Suppose we could prove the result for affine varieties and let $P \in t(V)$. If P is closed it will be closed in its affine neighborhood and hence the residue field at P will be k. Suppose the residue field at P is equal to k. Then P will correspond to a closed point in its affine neighborhood $(t(U_i), \alpha_{i*} \mathcal{O}_{U_i})$. Here P is a closed, irreducible subset of V, and $P \in t(U_i^c)^c \iff P \not\subseteq U_i^c \iff P \cap U_i \neq \emptyset$. Under the isomorphism $t(U_i^c)^c \longrightarrow t(U_i)$, P corresponds to the closed, irreducible subset $P \cap U_i$ of U_i. The fact that this point is closed in $t(U_i)$ (for any space X and $Q \in t(X)$, the closure of $\{Q\}$ in $t(X)$ is $t(Q)$, so $\{Q\}$ is closed iff. Q contains no proper closed irreducible subsets) means that there are no closed irreducible subsets of U_i properly contained in $P \cap U_i$.

Now suppose Z is a closed, irreducible proper subset of P. Since the U_i cover V, there is some U_i intersecting Z so that $U_i \cap Z$ is a closed irreducible subset of $P \cap U_i$. Hence $Z \supseteq P \cap U_i$. But $P \cap U_i$ is a nonempty open subset of the irreducible space P, hence dense, so $Z = P$, which is a contradiction. This shows P is a closed point in $t(V)$.

It remains now to prove the result for $\operatorname{Spec} A$ where A is a f.g. k-domain. A point $\mathfrak{p}$ is closed iff. it is maximal, and the local ring $A_\mathfrak{p}$ has residue field $A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} \cong Q(A/\mathfrak{p})$. The field k is a subfield of $Q(A/\mathfrak{p})$, and this subfield coincides with $Q(A/\mathfrak{p})$ iff. every coset of $\mathfrak{p}$ contains an element of k iff. the image of k in $A/\mathfrak{p}$ is the whole ring, which is iff. $\mathfrak{p}$ is maximal (since k is alg. closed). This completes the proof.

(b) Let $f: X \longrightarrow Y$ be a morphism of schemes over k, and suppose $P \in X$ has residue field k. (That is, the map $k \longrightarrow \mathcal{O}_X(X) \longrightarrow \mathcal{O}_{X,P} \longrightarrow \mathcal{O}_{X,P}/\mathfrak{m}_P$ is surjective). We have a commutative diagram

$$\begin{tikzcd} k \arrow[r] \arrow[dr] & \mathcal{O}_X(X) \arrow[r] & \mathcal{O}_{X,P} \\ & \mathcal{O}_Y(Y) \arrow[u] \arrow[r] & \mathcal{O}_{Y,f(P)} \arrow[u, "f_x"'] \end{tikzcd}$$

By assumption for $a \in \mathcal{O}_{Y,f(P)}$ there is $\ell \in k$ s.t. $f_x(a) - \ell \cdot 1 \in \mathfrak{m}_P$. Hence $a - \ell \cdot 1 \in \mathfrak{m}_{f(P)}$, and $\mathcal{O}_{Y,f(P)} = k$, also.

(c) Let V, W be any two varieties over k, and consider the map

$$\operatorname{Hom}_{\underline{Var}}(V, W) \longrightarrow \operatorname{Hom}_{\underline{Sch}/k}(t(V), t(W)) \qquad (1)$$

defined in Proposition 2.6. We show that this is a bijection. Recall that given $f: V \longrightarrow W$ the morphism $(g, \psi): t(V) \longrightarrow t(W)$ is defined by

$$g(Q) = \overline{f(Q)}$$
$$h \longmapsto hf$$
$$\psi_{t(U^c)^c} : (\beta_* \mathcal{O}_W)(t(U^c)^c) = \mathcal{O}_W(U) \longrightarrow \mathcal{O}_V(f^{-1}U) = (\alpha_* \mathcal{O}_V)(g^{-1} t(U^c)^c)$$

For $x \in V$, $\{x\} \in t(V)$ and $g(\{x\}) = \{f(x)\}$, so it is obvious that the map in (1) is injective. We now show that it is surjective. Let a morphism of schemes over k $(g, \psi): t(V) \longrightarrow t(W)$ be given. Using (a), (b) we see that g must map closed points to closed points. But $\alpha: V \longrightarrow t(V)$ and $\beta: W \longrightarrow t(W)$ give a bijection of V (resp. W) with the closed points of $t(V)$ (resp. $t(W)$). Thus for $P \in V$ we let $f(P) \in W$ be the unique element of $g(\{P\})$, so $g(\{P\}) = \{f(P)\}$ $\forall P \in V$. To see that f is continuous, let $Y \subseteq W$ be closed. Then

$$\begin{aligned} f^{-1} Y &= \{P \in V \mid f(P) \in Y\} \\ &= \{P \in V \mid \{f(P)\} \in t(Y)\} \\ &= \{P \in V \mid g(\{P\}) \in t(Y)\} \\ &= \alpha^{-1} g^{-1} t(Y) \end{aligned}$$

which is closed since g is continuous.

