3. First Properties of Schemes

In this section we will give some of the first properties of schemes. In particular, we will discuss open and closed subschemes, and products of schemes. In the exercises we introduce the notion of constructible subsets, and study the dimension of the fibres of a morphism.

DEFINITION A scheme is *connected* if its topological space is connected. A scheme is *irreducible* if its topological space is irreducible.

DEFINITION A scheme X is *reduced* if for every open set U, the ring $\mathcal{O}_X(U)$ has no nilpotent elements. Equivalently (Ex 2.3) X is reduced if and only if the local rings $\mathcal{O}_P$, for all $P \in X$, have no nilpotent elements. By convention the zero scheme is reduced.

DEFINITION A scheme X is *integral* if for every open set $\emptyset \neq U \subseteq X$ the ring $\mathcal{O}_X(U)$ is an integral domain (in particular $\mathcal{O}_X(U) \neq 0$ whenever $U \neq \emptyset$). By convention the zero scheme is not integral.

EXAMPLE 3.0.1 If $X = \operatorname{Spec} A$ is an affine scheme, then X is irreducible if and only if the nilradical $\mathfrak{N}$ of A is prime, since $\operatorname{Spec} A$ is irreducible $\Leftrightarrow$ $\forall \mathfrak{a}, \mathfrak{b}$ $V(\mathfrak{a}) \cup V(\mathfrak{b}) = \operatorname{Spec} A \Rightarrow V(\mathfrak{a}) = \operatorname{Spec} A$ or $V(\mathfrak{b}) = \operatorname{Spec} A$. But $\mathfrak{p} \supseteq \mathfrak{a}$ or $\mathfrak{p} \supseteq \mathfrak{b}$ iff. $\mathfrak{p} \supseteq \mathfrak{ab}$, so this says that $\forall \mathfrak{p}, \mathfrak{p} \supseteq \mathfrak{ab} \Rightarrow \forall \mathfrak{p}\ \mathfrak{p} \supseteq \mathfrak{a}$ or $\forall \mathfrak{p}\ \mathfrak{p} \supseteq \mathfrak{b}$. That is, $\mathfrak{N} \supseteq \mathfrak{ab} \Rightarrow \mathfrak{N} \supseteq \mathfrak{a}$ or $\mathfrak{N} \supseteq \mathfrak{b}$. Hence $\operatorname{Spec} A$ is irred. iff. $\mathfrak{N}$ is prime. (Even true for $A = 0$)

If $X = \operatorname{Spec} A$ then X is reduced if and only if $\mathfrak{N} = 0$. Since $A \cong \mathcal{O}_{\operatorname{Spec} A}(X)$ this condition is clearly necessary. If $\mathfrak{N} = 0$ then for every $\mathfrak{p} \in \operatorname{Spec} A$, $A_\mathfrak{p}$ is nilpotent free, so X is reduced. (0 is reduced, so holds $A = 0$)

If $X = \operatorname{Spec} A$ then X is integral if and only if A is an integral domain. This is trivial if $A = 0$. For $A \neq 0$ A is a domain $\Leftrightarrow$ 0 is prime $\Leftrightarrow$ $\mathfrak{N} = 0$ and $\mathfrak{N}$ is prime $\Leftrightarrow$ X is reduced and irreducible. So it suffices to show that a scheme X is integral if and only if it is both reduced and irreducible.

PROPOSITION 3.1 A scheme is integral if and only if it is both reduced and irreducible.

PROOF The zero scheme $(\emptyset, 0)$ is not integral, and is reduced but not irreducible. So the Proposition is true in this case. If X is a nonzero scheme, then if X is integral it is clearly reduced. If X is not irreducible, there are nonempty disjoint open $U_1, U_2 \subseteq X$. Then $\mathcal{O}(U_1 \cup U_2) \cong \mathcal{O}(U_1) \times \mathcal{O}(U_2)$ which is not an integral domain. Thus integral implies irreducible.

Conversely let X be a nonzero, reduced, irreducible scheme. Let $U \subseteq X$ be open and nonempty, and $f, g \in \mathcal{O}(U)$ with $fg = 0$. Let $Y = \{x \in U \mid f_x \in \mathfrak{m}_x\}$ and $Z = \{x \in U \mid g_x \in \mathfrak{m}_x\}$. By Ex 2.16a Y, Z are closed in the subspace topology on U. By assumption $Y \cup Z = U$. Since X is irreducible, so is U, so one of Y, Z is equal to U. Say $Y = U$. But then the restriction of f to any affine open subset of U will be nilpotent (Ex 2.18a), hence zero, so f is zero. This shows that X is integral. $\square$

nonempty cover. Equiv. every point has open affine neighborhood $\cong \operatorname{Spec} A$, A noetherian,

DEFINITION A scheme X is *locally noetherian* if it can be covered by open affine subsets $\operatorname{Spec} A_i$ where each A_i is a noetherian ring. X is *noetherian* if it is locally noetherian and quasi-compact. Equivalently, X is noetherian if it can be covered by a finite number of open affine subsets $\operatorname{Spec} A_i$ with each A_i a noetherian ring. (The spectrum of an noetherian ring is quasi-compact and a finite union of q.c. spaces is q.c.)

COROLLARY 3.1.1 If X is a noetherian scheme, then $sp(X)$ is a noetherian topological space, but not conversely (Ex 2.1.3 and 3.17) (see also Ex 2.5 of Ch. 1)

NOTE The zero scheme $(\emptyset, 0)$ is noetherian. Better def^N: X is locally noetherian iff. every point has an affine open neighborhood $\cong \operatorname{Spec} A$, A noetherian. This is equivalent to above (N.B. in the original def^N, doesn't matter if $\{\}$ is a cover)

Note that in this definition we do not require that every open subset which is affine be the spectrum of a noetherian ring. So while it is obvious from the definition that the spectrum of a noetherian ring is a noetherian scheme, the converse is not obvious. It is a question of showing that the noetherian property is a "local property". We will often encounter similar situations later in defining properties of a scheme or a morphism of schemes, so we will give a careful statement and proof of the local nature of the noetherian property, to illustrate this type of situation.

NOTE If X is integral then all the local rings $\mathcal{O}_{X,x}$ are domains ($x \in U = \operatorname{Spec} A$, $A_\mathfrak{p}$ domain)
If X is locally noetherian the local rings $\mathcal{O}_{X,x}$ are noetherian ($x \in U = \operatorname{Spec} A$, A noetherian $\Rightarrow A_\mathfrak{p}$ noetherian)

PROPOSITION 3.2 A scheme X is locally noetherian if and only if for every open affine subset $U = \operatorname{Spec} A$, A is a noetherian ring. In particular, an affine scheme $X = \operatorname{Spec} A$ is a noetherian scheme if and only if the ring A is a noetherian ring.

PROOF The conditions are equivalent for zero schemes, so assume $X \neq \emptyset$ throughout. The "if" part follows from the definition, so we have to show that if X is locally noetherian, and if $U = \operatorname{Spec} A$ is an open affine subset then A is a noetherian ring. First note that if B is a noetherian ring, so is any localisation B_f. The open subsets $D(f) \cong \operatorname{Spec} B_f$ form a base for the topology of $\operatorname{Spec} B$. Hence on a locally noetherian scheme X there is a base for the topology consisting of the spectra of noetherian rings. In particular, our open set U can be covered by spectra of noetherian rings.

So we have reduced to proving the following statement: let $X = \operatorname{Spec} A$ be an affine scheme, which can be covered by open subsets which are spectra of noetherian rings. Then A is noetherian. Let $U = \operatorname{Spec} B$ be an open subset of X, with B noetherian. Then for some $f \in A$, $D(f) \subseteq U$. Let $\bar{f}$ be the image of f in B. One checks that under $U \cong \operatorname{Spec} B$, $D(f) \cong D(\bar{f})$, so in particular $A_f \cong B_{\bar{f}}$, hence A_f is noetherian. So we can cover X by open subsets $D(f) \cong \operatorname{Spec} A_f$ with A_f noetherian. Since X is quasi-compact, a finite number will do.

So we have now reduced to a purely algebraic problem: A is a ring, $f_1, \dots, f_r$ are a finite number of elements of A, which generate the unit ideal, and each localisation A_{f_i} is noetherian. We have to show that A is noetherian. First we establish a lemma. Let $\mathfrak{a} \subseteq A$ be an ideal, and let $\varphi_i : A \longrightarrow A_{f_i}$ be the localisation map $i = 1, \dots, r$. Then

$$\mathfrak{a} = \bigcap \varphi_i^{-1}(\varphi_i(\mathfrak{a}) A_{f_i})$$

The inclusion $\subseteq$ is obvious. Conversely, given an element $b \in A$ in this intersection, we can write $\varphi_i(b) = a_i / f_i^{n_i}$ in A_{f_i} for each i, where $a_i \in \mathfrak{a}$ and $n_i > 0$. Increasing the n_i if necessary, we can make them all equal to a fixed n. This means that in A we have

$$f_i^{m_i}(f_i^n b - a_i) = 0$$

for some m_i. And as before, we can make all the $m_i = m$. Thus $f_i^{m+n} b \in \mathfrak{a}$ for each i. Since $f_1, \dots, f_r$ generate the unit ideal, the same is true of their Nth powers for any N. ($\{f_i\}$ generate (1) iff. $\bigcup D(f_i) = \operatorname{Spec} A$. But $D(f_i) = D(f_i^N)$ any $N > 0$). Take $N = n + m$. Then we have $1 = \sum c_i f_i^N$ for suitable $c_i \in A$. Hence $b = \sum c_i f_i^N b \in \mathfrak{a}$, as required.

Now we can easily show that A is noetherian. Let $\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \cdots$ be an ascending chain of ideals in A. Then for each i

$$\varphi_i(\mathfrak{a}_1) A_{f_i} \subseteq \varphi_i(\mathfrak{a}_2) A_{f_i} \subseteq \cdots$$

is an ascending chain of ideals in A_{f_i}, which must become stationary because A_{f_i} is noetherian. Since there are only finite many A_{f_i} we can find k so large that for $i = 1, \dots, r$ $\varphi_i(\mathfrak{a}_k) A_{f_i} = \varphi_i(\mathfrak{a}_{k+1}) A_{f_i} = \cdots$ But then by the Lemma $\mathfrak{a}_k = \bigcap \varphi_i^{-1}(\varphi_i(\mathfrak{a}_k) A_{f_i}) = \bigcap \varphi_i^{-1}(\varphi_i(\mathfrak{a}_{k+1}) A_{f_i}) = \mathfrak{a}_{k+1} = \cdots$ so the original chain is eventually stationary, and hence A is noetherian. $\square$

DEFINITION A morphism $f : X \longrightarrow Y$ of schemes is locally of finite type if there exists a covering of Y by open affine subsets $V_i = \operatorname{Spec} B_i$ such that for each i $f^{-1}(V_i)$ can be covered by open affine subsets $U_{ij} = \operatorname{Spec} A_{ij}$, where each A_{ij} is a finitely generated B_i-algebra (with the natural structure induced by $B_i \cong \mathcal{O}_Y(V_i) \longrightarrow \mathcal{O}_X(f^{-1}V_i) \longrightarrow \mathcal{O}_X(U_{ij}) \cong A_{ij}$). The morphism f is of finite type if in addition each $f^{-1}V_i$ can be covered by a finite number of the U_{ij}.

Coverings means a nonempty set of open sets — no empty covers,

Since 0 is trivially f.g. over any ring, we need only check the condition for nonempty $f^{-1}V_i$ and U_{ij}. In particular any morphism $f : (\emptyset, 0) \longrightarrow Y$ is of finite type, as is any morphism $g : X \longrightarrow (\emptyset, 0)$ (since then X is also zero).

DEFINITION A morphism $f : X \longrightarrow Y$ is finite if there is a covering of Y by open affine subsets $V_i = \operatorname{Spec} B_i$, such that for each i, $f^{-1}(V_i)$ is affine, equal to $\operatorname{Spec} A_i$, where A_i is a B_i-algebra which is a finitely generated B_i-module (with the canonical structure).

All three finiteness conditions are closed under isomorphisms on either end. Clearly an isomorphism is finite. Once again we need only check for nonempty $f^{-1}V_i$. In particular any morphism $f : (\emptyset, 0) \longrightarrow Y$ is finite, as is any morphism $e : X \longrightarrow (\emptyset, 0)$. The definitions above are also independent of which A_i (resp. A_{ij}) we choose to make $f^{-1}V_i \cong \operatorname{Spec} A_i$ (resp. $U_{ij} \cong \operatorname{Spec} A_{ij}$). Throughout we can also assume no V_i is empty. Clearly

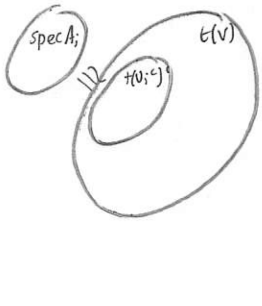
$$f : X \longrightarrow Y$$

$$\text{finite} \Longrightarrow \text{finite type} \Longrightarrow \text{locally of finite type}$$

In the above we can remove references to a particular B_i or A_{ij} by saying " $\mathcal{O}_X(U_{ij})$ is a f.g. $\mathcal{O}_Y(V_i)$-algebra"

Note in each of these definitions that a property of a morphism $f: X \longrightarrow Y$ is defined by the existence of an open affine cover of Y with certain properties. In fact in each case it is equivalent to require the given property for every open affine subset of Y (Ex 3.1–3.4).

<u>EXAMPLE 3.2.1</u> If V is a variety over an algebraically closed field k, then the associated scheme $t(V)$ (see 2.6) is an integral noetherian scheme of finite type over k. Indeed the structure sheaf is $\alpha_* \mathcal{O}_V$ where $\alpha: V \longrightarrow t(V)$, and $\mathcal{O}_V(U)$ is a domain $\forall\, \emptyset \neq U \subseteq V$ open, so $t(V)$ is integral. (if $Q \subseteq t(V)$ is nonempty, so is $\alpha^{-1}Q$). By (I,4.3) V can be covered by a finite number of open affine varieties, and as in the proof of Prop 2.6 this implies that $t(V)$ can be covered by a finite number of open affines of the form $\operatorname{Spec} A_i$, where each A_i is an integral domain which is a finitely generated k-algebra and hence noetherian. So the scheme $t(V)$ is noetherian.



Moreover under the isomorphism

$$\begin{array}{c} (t(U_i^c)^c, \alpha_* \mathcal{O}_V|_{t(U_i^c)^c}) \cong (t(U_i), \alpha_{i*}\mathcal{O}_{U_i}) \\ \cong \\ (t(Y), \alpha_* \mathcal{O}_Y) \\ \cong \\ \operatorname{Spec} A_i \end{array}$$

of Prop 2.6 it is straightforward to check that the map $\mathfrak{p} \mapsto e/1$ of $\mathcal{O}(\operatorname{Spec} A_i)$ is identified with the map $P \mapsto e$ which belongs to $\mathcal{O}_V(U_i) = (\alpha_* \mathcal{O}_V)(t(U_i^c)^c)$.

The morphism $t(V) \longrightarrow \operatorname{Spec} k$ is $k \longrightarrow \mathcal{O}_V(V)$, $e \longmapsto (P \mapsto e)$, so when we make $\mathcal{O}_V(U_i)$ into a k-algebra the isomorphism $A_i \cong \mathcal{O}(\operatorname{Spec} A_i) \cong \mathcal{O}_V(U_i)$ preserves this k-algebra structure. Since A_i is a f.g. k-algebra it follows that $t(V) \longrightarrow \operatorname{Spec} k$ has finite type, as required.

<u>EXAMPLE 3.2.2</u> If P is a point of a variety V, with local ring $\mathcal{O}_P$, then $\operatorname{Spec} \mathcal{O}_P$ is an integral noetherian scheme over k, which is not in general of finite type over k.

Next we come to open and closed subschemes.

<u>DEFINITION</u> An <u>open subscheme</u> of a scheme X is a scheme U, whose topological space is an open subset of X, and whose structure sheaf $\mathcal{O}_U$ is isomorphic to the restriction $\mathcal{O}_X|_U$ of the structure sheaf of X. An <u>open immersion</u> is a morphism $f: X \longrightarrow Y$ which induces an isomorphism of X with an open subscheme of Y. That is,

- f is injective and gives a homeomorphism $X \cong f(X)$, where $f(X)$ is open in Y.
- The induced morphism of schemes $X \longrightarrow (f(X), \mathcal{O}_Y|_{f(X)})$ is an isomorphism.

For any zero scheme $(\emptyset, 0) \longrightarrow Y$ is always an open immersion, as is the identity on Y. Any isomorphism of schemes $X \longrightarrow Y$ is an open immersion. The composite of open immersions are open immersions.

<u>NOTE</u> An open immersion $X \longrightarrow Y$ can be recovered from the isomorphism $X \cong f(X)$, since for $V \subseteq Y$ open $\mathcal{O}_Y(V) \longrightarrow \mathcal{O}_X(f^{-1}V)$ is $\mathcal{O}_Y(V) \longrightarrow \mathcal{O}_Y(V \cap f(X)) \longrightarrow \mathcal{O}_X(f^{-1}V)$. Moreover, given two schemes X, Y and an isomorphism of schemes $X \overset{g}{\cong} (U, \mathcal{O}_Y|_U)$ with $U \subseteq Y$ open, there is a unique morphism of schemes $X \longrightarrow Y$ which is an open immersion and induces the given isomorphism. For $V \subseteq Y$ we define $\mathcal{O}_Y(V) \longrightarrow \mathcal{O}_X(g^{-1}(U \cap V))$ to be the composite $\mathcal{O}_Y(V) \longrightarrow \mathcal{O}_Y(V \cap U) \longrightarrow \mathcal{O}_X(g^{-1}(V \cap U))$. Hence to give an open immersion $f: X \longrightarrow Y$ is precisely equivalent to giving an isomorphism of X with $(U, \mathcal{O}_Y|_U)$ as schemes for some open $U \subseteq Y$, and in this case f takes the form given above.

<u>DEFINITION</u> A <u>closed immersion</u> is a morphism $f: Y \longrightarrow X$ of schemes such that f induces a homeomorphism of $sp(Y)$ onto a closed subset of $sp(X)$, and furthermore the induced map $f^\#: \mathcal{O}_X \longrightarrow f_* \mathcal{O}_Y$ is an epimorphism of sheaves of abelian groups (i.e. is locally surjective). A <u>closed subscheme</u> of a scheme X is an equivalence class of closed immersions, where we say $f: Y \longrightarrow X$ and $f': Y' \longrightarrow X$ are equivalent if there is an isomorphism $i: Y' \longrightarrow Y$ such that $f' = f \circ i$. (The closed subschemes – i.e. the collection of equivalence classes – could be a class).

<u>NOTE</u> $(\emptyset, 0) \longrightarrow X$ is always a closed immersion, as is the identity on X. Any isomorphism of schemes $Y \longrightarrow X$ is a closed immersion. If $X \longrightarrow Y$ is a closed immersion and $X' \xrightarrow{\sim} X$ an isomorphism, the composite $X' \longrightarrow Y$ is a closed immersion. Similarly for an iso $Y \longrightarrow Y'$.

NOTE Closed Immersions and Closed Subschemes

Let X be a scheme and $Y \subseteq X$ a closed subset. Suppose $\mathcal{O}_Y$ is a sheaf of rings on Y which is a scheme, and suppose the inclusion $i: Y \longrightarrow X$ can be paired with a morphism of sheaves of rings $i^{\#}: \mathcal{O}_X \longrightarrow i_* \mathcal{O}_Y$ which is locally surjective and so that $(i, i^{\#})$ is a morphism of schemes. Then $i: Y \longrightarrow X$ is clearly a closed immersion, and in fact a closed subscheme structure on Y.

Conversely if $f: Z \longrightarrow X$ is a closed subscheme then put $Y = f(Z)$. There is an induced scheme structure $\mathcal{O}_Y$ on Y and an isomorphism of schemes $Y \Rightarrow Z$. Let $i: Y \longrightarrow X$ be the composite $Y \Rightarrow Z \xrightarrow{f} X$. The underlying map of i is the inclusion, and it is not hard to check that i is a closed immersion. The diagram

$$\begin{tikzcd} & X \\ Z \arrow[ur, "f"] & Y \arrow[l, Leftarrow] \arrow[u, "i"'] \end{tikzcd}$$

shows that i, f are actually the same closed subscheme. So any closed subscheme can be represented by a closed immersion of the form above.

NOTE Noetherian Schemes

If X is a locally noetherian scheme and $U \subseteq X$ open, then by (3.2) $(U, \mathcal{O}_X|_U)$ is also a locally noetherian scheme. Since every open subset of a noetherian space is quasi-compact, an open subset of a noetherian scheme is also noetherian.

NOTE Local Properties

A property P of morphisms is <u>local on the codomain</u> if whenever $f: X \longrightarrow Y$ has P, for any open $V \subseteq Y$, $f^{-1}V \longrightarrow V$ has P, and if whenever there is a cover of Y by open sets $\{V_i\}$ s.t. for each i $f^{-1}V_i \longrightarrow V_i$ has P, then f has P. As we will see later, being a closed immersion is a local property on the codomain.

NOTE Open Immersions

A morphism $f: X \longrightarrow Y$ is an open immersion if and only if f gives a homeomorphism of X with an open subset of Y, and if for all $x \in X$ the morphism $f_x: \mathcal{O}_{Y, f(x)} \longrightarrow \mathcal{O}_{X,x}$ is bijective.

NOTE $\mathbb{P}^n_A \longrightarrow \operatorname{Spec} A$ has finite type. $(n \geq 0)$ and is integral if A is.

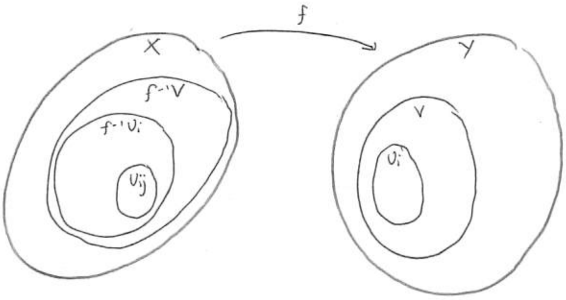
Since $\mathbb{P}^n_A = \operatorname{Proj} A[x_0, \dots, x_n]$ is covered by affine opens $D_+(x_i) \cong \operatorname{Spec} A[x_1, \dots, x_n]$ (if $n = 0$ $\mathbb{P}^n_A \longrightarrow \operatorname{Spec} A$ is an iso $\therefore$ f.t.) and $A[x_1, \dots, x_n]$ is clearly a f.g. A-algebra. Further if A is an integral domain then 0 is a homogeneous prime and thus $\mathbb{P}^n_A$ has a generic point, which implies that $\mathbb{P}^n_A$ is irreducible. (In fact for any graded domain S, $\operatorname{Proj} S$ is irreducible). Since $\mathbb{P}^n_A$ is covered by the schemes $\operatorname{Spec} A[x_1, \dots, x_n]$ it is also reduced (since this is a stalk-wise property). Hence by (3.1) if A is a domain, $\mathbb{P}^n_A$ is integral $\forall n \geq 0$.

NOTE FINITENESS IS LOCAL

We claim the following:

PROPOSITION If a morphism $f: X \longrightarrow Y$ is locally of finite type (of finite type, finite) then for any $V \subseteq Y$ open, $f^{-1}V \longrightarrow V$ has the same property. If $f: X \longrightarrow Y$ is a morphism and there is an open cover (nonempty) $\{V_i\}$ of Y such that $f^{-1}V_i \longrightarrow V_i$ is locally of finite type (of finite type, finite) for each i, then f has the same property.

Suppose f is locally of finite type and let $V \subseteq Y$ be open. Let $\{U_i\}$ be an affine open cover of V. By Ex 3.1 for each i $f^{-1}U_i$ is covered by affine open sets U_{ij} which are f.g. over $\mathcal{O}_Y(U_i)$.

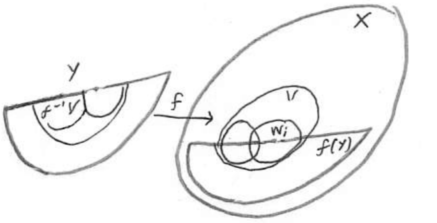


It follows that $f|_{f^{-1}V}: f^{-1}V \longrightarrow V$ is locally of finite type. Using Ex 3.3 and Ex 3.4 we see that the relevant statements for finite type and finite also hold.

The converse statements, where the finiteness of f follows from its local finiteness, are easily checked.

NOTE It is clear that if $V \subseteq Y$, and $f: X \longrightarrow Y$ is quasi-compact (Ex 3.2) so is $f^{-1}V \longrightarrow Y$.

NOTE If X is a topological space and F a sheaf of rings on X, and if $Z \subseteq X$ is closed, there is a sheaf of rings $F|_Z$ on Z made by locally patching restrictions of sections from X (see earlier in our notes). The local surjectivity condition on $f^{\#}: \mathcal{O}_X \longrightarrow f_* \mathcal{O}_Y$ means that for $V \subseteq Y$ open and $s \in \mathcal{O}_Y(f^{-1}V)$ there is an open cover $V = \cup_i W_i$ and $t_i \in \mathcal{O}_X(W_i)$ s.t. $\forall i$ $s|_{f^{-1}W_i} = f^{\#}(t_i)$. That is, s is locally a restriction (thinking of Y as a closed subset of X)



if $\mathfrak{a} = A$ is trivial
same if $\mathfrak{a} = 0$, so assume $0 \neq \mathfrak{a} \subset A$.

EXAMPLE 3.2.3 Let A be a ring, and let $\mathfrak{a}$ be an ideal of A. Let $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} A/\mathfrak{a}$. The ring morphism induces $f: Y \longrightarrow X$ which is a closed immersion. The map f is a homeomorphism of Y onto the closed subset $V(\mathfrak{a})$ of X, and the map of structure sheaves $\mathcal{O}_X \longrightarrow f_* \mathcal{O}_Y$ is an epimorphism of sheaves of abelian groups. To see this it suffices to show $(\mathcal{O}_X)_{\mathfrak{p}} \longrightarrow (f_* \mathcal{O}_Y)_{\mathfrak{p}}$ is surjective for all $\mathfrak{p} \in \operatorname{Spec} A$. But $(f_* \mathcal{O}_Y)_{\mathfrak{p}} \cong \mathcal{O}_{\operatorname{Spec} A/\mathfrak{a}, \mathfrak{p}}$ in such a way to make

$$\begin{tikzcd}
(\mathcal{O}_X)_{\mathfrak{p}} \arrow[r] \arrow[d, phantom, "\cong"] & (f_* \mathcal{O}_Y)_{\mathfrak{p}} \arrow[r, phantom, "\cong"] & \mathcal{O}_{\operatorname{Spec} A/\mathfrak{a}, \mathfrak{p}} \arrow[d, phantom, "\cong"] \\
A_{\mathfrak{p}} \arrow[rr, "a/f \mapsto (a+\mathfrak{a}, f+\mathfrak{a})"'] & & (A/\mathfrak{a})_{\mathfrak{p}}
\end{tikzcd}$$

commute. Since the bottom morphism is surjective, we have proved that $\operatorname{Spec} A/\mathfrak{a} \longrightarrow \operatorname{Spec} A$ is a closed immersion.

EXAMPLE 3.2.4 For some more specific examples, let $A = k[x,y]$, where k is a field. Then $\operatorname{Spec} A = \mathbb{A}^2_k$ is the affine plane over k. The ideal $\mathfrak{a} = (xy)$ gives a reducible subscheme, consisting of the union of the x and y-axis. The ideal $\mathfrak{a} = (x^2)$ gives a subscheme structure with nilpotents on the y-axis. The ideal $\mathfrak{a} = (x^2, xy)$ gives another subscheme structure on the y-axis, this one having nilpotents only in the local ring at the origin, since for $\mathfrak{p} \supseteq \mathfrak{a}$

$$\operatorname{nil}(A/\mathfrak{a})_{\mathfrak{p}} = (\operatorname{nil} A/\mathfrak{a})_{\mathfrak{p}} = (r(\mathfrak{a}) + \mathfrak{a}/\mathfrak{a})_{\mathfrak{p}}$$

But $\mathfrak{p} \supseteq (x^2, xy)$ iff. $\mathfrak{p} \supseteq (x)$, so $r(x^2, xy) = r(x) = (x)$. If $xf(x,y) + \mathfrak{a}/s + \mathfrak{a} \in \operatorname{nil}(A/\mathfrak{a})_{\mathfrak{p}}$ and $y \notin \mathfrak{p}$ then $yxf(x,y) \in \mathfrak{a}$, so provided $\mathfrak{p} \neq (x,y)$ $(A/\mathfrak{a})_{\mathfrak{p}}$ is nilpotent free. We say the origin is an <u>embedded point</u> for this subscheme.

EXAMPLE 3.2.5 Let V be an affine variety over the field k, and let W be a closed subvariety. Then W corresponds to a prime ideal $\mathfrak{p}$ in the affine coordinate ring A of V. Let $X = t(V)$ and $Y = t(W)$ be the associated schemes. Then $X \cong \operatorname{Spec} A$ and Y is the closed subscheme defined by $\mathfrak{p}$. To be precise, the coordinate ring of W is isomorphic to $A/\mathfrak{p}$, so $t(W) \cong \operatorname{Spec} A/\mathfrak{p}$. The morphism $\operatorname{Spec} A/\mathfrak{p} \longrightarrow \operatorname{Spec} A$ is a closed immersion, hence so is the composite

$$t(W) \xrightarrow{\sim} \operatorname{Spec} A/\mathfrak{p} \longrightarrow \operatorname{Spec} A \xrightarrow{\sim} t(V)$$

For each $n \geq 1$ let Y_n be the closed subscheme of X corresponding to the ideal $\mathfrak{p}^n$. Then $Y_1 = Y$ and the images (i.e. $\operatorname{Spec} A/\mathfrak{p}^n \longrightarrow \operatorname{Spec} A$) are all the same, but for $n > 1$, Y_n is a nonreduced scheme and so the subscheme structure on Y does not correspond to any subvariety of V. We call Y_n the nth <u>infinitesimal neighborhood</u> of Y in X. The schemes Y_n reflect properties of the embedding of Y in X. Later (§9) we will study the "formal" completion of Y in X, which is roughly the limit of the schemes Y_n as $n \longrightarrow \infty$.

EXAMPLE Let R be a ring, B a finitely generated R-algebra, $R \longrightarrow B$ canonical. Then $\operatorname{Spec} B \longrightarrow \operatorname{Spec} R$ is a morphism of finite type, since $\mathcal{O}_{\operatorname{Spec} B}(\operatorname{Spec} B) \longleftarrow \mathcal{O}_{\operatorname{Spec} R}(\operatorname{Spec} R)$ is $(\mathfrak{p} \mapsto r/1) \mapsto (\mathfrak{q} \mapsto r \cdot 1/1)$.

NOTE Included Reduced Schemes

Let X be a topological space, $\mathcal{O}_X$ a sheaf of rings on X and $\mathcal{I}$ a sheaf of ideals of $\mathcal{O}_X$. The $\mathcal{O}_X$-module $\mathcal{O}_X/\mathcal{I}$ is the sheafification of the presheaf $P(U) = \mathcal{O}_X(U)/\mathcal{I}(U)$, which is a presheaf of rings. Hence $\mathcal{O}_X/\mathcal{I}$ is a sheaf of rings. Since $0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_X \xrightarrow{\alpha} \mathcal{O}_X/\mathcal{I} \longrightarrow 0$ is exact we get an exact sequence

$$0 \longrightarrow \mathcal{I}_x \longrightarrow \mathcal{O}_{X,x} \xrightarrow{\alpha_x} (\mathcal{O}_X/\mathcal{I})_x \longrightarrow 0$$

So there is an isomorphism of rings $(\mathcal{O}_X/\mathcal{I})_x \cong \mathcal{O}_{X,x}/\mathcal{I}_x$, since $\mathcal{I}_x$ is an ideal of $\mathcal{O}_{X,x}$ consisting of all (U,a) with $a \in \mathcal{I}(U) \subseteq \mathcal{O}_X(U)$. This isomorphism is defined by $\mathcal{O}_{X,x}/\mathcal{I}_x \to (\mathcal{O}_X/\mathcal{I})_x$, $a + \mathcal{I}_x \mapsto \alpha_x(a)$, and α_U is the composite $\mathcal{O}_X(U) \longrightarrow \mathcal{O}_X(U)/\mathcal{I}(U) \longrightarrow (\mathcal{O}_X/\mathcal{I})(U)$.

Now suppose $\mathcal{O}_X$ is locally ringed, $Y \subseteq X$ closed and for $y \in X$ let $\varphi_y : \mathcal{O}_{X,y} \longrightarrow k(y)$ be canonical. Given $f \in \mathcal{O}_X(U)$ and $y \in U$, we say f <u>vanishes</u> on y if $\varphi_y(\text{germ}_y f) = 0$. Let

$$\mathcal{I}_Y(U) = \{ f \in \mathcal{O}_X(U) \mid f \text{ vanishes on } Y \cap U \}$$

It is not difficult to check that $\mathcal{I}_Y$ is a sheaf of ideals. Let $i : Y \longrightarrow X$ be the inclusion. Then $i^{-1}(\mathcal{O}_X/\mathcal{I}_Y)$ is a sheaf of rings on Y and for $y \in Y$

$$(1) \qquad i^{-1}(\mathcal{O}_X/\mathcal{I}_Y)_y \cong (\mathcal{O}_X/\mathcal{I}_Y)_y \cong \mathcal{O}_{X,y}/\mathcal{I}_{Y,y}$$

See our Section 5 notes.

Note that $\mathcal{O}_{X,y}$ is a local ring, $k(y)$ a field, so for all $U \subseteq X$, $\mathcal{I}_Y(U)$ is a proper ideal of $\mathcal{O}_X(U)$. In particular, $\mathcal{I}_{Y,y}$ is a proper ideal of $\mathcal{O}_{X,y}$, so it follows that $i^{-1}(\mathcal{O}_X/\mathcal{I}_Y)$ is a locally ringed space.

Now let X be a scheme, $Y \subseteq X$ closed, $\mathcal{O}_Y = i^{-1}(\mathcal{O}_X/\mathcal{I}_Y)$ the locally ringed space. From (1) and Ex 2.3 $\mathcal{O}_Y$ is a reduced scheme, provided we show it is a scheme, since for any $y \in Y$, the ideal $\mathcal{I}_{Y,y} \subseteq \mathcal{O}_{X,y}$ is radical.

To show that $i^{-1}(\mathcal{O}_X/\mathcal{I}_Y)$ is a scheme we need some preliminary facts: if $X = \operatorname{Spec} A$ is affine, $Y \subseteq X$ closed and $\mathfrak{a} = \bigcap_{\mathfrak{p} \in Y} \mathfrak{p}$, so $V(\mathfrak{a}) = Y$, then for all $\mathfrak{p} \in Y$ we claim there is an isomorphism of rings

$$\varphi : A_\mathfrak{p}/\mathfrak{a}A_\mathfrak{p} \longrightarrow (\mathcal{O}_X/\mathcal{I})_\mathfrak{p}$$

We already know that $\mathcal{O}_{X,\mathfrak{p}} \xrightarrow{\theta} A_\mathfrak{p}$, $(U,s) \longmapsto s(\mathfrak{p})$ is an isomorphism of rings. We need only show that $\theta(\mathcal{I}_{Y,\mathfrak{p}}) = \mathfrak{a}A_\mathfrak{p}$. Suppose $a \in \mathfrak{a}$ and $s \notin \mathfrak{p}$. Then it is clear that $a/s \in \mathcal{I}_Y(D(s))$, so $\theta^{-1}(\mathfrak{a}A_\mathfrak{p}) \subseteq \mathcal{I}_{Y,\mathfrak{p}}$. In the other direction, suppose $\mathfrak{p} \in U$ and $s \in \mathcal{I}_Y(U)$. Let $\mathfrak{p} \in V \subseteq U$ and $a, s \in A$ $s \notin \mathfrak{q}$ $\forall \mathfrak{q} \in V$ be such that $\forall \mathfrak{q} \in V$ $s(\mathfrak{q}) = a/s \in A_\mathfrak{q}$. Since V is open and X affine, let $f \in A$ be s.t. $\mathfrak{p} \in D(f) \subseteq V$. Then f belongs to every prime in $Y - V \cap Y$ and a belongs to every prime in $V \cap Y$, so $af \in \mathfrak{a}$ and hence $a/s = s(\mathfrak{p})$ belongs to $\mathfrak{a}A_\mathfrak{p}$, as required. So we get the desired isomorphism φ:

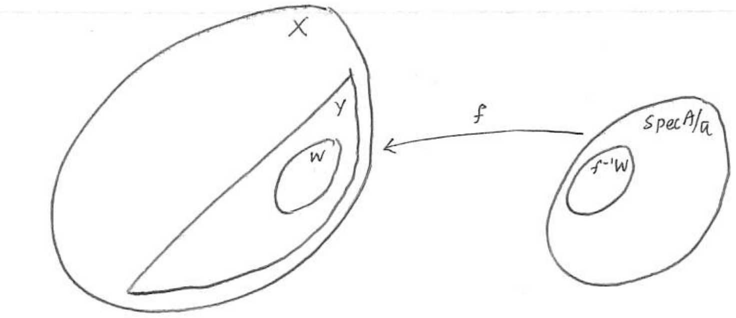
$$\begin{tikzcd}
 & \mathcal{O}_{X,\mathfrak{p}} \arrow[r, "\theta", "\sim"'] \arrow[d] & A_\mathfrak{p} \arrow[d] \\
(\mathcal{O}_X/\mathcal{I})_\mathfrak{p} \arrow[r, dashed, "\sim"] & \dfrac{\mathcal{O}_{X,\mathfrak{p}}}{\mathcal{I}_{Y,\mathfrak{p}}} \arrow[r, "\sim"] & A_\mathfrak{p}/\mathfrak{a}A_\mathfrak{p}
\end{tikzcd}$$

$$\varphi : A_\mathfrak{p}/\mathfrak{a}A_\mathfrak{p} \longrightarrow (\mathcal{O}_X/\mathcal{I})_\mathfrak{p}$$
$$a/s + \mathfrak{a}A_\mathfrak{p} \longmapsto (D(s), a/s + \mathcal{I}_Y(D(s)))$$

Hence there is an isomorphism of rings $\phi_\mathfrak{p} : (A/\mathfrak{a})_\mathfrak{p} \longrightarrow P_\mathfrak{p}$ where $P(V) = \varinjlim_{W \supseteq V} (\mathcal{O}_X/\mathcal{I}_Y)(W)$ sheafifies to give $\mathcal{O}_Y$, defined by

$$\phi_\mathfrak{p}\left(\frac{r+\mathfrak{a}}{s+\mathfrak{a}}\right) = (D(s) \cap Y, (D(s), r/s + \mathcal{I}_Y(D(s))))$$

Next we show that $\mathcal{O}_Y$ is isomorphic to structure sheaf induced on Y by the closed immersion $f : \operatorname{Spec} A/\mathfrak{a} \longrightarrow \operatorname{Spec} A$.



Let $W \subseteq Y$ be open and define $m_W : \mathcal{O}_{\operatorname{Spec} A/\mathfrak{a}}(f^{-1}W) \longrightarrow \mathcal{O}_Y(W)$ by

$$m_W(s)(\mathfrak{p}) = \phi_{\mathfrak{p}}(s(\mathfrak{p}/\mathfrak{a}))$$

It is not difficult to check that m is a morphism of sheaves of rings, and m is an isomorphism since for all $\mathfrak{p} \in Y$, the following diagram commutes:

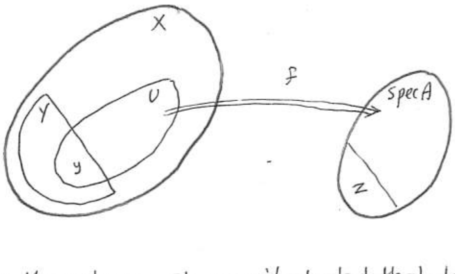
$$\begin{tikzcd}
\mathcal{O}_{\operatorname{Spec} A/\mathfrak{a},\, \mathfrak{p}/\mathfrak{a}} \arrow[r] \arrow[d, Rightarrow] & \mathcal{O}_{Y,\mathfrak{p}} \arrow[d, Rightarrow] \\
(A/\mathfrak{a})_{\mathfrak{p}} \arrow[r, "\phi_{\mathfrak{p}}"'] & P_{\mathfrak{p}}
\end{tikzcd}$$

It follows that $\mathcal{O}_Y$ is an affine scheme. So we have shown:

<u>LEMMA</u> Let X be an affine scheme, $X = \operatorname{Spec} A$ and $Y \subseteq X$ closed. If $\mathcal{J}_Y$ is the sheaf of ideals defined above, $i : Y \longrightarrow X$ the inclusion and $\mathfrak{a} = \bigcap_{\mathfrak{p} \in Y} \mathfrak{p}$ then the ringed space $\mathcal{O}_Y = (Y, i^{-1}(\mathcal{O}_X/\mathcal{J}_Y))$ is an affine scheme, isomorphic to $\operatorname{Spec} A/\mathfrak{a}$.

<u>PROPOSITION</u> Let X be a scheme, $Y \subseteq X$ closed, $i : Y \longrightarrow X$ the inclusion, and let $\mathcal{O}_Y = i^{-1}(\mathcal{O}_X/\mathcal{J}_Y)$. Then $(Y, \mathcal{O}_Y)$ is a reduced scheme,

<u>PROOF</u> We already know that $\mathcal{O}_Y$ is locally ringed. Let $y \in Y$ and let $y \in U \subseteq X$ be an affine open neighbourhood, $f : \mathcal{O}_X|_U \longrightarrow \operatorname{Spec} A$ an isomorphism.



Let $k : U \longrightarrow X$ be the inclusion. It is easily checked that $k^{-1}(\mathcal{O}_X/\mathcal{J}_Y) \cong \mathcal{O}_X|_U / \mathcal{J}_{Y \cap U}$. It is also not difficult to check that if $h : M \longrightarrow N$ is any isomorphism of schemes and $Z \subseteq M$ is closed, then $h_*(\mathcal{O}_M/\mathcal{J}_Z) \cong \mathcal{O}_N/\mathcal{J}_{h(Z)}$, since $h^{\#}$ induces isomorphisms $\mathcal{O}_N(V) \longrightarrow \mathcal{O}_M(h^{-1}V)$ which identify $\mathcal{J}_Z(h^{-1}V)$ and $\mathcal{J}_{h(Z)}(V)$, and h_* commutes with sheafification. Now let $f|_{Y \cap U} : Y \cap U \longrightarrow Z$ be the homeomorphism induced by f, $j : Y \cap U \longrightarrow U$ the inclusion, $t : Z \longrightarrow \operatorname{Spec} A$ the inclusion and $n : Y \cap U \longrightarrow U$ the inclusion, so the diagrams of continuous maps

$$\begin{tikzcd}
Y \arrow[r, "i"] & X \\
Y \cap U \arrow[u, "j"] \arrow[r, "n"'] & U \arrow[u, "k"']
\end{tikzcd}
\qquad
\begin{tikzcd}
Y \cap U \arrow[r, "n"] \arrow[d, "f|_{Y \cap U}"'] & U \arrow[d, "f"] \\
Z \arrow[r, "t"'] & \operatorname{Spec} A
\end{tikzcd}$$

commute. Using results from Section 5, we have

$$\begin{aligned}
\mathcal{O}_Y|_{Y\cap U} &\cong j^{-1}\mathcal{O}_Y \\
&\cong j^{-1}i^{-1}(\mathcal{O}_X/\mathcal{I}_Y) \\
&\cong (ij)^{-1}(\mathcal{O}_X/\mathcal{I}_Y) \\
&\cong (kn)^{-1}(\mathcal{O}_X/\mathcal{I}_Y) \\
&\cong n^{-1}k^{-1}(\mathcal{O}_X/\mathcal{I}_Y) \\
&\cong n^{-1}(\mathcal{O}_X|_U/\mathcal{I}_{Y\cap U}) \\
&\cong n^{-1}f^{-1}f_*(\mathcal{O}_X|_U/\mathcal{I}_{Y\cap U}) \\
&\cong (fn)^{-1}(\mathcal{O}_{\operatorname{Spec}A}/\mathcal{I}_Z) \\
&\cong (tf|_{Y\cap U})^{-1}(\mathcal{O}_{\operatorname{Spec}A}/\mathcal{I}_Z) \\
&\cong (f|_{Y\cap U})^{-1}t^{-1}(\mathcal{O}_{\operatorname{Spec}A}/\mathcal{I}_Z) \\
&\cong (f|_{Y\cap U})^{-1}\mathcal{O}_Z
\end{aligned}$$

⌜ If $g: \operatorname{Spec}A \longrightarrow \mathcal{O}_X|_U$ is f's inverse, then $f^{-1} \cong g_*$ and $g_*f_* = 1$. ⌟

Let $Z = V(\mathfrak{b})$, $\mathfrak{b} = \bigcap_{\mathfrak{p}\in Z}\mathfrak{p}$

Since $\mathcal{O}_Z$ is affine, and $(f|_{Y\cap U})^{-1}$ is isomorphic to $(f|_{Y\cap U}^{-1})_*$, $\mathcal{O}_Y|_{Y\cap U}$ is isomorphic to an affine scheme. Hence $(Y, \mathcal{O}_Y)$ is a scheme, and it is immediately reduced, as noted above.