Let $U \subseteq t(W)$ be open and for $P \in f^{-1}U$ consider the diagram

$$\begin{tikzcd}
k \arrow[rr, Rightarrow, no head] & & k \\
\mathcal{O}_{W,f(P)} \arrow[u] \arrow[rr, "\psi_P"] & & \mathcal{O}_{V,P} \arrow[u] \\
& \mathcal{O}_W(U) \arrow[ul] \arrow[rr, "\psi|_{t(U^c)^c}"] \arrow[d, equal] & & \mathcal{O}_V(f^{-1}U) \arrow[ul] \arrow[d, equal] \\
& (\beta_*\mathcal{O}_W)(t(U^c)^c) \arrow[uul] \arrow[rr] & & (\alpha_*\mathcal{O}_V)(g^{-1}t(U^c)^c) \arrow[uul]
\end{tikzcd}$$

It is easily checked that $(\beta_*\mathcal{O}_W)_{f(P)} = \mathcal{O}_{W,f(P)}$ and $\mathcal{O}_{V,P} = (\alpha_*\mathcal{O}_V)_P$. Since (g,ψ) is a morphism of schemes over k, ψ_P is a local morphism of local k-algebras, both with residue field k. Hence the induced morphism of the residue fields is the identity. Of course the composite $\mathcal{O}_V(f^{-1}U) \longrightarrow \mathcal{O}_{V,P} \longrightarrow k$ maps a regular function $g: f^{-1}U \to k$ to its value $g(P)$, so commutativity of the above diagram means that for a regular function $h: U \longrightarrow k$ on W,

$$(\psi|_{t(U^c)^c})(h)(Q) = h(f(Q)) \qquad \forall Q \in U$$

So $f: V \longrightarrow W$ is a morphism of varieties. It only remains to show that for all $Q \in t(V)$, $g(Q) = \overline{f(Q)}$. Let $Y \subseteq W$ be closed. Since g is continuous, $g^{-1}t(Y) = t(M)$ for some closed $M \subseteq V$. Then

$$P \in M \iff \{P\} \in t(M) \iff \{f(P)\} = g(\{P\}) \in t(Y) \iff f(P) \in Y$$

Hence $M = f^{-1}Y$ and $g^{-1}t(Y) = t(f^{-1}Y)$. Now let $Q \in t(V)$ be given and let $Y \subseteq W$ be closed. Then $Y \supseteq f(Q)$ iff. $Q \subseteq f^{-1}Y$ iff. $Q \in t(f^{-1}Y) = g^{-1}t(Y)$ iff. $g(Q) \in t(Y)$ iff. $g(Q) \subseteq Y$. Hence $g(Q) = \overline{f(Q)}$ as required, completing the proof.

Q2.16 Let X be a scheme, $f \in \Gamma(X, \mathcal{O}_X)$ and define X_f to be the set of points $x \in X$ such that the stalk f_x of f at x is not contained in the maximal ideal $\mathfrak{m}_x$ of $\mathcal{O}_{X,x}$.

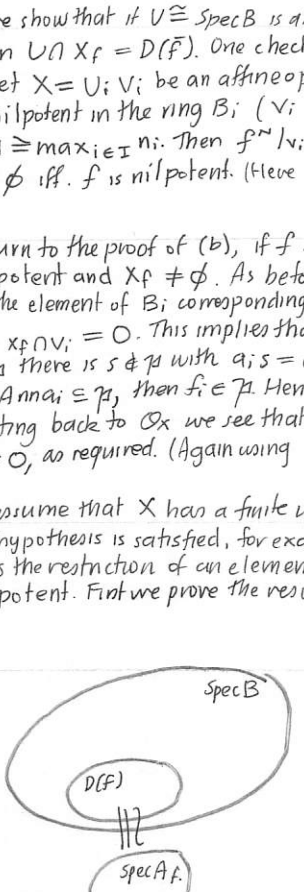
(a) We showed X_f was open in the proof of Ex 2.4. The other claim is in (b).

(b) Assume that X is quasi-compact. Let $A = \Gamma(X, \mathcal{O}_X)$ and let $a \in A$ be an element whose restriction to X_f is 0. We claim that for some $n > 0$ $f^n a = 0$.