Next we define a closed immersion $(Y,\mathcal{O}_Y) \longrightarrow (X, \mathcal{O}_X)$. For $y \in Y$ define a morphism of rings

$$\begin{aligned}
K_y : \mathcal{O}_{X,y} &\longrightarrow P_y \\
(U,s) &\longmapsto (Y\cap U, (U, s + \mathcal{I}_Y(U)))
\end{aligned}$$

Of course, this is the same as $\eta_U(s) = (U, s + \mathcal{I}_Y(U))$ and $t \in \mathcal{O}_Y(Y\cap U)$

Given $U \subseteq X$ we define $\eta_U : \mathcal{O}_X(U) \longrightarrow \mathcal{O}_Y(Y\cap U)$ by $\eta_U(s)(y) = K_y(U,s) = (Y\cap U, (U, s + \mathcal{I}_Y(U)))$. ⌟ It is not difficult to check that η is a morphism of schemes. To see that η is locally surjective, let $y \in Y\cap U$ be given and $y \in W\cap Y \subseteq Y$ an open neighborhood and $(V,t) \in P(W\cap Y)$ (so $V \supseteq W\cap Y$, $t \in (\mathcal{O}_X/\mathcal{I}_Y)(V)$) s.t. $\forall z \in W\cap Y$ $t(z) = (W\cap Y, (V,t))$. If Q sheafifies to give $\mathcal{O}_X/\mathcal{I}_Y$, let $y \in T \subseteq V$ be open and $s + \mathcal{I}_Y(T) \in \mathcal{O}_X(T)/\mathcal{I}_Y(T) = Q(T)$ such that for all $x \in T$, $t(x) = (T, s + \mathcal{I}_Y(T))$. Let $U_y = T$. Then T is an open neighorhood of y in U (can assume $W \subseteq U$, $V \subseteq W$) and for all $z \in T\cap Y$

$$\begin{aligned}
t(z) &= (W\cap Y, (V,t)) \\
&= (T\cap Y, (T, s + \mathcal{I}_Y(T))) \\
&= \eta_T(s)(z)
\end{aligned}$$

⌜ It is easier to note that η_y is surjective $\forall y \in Y$, hence $Y \to X$ is a closed immersion ⌟

Taking 0 on $U - Y$ we have a cover of U with the required property. Hence $(Y,\mathcal{O}_Y) \longrightarrow (X, \mathcal{O}_X)$ is a closed immersion. The morphism $\mathcal{O}_{X,y} \longrightarrow \mathcal{O}_{Y,y}$ on stalks is K_y followed by $P_y \Rightarrow \mathcal{O}_{Y,y}$. So $(U,s) \longmapsto (Y\cap U, (U, s + \mathcal{I}_Y(U)))$ Note that when we factor out $\mathcal{I}_{Y,y}$ we get the isomorphism in (1).

Next, let $f: Z \longrightarrow X$ be a morphism of schemes with Z reduced and $f(Z) \subseteq Y$. Assume $Z \neq \emptyset$. We claim that for all $U \subseteq X$, $\mathcal{I}_Y(U) \subseteq \operatorname{Ker} f^\#_U : \mathcal{O}_X(U) \longrightarrow \mathcal{O}_Z(f^{-1}U)$. Let $s \in \mathcal{I}_Y(U)$ be given. Then for all $x \in f^{-1}U$ the following diagram commutes:

$$\begin{tikzcd}
k(f(x)) \arrow[r] & k(x) \\
\mathcal{O}_{X,f(x)} \arrow[u] \arrow[r, "f_x"] & \mathcal{O}_{Z,x} \arrow[u] \\
\mathcal{O}_X(U) \arrow[u] \arrow[r, "f^\#_U"'] & \mathcal{O}_Z(f^{-1}U) \arrow[u]
\end{tikzcd}$$

So $(f^{-1}U, f^\#_U(s)) \in \mathfrak{m}_x \subseteq \mathcal{O}_{Z,x}$ for all $x \in f^{-1}U$. So the claim follows from the next Lemma:

<u>LEMMA</u> If Z is a reduced scheme, $V \subseteq Z$ open and $t \in \mathcal{O}_Z(V)$ is such that $\operatorname{germ}_x t \in \mathfrak{m}_x \subseteq \mathcal{O}_{Z,x}$ for all $x \in V$, then $t = 0$.

<u>PROOF</u> Let $x \in V$ be given $x \in W \subseteq Z$ an affine open set, $W \cong \operatorname{Spec}B$. Then $\operatorname{germ}_x t|_{W\cap V} = 0$ $\forall x \in W\cap V$ and so we reduce to case where $Z = \operatorname{Spec}B$ is affine, $t \in \mathcal{O}_Z(V)$. Let $x \in Q \subseteq V$ $a, s \in B$ with $s \notin \mathfrak{q}$ $\forall \mathfrak{q} \in Q$ be s.t. $t(\mathfrak{q}) = a/s \in B_\mathfrak{q}$ $\forall \mathfrak{q} \in Q$. Let $f \in B$ be s.t. $x \in D(f) \subseteq Q$. Then by assumption $\forall \mathfrak{q} \in D(f)$, $t(\mathfrak{q}) \in \mathfrak{q}A_\mathfrak{q}$, so $a \in \mathfrak{q}$. Hence af is contained in every prime of B, hence since B is nilpotent free, $af = 0$. But then $a|_{D(f)} = 0$ since $f|_{D(f)}$ is a unit. Hence $\operatorname{germ}_x t = 0$. Since $x \in V$ was arbitrary, we conclude that $t = 0$. □

Hence $\forall U \subseteq X$, $\mathscr{I}_Y(U) \subseteq \operatorname{Ker} f^{\#}_U$. Thus for each $x \in Z$ there is a ring morphism

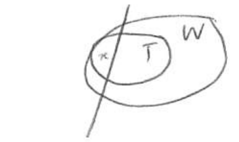
$$\varphi_x : \mathcal{O}_{X,f(x)} / \mathscr{I}_{Y,f(x)} \longrightarrow \mathcal{O}_{Z,x}$$

$$(U,s) + \mathscr{I}_{Y,f(x)} \longmapsto (f^{-1}U, f^{\#}_U(s))$$

Combined with the isomorphism $\mathcal{O}_{Y,f(x)} \xrightarrow{\sim} P_{f(x)} \xrightarrow{\sim} (\mathcal{O}_X/\mathscr{I}_Y)_{f(x)} \xrightarrow{\sim} \mathcal{O}_{X,f(x)}/\mathscr{I}_{Y,f(x)}$, we get a ring morphism

$$\phi_x : \mathcal{O}_{Y,f(x)} \longrightarrow \mathcal{O}_{Z,x}$$

If $W \subseteq X$ is open and $s \in \mathcal{O}_Y(W \cap Y)$, say



$x \in T \subseteq W$ and $t \in \mathcal{O}_X(T)$.
is s.t. $s|_{T \cap Y} = (Y \cap T, (T, t + \mathscr{I}_Y(T)))$
Then

$$\phi_x(W \cap Y, s) = (f^{-1}T, f^{\#}_T(t))$$

We use this to define a morphism of schemes $(Z, \mathcal{O}_Z) \xrightarrow{\alpha} (Y, \mathcal{O}_Y)$, by letting $\alpha^{\#}_{W \cap Y} : \mathcal{O}_Y(W \cap Y) \longrightarrow \mathcal{O}_Z(f^{-1}W)$ map s to the unique section $\alpha_{W \cap Y}(s)$ with

$$\operatorname{germ}_x \alpha_{W \cap Y}(s) = \phi_x(\operatorname{germ}_{f(x)} s) \qquad \forall x \in f^{-1}W$$

Since φ_x is a local morphism, so is ϕ_x, and consequently α is a morphism of schemes. Clearly the following diagram commutes:

$$\begin{tikzcd} & (X, \mathcal{O}_X) \\ (Z, \mathcal{O}_Z) \arrow[ur, "f"] \arrow[r, "\alpha"'] & (Y, \mathcal{O}_Y) \arrow[u, "\eta"'] \end{tikzcd}$$

This morphism α is unique since if α' makes the diagram commute,

$$\begin{aligned} \operatorname{germ}_x \alpha'_{W \cap Y}(s) &= \alpha'_x(\operatorname{germ}_{f(x)} s) \\ &= \alpha'_x \eta_x(T, t) \\ &= f_x(T, t) = \phi_x(\operatorname{germ}_{f(x)} s) = \operatorname{germ}_x \alpha_{W \cap Y}(s). \end{aligned}$$

Hence we have proven:

<u>PROPOSITION</u> Let $f : Z \longrightarrow X$ be a morphism of schemes, Z reduced, $Y \subseteq X$ closed and $f(Z) \subseteq Y$. Put the induced reduced scheme structure on Y. Then there is a unique morphism of schemes $Z \longrightarrow Y$ making the following diagram commute

$$\begin{tikzcd} & X \\ Z \arrow[ur, "f"] \arrow[r] & Y \arrow[u] \end{tikzcd}$$

(Trivial if $Z = 0$).

NOTE "Closed Immersion" is a local property.

LEMMA If $f: X \longrightarrow Y$ is a closed immersion, then $f^{-1}V \longrightarrow V$ is a closed immersion for any open $V \subseteq Y$.

PROPOSITION A morphism of schemes $f: X \longrightarrow Y$ is a closed immersion if and only if $f(X)$ is closed and for all $x \in X$ there is an open neighborhood V of $f(x)$ s.t. $f^{-1}V \longrightarrow V$ is a closed immersion.

PROOF The important point is that if $\varphi: R \longrightarrow S$ is locally surjective morphism of sheaves and $U \subseteq X$ is open then $\varphi|_U : R|_U \longrightarrow S|_U$ is locally surjective. If $f: X \longrightarrow Y$ is a closed immersion, it is clear that the condition is satisfied. So suppose the condition is satisfied, and $x \in X$. Then the morphism $\mathcal{O}_Y|_V \longrightarrow (f_* \mathcal{O}_X)|_V$ is $(\mathcal{O}_Y|_V) \longrightarrow (f|_{f^{-1}V})_* (\mathcal{O}_X|_{f^{-1}V})$ which is locally surjective. By a note in §2 it follows that f is a closed immersion. $\square$

PROPOSITION A morphism of schemes $f: X \longrightarrow Y$ is a closed immersion if and only if there is an open cover $\{V_i\}$ of Y such that $f^{-1}V_i \longrightarrow V_i$ is a closed immersion, for each i.

PROOF This follows from the previous proposition and the fact that if $f(X) \cap V_i$ is closed in V_i for all i then $V_i - f(X)$ is open in Y, hence $Y - f(X) = \bigcup_i V_i - f(X)$ is open, so $f(X)$ is closed. $\square$

NOTE A morphism of sheaves of rings is locally surjective iff. it is an epimorphism of sheaves of abelian groups iff. ϕ_x is surjective $R_x \longrightarrow S_x$ $\forall x \in X$. So a morphism $f: X \longrightarrow Y$ is a closed immersion iff. f is homeo, $f(X)$ is closed and for all $x \in X$ the morphism $\mathcal{O}_{Y,f(x)} \longrightarrow \mathcal{O}_{X,x}$ is surjective. (Since f gives a homeomorphism of X with a subspace of Y, so $(f_* \mathcal{O}_X)_{f(x)} \cong \mathcal{O}_{X,x}$ for $x \in X$)

NOTE This characterisation makes it clear that the composition of two closed immersions is a closed immersion.

Let X be a scheme, $Z \subseteq X$ closed, $(Z, \mathcal{O}_Z)$ a scheme and $i: Z \longrightarrow X$ a closed immersion whose underlying map is the inclusion. Let $\mathscr{I}$ be the kernel of the epimorphism of sheaves of abelian groups $i^\#: \mathcal{O}_X \longrightarrow i_* Z$, so we have an exact sequence

$$0 \longrightarrow \mathscr{I} \longrightarrow \mathcal{O}_X \longrightarrow i_* \mathcal{O}_Z \longrightarrow 0$$

Clearly $\mathscr{I}$ is a sheaf of ideals, and there is an isomorphism of sheaves of rings $\mathcal{O}_X/\mathscr{I} \cong i_* \mathcal{O}_Z$. (This is induced by the morphism from the presheaf $U \mapsto \mathcal{O}_X(U)/\mathscr{I}(U)$ to $i_* \mathcal{O}_Z$, which is a morphism of rings).

NOTE "Isomorphism" is a local property

It follows immediately from the characterisation of open immersions that a morphism of schemes $f: X \longrightarrow Y$ is an isomorphism iff. it is a homeomorphism and $f_x : \mathcal{O}_{Y,f(x)} \longrightarrow \mathcal{O}_{X,x}$ is bijective for all $x \in X$. Hence

LEMMA A morphism of schemes is an isomorphism iff. there is a nonempty open cover $\{V_i\}_{i \in I}$ of Y s.t. $f^{-1}V_i \longrightarrow V_i$ is an isomorphism $\forall i$. If $f: X \longrightarrow Y$ is an isomorphism so is $f^{-1}V \longrightarrow V$ for all open $V \subseteq Y$.

In any space closures of points are always irreducible, so if $X \neq \emptyset$ $\dim X \geq 0$,

$\dim \emptyset = -1$

DEFINITION The dimension of a scheme X, denoted $\dim X$, is its dimension as a topological space. If Z is an irreducible closed subset of X, then the codimension of Z in X, denoted $\operatorname{codim}(Z, X)$, is the supremum of integers n such that there exists a chain

$$Z = Z_0 \subset Z_1 \subset \cdots \subset Z_n$$

$\operatorname{codim}(Z,X)$ may be ∞ if there is no u.bound,

of distinct closed irreducible subsets of X, beginning with Z (so $\operatorname{codim}(Z,X) \geq 0$). If Y is any closed subset of X, we define

$$\operatorname{codim}(Y, X) = \inf_{Z \subseteq Y} \operatorname{codim}(Z, X)$$

where the infimum is taken over all closed irreducible subsets of Y. If Z' is closed and irreducible and $Z' \subset Z$ then $\operatorname{codim}(Z', X) \geq \operatorname{codim}(Z, X)$, so in this case the two definitions agree. If Y contains no closed irreducible subsets, put $\operatorname{codim}(Y, X)$ to be undefined (in particular if $Y = \emptyset$). (If $\operatorname{codim}(Z,X) = \infty$ for all c.irred $Z \subseteq Y$ set $\operatorname{codim}(Y,X) = \infty$,

EXAMPLE 3.2.7 If $X = \operatorname{Spec} A$ is an affine scheme, then the dimension of X is the same as the Krull dimension of A $(I, §1)$. Since $V(\mathfrak{a}) = V(\sqrt{\mathfrak{a}})$ for any ideal $\mathfrak{a}$, it suffices to consider radical ideals $\mathfrak{a}$. Given a subset $Y \subseteq \operatorname{Spec} A$ (not necc. closed) let $I(Y) = \bigcap_{\mathfrak{p} \in Y} \mathfrak{p}$. Then $I(Y)$ is an ideal of A and clearly $Y \subseteq V(I(Y))$. We claim that $\overline{Y} = V(I(Y))$. For let $W = V(\mathfrak{b})$ be closed and $Y \subseteq W$. Then $I(W) \subseteq I(Y)$. We may as well assume $\mathfrak{b}$ is radical, so that $IV(\mathfrak{b}) = \mathfrak{b}$. Hence $\mathfrak{b} \subseteq I(Y)$ and consequently $V(I(Y)) \subseteq W$. So if Y is closed, $V(I(Y)) = Y$. The operations $I(-)$ and $V(-)$ give a bijection between closed subsets of $\operatorname{Spec} A$ and radical ideals in A. Note $I(\emptyset) = A$ and $I(\operatorname{Spec} A) = \operatorname{nil} A$. This bijection reverses inclusions. Hence it is easily seen that $\dim X = \dim A$, once we show that under this bijection, irreducible closed sets correspond to prime ideals.

Let $\mathfrak{a}$ be radical and suppose $V(\mathfrak{a})$ is irreducible. Then $\mathfrak{a}$ is proper and if $bc \in \mathfrak{a}$, $(b)(c) \subseteq \mathfrak{a}$ so if $\mathfrak{p}$ is prime $\mathfrak{p} \supseteq \mathfrak{a} \Rightarrow \mathfrak{p} \supseteq (b)$ or $\mathfrak{p} \supseteq (c)$. That is, $V(\mathfrak{a}) \subseteq V((b)) \cup V((c))$. Since $V(\mathfrak{a})$ is irreducible, say $V((b)) \supseteq V(\mathfrak{a})$. Hence $\sqrt{(b)} \subseteq \mathfrak{a}$ and $b \in \mathfrak{a}$, as required.

Conversely suppose $\mathfrak{a}$ is prime. Then $V(\mathfrak{a})$ is nonempty and $V(\mathfrak{a}) = V(\mathfrak{b}) \cup V(\mathfrak{c}) \Leftrightarrow (\mathfrak{p} \supseteq \mathfrak{a} \Leftrightarrow \mathfrak{p} \supseteq \mathfrak{b}$ or $\mathfrak{p} \supseteq \mathfrak{c})$. Hence the radicals satisfy $\mathfrak{a} = r(\mathfrak{a}) = r(\mathfrak{b}\mathfrak{c}) = r(\mathfrak{b}) \cap r(\mathfrak{c}) = \mathfrak{b} \cap \mathfrak{c}$ (assume $\mathfrak{b}, \mathfrak{c}$ radical). Hence $\mathfrak{a} = \mathfrak{b}$ or $\mathfrak{a} = \mathfrak{c}$, as required. Also, if $Y \subseteq \operatorname{Spec} A$ is closed irreducible then $\dim Y = \operatorname{coht} I(Y)$ and $\operatorname{codim}(Y, X) = \operatorname{ht} I(Y)$.

CAUTION 3.2.8 Be careful in applying the concepts of dimension and codimension to arbitrary schemes. Our intuition is derived from working with schemes of finite type over a field, where these notions are well-behaved. For example, if X is an affine integral scheme of finite type over a field k, and if $Y \subseteq X$ is any closed irreducible subset, then $(I, 1.8A)$ implies that $\dim Y + \operatorname{codim}(X, Y) = \dim X$ (assuming that X is f.g. k-domain, not just covered by such). But on arbitrary (even noetherian) schemes, funny things can happen.

DEFINITION Let S be a scheme, and let X, Y be schemes over S, i.e. schemes with morphisms to S. We define the fibred product of X and Y over S, denoted $X \times_S Y$ to be a scheme together with morphisms $p_1 : X \times_S Y \to X$ and $p_2 : X \times_S Y \to Y$ which are a pullback in Sch. If X, Y are schemes without reference to any base scheme S, we take $S = \operatorname{Spec} \mathbb{Z}$ then the product of X and Y, denoted $X \times Y$ (the categorical product) is the pullback $X \times_{\operatorname{Spec} \mathbb{Z}} Y$.

THEOREM 3.3 For any two schemes X and Y over a scheme S, the fibred product $X \times_S Y$ exists, and is unique up to canonical isomorphism.

PROOF The idea is first to construct products for affine schemes and then glue. We proceed in seven steps.

Step 1. Let $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$, $S = \operatorname{Spec} R$. So we are given $\varphi : R \to A$, $\psi : R \to B$. The pushout in Rng is the diagram

$$\begin{tikzcd} R \arrow[r] \arrow[d] & A \arrow[d] \\ B \arrow[r] & A \otimes_R B \end{tikzcd} \qquad \begin{array}{l} a \mapsto a \otimes 1 \\ b \mapsto 1 \otimes b \end{array}$$

But Spec maps colimits to limits (see an earlier exercise) so the following diagram is a pullback in Sch:

$$\begin{tikzcd} \operatorname{Spec} R & \operatorname{Spec} A \arrow[l] \\ \operatorname{Spec} B \arrow[u] & \operatorname{Spec}(A \otimes_R B) \arrow[l] \arrow[u] \end{tikzcd}$$

Note that the pullback of any two morphisms into an initial object $(\emptyset, 0)$ exists trivially, so throughout we may assume $S \neq 0$ (since if $S = 0$ also $X = Y = 0$ and $0 \times_0 0 = 0$). So

$$\operatorname{Spec} A \times_{\operatorname{Spec} R} \operatorname{Spec} B = \operatorname{Spec}(A \otimes_R B)$$

<u>Step 2</u>. It follows immediately from the universal product property that is unique up to unique isomorphism, if it exists. We will need this uniqueness for those products already constructed as we go along.

<u>Step 3 Glueing morphisms</u> We have already seen how to glue sheaves (Ex 1.22) and how to glue schemes (Ex 2.12). Now we glue morphisms. If X, Y are schemes, then to give a morphism from X to Y, it is equivalent to give an open cover $\{U_i\}$ of X, together with morphisms $f_i : U_i \longrightarrow Y$, where U_i has the induced open subscheme structure, such that the restrictions of f_i and f_j to $U_i \cap U_j$ are the same, for each i, j. That is, for all open $W \subseteq Y$ the diagram

$$\begin{tikzcd} \mathcal{O}_Y(W) \arrow[r, "f_i^{\#}|_W"] \arrow[d, "f_j^{\#}|_W"'] & \mathcal{O}_X(f_i^{-1}W) \arrow[d] \\ \mathcal{O}_X(f_j^{-1}W) \arrow[r] & \mathcal{O}_X(f_i^{-1}W \cap f_j^{-1}W) \end{tikzcd}$$

commutes. Given $s \in \mathcal{O}_Y(W)$ we define $f^{\#}|_W(s)$ to be the amalgamation of the $f_i^{\#}|_W(s)$. This gives a morphism of ringed spaces $X \longrightarrow Y$ restricting to f_i on U_i. This is easily checked to be a morphism of schemes. Note that if $f_i = g|_{U_i}$ $\forall i$ then the induced f is just g.

<u>Step 4</u> If X, Y are schemes over a scheme S and $U \subseteq X$ is open, then if the product $X \times_S Y$ exists, then $p_1^{-1}(U) \subseteq X \times_S Y$ is a product for U and Y over S, where U has the canonical structure of a scheme over S.

$$\begin{tikzcd} p_1^{-1}(U) \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathrm{II}"] & U \arrow[d] \\ X \times_S Y \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, phantom, "\mathrm{I}"] & X \arrow[d] \\ Y \arrow[r] & S \end{tikzcd}$$

In detail, to show that the outer square is a pullback, it suffices to show that II is a pullback, by pasting. This follows from the following more general result:

<u>LEMMA</u> If $f : X \longrightarrow Y$ is a morphism of schemes and $V \subseteq Y$ is open, then the following diagram is a pullback (open sets having the canonical scheme structures and all maps canonical)

$$\begin{tikzcd} f^{-1}V \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

<u>PROOF</u> The map $f^{-1}V \longrightarrow V$ is induced by f and is unique making the diagram commute. If $Z \longrightarrow X$ and $Z \longrightarrow V$ make the diagram commute, $\operatorname{Im}(Z \longrightarrow X) \subseteq f^{-1}V$ so there is unique $Z \longrightarrow f^{-1}V$ s.t. $Z \longrightarrow f^{-1}V \longrightarrow X$ is $Z \longrightarrow X$. Since the factoring of $Z \longrightarrow X \longrightarrow Y$ is unique, it follows that $Z \longrightarrow f^{-1}V \longrightarrow V = Z \longrightarrow V$, as required. $\square$

(If $p_1^{-1}U$ is empty the above is trivial) In particular if X is a scheme and $U, V \subseteq X$ are open, the square

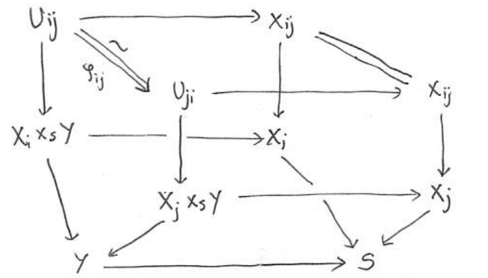
$$\begin{tikzcd} U \cap V \arrow[r] \arrow[d] & U \arrow[d] \\ V \arrow[r] & X \end{tikzcd}$$

is a pullback, where all morphisms are inclusions.