First we show that if $V \cong \operatorname{Spec} B$ is an affine open subscheme of X, and if $\bar{f} \in B = \Gamma(V, \mathcal{O}_X|_V)$ is the restriction of f, then $V \cap X_f = D(\bar{f})$. One checks this easily. If f is nilpotent then certainly $X_f = \emptyset$. Conversely, if $X_f = \emptyset$ then let $X = \bigcup_i V_i$ be an affine open cover of X. By assumption, if $\bar{f}_i = f|_{V_i}$, we have $D(\bar{f}_i) = \emptyset$. Hence $\bar{f}_i$ is nilpotent in the ring B_i ($V_i \cong \operatorname{Spec} B_i$). Say $(f|_{V_i})^{n_i} = 0$ $n_i \geq 1$ in $\mathcal{O}_X(V_i)$. Equivalently, $f^{n_i}|_{V_i} = 0$. Let $N \geq \max_{i \in I} n_i$. Then $f^N|_{V_i} = 0 \; \forall i$ and consequently $f^N = 0$ in $\mathcal{O}_X(X)$. Hence for $f \in \Gamma(X, \mathcal{O}_X)$, $X_f = \emptyset$ iff. f is nilpotent. (Here we use quasi-compactness to get N finite)

To return to the proof of (b), if f is nilpotent clearly $n > 0$ exists with $f^n a = 0$. So we may assume f is not nilpotent and $X_f \neq \emptyset$. As before let $X = \bigcup_i V_i$ be an affine cover, with $(V_i, \mathcal{O}_X|_{V_i}) \cong \operatorname{Spec} B_i$. Denote by f_i the element of B_i corresponding to $f|_{V_i}$ and similarly a_i for a. The fact that $a|_{X_f} = 0$ means that $\forall i$ $(a|_{V_i})|_{X_f \cap V_i} = 0$. This implies that $\dot{a}_i|_{D(f_i)} = 0$ where $\dot{a}_i$ maps $\mathfrak{p} \in \operatorname{Spec} B_i$ to $a_i/1 \in A_\mathfrak{p}$. But this means that if $f_i \notin \mathfrak{p}$ there is $s \notin \mathfrak{p}$ with $a_i s = 0$. Hence if $f_i \notin \mathfrak{p}$ then the ideal $\operatorname{Ann} a_i$ in B_i also is not contained in $\mathfrak{p}$. This means that if $\operatorname{Ann} a_i \subseteq \mathfrak{p}$, then $f_i \in \mathfrak{p}$. Hence $f \in \sqrt{\operatorname{Ann} a_i}$ and consequently there is $n_i \geq 1$ with $f^{n_i} a_i = 0$ in B_i. Translating back to $\mathcal{O}_X$ we see that $(f^{n_i} a)|_{V_i} = 0$. Setting $N \geq \max_i n_i$ we see that since $\mathcal{O}_X$ is a sheaf $f^N a = 0$, as required. (Again using quasi-compactness to get N finite).

(c) Now assume that X has a finite cover by open affines U_i such that each intersection $U_i \cap U_j$ is quasi-compact (This hypothesis is satisfied, for example, if $sp(X)$ is noetherian). Let $b \in \mathcal{O}(X_f)$. We claim that for some $n > 0$ $f^n b$ is the restriction of an element of A. If $b = 0$ (hence if $X_f = \emptyset$) this is trivial. So assume $b \neq 0$ and f not nilpotent. First we prove the result in the case where $X = \operatorname{Spec} B$ is affine (with none of the above assumptions)



Let $f \in B$. By Exercise 2.1 $(D(f), \mathcal{O}_{\operatorname{Spec} B}|_{D(f)}) \cong \operatorname{Spec} A_f$, so any $b \in \mathcal{O}_{\operatorname{Spec} B}(D(f))$ has the form $b(\mathfrak{p}) = a/f^n \in A_\mathfrak{p}$ $\forall \mathfrak{p} \in D(f)$ for some fixed $a \in A$ and $n \geq 0$. Then $f|_{D(f)}$ is $\mathfrak{p} \mapsto f/1$, so

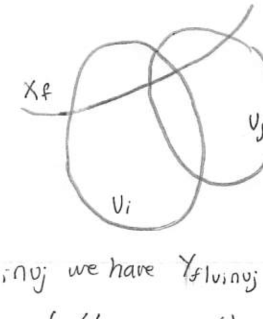
$$(bf^n)(\mathfrak{p}) = a/1 \in A_\mathfrak{p} \quad \forall \mathfrak{p} \in D(f)$$

Hence $b\,(f|_{D(f)})^n$ is the restriction of $a \in B$, as required.

Returning now to the general case, let $U_i \cong \operatorname{Spec} B_i$ $\forall i$, so $X_f \cap U_i$ is identified with $D(f_i)$, $f_i \in B_i$ corresponding to $f|_{U_i}$. It follows that there is $a_i \in \mathcal{O}(U_i)$ and $n_i \geq 0$ with $(f|_{X_f \cap U_i})^{n_i}\, b|_{X_f \cap U_i} = a_i|_{X_f \cap U_i}$. (if $X_f \cap U_i = \emptyset$ then we can take $a_i = 0$). Since the cover $\{U_i\}$ is finite there is $N_1 > 0$ and $a_i' \in \mathcal{O}(U_i)$ with

$$\left((f|_{X_f})^{N_1} b\right)|_{X_f \cap U_i} = a_i'|_{X_f \cap U_i} \qquad \forall i \in I$$

we want to turn the a_i' into a matching family. For this we use (b). By assumption $\mathcal{O}|_{U_i \cap U_j}$ is a quasi-compact scheme for all i, j. (we need only consider $U_i \cap U_j \neq \emptyset$)



Now in the scheme $Y = \mathcal{O}|_{U_i \cap U_j}$ we have $Y_{f|_{U_i \cap U_j}} = X_f \cap U_i \cap U_j$, and

$$\begin{aligned}&(a_i'|_{U_i \cap U_j} - a_j'|_{U_i \cap U_j})|_{X_f \cap U_i \cap U_j} \\ &= ((f|_{X_f})^N b)|_{X_f \cap U_i \cap U_j} - ((f|_{X_f})^N b)|_{X_f \cap U_i \cap U_j} = 0\end{aligned}$$

Hence from (b) we conclude that $(f|_{U_i \cap U_j})^{m_{ij}}\,(a_i'|_{U_i \cap U_j} - a_j'|_{U_i \cap U_j}) = 0$ for some $m_{ij} > 0$.

that is, for all i,j $\left((f|_{U_i})^{m_{ij}} a_i'\right)|_{U_i\cap U_j} = \left((f|_{U_j})^{m_{ij}} a_j'\right)|_{U_i\cap U_j}$. Let N_2 be the maximum of N_1, and m_{ij} for all i,j. Set $a_i'' = (f|_{U_i})^{N_2} a_i'$. Then

$$((f|_{X_f})^N b)|_{X_f\cap U_i} = a_i''|_{X_f\cap U_i} \qquad \forall i$$

$$a_i''|_{U_i\cap U_j} = a_j''|_{U_i\cap U_j} \qquad \forall ij$$

where $N = N_1 + N_2$. Since the a_i'' are a matching family and $\mathcal{O}$ is a sheaf, there is $a \in A$ with $a|_{U_i} = a_i''$ $\forall i$. Hence $\forall i$ $a|_{X_f\cap U_i} = ((f|_{X_f})^N b)|_{X_f\cap U_i}$. Since the $X_f \cap U_i$ cover X_f, it follows that $(f|_{X_f})^N b = a|_{X_f}$, as required.

(d) With the hypothesis of (c) and assuming X is quasi-compact (which follows from being covered by a finite no. of q.c sets) we claim that $\Gamma(X_f, \mathcal{O}) \cong A_f$. Since $X_f = \emptyset$ iff. f is nilpotent iff. $A_f = 0$ this is true if $X_f = \emptyset$. So assume $X_f \neq \emptyset$. First we show that $f|_{X_f}$ is a unit in $\mathcal{O}(X_f)$. For $x \in X$ let V_x be an open neighborhood of x and $s_x \in \mathcal{O}(V_x)$ with $s_x f|_{V_x} = 1$ in $\mathcal{O}(V_x)$. (These exist since f is a unit in $\mathcal{O}_{X,x}$). For $x,y \in X$ and $z \in V_x \cap V_y$ $(V_x, s_x) = (V_y, s_y)$ in $\mathcal{O}_{X,z}$ since both are inverse to $(X_f, f|_{X_f})$. Hence $s_x|_{V_x\cap V_y}$ and $s_y|_{V_x\cap V_y}$ agree on a cover of $V_x \cap V_y$ and are thus equal. Let $s \in \mathcal{O}(X_f)$ be the amalgamation of the s_x. Then $sf|_{X_f} = 1$ in $\mathcal{O}(X_f)$, as required.

Let $\varphi: A_f \longrightarrow \mathcal{O}(X_f)$ be the induced map, $\varphi(a/f^n) = a|_{X_f} s^n$. Suppose $\varphi(a/f^n) = 0$. Then $a|_{X_f} = 0$ and so by (b) since X is quasi-compact for some $n > 0$ $f^n a = 0$. Hence $a/f = 0$ in A_f, so φ is injective. To see that φ is surjective let $b \in \mathcal{O}(X_f)$ be given. By (c) for some $n > 0$ $(f|_{X_f})^n b = a|_{X_f}$ with $a \in A$. But then

$$b = a|_{X_f} s^n = \varphi(a/f^n)$$

Hence φ is an isomorphism, as claimed.

<u>NOTE</u> If $X = \operatorname{Spec} A$ and $f \in A$ then $X_f = D(f)$.

Q2.17 A Criterion for Affineness

(a) Let $f: X \longrightarrow Y$ be a morphism of schemes and suppose that Y can be covered by open subsets U_i such that for each i, the induced map $f^{-1}U_i \longrightarrow U_i$ is an isomorphism. We claim that f is an isomorphism. If $Y = \emptyset$ then $X = \emptyset$ and this is trivial. So assume $Y \neq \emptyset$. The underlying map of spaces is injective since if $f(x) = f(x') = y$, let $y \in U_i$ so $x, x' \in f^{-1}U_i$, on which f is injective. Hence $x = x'$. Surjectivity is also easily checked. The map f is a homeomorphism since for $U \subseteq X$ open, $f(U) = f(\bigcup_i U \cap f^{-1}U_i) = \bigcup_i f(U \cap f^{-1}U_i)$.