Step 5 Let X, Y be schemes over S, and let $\{X_i\}$ be an open covering of X, and suppose that for each i the product $X_i \times_S Y$ exists. For each i,j let $U_{ij} \subseteq X_i \times_S Y$ be $p_i^{-1}(X_{ij})$ where $X_{ij} = X_i \cap X_j$:

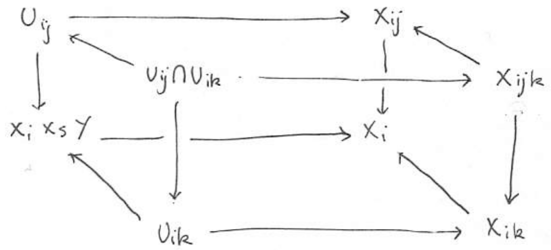
$$
\begin{tikzcd}
U_{ij} \arrow[r] \arrow[d] & X_i \cap X_j \arrow[d] & \\
X_i \times_S Y \arrow[r] \arrow[d] & X_i \arrow[d] \arrow[r] & X \arrow[dl] \\
Y \arrow[r] & S &
\end{tikzcd}
$$

Then the outer square is a product for X_{ij} and Y over S. By the uniqueness of pullback there are isomorphisms $\varphi_{ij} : U_{ij} \longrightarrow U_{ji}$ for each i,j unique making the following diagram commute:



(0)

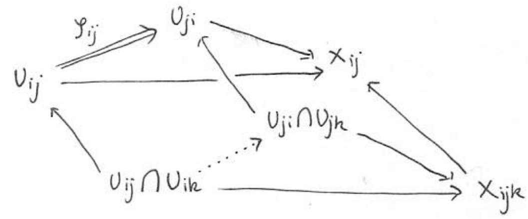
We now make some observations about the U_{ij}. To begin with, let i,j,k be given and consider the following diagram:



Here $X_{ijk} = X_i \cap X_j \cap X_k$, so the faces are all pullbacks, except for the top and forward ones, and $U_{ij} \cap U_{ik} \longrightarrow X_{ijk}$ is induced via these pullbacks, so that the cube commutes. Since we can paste pullbacks, the square formed by the left and bottom faces is a pullback. Hence in the diagram

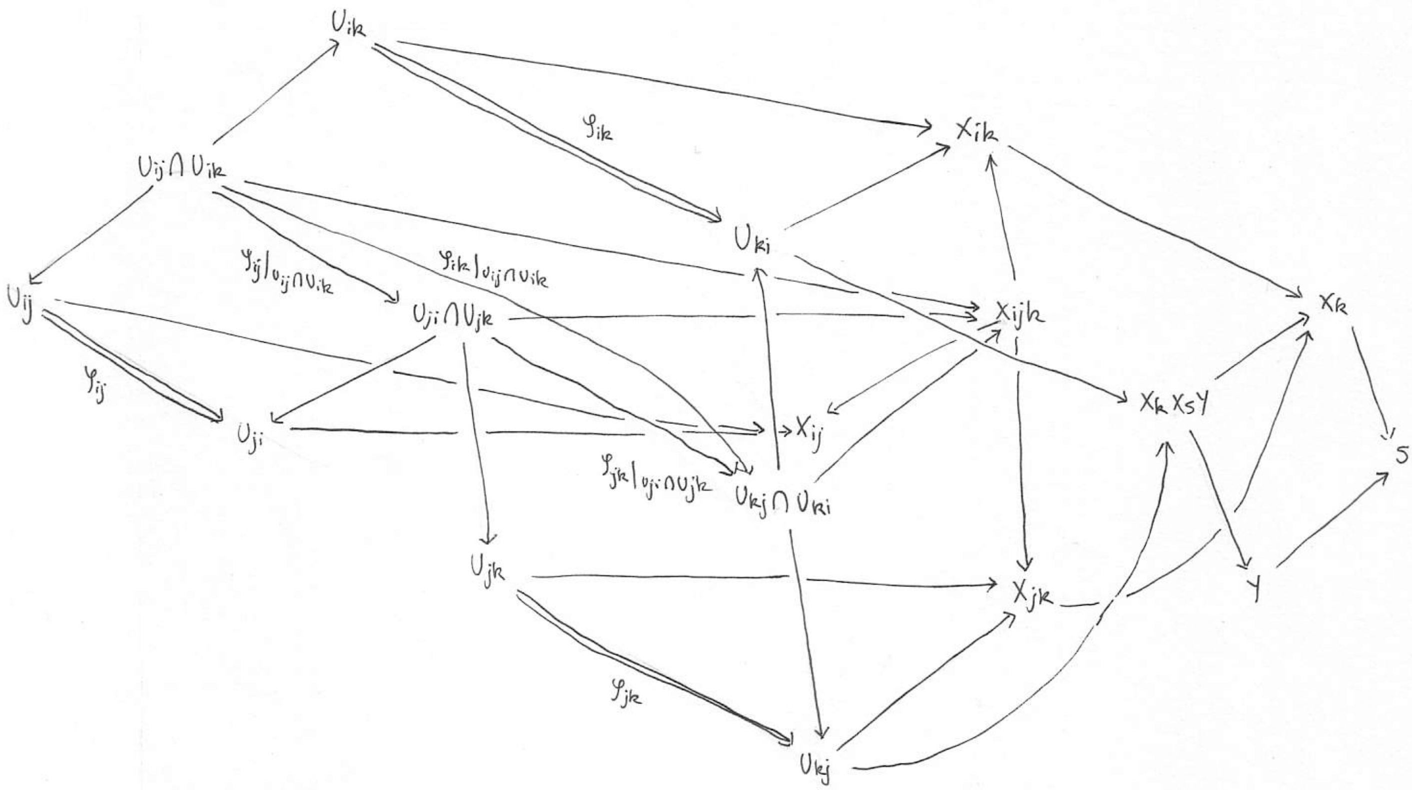
$$
\begin{tikzcd}
U_{ij} \arrow[r] & X_{ij} \arrow[r] & X_i \\
U_{ij} \cap U_{ik} \arrow[r] \arrow[u] \arrow[ur, phantom, "\mathrm{I}"] & X_{ijk} \arrow[r] \arrow[u] \arrow[ur, phantom, "\mathrm{II}"] & X_{ik} \arrow[u]
\end{tikzcd}
$$

The outer and right hand squares are both pullbacks. Hence also is the left hand square I. Now form the diagram



Again pasting pullbacks shows that the left hand square is a pullback, and it follows immediately that $\varphi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$.

Next we have to show that $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ on $U_{ij} \cap U_{ik}$. Consider the following diagram:

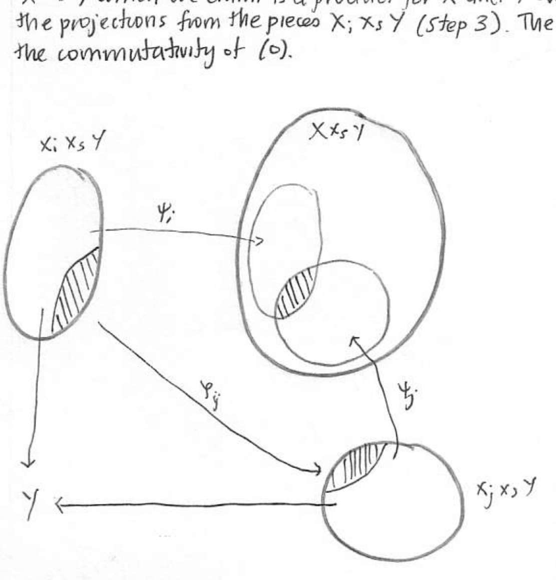


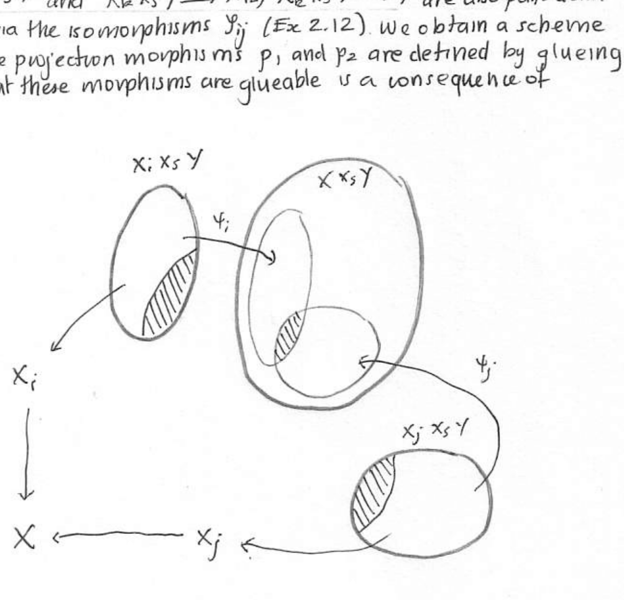
In this diagram all squares commute, many are pullbacks, and we use these pullbacks to show that

$$\varphi_{ik}|_{U_{ij} \cap U_{ik}} = \varphi_{jk}|_{U_{ji} \cap U_{jk}} \circ \varphi_{ij}|_{U_{ij} \cap U_{ik}} \qquad (1)$$

Since $U_{kj} \cap U_{ki} \longrightarrow X_{ijk}$ and $U_{kj} \cap U_{ki} \longrightarrow U_{kj}$ are a pullback it suffices to show the two maps in (1) give the same morphism on composition with these two morphisms.

To do this we use the fact that $U_{kj} \longrightarrow X_{jk}$, $U_{kj} \longrightarrow X_k \times_S Y$ and $X_k \times_S Y \longrightarrow X_k$, $X_k \times_S Y \longrightarrow Y$ are also pullbacks.
We are now in a position to glue the schemes $X_i \times_S Y$ via the isomorphisms φ_{ij} (Ex 2.12). We obtain a scheme $X \times_S Y$ which we claim is a product for X and Y over S. The projection morphisms p_1 and p_2 are defined by glueing the projections from the pieces $X_i \times_S Y$ (Step 3). The fact that these morphisms are glueable is a consequence of the commutativity of (0).





It follows that for each i the following diagram commutes (except possibly for Ⓘ)

$$\begin{tikzcd}
X_i \times_S Y \arrow[r, "\psi_i"] \arrow[d] \arrow[rr, bend left, "p_2"] & X \times_S Y \arrow[r] \arrow[d, "p_1"] \arrow[dr, phantom, "\mathrm{I}"] & Y \arrow[d] \\
X_i \arrow[r] & X \arrow[r] & S
\end{tikzcd}$$

Since a morphism of schemes is determined by its restrictions to a cover, it follows that Ⓘ commutes, since the $\psi_i(X_i \times_S Y)$ cover $X \times_S Y$.

Next we show that $X \times_S Y$ is a pullback. First we show uniqueness. Suppose $f, g : Z \longrightarrow X \times_S Y$ are s.t. $p_2 f = p_2 g$ and $p_1 f = p_1 g$. The fact that the inverse image of X_{ij} under $X_j \times_S Y \longrightarrow X_j$ is U_{ij} means that for all $i \in I$, $p_1^{-1} X_i = \psi_i(X_i \times_S Y)$. Hence let

$$Q_i = f^{-1}\psi_i(X_i \times_S Y) = g^{-1}\psi_i(X_i \times_S Y)$$

Then there are unique $f_i, g_i : Q_i \longrightarrow X_i \times_S Y$ making the following diagram commute

$$\begin{tikzcd}
Q_i \arrow[r, shift left, "f_i"] \arrow[r, shift right, "g_i"'] \arrow[d] & X_i \times_S Y \arrow[d, "\psi_i"] \arrow[dr] \arrow[ddr] & & \\
Z \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & X \times_S Y \arrow[rr, "p_2"] \arrow[dd, "p_1"] & & Y \arrow[dd] \\
& & X_i \arrow[dl] & \\
& X \arrow[rr] & & S
\end{tikzcd} \qquad (2)$$

Using the fact that $X_i \times_S Y$ is a pullback and $X_i \longrightarrow X$ is a monomorphism we see that $f_i = g_i$. Hence $f|_{Q_i} = g|_{Q_i}$. But the Q_i cover Z, so it follows that $f = g$ as required.

Now for the existence part. Let Z be a scheme and $f : Z \longrightarrow X$, $g : Z \longrightarrow Y$ morphisms of schemes over S. For $i \in I$ let $f^{-1} X_i = Z_i$. Then $Z_i \longrightarrow X_i$ and $Z_i \longrightarrow Z \longrightarrow Y$ induce $\theta_i : Z_i \longrightarrow X \times_S Y$ making

$$\begin{tikzcd}
Z_i \arrow[rr] \arrow[dr] \arrow[dd] & & X_i \arrow[dd] \\
& X_i \times_S Y \arrow[ur] \arrow[dd] \arrow[ddl, bend right] & \\
Z \arrow[rr] \arrow[dd] & & X \arrow[dd] \\
& X \times_S Y \arrow[dl] & \\
Y \arrow[rr] & & S
\end{tikzcd}$$

Using the uniqueness proved above, it is not hard to check that $\theta_i|_{Z_i \cap Z_j} = \theta_j|_{Z_i \cap Z_j}$. Since the Z_i cover Z, we can glue the θ_i to form a morphism $\theta : Z \longrightarrow X \times_S Y$. Using a diagram similar to (2) we see that $(p_1\theta)|_{Z_i} = f|_{Z_i}$ and $(p_2\theta)|_{Z_i} = g|_{Z_i}$ $\forall i$. Hence $p_1\theta = f$ and $p_2\theta = g$, as required. Since we have already checked uniqueness, this completes the proof that $X \times_S Y$ is a pullback.

Step 6 We know from Step 1 that if X, Y, S are all affine, then $X \times_S Y$ exists. Thus using Step 5 we conclude that for any X, but Y, S affine, the product exists. Using Step 5 again, with X and Y interchanged, we find that the product exists for any X and any Y over an affine S.

Step 7 Given arbitrary X, Y, S let $q : X \longrightarrow S$ and $r : Y \longrightarrow S$ be the given morphisms. Let S_i be an affine open cover of S. Let $X_i = q^{-1}S_i$ and $Y_i = r^{-1}S_i$. Then by Step 6, $X_i \times_{S_i} Y_i$ exists. Note that this same scheme is a product for X_i and Y over S. Indeed, given morphisms $f : Z \longrightarrow X_i$ and $f : Z \longrightarrow Y$ over S, the image of g must land inside Y_i. Thus, using the fact that open immersions are monomorphisms, we see that $X_i \times_S Y$ exists for each i, and one more application of Step 5 gives us $X \times_S Y$. This completes the proof. □

NOTE Glueing morphisms

assume I is nonempty

Let X, Y be schemes, $\{U_i\}$ an open cover of X. There is a bijection between morphisms $X \longrightarrow Y$ and collections of morphisms $U_i \longrightarrow Y$ which "agree" on $U_i \cap U_j$. In step 3 of the previous proof we defined what we mean by "agree". But there is a more elegant way. If $U \subseteq X$ is open and $f : X \longrightarrow Y$ the composition $U \longrightarrow X \longrightarrow Y$ is defined to be the restriction $f|_U$. So $f_i : U_i \longrightarrow Y$ agree iff. $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ $\forall i,j$. That is, $U_i \cap U_j \longrightarrow U_i \xrightarrow{f_i} Y = U_i \cap U_j \longrightarrow U_j \xrightarrow{f_j} Y$ $\forall i,j$. So the glueing property is:

LEMMA Let X, Y be schemes, $\{U_i\}$ a nonempty cover of X. If for each i there is a morphism of schemes $f_i : U_i \longrightarrow Y$ and if the diagram

$$\begin{tikzcd} U_i \cap U_j \arrow[r] \arrow[d] & U_i \arrow[d, "f_i"] \\ U_j \arrow[r, "f_j"'] & Y \end{tikzcd}$$

commutes for all i, j there is a unique morphism of schemes $f : X \longrightarrow Y$ with the property that

$$\begin{tikzcd} U_i \arrow[r] \arrow[dr, "f_i"'] & X \arrow[d, "f"] \\ & Y \end{tikzcd}$$

commutes for all i.

LEMMA If $f, g : X \longrightarrow Y$ are morphisms of schemes and $\{U_i\}$ is a nonempty open cover of X, and if $f|_{U_i} = g|_{U_i}$ $\forall i$, then $f = g$.

NOTE Open Immersions are Monomorphisms

If X is a scheme and U is an open subset, the morphism $U \longrightarrow X$ is a monomorphism in Sch. An open immersion is a morphism $f : X \longrightarrow Y$ whose image is open and for which the induced morphism $X \longrightarrow (f(X), \mathcal{O}_Y|_{f(X)})$ is an isomorphism of schemes. So it is obvious that f is a monomorphism.

NOTE Closed Immersions are Monomorphisms

Let $f : X \longrightarrow Y$ be a closed immersion, $g, h : Z \longrightarrow X$ and suppose $fg = fh$. Since f is injective, the underlying maps of g, h are the same, and $fg = fh$ shows that $(f_* g^\#) f^\# = (f_* h^\#) f^\#$. By assumption $f^\#$ is a locally surjective morphism of sheaves of abelian groups, hence an epimorphism, so $f_* g^\# = f_* h^\#$. But f is a homeomorphism onto its image, so $g^\# = h^\#$. Hence f is a monomorphism.

NOTE Dimension for Noetherian spaces

Let $Y \subseteq X$ be a nonempty closed subset of a noetherian space X. Let $Y = Y_1 \cup \dots \cup Y_n$ be the irreducible components of Y. Then clearly $\operatorname{codim}(Y, X)$ is defined and in fact

$$\operatorname{codim}(Y, X) = \min\{\operatorname{codim}(Y_1, X), \dots, \operatorname{codim}(Y_n, X)\}$$

NOTES

- Let $f : X \longrightarrow S$ and $g : Y \longrightarrow S$ be schemes over S, and suppose $f(X)$ and $g(Y)$ are both contained in an open set $V \subseteq S$. Then there are induced morphisms $X \longrightarrow V$ and $Y \longrightarrow V$:

$$\begin{tikzcd}
X \times_S Y \arrow[r] \arrow[d] & X \arrow[d] \arrow[dr] & \\
Y \arrow[r] \arrow[rr, bend right] & S & V \arrow[l]
\end{tikzcd}$$

It is not difficult to check, since $V \longrightarrow S$ is monic, that the outside square is also a pullback. Hence $X \times_S Y = X \times_V Y$.

- Let $f : X \longrightarrow S$, $g : Y \longrightarrow S$ be schemes over S, $U \subseteq X$ and $W \subseteq Y$ open. Let $p_1 : X \times_S Y \longrightarrow X$ and $p_2 : X \times_S Y \longrightarrow Y$ be given. Consider

$$\begin{tikzcd}
p_1^{-1}U \cap p_2^{-1}W \arrow[r] \arrow[d] & p_1^{-1}U \arrow[r] \arrow[d] & U \arrow[d] \\
p_2^{-1}W \arrow[r] \arrow[d] & X \times_S Y \arrow[r, "p_1"] \arrow[d, "p_2"] & X \arrow[d] \\
W \arrow[r] & Y \arrow[r] & S
\end{tikzcd}$$

Since all these squares are pullbacks, the outer square is a pullback, so $p_1^{-1}U \cap p_2^{-1}W = W \times_S U$.

- If $X \cong \mathrm{Spec}\,A$, $Y \cong \mathrm{Spec}\,B$ and $Z \cong \mathrm{Spec}\,R$, and if $X \longrightarrow Z$ and $Y \longrightarrow Z$ are given, $\mathrm{Spec}(A \otimes_R B)$ is the pullback of $X \longrightarrow Z$, $Y \longrightarrow Z$ in the natural way (i.e. the following commutes and outer square is pullback)

$$\begin{tikzcd}
\mathrm{Spec}(A \otimes_R B) \arrow[r] \arrow[d] & \mathrm{Spec}A \cong X \arrow[d] \arrow[dd, bend left] \\
\mathrm{Spec}B \arrow[r] \arrow[d, phantom, "\cong"] & \mathrm{Spec}R \arrow[d, phantom, "\cong"] \\
Y \arrow[r] & Z
\end{tikzcd}$$

So the pullback of affine schemes is affine. (over an affine scheme)

- Start with a pullback of schemes (on the left) and assume $V' \subseteq Y'$, $V \subseteq Y$ are open and the image of V' is contained in V. Then the image of $^{-1}V'$ is contained in $f^{-1}V$, and standard results on pullbacks show that the outer square on the right is a pullback ($\boxed{\mathrm{I}\,|\,\mathrm{II}}$ outer and $\mathrm{II} \Rightarrow \mathrm{I}$)

$$\begin{tikzcd}
X' \arrow[r, "g"] \arrow[d] & Y' \arrow[d] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
g^{-1}V' \arrow[rrr] \arrow[dr] \arrow[ddd] & & & V' \arrow[dl] \arrow[ddd] \\
& X' \arrow[r] \arrow[d] & Y' \arrow[d] & \\
& X \arrow[r] & Y & \\
f^{-1}V \arrow[ur] \arrow[rrr] & & & V \arrow[ul]
\end{tikzcd}$$

NOTE Products are Local

LEMMA Let $f, g : X \longrightarrow Y$ be morphisms of schemes. Suppose that there is an open cover $\{V_\alpha\}$ of Y such that $f^{-1}V_\alpha = g^{-1}V_\alpha$ for all α and the induced morphisms of schemes

$$f|_{f^{-1}V_\alpha} : f^{-1}V_\alpha \longrightarrow V_\alpha$$
$$g|_{g^{-1}V_\alpha} : g^{-1}V_\alpha \longrightarrow V_\alpha$$

coincide. Then $f = g$.

PROOF It is clear that the underlying maps of f, g must be the same. Let $V \subseteq Y$ be open, $s \in \mathcal{O}_Y(V)$. Then the open sets $f^{-1}(V \cap V_\alpha)$ cover $f^{-1}V$ and

$$\begin{aligned} f^\#_V(s)|_{f^{-1}(V\cap V_\alpha)} &= f^\#_{V\cap V_\alpha}(s|_{V\cap V_\alpha}) \\ &= (f|_{f^{-1}V_\alpha})^\#_{V\cap V_\alpha}(s|_{V\cap V_\alpha}) \\ &= (g|_{f^{-1}V_\alpha})^\#_{V\cap V_\alpha}(s|_{V\cap V_\alpha}) \\ &= g^\#_V(s)|_{f^{-1}(V\cap V_\alpha)} \end{aligned}$$

Hence $f = g$, as required. □

PROPOSITION (EGA I, 3.2.6.2) Suppose there is a commutative diagram of schemes

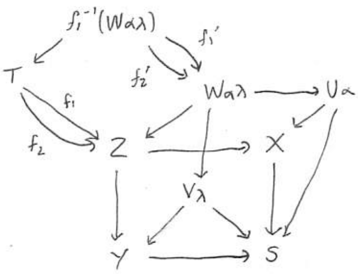
$$\begin{tikzcd} Z \arrow[r, "p"] \arrow[d, "q"'] & X \arrow[d] \\ Y \arrow[r] & S \end{tikzcd} \qquad (1)$$

And that there are open covers $\{U_\alpha\}$ of X and $\{V_\lambda\}$ of Y such that if $W_{\alpha\lambda} = p^{-1}U_\alpha \cap q^{-1}V_\lambda$ then $W_{\alpha\lambda}$ with the induced morphisms $W_{\alpha\lambda} \longrightarrow U_\alpha$, $W_{\alpha\lambda} \longrightarrow V_\lambda$ is a product of schemes over S for all α, λ. Then (1) is a product of schemes over S.

PROOF If either of X, Y is zero, so is Z, and (1) is trivially a pullback. So suppose $X, Y \neq \emptyset$ and the covers $\{U_\alpha\}, \{V_\lambda\}$ are nonempty. First we show uniqueness of factorisation. Suppose $f_1, f_2 : T \longrightarrow Z$ are morphisms of schemes with $pf_1 = pf_2$, $qf_1 = qf_2$. Then

$$\begin{aligned} f_1^{-1}(W_{\alpha\lambda}) &= f_1^{-1}(p^{-1}U_\alpha \cap q^{-1}V_\lambda) \\ &= (pf_1)^{-1}U_\alpha \cap (qf_1)^{-1}V_\lambda \\ &= (pf_2)^{-1}U_\alpha \cap (qf_2)^{-1}V_\lambda = f_2^{-1}(W_{\alpha\lambda}) \end{aligned}$$

The $W_{\alpha\lambda}$ clearly cover Z, so it suffices to show f_1, f_2 are equal when restricted to $W_{\alpha\lambda}$. Consider:



Since by assumption $W_{\alpha\lambda} \longrightarrow U_\alpha$, $W_{\alpha\lambda} \longrightarrow V_\lambda$ is a pullback, one checks easily that $f_1' = f_2'$. Hence $f_1 = f_2$, as claimed.

Now let morphisms $g : T \longrightarrow X$, $h : T \longrightarrow Y$ be given making a commutative diagram with $X \longrightarrow S$, $Y \longrightarrow S$. Let $T_{\alpha\lambda} = g^{-1}U_\alpha \cap h^{-1}V_\lambda$. Since by assumption $W_{\alpha\lambda} = U_\alpha \times_S V_\lambda$ we get a commutative diagram:

$$\begin{tikzcd}
 & T_{\alpha\lambda} \arrow[dl] \arrow[dr, "f_{\alpha\lambda}"] \arrow[drrr, bend left] \arrow[dddr, bend right] & & & \\
T \arrow[drrr, "h"] \arrow[dddr, "g"'] & & W_{\alpha\lambda} \arrow[rr] \arrow[dl] \arrow[dd] & & V_\lambda \arrow[dl] \arrow[dddl] \\
 & Z \arrow[rr] \arrow[dd] & & Y \arrow[dd] & \\
 & & U_\alpha \arrow[dl] \arrow[dr] & & \\
 & X \arrow[rr] & & S &
\end{tikzcd}$$

where $f_{\alpha\lambda} : T_{\alpha\lambda} \longrightarrow W_{\alpha\lambda}$ is induced by the restrictions of g, h. The $T_{\alpha\lambda}$ are an open cover of T, and so to complete the proof it suffices to show $f_{\alpha\lambda}$ and $f_{\beta\mu}$ coincide on $T_{\alpha\lambda} \cap T_{\beta\mu}$.

$$\begin{tikzcd}
T_{\alpha\lambda} \cap T_{\beta\mu} \arrow[rr] \arrow[dd] \arrow[dr, phantom, "\mathrm{I}"] & & T_{\beta\mu} \arrow[dl] \arrow[ddd, "f_{\beta\mu}"] & & \\
 & T \arrow[ddrrr, bend left, "g"] \arrow[dddr, bend right, "h"'] & & & \\
T_{\alpha\lambda} \arrow[ur] \arrow[drr, "f_{\alpha\lambda}"'] & & & & \\
 & & Z \arrow[rr, "p"] \arrow[d, "g"] & & X \arrow[d] \\
 & & Y \arrow[rr] & & S
\end{tikzcd} \qquad (2)$$

It suffices to show $T_{\alpha\lambda} \cap T_{\beta\mu} \longrightarrow T_{\beta\mu} \longrightarrow Z = T_{\alpha\lambda} \cap T_{\beta\mu} \longrightarrow T_{\alpha\lambda} \longrightarrow Z$ agree when composed with p and q. But by definition $q f_{\alpha\lambda} = T_{\alpha\lambda} \longrightarrow T \xrightarrow{h} Y$, $p f_{\alpha\lambda} = T_{\alpha\lambda} \longrightarrow T \xrightarrow{g} X$, and similarly for $f_{\beta\mu}$, so the claim follows immediately from (2) and the fact that I commutes. This completes the proof. $\square$

NOTE The Graph of a Morphism

DEFINITION Let $f : X \longrightarrow Y$ be a morphism of schemes over a scheme S. The graph of f, denoted Γ_f, is the morphism $\Gamma_f : X \longrightarrow X \times_S Y$ which gives $1_X, f$ upon composing with the first and second projections respectively.

By definition Γ_f is a coretraction and hence a monomorphism.

LEMMA Let f be a morphism of schemes $X \longrightarrow Y$ over S. Then the following diagram is a pullback:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\Gamma_f"'] & Y \arrow[d, "\Delta"] \\ X \times_S Y \arrow[r, "f \times 1"'] & Y \times_S Y \end{tikzcd} \qquad (1)$$

PROOF Denote the projections from $X \times_S Y$ by p_1, q_1 resp. and the projections from $Y \times_S Y$ by p_2, q_2. Then

$$\begin{aligned} p_2 \Delta f &= f = q_1 \Gamma_f = \\ &= p_2 (f \times 1) \Gamma_f \\ p_1 \Delta f &= f = f p_1 \Gamma_f \\ &= p_2 (f \times 1) \Gamma_f \end{aligned}$$

So (1) at least commutes. Now suppose T is a scheme and $a : T \longrightarrow X \times_S Y$, $b : T \longrightarrow Y$ are s.t. $\Delta b = (f \times 1) a$. Let $c : T \longrightarrow X$ be $c = p_1 a$. Since Γ_f is monic it suffices to show $\Gamma_f c = a$ and $fc = b$. But

$$\begin{aligned} p_1 \Gamma_f c &= c \quad \text{and} \quad p_1 a = c \\ p_2 \Gamma_f c &= p_2 \Gamma_f p_1 a = f p_1 a = p_2 (f \times 1) a \\ &= p_2 \Delta b = b = q_2 (f \times 1) a = p_2 a \qquad (\text{since } q_2 \Delta b = q_2 (f \times 1) a) \end{aligned}$$

Hence $\Gamma_f c = a$. Also $fc = f p_1 a = b$, so we are done. $\square$

DEFINITION Let $f: X \longrightarrow Y$ be a morphism of schemes, and let $y \in Y$ be a point. Let $k(y)$ be the residue field of y, and $\operatorname{Spec} k(y) \longrightarrow Y$ be the natural morphism (Ex 2.7). Then we define the fibre of the morphism f over the point y to be the scheme

$$X_y = X \times_Y \operatorname{Spec} k(y)$$

$$\begin{array}{ccc} X_y & \longrightarrow & X \\ \downarrow & & \downarrow f \\ \operatorname{Spec} k(y) & \longrightarrow & Y \end{array}$$

The fibre X_y is a scheme over $k(y)$, and one can show that its underlying topological space is homeomorphic to the subset $f^{-1}(y)$ of X. (Ex 3.10).

The notion of a fibre of a morphism allows us to regard a morphism as a family of schemes (namely its fibres) parametrised by the points of the image scheme. Conversely, this notion of family is a good way of making sense of the idea of a family of schemes varying algebraically. For example, given a scheme X_0 over a field k, we define a family of deformations of X_0 to be a morphism $f: X \longrightarrow Y$ with Y connected, together with a point $y_0 \in Y$ s.t. $k(y_0) \cong k$ and $X_{y_0} \cong X_0$ with respect to this iso of fields. I.e,

$$\begin{array}{ccccc} X_0 & \cong & X_{y_0} & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow f \\ \operatorname{Spec} k & \cong & \operatorname{Spec} k(y_0) & \longrightarrow & Y \end{array}$$

commutes. The other fibres X_y of f are called deformations of X_0.

An interesting kind of family arises when we have a scheme over $\operatorname{Spec} \mathbb{Z}$. In this case, taking the fibre over the generic point gives a scheme $X_{\mathbb{Q}}$ over $\mathbb{Q}$, while taking the fibre over a closed point, corresponding to a prime number p, gives a scheme X_p over $\mathbb{F}_p$. We say that X_p arises by reduction mod p of the scheme X.

Another important application of fibred products is to the notion of base extension. Let S be a fixed scheme which we think of as a base scheme, meaning that we are interested in the category of schemes over S. For example, think of $S = \operatorname{Spec} k$, where k is a field. If S' is another base scheme, and if $S' \longrightarrow S$ is a morphism, then for any scheme X over S, we let $X' = X \times_S S'$, which will be a scheme over S'. This actually defines a functor $\mathfrak{Sch}/S \longrightarrow \mathfrak{Sch}/S'$. We say that X' is obtained from X by making a base extension $S' \longrightarrow S$. For example, think of $S' = \operatorname{Spec} k'$ where k' is an extension field of k. Note, by the way, that base extension is a transitive operation: if $S'' \longrightarrow S' \longrightarrow S$ are two morphisms, then $(X \times_S S') \times_{S'} S'' \cong X \times_S S''$ since you can paste pullbacks.

This ties in with a general philosophy, emphasised by Grothendieck in EGA, that one should try to develop all concepts of algebraic geometry in a relative context. Instead of always working over a fixed base field, and considering properties of one variety at a time, one should consider a morphism of schemes $f: X \longrightarrow S$, and study properties of the morphism. It then becomes important to study the behaviour of properties of f under base extension, and in particular, to relate properties of f to properties of the fibres of f. For example, if $f: X \longrightarrow S$ is a morphism of finite type, and if $S' \longrightarrow S$ is any base extension, then $f': X' \longrightarrow S'$ is also a morphism of finite type, where $X' = X \times_S S'$. Hence we say the property of a morphism f being of finite type is stable under base extension. On the other hand, if for example $f: X \longrightarrow S$ is a morphism of integral schemes, the fibres of f may be neither irreducible nor reduced. So the property of a scheme being integral is not stable under base extension.

EXAMPLE 3.3.1 Let k be an algebraically closed field,

$$X = \operatorname{Spec} k[x,y,t]/(ty - x^2)$$

$$Y = \operatorname{Spec} k[t]$$

Let $f: X \longrightarrow Y$ be the morphism determined by $k[t] \longrightarrow k[x,y,t]/(ty - x^2)$. Then X and Y are integral schemes of finite type over k ($y - x^2$ is irred. by Eisenstein, hence $ty - x^2$ by the Proj. closure trick). We identify the closed points of Y with elements of k via $a \longleftrightarrow \mathfrak{m}_a = (t - a)$.

ASIDE

If $Z = \operatorname{Spec} A$ is an affine scheme and $\mathfrak{p} \in \operatorname{Spec} A$, then the morphism of schemes $\operatorname{Spec} k(\mathfrak{p}) \longrightarrow \operatorname{Spec} A$ arising from Ex 2.7 is in fact the morphism corresponding to $A \to k(\mathfrak{p})$ given by

$$\begin{tikzcd} A \arrow[r] & A_{\mathfrak{p}} \arrow[r] \arrow[d, phantom, "\cong"] & A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \cong k(\mathfrak{p}) \arrow[d, phantom, "\cong"] \\ & \mathcal{O}_{Z,\mathfrak{p}} \arrow[r] & \frac{\mathcal{O}_{Z,\mathfrak{p}}}{\mathfrak{m}} \end{tikzcd}$$

In the case where $\mathfrak{m}$ is maximal, $A_{\mathfrak{m}}/\mathfrak{m}A_{\mathfrak{m}} \cong A/\mathfrak{m}$ and hence $k(\mathfrak{m}) \cong A/\mathfrak{m}$. So there is a commutative diagram of schemes

$$\begin{tikzcd} \operatorname{Spec} k(\mathfrak{m}) \arrow[r] \arrow[d, phantom, "\cong"] & \operatorname{Spec} A \\ \operatorname{Spec} A/\mathfrak{m} \arrow[ur] & \end{tikzcd}$$

So in forming the fibre $Z_{\mathfrak{m}}$ we can pullback along $\operatorname{Spec} A/\mathfrak{m} \longrightarrow \operatorname{Spec} A$ and get a scheme isomorphic to the fibre defined by $\operatorname{Spec} k(\mathfrak{m}) \longrightarrow \operatorname{Spec} A$.

Next we examine the fibers of $f : X \longrightarrow Y$. Let $\mathfrak{m}_a$ be a closed point of Y. The fibre X_a is the pullback of

$$\begin{tikzcd} & \operatorname{Spec} k[x,y,t]/(ty-x^2) \arrow[d] \\ \operatorname{Spec} \frac{k[t]}{(t-a)} \arrow[r] & \operatorname{Spec} k[t] \end{tikzcd}$$

Since Spec maps colimits to limits, X_a is the spectrum of the pushout of

$$\begin{tikzcd} & \frac{k[x,y]}{(ay-x^2)} \arrow[dl, "\gamma"'] & k[x,y,t]/(ty-x^2) \arrow[l, "\theta'"'] \arrow[dll, dashed, bend right, "\epsilon'"'] \\ Z & & \\ & \frac{k[t]}{(t-a)} \arrow[uu, "\theta"] \arrow[ul, dashed, bend left, "\epsilon"] & k[t] \arrow[l, "\varphi"] \arrow[uu, "\psi"'] \end{tikzcd}$$

$\psi(t) = t + (ty - x^2)$
$\varphi(t) = t + (t-a)$

We claim that $k[x,y]/(ay-x^2)$ together with $\theta : k[t]/(t-a) \longrightarrow k[x,y]/(ay-x^2)$, $\theta' : k[x,y,t]/(ty-x^2) \longrightarrow k[x,y]/(ay-x^2)$ defined by $\theta(t) = a$ and $\theta'(x) = x$, $\theta'(y) = y$, $\theta'(t) = a$ is such a pushout. It clearly makes the above diagram commute, and if Z is a ring and ϵ, ϵ' morphisms, Z becomes a k-algebra and ϵ, ϵ' morphisms of k-algebras. Define $\gamma(x + (ay-x^2)) = \epsilon'(x + (ty-x^2))$, $\gamma(y + (ay-x^2)) = \epsilon(y + (ty-x^2))$ (using the k-algebra structure on Z given by $\epsilon'\psi$). It is clear that γ is unique. Hence

$$X_a = \operatorname{Spec} \frac{k[x,y]}{(ay-x^2)}$$

For $a \neq 0$ X_a is the plane curve $ay - x^2$ in $\mathbb{A}^2_k$ which is an irreducible, reduced curve. (That is, X_a is an irreducible reduced scheme of dimension 1. It is irreducible, reduced since it is integral, $(ay - x^2)$ is prime, and $\dim X_a = \dim k[x,y]/(ay-x^2) = \dim Y$ where $Y = Z(ay-x^2)$. We know from Ch.1 that $\dim Y = 1$, so $\dim X_a = 1$) But for $a = 0$, the fibre X_0 is the nonreduced scheme given by $x^2 = 0$ in $\mathbb{A}^2_k$. Thus we have a family in which most members are irreducible curves, but one is nonreduced. This show how nonreduced schemes occur naturally, even as one in is primarily interested in varieties. We can say that the nonreduced scheme $x^2 = 0$ in $\mathbb{A}^2_k$ is a deformation of the irreducible parabola $ay = x^2$ as $a \to 0$. (See Fig 7. in Hartshorne).

EXAMPLE 3.3.2 Similarly, if $X = \operatorname{Spec} k[x,y,t]/(xy - t)$, we get a family whose general member X_a is an irreducible hyperbola $xy = a$, when $a \neq 0$, whose special member X_0 is the reducible scheme consisting of two lines.

NOTE Fibers

Let $f: X \longrightarrow Y$ be a morphism, $y \in Y$. The morphism $\operatorname{Spec} k(y) \longrightarrow Y$ of Ex 2.7 is actually the composite

$$\operatorname{Spec} k(y) \longrightarrow \operatorname{Spec} \mathcal{O}_{Y,y} \longrightarrow Y$$

Since both maps agree on points and have the same local maps. By definition the *fibre* of f at y is $X_y = X \times_Y \operatorname{Spec} k(y)$. We have shown in our A&M notes that if $\varphi: A \longrightarrow B$ is a morphism of rings and $\mathfrak{p} \subseteq A$ prime, $B_\mathfrak{p}$ denoting $(A-\mathfrak{p})^{-1}B$ then $B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p} \cong k(\mathfrak{p}) \otimes_A B$ in such a way that the following commutes:

$$\begin{tikzcd} B \arrow[r] \arrow[drr, "b \mapsto 1 \otimes b"'] & B_\mathfrak{p} \arrow[r] & B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p} \arrow[d, Rightarrow] \\ & & k(\mathfrak{p}) \otimes_A B \end{tikzcd} \qquad \begin{matrix} b/t + \mathfrak{p}B_\mathfrak{p} \\ \downarrow \\ 1/t \otimes b \end{matrix}$$

But $\operatorname{Spec}(B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}) \longrightarrow \operatorname{Spec} B$ gives a homeomorphism of $\operatorname{Spec}(B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p})$ with the subspace $\{\mathfrak{q} \in \operatorname{Spec} B \mid \varphi^{-1}\mathfrak{q} = \mathfrak{p}\}$ of $\operatorname{Spec} B$. Hence the canonical ring morphism $B \longrightarrow k(\mathfrak{p}) \otimes B$ gives rise to $\operatorname{Spec}(k(\mathfrak{p}) \otimes B) \longrightarrow \operatorname{Spec} B$ which is a homeomorphism onto the subspace $\Phi^{-1}\mathfrak{p}$ of $\operatorname{Spec} B$, where $\Phi: \operatorname{Spec} B \longrightarrow \operatorname{Spec} A$ is induced by φ.

PROPOSITION Let $f: X \longrightarrow Y$ be a morphism of schemes, $y \in Y$ and suppose we have a pullback

$$\begin{tikzcd} X_y \arrow[r, "j"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ \operatorname{Spec} k(y) \arrow[r, "i"'] & Y \end{tikzcd} \qquad (1)$$

Then the underlying morphism of spaces $j: X_y \longrightarrow X$ is a homeomorphism of X_y with the subspace $f^{-1}y$ of X.

PROOF Commutativity of (1) shows that $j(X_y) \subseteq f^{-1}y$, so it only remains to show that $j': X_y \longrightarrow f^{-1}y$ is a homeomorphism. But being a homeomorphism is a local property, so it suffices to show there is an open cover of $f^{-1}y$ s.t. the restriction of j' is a homeomorphism on every element of the cover. So it suffices to find an open cover $\{U_i\}$ of $f^{-1}y$ in X (U_i open, $f^{-1}y \subseteq \bigcup_i U_i$) s.t. for each i, $j^{-1}U_i \longrightarrow U_i$ is a homeomorphism of $j^{-1}U_i$ with $f^{-1}y \cap U_i$. Let $x \in f^{-1}y$ be given, $V \cong \operatorname{Spec} A$ an affine open neighborhood of y and $U \cong \operatorname{Spec} B$ an affine open neighborhood of x contained in $f^{-1}V$. Then $j^{-1}U = \operatorname{Spec} k(y) \times_V U \cong \operatorname{Spec} k(\mathfrak{p}) \times_{\operatorname{Spec} A} \operatorname{Spec} B = \operatorname{Spec}(k(\mathfrak{p}) \otimes_A B)$:

$$\begin{tikzcd} j^{-1}U = \operatorname{Spec}(k(\mathfrak{p}) \otimes_A B) \arrow[rr] \arrow[d] \arrow[ddd] & & U = \operatorname{Spec} B \arrow[dl] \arrow[ddd] \\ X_y \arrow[r, "j"] \arrow[d, "g"'] & X \arrow[d, "f"] & \\ \operatorname{Spec} k(y) \arrow[r, "i"'] & Y & \\ \operatorname{Spec} k(\mathfrak{p}) \arrow[u, equal] \arrow[rr] & & V = \operatorname{Spec} A \arrow[ul] \end{tikzcd}$$

As discussed $\operatorname{Spec}(k(\mathfrak{p}) \otimes B) \longrightarrow \operatorname{Spec} B$ is a homeomorphism onto the image of $f^{-1}y \cap U$ in $\operatorname{Spec} B$, and $\mathfrak{p} \in \operatorname{Spec} A$ corresponds to $y \in V$. Here $k(\mathfrak{p}) = A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} \cong k(y)$ via $\mathcal{O}_{Y,y} \cong A_\mathfrak{p}$. One checks the diagram commutes and so $j^{-1}U \longrightarrow U$ is a homeomorphism onto $f^{-1}y \cap U$, as required. $\square$

NOTE Open Immersions of Affine Schemes

Let A be a ring, $S \subseteq A$ a multiplicatively closed set (so $1 \in S$). We say S is <u>saturated</u> if whenever $s \in S$ and $a \in A$ divides s, then also $a \in S$. If S is multiplicatively closed let $T = \{ a \in A \mid a \text{ divides an element of } S \}$. Then T is a saturated multiplicatively closed set. If $\mathfrak{p}$ is prime then $\mathfrak{p} \cap S = \emptyset$ iff. $\mathfrak{p} \cap T = \emptyset$.

<u>LEMMA</u> If S is a multiplicatively closed set with saturation T, $S^{-1}A \cong T^{-1}A$.