The injectivity of $\mathcal{O}_Y(V) \longrightarrow \mathcal{O}_X(f^{-1}V)$ on open $V \subseteq U_i$ easily implies injectivity for any $V \subseteq Y$. As for surjectivity, given $t \in \mathcal{O}_X(f^{-1}U)$, the maps $\mathcal{O}_Y(U\cap U_i) \longrightarrow \mathcal{O}_X(f^{-1}U \cap f^{-1}U_i)$ are all surjective, say s_i maps to $t|_{f^{-1}U \cap f^{-1}U_i}$. Since f is injective on sections the s_i match on the cover $U_i \cap U$ of U, to give $s \in \mathcal{O}_Y(U)$ with $s \longmapsto t$. Hence f is an isomorphism.

(b) We claim that a scheme X is affine iff. there is a finite set $f_1, \dots, f_r$ of elements of $A = \Gamma(X, \mathcal{O}_X)$ such that the open subsets X_{f_i} are affine, and $f_1, \dots, f_r$ generate the unit ideal in A.

Let X be a scheme, $A = \Gamma(X, \mathcal{O}_X)$. First we show that X is affine iff. the canonical map $X \xrightarrow{\phi} \operatorname{Spec} A$ (Ex 2.4) is an isomorphism. Suppose X is affine and $X \xrightarrow{\psi} \operatorname{Spec} B$ an isomorphism. Since $\operatorname{Spec} \dashv \Gamma$ there is a unique morphism of rings $B \xrightarrow{\alpha} A$ making

$$\begin{tikzcd} X \arrow[r, "\psi"] \arrow[dr, "\phi"'] & \operatorname{Spec} B \arrow[d, shift left, "\operatorname{Spec}\alpha"] \\ & \operatorname{Spec} A \arrow[u, dashed, shift left, "\operatorname{Spec}\beta"] \end{tikzcd}$$

commute. Since Spec is full, there is $\beta: A \longrightarrow B$ with $\operatorname{Spec}\beta = \phi\psi^{-1}$. Using the uniqueness property of the unit, the fact that ψ is epi, and the fact that Spec is faithful, we see that $\alpha\beta = 1$ and $\beta\alpha = 1$. Since $\phi = \operatorname{Spec}\beta\, \psi$ and $\operatorname{Spec}\beta$, ψ are both isomorphisms, it follows that ϕ is an isomorphism, as claimed.

We showed in Ex 2.4 that under $f: X \longrightarrow \operatorname{Spec} A$, $f^{-1}D(f) = X_f$, so if X is affine the conditions are trivially satisfied (put $f_1 = 1$).

For the converse, suppose the $f_1, \dots, f_r$ are given and $\phi : X \longrightarrow \operatorname{Spec} A$ is canonical. Since the f_i generate the unit ideal in A, the X_{f_i} cover X. These open sets are quasi-compact, and for $i \neq j$ by Ex 2.16 a) the intersection $X_{f_i} \cap X_{f_j}$ corresponds under $X_{f_i} \cong \operatorname{Spec} B_i$ to a distinguished open set $D(\bar{f_j})$, which is thus quasi-compact. Since X is covered by a finite number of quasi-compact open sets, X is quasi-compact, so we are in position to apply Ex 2.16 (d). That is, there is an isomorphism $A_{f_i} \longrightarrow \mathcal{O}(X_{f_i})$ making the following diagram commute

$$\begin{tikzcd}
A \arrow[r, equal] \arrow[d] & \mathcal{O}(X) \arrow[d] \\
A_{f_i} \arrow[r, "\sim"] & \mathcal{O}(X_{f_i})
\end{tikzcd} \qquad (1)$$

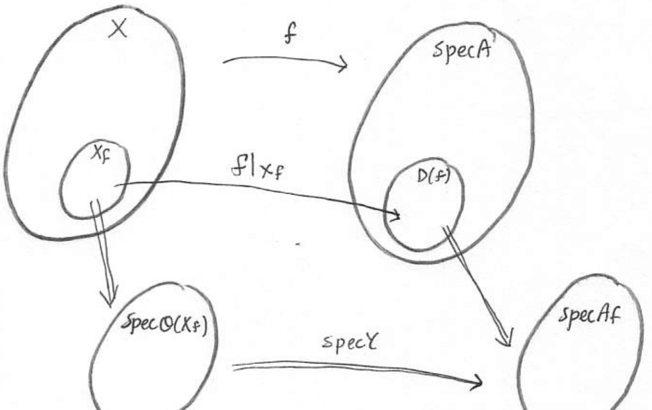
Since by assumption X_{f_i} is affine, the canonical morphism $X_{f_i} \longrightarrow \operatorname{Spec} \mathcal{O}(X_{f_i})$ is an isomorphism. The value of this morphism on sections is determined by the morphism of rings $(\mathcal{O}(X_{f_i}))_{\varphi_x'^{-1} m_x'} \longrightarrow (\mathcal{O}|_{X_{f_i}})_x$ for $x \in X_{f_i}$, where $\varphi_x' : \mathcal{O}(X_f) \longrightarrow (\mathcal{O}|_{X_f})_x$ and m_x' are canonical, and the following diagram commutes:

$$\begin{tikzcd}
\mathcal{O}(X_f) \arrow[r, equal] \arrow[d] & \mathcal{O}(X_f) \arrow[d] \\
(\mathcal{O}(X_f))_{\varphi_x'^{-1} m_x'} \arrow[r] & (\mathcal{O}|_{X_f})_x
\end{tikzcd} \qquad (2)$$

Similarly the open subset $D(f_i) \subseteq \operatorname{Spec} A$ is isomorphic to $\operatorname{Spec} A_f$ via the morphism $(A_f)_{\mathfrak{p} A_f} \longrightarrow A_{\mathfrak{p}}$ which makes

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & A_f \arrow[d] \\
A_{\mathfrak{p}} & (A_f)_{\mathfrak{p} A_f} \arrow[l]
\end{tikzcd} \qquad (3)$$

commute. By part (a) it suffices to show that the induced morphism $f^{-1}(D(f_i)) = X_{f_i} \longrightarrow D(f_i)$ is an isomorphism for all i. Consider the diagram (pick i and put $f = f_i$)



$A_f \xrightarrow{\gamma} \mathcal{O}(X_f)$

(Safe if $X_f = \emptyset$ since 2.16 (d) considers this case)

It suffices to show that the square of morphisms out of X_f commutes. That the underlying maps of spaces commute is easily checked. That the diagram of morphisms of ringed spaces commutes comes down to the commutativity of the following diagram of rings:

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & A_{\mathfrak{p}} \arrow[r] & \mathcal{O}_{X,x} \arrow[rr, equal, bend left] & & \\
A_f \arrow[r] & (A_f)_{\mathfrak{p} A_f} \arrow[u] \arrow[r] & (\mathcal{O}(X_f))_{\varphi_x'^{-1} m_x'} \arrow[r] & (\mathcal{O}|_{X_f})_x
\end{tikzcd}$$

This follows from (1), (2), (3) and the fact that $A \to A_f$, $A_f \to (A_f)_{\mathfrak{p} A_f}$ are epimorphisms. $\square$

NOTE

Let X be a scheme, $U \subseteq X$ open and $s \in \mathcal{O}_X(U)$. We claim that

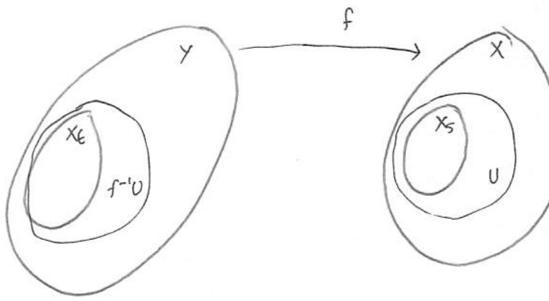
$$X_s = \{x \in U \mid \mathrm{germ}_x s \notin \mathfrak{m}_x\}$$

is an open set. We can immediately reduce to the case where $X = \operatorname{Spec} A$ is affine. Then we can cover U by open sets $D(g_i)$, $g_i \in A$, then $s|_{D(g_i)}$ corresponds to a global section $a_i/s_i \in A_{g_i}$ and

$$X_s \cap D(g_i) = D(a_i/s_i) \subseteq \operatorname{Spec} A_{g_i}$$

Proving that X_s is open. If $f : Y \longrightarrow X$ is any morphism of schemes and $t = f^\#_U(s) \in \mathcal{O}_Y(f^{-1}U)$ then we claim that $f^{-1}X_s = X_t$, which follows from

$$\begin{aligned} f^{-1}X_s &= \{y \in Y \mid f(y) \in X_s\} \\ &= \{y \in f^{-1}U \mid \mathrm{germ}_{f(y)} s \notin \mathfrak{m}_{f(y)}\} \\ &= \{y \in f^{-1}U \mid \mathrm{germ}_{f(y)} s \notin f_y^{-1}(\mathfrak{m}_y)\} \\ &= \{y \in f^{-1}U \mid \mathrm{germ}_y t \notin \mathfrak{m}_y\} \\ &= X_t. \end{aligned}$$



$\boxed{Q2.18}$ In this exercise, we compare some properties of a ring homomorphism to the induced morphism of the spectra of rings.