<u>PROOF</u> Let $\psi : S^{-1}A \longrightarrow T^{-1}A$, $\psi(a/s) = a/s$ be the ring morphism induced by the fact that $S \subseteq T$. Then $\psi(a/s) = 0$ means there $\exists t \in T$ with $ta = 0$. But there is $m \in A$ s.t. $tm \in S$, so $(tm)a = 0$ implies $a/s = 0$ in $S^{-1}A$. Hence ψ is injective. If $a/t \in T^{-1}A$ and $tm \in S$ then $a/t = am/tm = \psi(am/tm)$ so ψ is an isomorphism. □

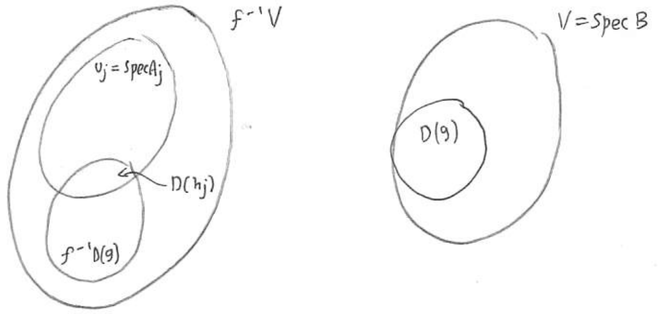
a saturated

<u>LEMMA</u> Let S be multiplicatively closed and suppose $U = \{ \mathfrak{p} \mid \mathfrak{p} \cap S = \emptyset \}$ is open in $\operatorname{Spec} A$. Then S is the saturation of $\{1, g, g^2, \ldots\}$ for some $g \in A$.

<u>PROOF</u> If ~~$U = \emptyset$~~ then $0 \in S$ and hence $S = A$, so S is the saturation of $\{0, 0^2, \ldots\}$. Otherwise let $g \in A$ be s.t. $D(g) \subseteq U$. Let $S_g = \{1, g, \ldots\}$ have saturation T_g. To show $S \subseteq T_g$, suppose $s \in S$ with $s \notin T_g$. That is, the ideal (s) does not meet S_g. Then we can expand (s) to a prime $\mathfrak{q}$ containing (s) but with $\mathfrak{q} \cap S_g = \emptyset$. Thus $g \notin \mathfrak{q}$, or $\mathfrak{q} \in D(g)$. But then $\mathfrak{q} \in U \Longleftrightarrow \mathfrak{q} \cap S = \emptyset$ is a contradiction. Hence $S \subseteq T_g$. In the other direction,

NOTE

Let $f: X \longrightarrow Y$ be a morphism of schemes and $V \cong \operatorname{Spec} B$ an affine open subset of Y with an affine open cover $f^{-1}V = \bigcup_j U_j$, $U_j \cong \operatorname{Spec} A_j$. Let $g_1, \dots, g_n$ be elements of B s.t. $V = \bigcup D(g_i)$.

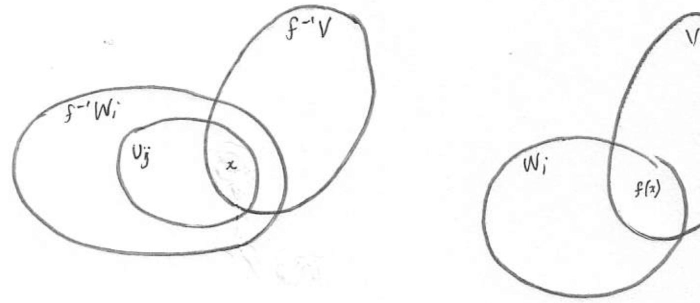


Let $h_j \in A_j$ be the image of g under $B \cong \mathcal{O}_Y(V) \longrightarrow \mathcal{O}_X(f^{-1}V) \longrightarrow \mathcal{O}_X(U_j) \cong A_j$. Then $f^{-1}D(g) = X_{\bar{g}}$ where $\bar{g}$ = image g in $\mathcal{O}_X(f^{-1}V)$ and $D(h_j)$ corresponds to $f^{-1}D(g) \cap U_j$. Hence $f^{-1}D(g)$ is covered by $\operatorname{Spec}(A_j)_{h_j}$.

EXERCISES §3

Q3.1 Let $f: X \to Y$ be a morphism of schemes. We claim that f is locally of finite type if and only if for <u>every</u> open affine subset $V = \operatorname{Spec} B$ of Y, $f^{-1}V$ can be covered by open affine subsets $U_j = \operatorname{Spec} A_j$ where each A_j is a finitely generated B-algebra.

The stated condition is clearly sufficient. To see that it is also necessary, let $V = \operatorname{Spec} B$ be given, and suppose f is locally of finite type. Let $Y = \bigcup_i W_i$ be $W_i = \operatorname{Spec} B_i$ open affine cover of Y s.t. every $f^{-1}W_i$ has the necessary property. Let $x \in f^{-1}V$ be given and find i s.t. $x \in f^{-1}W_i$. Let $U_{ij} \subseteq f^{-1}W_i$ be one of the affine opens covering $f^{-1}W_i$ with $x \in U_{ij}$.



Since $V \cap W_i$ is open there is $b \in B$ s.t. $D(b) \subseteq V \cap W_i$ and $f(x) \in D(b)$. Then $f^{-1}D(b) \subseteq f^{-1}V \cap f^{-1}W_i$ contains x, so $f^{-1}D(b) \cap U_{ij}$ is open and since U_{ij} is affine, say $U_{ij} \cong \operatorname{Spec} A_{ij}$, there is $c \in A_{ij}$ s.t. $D(c) \subseteq f^{-1}D(b) \cap U_{ij}$. We need the following two algebraic facts:

1. If a ring A is a finitely generated R-algebra, say $A = R[a_1, \dots, a_n]$, then for any $f \in A$, A_f is also finitely generated as an R-algebra, since $A_f = R[a_1/1, \dots, a_n/1, 1/f]$.

2. Let B be a ring, $f \in B$ and $B \longrightarrow B_f \xrightarrow{\varphi} C$ morphisms of rings. If via φ, C is a finitely-generated B_f-algebra, then C is a finitely generated B-algebra, since if $C = B_f[c_1, \dots, c_n]$ then $C = B[\varphi(1/f), c_1, \dots, c_n]$.

Since $D(c) \subseteq f^{-1}V$ is an open affine neighborhood of x in $f^{-1}V$, it only remains to show $\mathcal{O}_X(D(c))$ is a f.g. $\mathcal{O}_Y(V)$-algebra. By assumption $\mathcal{O}_X(U_{ij})$ is a f.g. $\mathcal{O}_Y(W_i)$-algebra, so by (1), $D(c)$ is a f.g. $\mathcal{O}_Y(W_i)$ algebra via

$$
\begin{aligned}
& \mathcal{O}_Y(W_i) \longrightarrow \mathcal{O}_X(f^{-1}W_i) \longrightarrow \mathcal{O}_X(D(c)) \\
= \; & \mathcal{O}_Y(W_i) \longrightarrow \mathcal{O}_Y(W_i \cap V) \longrightarrow \mathcal{O}_X(f^{-1}(W_i \cap V)) \longrightarrow \mathcal{O}_X(D(c)) \\
= \; & \mathcal{O}_Y(W_i) \longrightarrow \mathcal{O}_Y(D(b)) \longrightarrow \mathcal{O}_X(f^{-1}D(b)) \longrightarrow \mathcal{O}_X(D(c))
\end{aligned}
$$

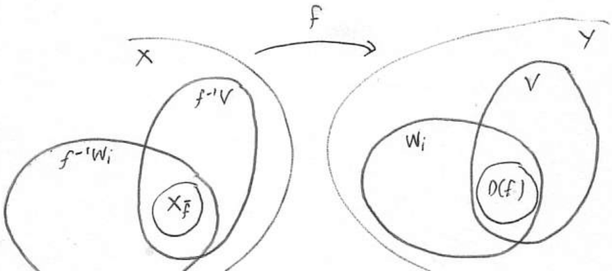
It follows that $\mathcal{O}_X(D(c))$ is a f.g. $\mathcal{O}_Y(D(b))$ algebra. It follows from (2) that $\mathcal{O}_X(D(c))$ is a f.g. $\mathcal{O}_Y(V)$-algebra via $\mathcal{O}_Y(V) \longrightarrow \mathcal{O}_Y(D(b)) \longrightarrow \mathcal{O}_X(f^{-1}D(b)) \longrightarrow \mathcal{O}_X(D(c)) = \mathcal{O}_Y(V) \longrightarrow \mathcal{O}_Y(f^{-1}V) \longrightarrow \mathcal{O}_Y(D(c))$, as required.

(allow empty covers or not, makes no difference)

Q3.2 A morphism $f: X \to Y$ of schemes is <u>quasi-compact</u> if there is a cover of Y by open affines V_i s.t. $f^{-1}V_i$ is quasi-compact for each i. We claim that f is quasi-compact iff. for <u>every</u> open affine subset $V \subseteq Y$, $f^{-1}V$ is quasi-compact. (Equiv. every $y \in Y$ has open affine neigh. V s.t. $f^{-1}V$ is q.c.)

If we assume only V q.c. need $f \in \mathcal{O}_Y(W_i)$ some i

Let an affine open subset $V \subseteq Y$ be given (in fact, it suffices to assume V quasi-compact) and let $V = \bigcup_i D(f_i)$ be a finite cover of V by $D(f_1), \dots, D(f_n)$. If $Y = \bigcup_i W_i$ is the cover by open affines with $f^{-1}W_i$ quasi-compact for all i, then we can arrange for each $D(f_j)$ to be contained in some W_i. Since $f^{-1}V$ is covered by $f^{-1}D(f_i)$ and the finite union of quasi-compact sets is quasi-compact, it suffices to show that $f^{-1}D(f_i)$ is quasi-compact for all i.



$f \in \mathcal{O}_Y(W_i) \cong B_i$
$W_i \cong \operatorname{Spec} B_i$

Let f be some f_i and say $D(f) \subseteq W_i$. Let $\bar{f} \in \mathcal{O}_X(f^{-1}W_i)$ be the image of f. Then $f^{-1}D(f) = X_{\bar{f}}$. Since by assumption $f^{-1}W_i$ is quasi-compact, there is a finite affine open cover $U_{i1}, \dots, U_{in}$ of $f^{-1}W_i$. By Ex 2.16 a) for each k, $X_{\bar{f}} \cap U_{ik}$ is open affine. Hence $X_{\bar{f}}$ is covered by a finite number of quasi-compact open sets, and is thus quasi-compact, as required. NOTE Any morphism out of a noetherian scheme is quasi-compact.

$\boxed{Q3.3}$ (a) We claim that a morphism $f: X \longrightarrow Y$ is of finite type if and only if it is locally of finite type and quasi-compact.

If f is of finite type then it is clearly of locally finite type and quasi-compact. The converse follows from Ex 3.2.

(b) It follows from 3.1 and 3.2 that a morphism $f: X \longrightarrow Y$ is of finite type if and only if for every open affine subset $V = \operatorname{Spec} B$ of Y, $f^{-1}V$ can be covered by a finite number of open affines $U_j = \operatorname{Spec} A_j$, where each A_j is a finitely-generated B-algebra.

(c) Let X be a scheme, $U \subseteq V$ two open affine subsets. Say $U \cong \operatorname{Spec} A$, $V \cong \operatorname{Spec} B$. If $g \in B$ is such that $D(g) \subseteq U$ then $D(g)$ corresponds to the open set $D(\varphi(g))$ of $\operatorname{Spec} A$, where $\varphi: B \longrightarrow A$ is induced by $U \subseteq V$. This is a very useful fact.

Let $f: X \longrightarrow Y$ be a morphism of schemes locally of finite type and suppose $V \subseteq Y$ is open affine and $U \subseteq f^{-1}V$ is open affine. We claim that if $V \cong \operatorname{Spec} B$, $U \cong \operatorname{Spec} A$ then A is a f.g. B-algebra. By 3.1 $f^{-1}V$ is covered by open affine sets $W_i = \operatorname{Spec} C_i$ with each C_i a f.g. B-algebra. Let $x \in U$ be given and find i s.t. $x \in W_i \cap U$. Let $h \in C_i$ be such that $x \in D(h) \subseteq W_i \cap U$. Then $D(h) \cong \operatorname{Spec}(C_i)_h$ and $(C_i)_h$ is a f.g.-B-algebra. Now let $k \in A$ be s.t. $D(k) \subseteq D(h)$. By the above comment $D(k)$ corresponds to $D(k') \subseteq \operatorname{Spec}(C_i)_h$ for some $k' \in (C_i)_h$, and so A_k is a f.g. B-algebra. Since x was arbitrary and U quasi-compact we can cover $\operatorname{Spec} A$ by finitely many $D(k_i)$ s.t. $k_1, \dots, k_n$ generates the unit ideal in A and each A_{k_i} is a f.g. B-algebra. So we have reduced to the following algebra problem:

LEMMA Let A be a B-algebra and $k_1, \dots, k_n \in A$ elements generating the unit ideal, with the property that A_{k_i} is a f.g. B-algebra for all i. Then A is a f.g. B-algebra.

PROOF Pick generators $c_{i1}, c_{i2}, \dots, c_{ie_i}$ for A_{k_i} over B. Then each c_{ij} has the form a_{ij}/k_i^n for some $a_{ij} \in A$ (since finitely many k_i we can assume n fixed). We claim the $\{a_{ij}\}_{1 \le i \le n, 1 \le j \le e_i}$ generate A as a B-algebra. Pick $a \in A$ and let $\phi_i: A \longrightarrow A_i$ be canonical. Suppose

$$\phi_i(a) = p(c_{i1}, \dots, c_{ie_i})$$ NO. START AGAIN

where p is a polynomial with coefficients in B. Clearing denominators we have

PROOF Let $h_1, \dots, h_n \in A$ be s.t. $\sum h_i k_i = 1$. For each i pick generators $c_{i1}, \dots, c_{ie_i}$ for A_{k_i} over B. Each c_{ij} has the form a_{ij}/k_i^n for $a_{ij} \in A$ and $n > 0$ fixed for all i. Let

$$F = \{a_{ij}\}_{1 \le i \le n, 1 \le j \le e_i} \cup \{h_1, \dots, h_n\} \cup \{k_1, \dots, k_n\}$$

We claim that F generates A as a B-algebra. Let $A' = B[F]$ be the B-subalgebra of A generated by F, and let $a \in A$ be given. Then in A_{k_i} we have

$$a/1 = p(c_{i1}, \dots, c_{ie_i})$$

For a polynomial p with coefficients in B. Clearing denominators and translating back into A we find

$$k_i^N a = q_i(a_{i1}, \dots, a_{ie_i}, k_i)$$

where q has coefficients in B, and we can make N large enough to work for all i. Hence $k_i^N a \in A'$ for all i. Since the h_i belong to A', $\{k_i\}$ generates the unit ideal in A' and thus there are $c_i \in A'$ with $\sum c_i k_i^N = 1$. Consequently

$$a = \sum c_i k_i^N a \in A'$$

as required. $\square$

COROLLARY A morphism $f: \operatorname{Spec} B \longrightarrow \operatorname{Spec} A$ of affine schemes is locally of finite type if and only if B is a finitely generated A-algebra via $A \longrightarrow B$. Hence for morphisms of affine schemes locally of finite type $\Leftrightarrow$ finite type (also true for $X \cong \operatorname{Spec} A$ kind of affine schemes)

Q3.4 Let $f: X \longrightarrow Y$ be a finite morphism of schemes and let $V \cong \operatorname{Spec} A$ be an affine open subset of Y. We claim that $f^{-1}V$ is an affine open subset of X with $f^{-1}V \cong \operatorname{Spec} B$ and B a f.g. A-module.
(equiv $\mathcal{O}_X(f^{-1}V)$ is a f.g. $\mathcal{O}_Y(V)$-module)

Let $\{W_\alpha\}$ be a nonempty open cover of Y by open affines $W_\alpha \cong \operatorname{Spec} A_\alpha$ s.t. $f^{-1}W_\alpha \cong \operatorname{Spec} B_\alpha$ for a f.g. A_α-module B_α. For $x \in V$ find an index α and $g \in A_\alpha$ s.t. $x \in D(g) \subseteq W_\alpha \cap V$. Then find $g' \in A$ with $x \in D(g') \subseteq D(g)$. The inclusion $D(g) \longrightarrow V$ induces a ring morphism $\gamma: A \longrightarrow (A_\alpha)_g$ and $D(g')$ corresponds to $D(\gamma(g')) \subseteq \operatorname{Spec}(A_\alpha)_g$, and thus if we put $\gamma(g') = a/g^n$ the open set $D(g') \subseteq W_\alpha \cap V$ corresponds to $D(ag) \subseteq \operatorname{Spec} A_\alpha$. The whole point being that we need an open neighborhood of x which is distinguised for both W_α and V. Since $f^{-1}W_\alpha$ is affine, $f^{-1}D(g')$ is affine, equal to $\operatorname{Spec}(B_\alpha)_h$ some $h \in B_\alpha$. Let $\bar{g}' \in \mathcal{O}_X(f^{-1}V)$ denote the image of g. Then also $f^{-1}D(g') = X_{\bar{g}'}$.

Since V is covered by such $D(g')$ we can find $g_1, \dots, g_n \in A$ which generate the unit ideal in A and s.t. $X_{\bar{g}_i}$ are all affine. Since $\bar{g}_1, \dots, \bar{g}_n$ must also generate the unit ideal in $\mathcal{O}_X(f^{-1}V)$ it follows from Ex 2.17 that $f^{-1}V$ is affine, say $f^{-1}V \cong \operatorname{Spec} B$.

Let $\varphi: A \longrightarrow B$ be the ring morphism corresponding to $f^{-1}V \longrightarrow V$. Then the $\varphi(g_i)$ are such that $B_{\varphi(g_i)}$ is a f.g. A_{g_i}-module by the assumption on the $f^{-1}W_\alpha$. In more detail: let $\varphi_\alpha: A_\alpha \longrightarrow B_\alpha$ correspond to $f^{-1}W_\alpha \longrightarrow W_\alpha$. Then for some α by construction $D(g_i) \subseteq W_\alpha \cap V$, and $D(\varphi(g_i)) = f^{-1}D(g_i)$ $\cong \operatorname{Spec}(B_\alpha)_{\varphi_\alpha(h_i)}$ where $h_i \in A_\alpha$ is s.t. $D(h_i) = D(g_i)$. Since B_α is a f.g. A_α-module, $(B_\alpha)_{\varphi_\alpha(h_i)}$ is a f.g. $(A_\alpha)_{h_i}$-module. A bit of careful thinking shows that this implies $B_{\varphi(g_i)}$ is a f.g. A_{g_i}-module. We need to show B is a f.g. A-module – so we have reduced to the following algebra problem:

LEMMA Let $\varphi: A \longrightarrow B$ be a morphism of rings, $g_1, \dots, g_n$ elements of A which generate the unit ideal and suppose via $\varphi_{g_i}: A_{g_i} \longrightarrow B_{\varphi(g_i)}$ that $B_{\varphi(g_i)}$ is a finitely generated A_{g_i}-module for all i. Then B is a f.g. A-module.

PROOF For each i suppose $c_{i1}, \dots, c_{ie_i}$ generate $B_{\varphi(g_i)}$ as a A_{g_i}-module. We may assume that for $1 \le i \le n, 1 \le j \le e_i$ $c_{ij} = a_{ij}/\varphi(g_i)^n$ for $a_{ij} \in B$ and $n > 0$ independent of i. Put $F = \{a_{ij}\}_{1 \le i \le n, 1 \le j \le e_i}$ and let B' be the A-submodule generated by F. We claim that $B' = B$. Let $b \in B$ be given. Then in $B_{\varphi(g_i)}$ we have

$$\begin{aligned} b/1 &= \frac{\alpha_1}{g_i^{n_i}} \cdot \frac{a_{i1}}{\varphi(g_i)^n} + \dots + \frac{\alpha_n}{g_i^{n_i}} \cdot \frac{a_{ie_i}}{\varphi(g_i)^n} \\ &= \frac{\alpha_1 \cdot a_{i1} + \dots + \alpha_n \cdot a_{ie_i}}{\varphi(g_i)^{n+n_i}} \end{aligned}$$

(can assume n_i constant for the coefficients)

Consequently $\varphi(g_i)^N b \in B'$ for $N > 0$ independent of i. The elements g_i^N generate the unit ideal of A so for some $h_i \in A$ we have $\sum h_i g_i^N = 1$ and thus

$$\begin{aligned} b &= b \sum \varphi(h_i)\varphi(g_i)^N \\ &= \sum \varphi(h_i)\{\varphi(g_i)^N b\} \\ &= \sum h_i \cdot \{\varphi(g_i)^N b\} \in B' \end{aligned}$$

as required. $\square$

COROLLARY A morphism of affine schemes $\operatorname{Spec} A \longrightarrow \operatorname{Spec} B$ is finite if and only if A is a finitely generated B-module via $B \longrightarrow A$. (So $f: X \longrightarrow Y$ is finite for $X \cong \operatorname{Spec} A$, $Y \cong \operatorname{Spec} B$ iff. $\mathcal{O}_X(X)$ is a f.g. $\mathcal{O}_Y(Y)$-module).

$\boxed{Q3.5}$ A morphism $f: X \longrightarrow Y$ is quasi-finite if for every point $y \in Y$, $f^{-1}(y)$ is a finite set.

(a) A finite morphism is quasi-finite. Suppose $f: X \longrightarrow Y$ is finite and $y \in Y$. Let V be an affine open neighborhood of y. By Ex 3.4 $f^{-1}V$ is affine and $\mathcal{O}_X(f^{-1}V)$ is a f.g. $\mathcal{O}_Y(V)$-module. So we can reduce to $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$ since if f is finite so is $f^{-1}V \longrightarrow V$. Let $\varphi: B \longrightarrow A$ correspond to $f: \operatorname{Spec} A \longrightarrow \operatorname{Spec} B$. We have shown in our notes on fibers that for $\mathfrak{p} \in \operatorname{Spec} B$

$$f^{-1}\mathfrak{p} \cong \operatorname{Spec}(k(\mathfrak{p}) \otimes_B A) \qquad \text{"bijection"}$$

where $k(\mathfrak{p}) = B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}$. So it would suffice to show that $k(\mathfrak{p}) \otimes_B A$ is artinian. By assumption A is a f.g. B-module and so $k(\mathfrak{p}) \otimes_B A$ is a f.g. $k(\mathfrak{p})$-vector space, hence finite dimensional, and thus has DCC on $k(\mathfrak{p})$-submodules. But an ideal is a $k(\mathfrak{p})$-submodule, so $k(\mathfrak{p}) \otimes_B A$ is Artinian.