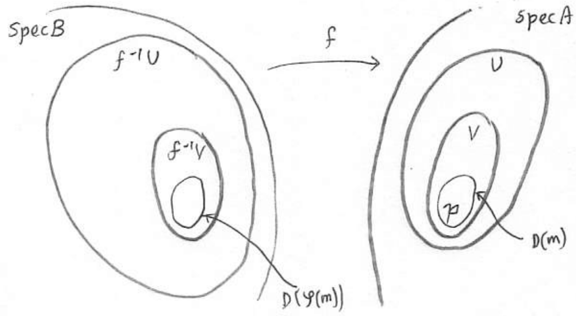
(a) Trivial, since $\sqrt{f} = \bigcap_{\mathfrak{p} \ni f} \mathfrak{p}$.

(b) Let $\varphi : A \longrightarrow B$ be a morphism of rings, and let $f : \operatorname{Spec} B \longrightarrow \operatorname{Spec} A$ be the induced morphism of schemes. Then φ is injective if and only if $f^{\#} : \mathcal{O}_{\operatorname{Spec} A} \longrightarrow f_* \mathcal{O}_{\operatorname{Spec} B}$ is injective.

Suppose φ is injective and for $\mathfrak{q} \in \operatorname{Spec} B$ let $\varphi_{\mathfrak{q}} : A_{\varphi^{-1}\mathfrak{q}} \longrightarrow B_{\mathfrak{q}}$ be canonical. Let $U \subseteq \operatorname{Spec} A$ be open. If U is nonempty, there is non-nilpotent $g \in A$ s.t. $D(g) \subseteq U$. Hence $D(\varphi(g)) = f^{-1}D(g) \subseteq f^{-1}U$. Since φ is injective, $\varphi(g)$ is also non-nilpotent, so $D(\varphi(g))$ — and hence $f^{-1}U$ — is nonempty. Let $t \in \mathcal{O}_{\operatorname{Spec} A}(U)$ be given, $t : U \longrightarrow \bigcup_{\mathfrak{p} \in U} A_{\mathfrak{p}}$, and say $f^{\#}|_U(t) = 0$. That is, for all $\mathfrak{q} \in f^{-1}U$,

$$\varphi_{\mathfrak{q}}(t(\varphi^{-1}\mathfrak{q})) = 0$$

We show $t = 0$ by showing that $t(\mathfrak{p}) = 0 \ \forall \mathfrak{p} \in U$. Let $\mathfrak{p} \in U$ be given. Since t is regular there is an open neighborhood V of $\mathfrak{p}$ in U and $a, s \in A$ with $s \notin \mathfrak{n} \ \forall \mathfrak{n} \in V$ and $t(\mathfrak{n}) = a/s \in A_{\mathfrak{n}} \ \forall \mathfrak{n} \in V$. Then for $\mathfrak{q} \in f^{-1}V$ we have $\varphi(a)/\varphi(s) = \varphi_{\mathfrak{q}}(a/s) = \varphi_{\mathfrak{q}}(t(\varphi^{-1}\mathfrak{q})) = 0$ in $B_{\mathfrak{q}}$.



So there is $t \notin \mathfrak{q}$ with $t\varphi(a) = 0$ in B. Let $m \in A$ be s.t. $\mathfrak{p} \in D(m) \subseteq V$. Then $D(\varphi(m)) \subseteq f^{-1}V$ and so for all $\mathfrak{q} \in \operatorname{Spec} B$ if $\varphi(m) \notin \mathfrak{q}$, there is $t \notin \mathfrak{q}$ with $t\varphi(a) = 0$. That is, $\operatorname{Ann}(\varphi(a)) \subseteq \mathfrak{q} \Rightarrow \varphi(m) \in \mathfrak{q}$. Hence $\varphi(m) \in r(\operatorname{Ann}(\varphi(a)))$, say $\varphi(m)^n \varphi(a) = 0$. Since φ is injective, $m^n a = 0$. Since $m \notin \mathfrak{p}$, $m^n \notin \mathfrak{p}$ and it follows that $a/s = 0$ in $A_{\mathfrak{p}}$, so $t(\mathfrak{p}) = 0$, as required.

The converse is trivial, since if $f^{\#}$ is injective, in particular $\mathcal{O}_{\operatorname{Spec} A}(\operatorname{Spec} A) \longrightarrow \mathcal{O}_{\operatorname{Spec} B}(\operatorname{Spec} B)$ is injective. But this is just φ (up to isomorphism), so φ is injective.

Suppose that φ is injective. Then

$$f(\operatorname{Spec} B) = \{ \varphi^{-1}\mathfrak{q} \mid \mathfrak{q} \in \operatorname{Spec} B \}$$

The closure in $\operatorname{Spec} A$ of this set is $V(\mathfrak{a})$, where

$$\begin{aligned} \mathfrak{a} &= \textstyle\bigcap_{\mathfrak{q} \in \operatorname{Spec} B} \varphi^{-1}\mathfrak{q} \\ &= \varphi^{-1}\left(\textstyle\bigcap_{\mathfrak{q}} \mathfrak{q}\right) \\ &= \varphi^{-1}(\operatorname{nil} B) \\ &= \operatorname{nil} A \qquad (\text{since } \varphi \text{ injective}) \end{aligned}$$

Thus $V(\mathfrak{a}) = \operatorname{Spec} A$, so $f(\operatorname{Spec} B)$ is dense.