(b) A finite morphism is closed, i.e. $f(Q)$ is closed in Y $\forall Q \subseteq X$ closed. Let $f: X \longrightarrow Y$ be finite and $Q \subseteq X$ closed. Cover Y with affine opens $\{V_\alpha\}$ s.t. $f^{-1}V_\alpha$ is affine. It suffices to show $f(Q) \cap V_\alpha$ is closed in V_α for all α. Hence we reduce to the case $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$, $f: \operatorname{Spec} A \longrightarrow \operatorname{Spec} B$ finite, so $\varphi: B \longrightarrow A$ and A a f.g. B-module. Let $V(\mathfrak{a}) \subseteq \operatorname{Spec} A$ be closed. We show that $f(V(\mathfrak{a})) = V(\varphi^{-1}\mathfrak{a})$. Clearly $\subseteq$. For the reverse inclusion, notice that is integral over $\varphi(B)$ (A&M 5.1). Let $\mathfrak{q}$ be a prime of B containing $\varphi^{-1}\mathfrak{a}$. If B or A are 0 there is nothing to prove. Otherwise $\varphi(\mathfrak{q})$ is a prime ideal of $\varphi(B)$ containing $\mathfrak{a} \cap \varphi(B)$. Using A&M 5.10 there is $\mathfrak{p} \in \operatorname{Spec} A$ with $\mathfrak{p} \cap \varphi(B) = \varphi(\mathfrak{q})$. Moreover by A&M 5.6 (i) we can arrange for $\mathfrak{p} \supseteq \mathfrak{a}$. Hence $V(\varphi^{-1}\mathfrak{a}) \subseteq f(V(\mathfrak{a}))$ and so f is closed.

$\boxed{Q3.6}$ Let X be an integral scheme. Then X is irreducible, so by Ex 2.9, X has a unique generic point ξ. Let U be an affine open neighborhood of ξ, say $U \cong \operatorname{Spec} A$. The prime of A corresponding to ξ is then the generic point of $\operatorname{Spec} A$, which is (0), since $A \cong \mathcal{O}_X(U)$ is a domain. So the local ring $\mathcal{O}_\xi \cong A_{(0)}$ which is the quotient field of A. This field is denoted by $K(X)$ and is called the <u>function field</u> of X. Any nonempty open subset of X must contain ξ, so the second claim is trivial.

Notice that for any point $x \in X$ there is a ring morphism $\mathcal{O}_{X,x} \longrightarrow K$ $(= \mathcal{O}_{X,\xi})$, given by $(U,s) \mapsto (U,s)$. Since X is integral $\mathcal{O}_{X,x}$ is a domain, so we can consider $\mathcal{O}_{X,x}$ as a subring of K. To see that $\mathcal{O}_{X,x} \longrightarrow K$ is injective, let U be an open affine neighborhood of x, $U \cong \operatorname{Spec} A$, and note that the following diagram commutes:

$$\begin{array}{ccc}
\mathcal{O}_{X,x} & \longrightarrow & K \\
\wr\Vert & & \wr\Vert \\
(\mathcal{O}_X|_U)_x & \longrightarrow & (\mathcal{O}_X|_U)_\xi \\
\wr\Vert & & \wr\Vert \\
\mathcal{O}_{\operatorname{Spec} A, \mathfrak{p}} & \longrightarrow & \mathcal{O}_{\operatorname{Spec} A, (0)} \\
\wr\Vert & & \wr\Vert \\
A_\mathfrak{p} & \longrightarrow & Q
\end{array}$$

Since the bottom map is injective, so is the top. Since Q is the quotient field of $A_\mathfrak{p}$, it follows that K is also the smallest subfield containing $\mathcal{O}_{X,x}$, hence K is the quotient field of $\mathcal{O}_{X,x}$ for all $x \in X$.

<u>LEMMA</u> Let X be a scheme, $Y \subseteq X$ a closed irreducible subset with generic point ξ. Then

$$\dim \mathcal{O}_{X,\xi} = \operatorname{codim}(Y,X)$$

<u>PROOF</u> Let $U \cong \operatorname{Spec} A$ be an affine open neighborhood of ξ in X. Then let $\mathfrak{p} \in \operatorname{Spec} A$ correspond to ξ. Clearly $Y \cap U$ is the closure in U of ξ, so $Y \cap U$ corresponds to $V(\mathfrak{p}) \subseteq \operatorname{Spec} A$. If Z, Z' are closed irreducible subsets of X meeting U with $Z \subset Z'$, then $Z \cap U \subset Z' \cap U$. If $Z \cap U = Z' \cap U$ then we would have $Z' = Z \cup (Z' \cap U^c)$ contradicting $Z \subset Z'$ on the one hand and $Z' \cap U \neq \emptyset$ on the other, since Z' irreducible. It follows that if $Y \subset Z_1 \subset \cdots \subset Z_n$ is a chain of irreducible closed subsets of X then $Y \cap U \subset Z_1 \cap U \subset \cdots \subset Z_n \cap U$ is a chain of irreducible closed subsets of U. Hence $\operatorname{codim} V(\mathfrak{p}) \geq \operatorname{codim}(Y,X)$.

On the other hand if $Q \subset Q'$ are closed irreducible subsets of U then $\overline{Q} \subset \overline{Q'}$ since $\overline{Q} \cap U = Q$, so a chain $Y \cap U \subset Q_1 \subset \cdots \subset Q_n$ becomes $Y \subset \overline{Q}_1 \subset \cdots \subset \overline{Q}_n$ so $\operatorname{codim} V(\mathfrak{p}) \leq \operatorname{codim}(Y,X)$ and consequently

$$\begin{aligned}
\dim \mathcal{O}_{X,\xi} &= \dim \mathcal{O}_{\operatorname{Spec} A, \mathfrak{p}} \\
&= \dim A_\mathfrak{p} \\
&= \operatorname{ht.} \mathfrak{p} \\
&= \operatorname{codim} V(\mathfrak{p}) \\
&= \operatorname{codim}(Y,X)
\end{aligned}$$

Since there is an inclusion reversing bijection between closed sets of $\operatorname{Spec} A$ and radical ideals of A, which identifies prime ideals and irreducible sets. $\square$

<u>LEMMA</u> Let X be an integral scheme with generic point ξ. Sections $s \in \mathcal{O}_X(U)$, $t \in \mathcal{O}_X(V)$ $(U, V \neq \emptyset \therefore \text{contain } \xi)$ then s,t have the same germ at ξ if and only if $s|_{U\cap V} = t|_{U \cap V}$.

<u>PROOF</u> By the above for any $x \in U \cap V$, $\mathcal{O}_{X,x} \longrightarrow \mathcal{O}_{X,\xi}$ is injective, so since $\operatorname{germ}_\xi s = \operatorname{germ}_\xi t$ we have $\operatorname{germ}_x s = \operatorname{germ}_x t$ also. Hence $s|_{U\cap V} = t|_{U\cap V}$. $\square$

For exactly the same reason, if X is integral then $\mathcal{O}_X(U) \longrightarrow K$ is injective for all open U $(K = \mathcal{O}_{X,\xi})$ and two sections $s,t \in \mathcal{O}_X(U), \mathcal{O}_X(V)$ agree on $U \cap V$ iff. they agree on some nonempty open $W \subseteq U \cap V$.

$\boxed{Q3.11}$ We do b) first and return to a)

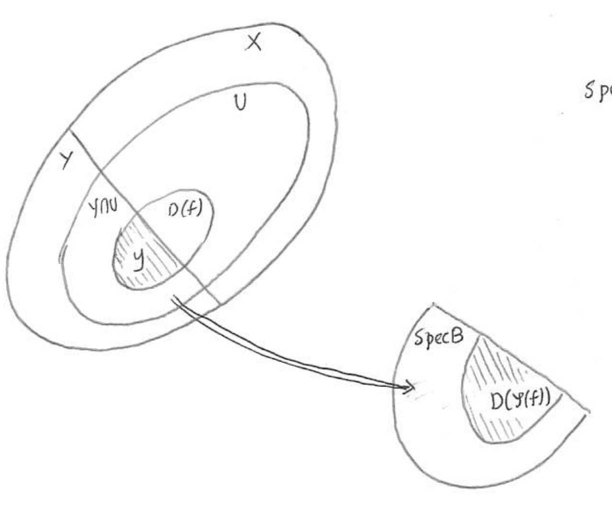
b) We prove the following: every closed subscheme $Y \longrightarrow X$ of an affine scheme $X = \operatorname{Spec} A$ is of the form $\operatorname{Spec}(A/\mathfrak{a}) \longrightarrow \operatorname{Spec}(A)$ for some ideal $\mathfrak{a} \subseteq A$. That is, if $f : Y \longrightarrow X$ is a closed immersion, there is an ideal $\mathfrak{a} \subseteq A$ and an isomorphism $Y \cong \operatorname{Spec}(A/\mathfrak{a})$ fitting into the following commutative diagram

$$\begin{tikzcd}
Y \arrow[rr, "f"] & & \operatorname{Spec} A \\
& \operatorname{Spec}(A/\mathfrak{a}) \arrow[ul, Rightarrow, "\sim"] \arrow[ur] &
\end{tikzcd}$$

<u>PROOF</u> First of all, any closed immersion is equivalent to the inclusion of a scheme structure on a closed subset, so we may assume $Y \subseteq X$ is closed, and the underlying map of f is the inclusion. If $Y = \emptyset$ there is nothing to prove. Otherwise, for $y \in Y$ since Y is a scheme there is an open set $U \subseteq X$ s.t. $y \in Y \cap U$ and $Y \cap U$ is affine in Y, say $Y \cap U \cong \operatorname{Spec} B$. Since the open sets $D(f)$ form a base for the topology on X, there is $f \in A$ with $y \in D(f) \subseteq U$. Let $g \in \mathcal{O}_Y(Y)$ correspond to f and $g' \in B$ correspond to g:

$$\begin{tikzcd}
Y \cap U \arrow[r] & Y \arrow[d] \\
\operatorname{Spec} B \arrow[u, Rightarrow] \arrow[r] & \operatorname{Spec} A
\end{tikzcd}$$

$\varphi : A \longrightarrow B$



It follows that the open subset $D(f) \cap Y$ of Y is affine, isomorphic to $B_{\varphi(f)}$. Notice that $D(f) \cap Y = X_g$. Let $\{f_i\}$ be a collection of elements of A so that each $D(f_i) \cap Y = X_{g_i}$, $g_i \in \mathcal{O}_Y(Y)$ is affine and the X_{g_i} cover Y. By adding f_i with $D(f_i) \cap Y = \emptyset$ if necessary we may assume the $D(f_i)$ cover X (so some $X_{g_i} = \emptyset$, but this is fine). Hence the f_i generate the unit ideal in A, and thus the g_i generate the unit ideal in $\mathcal{O}_Y(Y)$ (we can assume finitely many f_i). It follows from Ex 2.17 that $(Y, \mathcal{O}_Y)$ is affine, say $Y \cong \operatorname{Spec} C$. Then $\operatorname{Spec} C \Rightarrow Y \longrightarrow X$ is a closed immersion, from which it follows By Ex 2.18 d) that there is a surjective ring morphism $\alpha : A \longrightarrow C$ inducing $\operatorname{Spec} C \longrightarrow \operatorname{Spec} A$. Let $\mathfrak{a}$ be the kernel of this morphism. Then we have commutative diagrams

$$\begin{tikzcd}
A \arrow[r] \arrow[dr] & C \arrow[d, Rightarrow] \\
& A/\mathfrak{a}
\end{tikzcd}
\qquad
\begin{tikzcd}
& Y \arrow[dl] & \\
X & & \operatorname{Spec} C \arrow[ll] \arrow[ul, Rightarrow] \\
& \operatorname{Spec}(A/\mathfrak{a}) \arrow[ul] \arrow[ur, Rightarrow] &
\end{tikzcd}$$

which completes the proof. $\square$

a) Consider a pullback diagram

$$\begin{tikzcd}
X' \arrow[r, "g"] \arrow[d] & Y' \arrow[d] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

in which f is a closed immersion. Since being a closed immersion is local in Y', we may simply show that every point $y \in Y'$ has an open neighborhood V' for which $g^{-1}V' \longrightarrow V'$ is a closed immersion. Let $y \in Y'$ be given, let V be an affine open neighborhood of the image of y in Y, and V' an affine open neighborhood of y whose image is contained in V. Then $f^{-1}V \longrightarrow V$ is a closed immersion and the following diagram is a pullback (see earlier notes)

$$\begin{tikzcd} g^{-1}V' \arrow[r] \arrow[d] & V' \arrow[d] \\ f^{-1}V \arrow[r] & V \end{tikzcd}$$

which means we can reduce to the case where Y, Y' are affine. But then by (a) we can reduce to the case where X is affine. Since the pullback of affine schemes, we can reduce to showing that in a pullback

$$\begin{tikzcd} \mathrm{Spec}(B \otimes_A A/\mathfrak{a}) \arrow[r] \arrow[d] & \mathrm{Spec}\,B \arrow[d] \\ \mathrm{Spec}(A/\mathfrak{a}) \arrow[r] & \mathrm{Spec}\,A \end{tikzcd}$$

the top morphism is a closed immersion, which by Ex 2.18 is equivalent to showing that $B \longrightarrow B \otimes_A A/\mathfrak{a}$ is surjective, which is obvious. Thus closed immersions are stable under base change.

Q3.12 Closed Subschemes of Proj S

(a) First we note the following facts: the kernel of a morphism of graded rings is homogenous, if S is graded, $\mathfrak{a} \subseteq S$ homogenous the correspondence between ideals containing $\mathfrak{a}$ and the ideals of $S/\mathfrak{a}$ identifies homogenous ideals, where $S/\mathfrak{a}$ is canonically graded. If $\psi: S \longrightarrow T$ is a surjection of graded rings, the map $S/\mathfrak{a} \Longrightarrow T$ is an isomorphism of graded rings ($\mathfrak{a} = \operatorname{Ker}\psi$).

Let $\varphi: S \longrightarrow T$ be a surjective morphism of graded rings, with kernel $\mathfrak{a}$. The open set U of Ex 2.14 is clearly all of $\operatorname{Proj} T$, so we have a morphism $\Phi: \operatorname{Proj} T \longrightarrow \operatorname{Proj} S$. We claim Φ is a closed immersion. Firstly we show $\operatorname{Im}\Phi = V(\mathfrak{a})$. This follows immediately from the fact that $S/\mathfrak{a} \cong T$ as graded rings. This also implies Φ is injective, and for a homogenous ideal $\mathfrak{b} \subseteq T$ we have $\Phi(V(\mathfrak{b})) = V(\varphi^{-1}\mathfrak{b})$. The inclusion $\subseteq$ is clear, and if $\mathfrak{q} \in V(\varphi^{-1}\mathfrak{b}) \subseteq V(\mathfrak{a})$ and say $\mathfrak{q} = \varphi^{-1}\mathfrak{p}$, then $\mathfrak{p} \supseteq \mathfrak{b}$ since otherwise the surjectivity of φ means $\exists \varphi(s) \notin \mathfrak{p}$ with $\varphi(s) \in \mathfrak{b}$, contradicting $\varphi^{-1}\mathfrak{p} \supseteq \varphi^{-1}\mathfrak{b}$. Hence Φ is a homeomorphism onto the closed set $V(\mathfrak{a}) \subseteq \operatorname{Proj} S$.

To show that Φ is a closed immersion, it suffices to show that for all $\mathfrak{p} \in \operatorname{Proj} T$, $\Phi_\mathfrak{p}: \mathcal{O}_{\operatorname{Proj} S, \Phi(\mathfrak{p})} \longrightarrow \mathcal{O}_{\operatorname{Proj} T, \mathfrak{p}}$ is surjective. But the following commutes:

$$\begin{array}{ccc} \mathcal{O}_{\operatorname{Proj} S, \Phi(\mathfrak{p})} & \xrightarrow{\Phi_\mathfrak{p}} & \mathcal{O}_{\operatorname{Proj} T, \mathfrak{p}} \\ \Downarrow & & \| \\ S_{(\varphi^{-1}\mathfrak{p})} & \xrightarrow[\varphi_{(\mathfrak{p})}]{} & T_{(\mathfrak{p})} \end{array}$$

So it suffices to show $\varphi_{(\mathfrak{p})}$ is surjective. Let $b \in T$ and $t \in T$ with $t \notin \mathfrak{p}$ and b, t homogenous of the same degree be given. Say $a, s \in A$ with $\varphi(a) = b$, $\varphi(s) = t$. We may assume no homogenous part of a or s belongs to $\mathfrak{a}$, in which case $\varphi(a) = b$, $\varphi(s) = t$ implies a, s are homogenous of the same degree and $s \notin \varphi^{-1}\mathfrak{p}$. Hence $\varphi_{(\mathfrak{p})}(a/s) = b/t$ and so $\varphi_{(\mathfrak{p})}$ is surjective, as required.

<u>LEMMA</u> Let S be a graded ring, $\mathfrak{a}$ a homogonous ideal and $\mathfrak{p} \in \operatorname{Proj} S$. Suppose there is $d_0 \geq 0$ s.t. $\forall d \geq d_0$ $\mathfrak{a}_d \subseteq \mathfrak{p}_d$. Then $\mathfrak{a} \subseteq \mathfrak{p}$.

<u>PROOF</u> By recursion on d_0. If $d_0 = 0$ we are done, since $\mathfrak{a} = \bigoplus_{d \geq 0} \mathfrak{a}_d$. For the recursive step we assume $d_0 > 0$ and $\forall d \geq d_0$ $\mathfrak{a}_d \subseteq \mathfrak{p}_d$, and we show $\mathfrak{a}_{d_0 - 1} \subseteq \mathfrak{p}$. Since $\mathfrak{p} \not\supseteq S_+$ there is some $e > 0$ and $f \in S_e$ with $f \notin \mathfrak{p}$. If $a \in \mathfrak{a}_{d_0-1}$ then $af \in \mathfrak{a}_{d_0-1+e} \subseteq \mathfrak{p}$. Hence since $f \notin \mathfrak{p}$ we have $a \in \mathfrak{p}$ as required. □

So if $\varphi: S \longrightarrow T$ is a surjective morphism of graded rings and $\exists d_0 \geq 0$ s.t. φ_d is an isomorphism for all $d \geq d_0$ then $f: \operatorname{Proj} T \longrightarrow \operatorname{Proj} S$ is an isomorphism. By (a) it suffices to show that $V(\mathfrak{a}) = \operatorname{Proj} S$ where $\mathfrak{a} = \operatorname{Ker}\varphi$ and $\varphi_{(\mathfrak{p})}: S_{(\varphi^{-1}\mathfrak{p})} \longrightarrow T_{(\mathfrak{p})}$ is injective for all $\mathfrak{p} \in \operatorname{Proj} T$.

To show $V(\mathfrak{a}) = \operatorname{Proj} S$ let $\mathfrak{p} \in \operatorname{Proj} S$ be given. Since φ_d is injective $\forall d \geq d_0$ it follows that $\mathfrak{a}_d = 0$ $\forall d \geq d_0$. Hence $\mathfrak{a}_d \subseteq \mathfrak{p}$ $\forall d \geq d_0$ and by the Lemma $\mathfrak{a} \subseteq \mathfrak{p}$. Hence $V(\mathfrak{a}) = \operatorname{Proj} S$. To see that $\varphi_{(\mathfrak{p})}$ is injective, let $a/s \in S_{(\varphi^{-1}\mathfrak{p})}$ be given. Since $\varphi^{-1}\mathfrak{p} \not\supseteq S_+$ we can find $f \in S_e$, $e > 0$ with $f \notin \varphi^{-1}\mathfrak{p}$. Then $a/s = af^n/sf^n$ for $n > 0$ so by making n sufficiently large we may assume a, s homogenous of degree $> d_0$. Then $\varphi_{(\mathfrak{p})}(a/s) = 0$ means $t\varphi(a) = 0$ for some homogenous $t \notin \mathfrak{p}$. Since φ is surjective $t = \varphi(q)$ for some homogenous $q \notin \varphi^{-1}\mathfrak{p}$. But $t\varphi(a) = 0 \Longrightarrow \varphi(qa) = 0 \Longrightarrow aq = 0$ since $\varphi_d: S_d \longrightarrow T_d$ is iso $\forall d \geq d_0$. Hence $a/s = 0$ in $S_{(\varphi^{-1}\mathfrak{p})}$ as required.

(b) If $I \subseteq S$ is a homogenous ideal, $S \longrightarrow S/I$ gives rise to a closed immersion $f: \operatorname{Proj} S/I \longrightarrow \operatorname{Proj} S$. In fact, distinct ideals can give rise to the same closed subscheme. For example if $d_0 \geq 0$ and $I' = \bigoplus_{d \geq d_0} I_d$ then I' is a homogenous ideal. Let $\alpha: S/I' \longrightarrow S/I$ be the induced morphism of graded rings. Then α is surjective and for $d \geq d_0$, α_d is an isomorphism, so $\operatorname{Proj} S/I \longrightarrow \operatorname{Proj} S/I'$ is an isomorphism of schemes. The following diagram commutes:

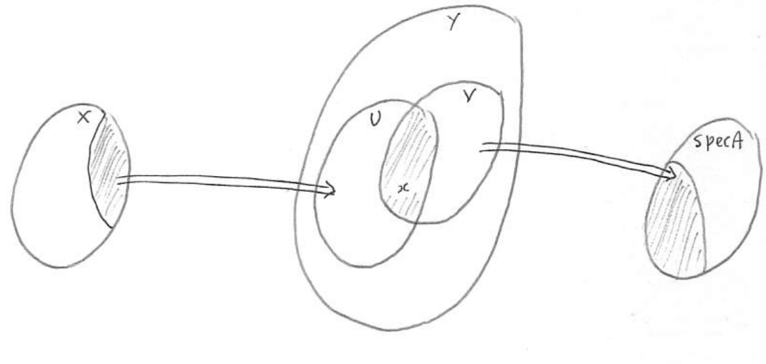
$$\begin{array}{ccc} \operatorname{Proj} S/I & \Longrightarrow & \operatorname{Proj} S/I' \\ & \searrow \quad \swarrow & \\ & \operatorname{Proj} S & \end{array}$$

So I, I' determine the same closed subscheme.

Q3.13 Properties of Morphisms of Finite Type

(a) The property of being a closed immersion is local on the codomain, and the only closed immersions into $\operatorname{Spec} A$ are of the form $\operatorname{Spec}(A/\mathfrak{a}) \longrightarrow \operatorname{Spec}(A)$, so we may reduce to the case of a closed immersion $\operatorname{Spec}(A/\mathfrak{a}) \longrightarrow \operatorname{Spec}(A)$, since being of finite type is also a local property. But $\operatorname{Spec}(A/\mathfrak{a}) \longrightarrow \operatorname{Spec}(A)$ is clearly of finite type.

(b) By Ex 3.3a) it suffices to show that an open immersion is locally of finite type. Let $f: X \longrightarrow Y$ be an open immersion factored as $X \Rightarrow U \longrightarrow Y$ where $U \subseteq Y$ is open. For each $x \in U$ let $x \in V \subseteq Y$ be an affine open neighbourhood of x in Y, say $U \cong \operatorname{Spec} A$.



The image of $U \cap V$ in $\operatorname{Spec} A$ is open, and hence can be covered by open affines A_{f_i}, $f_i \in A$. But $A \longrightarrow A_{f_i}$ clearly exhibits $A_{f_i} = A[1/f_i]$ as a f.g. A-algebra, so $X \longrightarrow Y$ is of locally finite type.

(c) A bit of thought shows that the composition of two quasi-compact morphisms is quasi-compact, so it suffices to show the composition of two morphisms of locally finite type as locally of finite type. Suppose $f: X \longrightarrow Y$ and $g: Y \longrightarrow Z$ are locally of finite type, and let $V \subseteq Z$ be an open affine, $g^{-1}V = \cup_i V_i$ a cover by open affines s.t. each $\mathcal{O}_Y(V_i)$ is a f.g. $\mathcal{O}_Z(V)$ algebra, and $f^{-1}V_i = \cup_j V_{ij}$ an affine open cover with each $\mathcal{O}_X(V_{ij})$ a f.g. $\mathcal{O}_Y(V_i)$-algebra. Then the U_{ij} cover $(gf)^{-1}V$ and it suffices to note that if $A \longrightarrow B$ makes B a f.g. A-algebra on generators $b_1, \dots, b_r$ and $B \longrightarrow C$ makes C a f.g. B-algebra on generators $c_1, \dots, c_s$ then the images of the b_i together with the c_j make C f.g. as an A-algebra.

(d) Let $f: X \longrightarrow Y$ be of finite type, $i: Y' \longrightarrow Y$ another morphism. Consider a pullback diagram

$$\begin{tikzcd} X' \arrow[r, "g"] \arrow[d, "h"'] & Y' \arrow[d, "i"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

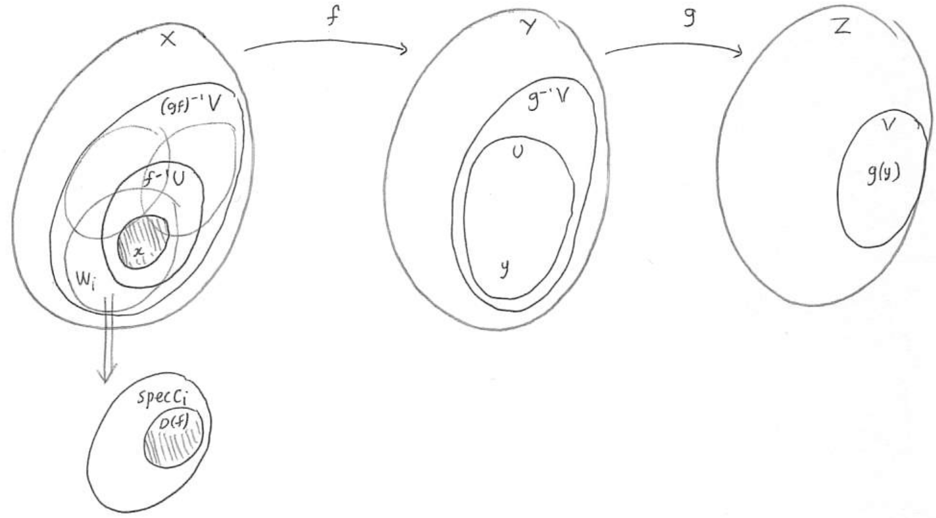
Since being of finite type is local, let $y \in Y$, $i(y) \in V \subseteq Y$ an affine open neighbourhood of $i(y)$ and $y \in V' \subseteq i^{-1}V$ an affine neighbourhood of y. Then we get a pullback involving $g^{-1}V'$, $f^{-1}V$, V, V' by which we reduce to the case where $Y' = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$ are affine. Since f is of finite type, there is an open cover of X by affine open sets $V_i \cong \operatorname{Spec} C_i$ with each C_i a f.g. A-algebra. Pulling back along h we obtain

$$\begin{tikzcd} \operatorname{Spec}(C_i \otimes_B A) \arrow[r, Rightarrow] \arrow[d] \arrow[rrr, bend left] & h^{-1}V_i \arrow[r] \arrow[d] & X' \arrow[r, "g"] \arrow[d, "h"'] & Y' = \operatorname{Spec} A \arrow[d, "i"] \\ \operatorname{Spec} C_i \arrow[r, Rightarrow] \arrow[rrr, bend right] & V_i \arrow[r] & X \arrow[r, "f"'] & Y = \operatorname{Spec} B \end{tikzcd}$$

It thus suffices to show that for each i, $C_i \otimes_B A$ is a f.g. A-algebra. But if $C_i = B[c_1, \dots, c_r]$ then it is easily checked that $C_i \otimes_B A = A[c_1 \otimes 1, \dots, c_r \otimes 1]$, which completes the proof.

(e) Follows immediately from (c), (d).

(f) Let $f: X \longrightarrow Y$ and $g: Y \longrightarrow Z$ be morphisms, and assume f is quasi-compact, $g \circ f$ of finite type. We show f is of finite type, for which it suffices to show that f is locally of finite type. Let $y \in Y$ be given, and let $V \subseteq Y$ be an affine open neighborhood of $g(y)$, U an affine open neighborhood of y contained in $g^{-1}V$. By assumption $f^{-1}g^{-1}V$ is covered by open affines W_i s.t. $\mathcal{O}_X(W_i)$ is a f.g. $\mathcal{O}_Z(V)$-algebra



If $W_i \cong \operatorname{Spec} C_i$ then for $x \in f^{-1}U$ find W_i s.t. $x \in f^{-1}U \cap W_i$ and let $f \in C_i$ given an affine open set $D(f)$ which is a neighborhood of x in $f^{-1}U$. Since C_i is a f.g. $\mathcal{O}_Z(V)$-algebra, so is C_{if}. It follows that $\mathcal{O}_Z(V) \longrightarrow \mathcal{O}_Y(U) \longrightarrow \mathcal{O}_X(f^{-1}U) \longrightarrow D(f)$ gives $D(f)$ as a f.g. $\mathcal{O}_Z(V)$-algebra, and hence $\mathcal{O}_Y(U) \longrightarrow O(f)$ gives $D(f)$ as a f.g. $\mathcal{O}_Y(U)$-algebra, as required. Hence f is locally of finite type, and hence of finite type, since it is compact.

(g) Let $f: X \longrightarrow Y$ be a morphism of finite type and assume Y is noetherian. First of all if $f: X \longrightarrow Y$ is a morphism of finite type and Y is quasi-compact then f is quasi-compact and hence X is quasi-compact. So it suffices to show that if Y is locally noetherian then so is X. Let $Y = \bigcup_i V_i$ be a cover of Y by affine open sets, $V_i \cong \operatorname{Spec} A_i$ with A_i noetherian. Then $f^{-1}V_i$ is a union of affine open sets $U_{ij} \cong \operatorname{Spec} B_{ij}$ with B_{ij} a f.g. A_i-algebra. Hence B_{ij} is noetherian, and since the U_{ij} cover X, X is locally noetherian.

(h) Let $f: X \longrightarrow Y$ and $f': X' \longrightarrow Y'$ be morphisms of schemes over S, both f and f' of finite type. We show that the product $f \times f': X \times_S X' \longrightarrow Y \times_S Y'$ is also of finite type:

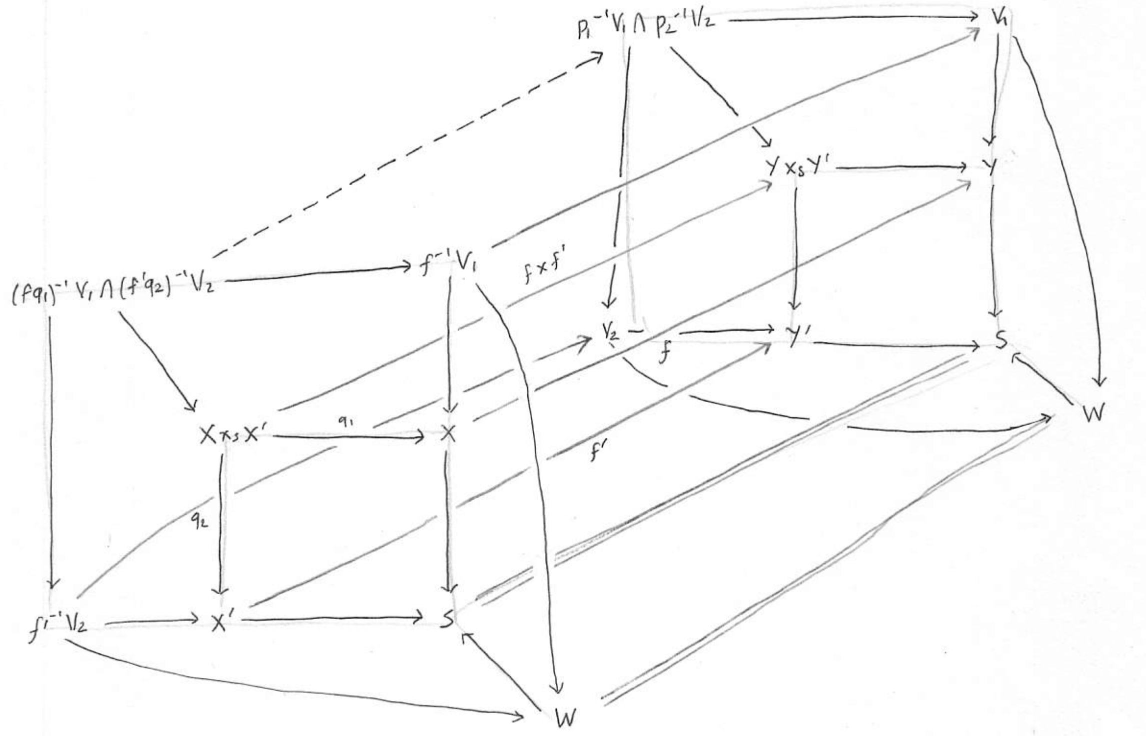
$$\begin{tikzcd}
X \times_S X' \arrow[r, "q_1"] \arrow[d, "q_2"'] \arrow[ddr, "f \times f'"] & X' \arrow[d] \arrow[ddr, "f'"] & \\
X \arrow[r] \arrow[dddr, "f"'] & S \arrow[ddr, equal] & \\
& Y \times_S Y' \arrow[r, "p_1"] \arrow[d, "p_2"] & Y' \arrow[d] \\
& Y \arrow[r] & S
\end{tikzcd}$$

[For any choice of the two pullbacks]

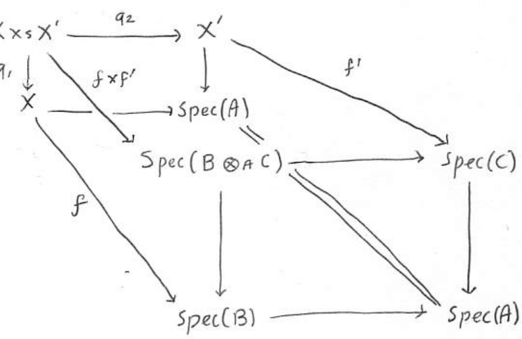
First we reduce to the case where Y, S, Y' are all affine. Since "finite type" is local on the base scheme, it suffices to show $f \times f'$ is of finite type on a neighborhood of each $z \in Y \times_S Y'$. Let such z be given and let $y_1 = p_1(z)$, $y_2 = p_2(z)$. Let q be the image of z in S. Let W be an affine open neighborhood of q in S, V_1 and V_2 affine open neighborhoods of y_1, y_2 respectively, contained in the inverse image of W. Then the outside square in the following diagram is a pullback:

$$\begin{tikzcd}
p_1^{-1}V_1 \cap p_2^{-1}V_2 \arrow[rrr] \arrow[dr] \arrow[ddd] & & & V_1 \arrow[dl] \arrow[ddd] \\
& Y \times_S Y' \arrow[r] \arrow[d] & Y' \arrow[d] & \\
& Y \arrow[r] & S & \\
V_2 \arrow[ur] \arrow[rrr] & & & W \arrow[ul]
\end{tikzcd}$$

We construct the following commutative diagram:



Now $p_1^{-1}V_1 \cap p_2^{-1}V_2$ is an open neighborhood of $z \in Y \times_S Y'$ and $(fq_1)^{-1}V_1 \cap (f'q_2)^{-1}V_2 = (f \times f')^{-1}(p_1^{-1}V_1 \cap p_2^{-1}V_2)$, and moreover one checks that the restriction of $f \times f'$ $(fq_1)^{-1}V_1 \cap (f'q_2)^{-1}V_2 \longrightarrow p_1^{-1}V_1 \cap p_2^{-1}V_2$ is actually the morphism induced between pullbacks by $f^{-1}V_1 \longrightarrow V_1$ and $f'^{-1}V_2 \longrightarrow V_2$. So it suffices to show the dashed morphism is of finite type, whence we may reduce to the case $Y = \operatorname{Spec} B$, $Y' = \operatorname{Spec} C$, $S = \operatorname{Spec} A$, and $Y \times_S Y' = \operatorname{Spec}(B \otimes_A C)$, so we must show $f \times f'$ is of finite type in the following diagram:



Since f is of finite type, there is a finite cover of X by open affines $U_1, \dots, U_n$ with $U_n \cong \operatorname{Spec} B_i$ and each B_i a f.g. B-algebra. Similarly X' is covered by $V_1, \dots, V_m$ with $V_m \cong \operatorname{Spec} C_j$ and each C_j a f.g. C-algebra. The open sets $q_1^{-1}U_i \cap q_2^{-1}V_j$ cover $X \times_S X'$. But $q_1^{-1}U_i \cap q_2^{-1}V_j \cong \operatorname{Spec}(B_i \otimes_A C_j)$, and after a little thought we are reduced to showing that the ring morphism $B \otimes_A C \longrightarrow B_i \otimes_A C_j$, $b \otimes c \longmapsto (b \cdot 1) \otimes (c \cdot 1)$ makes $B_i \otimes_A C_j$ into a f.g. $B \otimes_A C$-algebra, given that B_i, C_j are f.g. B and C-algebras resp. But it is easily checked that if $B_i = B[b_1, \dots, b_r]$ and $C_j = C[c_1, \dots, c_s]$ then $B_i \otimes_A C_j = (B \otimes_A C)[b_1 \otimes 1, \dots, b_r \otimes 1, 1 \otimes c_1, \dots, 1 \otimes c_s]$, which completes the proof. □

$\boxed{\text{Q3.14}}$ See our Notes on Prop 4.10

Q3.17 <u>Zariski Spaces</u> A topological space X is a <u>Zariski space</u> if it is noetherian and every (nonempty) closed irreducible subset has a unique generic point (Ex 2.9). For example, let R be a discrete valuation ring, and let $T = sp(\operatorname{Spec} R)$. Then T consists of two points $t_0 =$ the maximal ideal, $t_1 =$ the zero ideal. The open subsets are $\emptyset$, $\{t_1\}$ and T. This is an irreducible Zariski space with generic point t_1. $\emptyset$ is a Zariski space.

(a) If X is a noetherian scheme, then $sp(X)$ is noetherian, and by (Ex 2.9) it is a Zariski space.

(b) Let $Z \subseteq X$ be a minimal nonempty closed set. Since X is noetherian, Z can be written as a finite union of irreducible closed subsets. Hence Z is irreducible, with generic point ξ. We claim $Z = \{\xi\}$. Otherwise if $y \neq \xi$ belongs to Z, the closure of $\{y\}$ is a proper closed subset of Z (generic points are unique), which is a contradiction. Hence $Z = \{\xi\}$. We call these <u>closed points</u>.

(c) Let x, y be distinct points of X. Since generic points are unique, and we can assume $x \in y^-$, $y \in x^-$ (since otherwise the result is trivial) it follows that $x^- = y^-$ and hence $x = y$ (x^- is a nonempty irreducible closed set) which is a contradiction. Hence either $x \notin y^-$ or $y \notin x^-$, and X is T_0.

(d) Let X be an irreducible Zariski space, with generic point ξ. If $U \subseteq X$ is a nonempty open set, then $\xi \in U$ since otherwise $\xi \in U^c \subset X$, a contradiction.

(e) If x_0, x_1 are points of a space X, and if $x_0 \in \{x_1\}^-$, then we say that x_1 <u>specialises</u> to x_0, written $x_1 \rightsquigarrow x_0$. We also say x_0 is a <u>specialisation</u> of x_1, or that x_1 is a <u>generisation</u> of x_0. Let X be a Zariski space and define a partial ordering by $x_1 \geqslant x_0$ iff. $x_1 \rightsquigarrow x_0$. iff. $x_0 \in \{x_1\}^-$. Note that $x_0 \in \{x_1\}^-$ iff. $\{x_0\}^- \subseteq \{x_1\}^-$. Clearly the points minimal under this ordering are precisely the closed points. Let $X = Y_1 \cup \cdots \cup Y_r$ be the irreducible components of X. Then if ξ_i is the generic point of Y_i, clearly ξ_i is maximal under $\geqslant$, since if $y \geqslant \xi_i$ then $\xi_i \in \{y\}^-$ whence $Y_i \subseteq \{y\}^-$. But $\{y\}^-$ is an irreducible closed set, which must be contained in some Y_j. Thus $Y_i \subseteq Y_j \Rightarrow Y_i = Y_j$ since we can arrange for the Y_i to not be contained in one another. Hence $Y_i = \{y\}^-$ and by uniqueness $y = \xi_i$ as desired. Conversely if $x \in X$ is maximal under $\geqslant$, say $x \in Y_i$, then $Y_i \supseteq \{x\}^-$ so $x \in \{\xi_i\}^- \Longleftrightarrow x \leq \xi_i \Rightarrow x = \xi_i$. So the points maximal under $\geqslant$ are precisely the generic points of the irreducible components of X.

If $x_1 \rightsquigarrow x_0$ and $x_1 \in V$ for a closed $V \subset X$, then $x_0 \in \{x_1\}^- \subseteq V$, so closed sets are closed under specialisation. Similarly if $x_0 \in U$ for U open, and $x_1 \in U^c$ then $x_0 \in \{x_1\}^- \subseteq U^c$, a contradiction. Hence open sets are closed under generisation.

(f)

Q3.20 <u>Dimension</u> Let X be an integral scheme of finite type over a field k (not necessarily algebraically closed). Then there is a nonempty cover of X by nonempty open sets U_i with $U_i \cong \operatorname{Spec} A_i$ for finitely generated k-domains A_i. Moreover we can assume this cover is <u>finite</u>.

(a) For a closed point P, we claim that $\dim X = \dim \mathcal{O}_P$ (Krull dimension). Let ξ be the generic point of X. Then the function field $K(X) = \mathcal{O}_{X,\xi}$ is also the quotient field of any A_i (see Ex 3.6). In fact, this is an isomorphism of k-algebras. But since the A_i are affine k-algebras,

$$\dim A_i = \text{tr.deg. } Q(A_i)/k = \text{tr.deg. } K(X)/k$$

But by Ex I.10 of Ch. I since the $\{U_i\}$ cover X we have

$$\begin{aligned}\dim X &= \sup \dim U_i \\ &= \sup \dim \operatorname{Spec} A_i \\ &= \sup \dim A_i = \text{tr.deg. } K(X)/k \qquad (= \dim A_i \ \forall i) \\ &\qquad\qquad (\text{hence } \dim X \text{ finite})\end{aligned}$$

Let $P \in X$ be a closed point, and say $P \in U_i \subseteq X$. Then P corresponds to a maximal ideal $\mathfrak{m}$ of A_i, and

$$\begin{aligned}\dim \mathcal{O}_{X,P} &= \dim \mathcal{O}_{\operatorname{Spec} A_i, \mathfrak{m}} = \dim A_{i\mathfrak{m}} \\ &= \text{ht.}\, \mathfrak{m} = \dim A_i - \text{coht.}\, \mathfrak{m} \qquad (\text{since } A_i \text{ affine } k\text{-alg}) \\ &= \dim A_i = \dim X\end{aligned}$$

Note that the $\dim A_i$ are all finite, hence so is $\dim X$. We have also proven (b) above.

(c) Let $U \subseteq X$ be a nonempty open subset. Then U is an integral scheme of finite type over k (X f.t. over $k \Rightarrow X$ noetherian $\therefore U$ noetherian $\therefore U \longrightarrow X$ quasi-compact open immersion $\therefore$ f.t. $\therefore$ composite $U \longrightarrow X \longrightarrow k$ is f.t.). Hence $\dim U = \dim X$ by (a) and Ex 3.14.

(d) If $Y \subseteq X$ is an <u>irreducible</u> closed subset we show that $\dim Y + \operatorname{codim}(Y, X) = \dim X$. Now $\dim Y \leq \dim X$ so $\dim Y$ is a finite positive integer, and also $0 \leq \operatorname{codim}(Y, X) \leq \dim X$. Clearly $\dim Y + \operatorname{codim}(Y, X) \leq \dim X$. For any open set $U \subseteq X$ with $Y \cap U \neq \emptyset$ it is easy to see that $\operatorname{codim}(Y, X) = \operatorname{codim}(Y \cap U, U)$, since a chain $Y \cap U \subset Q_1 \subset \cdots \subset Q_n$ becomes $Y \subset \overline{Q_1} \subset \cdots \subset \overline{Q_n}$ and $Y \subset Z_1 \subset \cdots \subset Z_n$ becomes $Y \cap U \subset Z_1 \cap U \subset \cdots \subset Z_n \cap U$. ($\overline{Y \cap U} = Y$ since the generic point of Y is in $Y \cap U$). Since $\dim Y$ is finite there is a maximal chain $Z_0 \subset \cdots \subset Z_n = Y$ with $n = \dim Y$. By 3.17b) Z_0 is a closed point (since if it properly contained a closed set (other than $\emptyset$) then it would contain the irred. components of this set). Suppose $Z_0 \subseteq U_i$. Then intersecting the chain with U_i shows that in U_i, $\dim(Y \cap U_i) = \dim Y$. But since $\text{ht.}\,\mathfrak{p} + \text{coht.}\,\mathfrak{p} = \dim A$ in an affine k-algebra A, and $U_i \cong \operatorname{Spec} A_i$, we have

$$\begin{aligned}\dim Y &= \dim(Y \cap U_i) \\ &= \dim U_i - \operatorname{codim}(Y \cap U_i, U_i) \\ &= \dim X - \operatorname{codim}(Y, X)\end{aligned}$$

as required.

<u>NOTE</u> If X is an integral scheme of finite type over a field k, then the above shows $0 \leq \dim X < \infty$. Then $\dim X = 0 \Rightarrow X$ is a finite number of points.

<u>PROOF</u> We can cover X by $U_1, \ldots, U_n$ where $U_i \cong \operatorname{Spec} A_i$ where A_i are f.g. k-domains with $\dim A_i = \dim X = 0$. Hence the A_i are artinian domains, that is, they are fields. So $U_i = \{P_i\}$ and so $X = \{P_1, \ldots, P_n\}$. □