(c) If $\varphi : A \longrightarrow B$ is surjective then there is an ideal $\mathfrak{a} \subseteq A$ and a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[dr] & B \\ & A/\mathfrak{a} \arrow[u, "\cong"'] \end{tikzcd}$$

and hence a commutative diagram of schemes

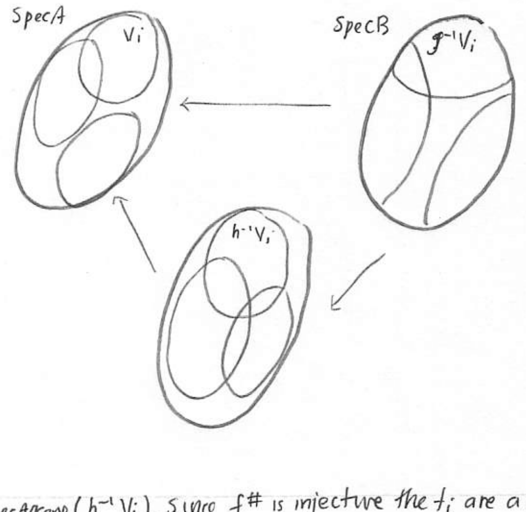
$$\begin{tikzcd} \operatorname{Spec} A & \operatorname{Spec} B \arrow[l] \arrow[d, "\cong"] \\ & \operatorname{Spec} A/\mathfrak{a} \arrow[ul] \end{tikzcd}$$

We check in Example 3.2.3 that $\operatorname{Spec} A/\mathfrak{a} \longrightarrow \operatorname{Spec} A$ is a closed immersion, and upon composing with the isomorphism $\operatorname{Spec} A/\mathfrak{a} \cong \operatorname{Spec} B$ we see that $\operatorname{Spec} B \longrightarrow \operatorname{Spec} A$ is also a closed immersion.

(d) We now prove the converse to (c). Namely, if $\operatorname{Spec}\varphi : \operatorname{Spec} B \longrightarrow \operatorname{Spec} A$ is a closed immersion, then φ is surjective. Consider the commutative diagram of schemes:

$$\begin{tikzcd} \operatorname{Spec} A & & \operatorname{Spec} B \arrow[ll, "\operatorname{Spec}\varphi"'] \arrow[dl, "f"] \\ & \operatorname{Spec} A/\operatorname{Ker}\varphi \arrow[ul, "h"] & \end{tikzcd}$$

Since $A/\operatorname{Ker}\varphi \longrightarrow B$ is injective, $f^{\#} : \mathcal{O}_{\operatorname{Spec} A/\operatorname{Ker}\varphi} \longrightarrow f_* \mathcal{O}_{\operatorname{Spec} B}$ is injective by (b). Let $b \in B$ be given, and denote by $\tilde{b}$ the corresponding global section of $\operatorname{Spec} B$. Since $\operatorname{Spec}\varphi$ is a closed immersion, there is an open cover $\operatorname{Spec} A = \bigcup_i V_i$ and sections $s_i \in \mathcal{O}_{\operatorname{Spec} A}(V_i)$ s.t. $\tilde{b}|_{g^{-1}V_i} = g^{\#}|_{V_i}(s_i)$ where $\operatorname{Spec}\varphi = (g, g^{\#})$.



Let $t_i = h^{\#}|_{V_i}(s_i) \in \mathcal{O}_{\operatorname{Spec} A/\operatorname{Ker}\varphi}(h^{-1}V_i)$. Since $f^{\#}$ is injective the t_i are a matching family, so there is $t \in \mathcal{O}_{\operatorname{Spec} A/\operatorname{Ker}\varphi}(\operatorname{Spec} A/\operatorname{Ker}\varphi)$ with $t|_{h^{-1}V_i} = t_i$ $\forall i$. Hence there is $t' \in A/\operatorname{Ker}\varphi$ mapping to $b \in B$, showing that φ is surjective.

(e) Suppose A, B are reduced $\varphi : B \longrightarrow A$ a morphism of rings s.t. the induced $f : \operatorname{Spec} A \longrightarrow \operatorname{Spec} B$ is dominant. Then φ is injective, since for $b \in B$

$$\begin{aligned} \varphi(b) = 0 \quad &\text{iff. } \varphi(b) \in \mathfrak{p} \ \ \forall \mathfrak{p} \in A \\ &\text{iff. } b \in \varphi^{-1}\mathfrak{p} \ \ \forall \mathfrak{p} \in A \\ &\text{iff. } b \in f(\mathfrak{p}) \ \ \forall \mathfrak{p} \in A \\ &\text{iff. } b \in \mathfrak{a} \text{ where } V(\mathfrak{a}) = \overline{f(\operatorname{Spec} A)} = \operatorname{Spec} B \end{aligned}$$

Hence $\mathfrak{a} = 0$, so $b = 0$.