4. Separated and Proper Morphisms

We now come to two properties of schemes, or rather of morphisms between schemes, which correspond to well-known properties of ordinary topological spaces. Separatedness corresponds to the Hausdorff axiom for a topological space. Properness corresponds to the usual notion of properness, namely the inverse image of a compact set is compact. However, the usual definitions are not suitable in algebraic geometry, because the Zariski topology is never Hausdorff, and the underlying topological space of a scheme does not accurately reflect all its properties. So instead we will use definitions which reflect the functional behaviour of the morphism within the category of schemes. For schemes of finite type over $\mathbb{C}$, one can show that these notions, defined abstractly, are in fact the same as the usual notions if we consider these schemes as complex analytic spaces in the ordinary topology.

In this section we will define separated and proper morphisms. We will give criteria for a morphism to be separated or proper using valuation rings. Then we will show that projective space over any scheme is proper.

<u>DEFINITION</u> Let $f: X \longrightarrow Y$ be a morphism of schemes. The <u>diagonal morphism</u> is the unique morphism $\Delta: X \longrightarrow X \times_Y X$ whose composition with both projections $p_1, p_2 : X \times_Y X \longrightarrow X$ is the identity. We say that the morphism f is <u>separated</u> if the diagonal morphism Δ is a closed immersion. In that case we also say X is <u>separated</u> over Y. A scheme X is separated if it is separated over $\operatorname{Spec}\mathbb{Z}$. (Note the definition is independent of the chosen pullback)

<u>PROPOSITION 4.1</u> If $f: X \longrightarrow Y$ is any morphism of affine schemes, then f is separated.

<u>PROOF</u> Let $X \cong \operatorname{Spec} A$ and $Y \cong \operatorname{Spec} B$ and let $\varphi: B \longrightarrow A$ be the induced morphism of rings. The diagonal morphism $\operatorname{Spec} A \longrightarrow \operatorname{Spec}(A \otimes_B A)$ corresponds to $A \otimes_B A \longrightarrow A$, $a \otimes b$, which is surjective. Hence by Ex 2.18 $\operatorname{Spec} A \longrightarrow \operatorname{Spec}(A \otimes_B A)$ is a closed immersion, as required. $\square$

<u>COROLLARY 4.2</u> An arbitrary morphism $f: X \longrightarrow Y$ is separated if and only if the image of the diagonal morphism is a closed subset of $X \times_Y X$.

<u>PROOF</u> Note the condition is independent of the chosen pullback. One implication is obvious, so we only have to prove that if $\Delta(X)$ is a closed subset, then $\Delta: X \longrightarrow X \times_Y X$ is a closed immersion. In other words, we have to check that $\Delta: X \longrightarrow \Delta(X)$ is a homeomorphism, and that the morphism of sheaves $\mathcal{O}_{X \times_Y X} \longrightarrow \Delta_* \mathcal{O}_X$ is surjective. Let $p_1 : X \times_Y X \longrightarrow X$ be the first projection. Since $p_1 \Delta = \mathrm{id}_X$, it follows immediately that Δ gives a homeomorphism onto $\Delta(X)$. Since $\Delta(X)$ is closed, to show $\mathcal{O}_{X \times_Y X} \longrightarrow \Delta_* \mathcal{O}_X$ is surjective (that is, locally surjective), it suffices to show that for all $P \in X$ there is an open neighborhood $\Delta(P) \in V \subseteq X \times_Y X$ s.t. $\mathcal{O}_{X \times_Y X}|_V \longrightarrow (\Delta_* \mathcal{O}_X)|_V$ is locally surjective (see a note in Section 1).

Let $P \in X$ be given. Let U be an open affine neighborhood of P which is small enough so that $f(U)$ is contained in an open affine subset V of Y. Then the open subscheme $p_1^{-1}U \cap p_2^{-1}U$ of $X \times_Y X$, together with the induced morphisms into U, is a pullback of $U \longrightarrow V$, $U \longrightarrow V$. That is,

$$p_1^{-1}U \cap p_2^{-1}U = U \times_V U$$

Since U, V are both affine, this means that $p_1^{-1}U \cap p_2^{-1}U$ is also affine. Note that $\Delta(P) \in U \times_V U$ and $\Delta^{-1}(U \times_V U) = U$. Hence there is an induced morphism $\Delta' : U \longrightarrow U \times_V U$. It is not difficult to check that Δ' is in fact the diagonal, which is a closed immersion by (4.4). But $\Delta'^{\#} : \mathcal{O}_{U \times_V U} \longrightarrow \Delta'_* \mathcal{O}_U$ is none other than $\mathcal{O}_{X \times_Y X}|_{U \times_V U} \longrightarrow (\Delta_* \mathcal{O}_X)|_{U \times_V U}$, completing the proof. $\square$

<u>NOTE</u> A morphism $f: X \longrightarrow Y$ is a monomorphism of schemes iff.

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, equal] & X \arrow[d, "f"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

is a pullback, so if f is monic the diagonal Δ is the identity and so f is separated. In particular any open or closed immersion is separated, any isomorphism is separated, and it is easily checked that if $X \longrightarrow Y$ is separated and $Z' \Longrightarrow X$, $Y \Longrightarrow Z$ are isomorphisms, both $Z' \longrightarrow X \longrightarrow Y$ and $X \longrightarrow Y \longrightarrow Z$ are separated.

NOTE Separatedness is Local

Let $f : X \longrightarrow Y$ be a separated morphism of schemes, $V \subseteq Y$ open and $f^{-1}V \longrightarrow V$ the induced morphism. We claim that $f^{-1}V \longrightarrow V$ is separated. Using the usual properties of pullbacks (and the fact that $p_1^{-1}f^{-1}(V) = p_2^{-1}f^{-1}(V)$) the outside square in the following diagram is a pullback:

$$\begin{tikzcd}
(fp_1)^{-1}V \arrow[rrr, dashed] \arrow[dr] \arrow[ddd] & & & f^{-1}V \arrow[dl] \arrow[ddd] \\
& X \times_Y X \arrow[r] \arrow[d] & X \arrow[d] & \\
& X \arrow[r] & Y & \\
f^{-1}V \arrow[ur] \arrow[rrr] & & & V \arrow[ul]
\end{tikzcd}$$

Let $\Delta : X \longrightarrow X \times_Y X$ be the diagonal, and note that $\Delta^{-1}(fp_1)^{-1}V = f^{-1}V$ and the induced map $f^{-1}V \longrightarrow (fp_1)^{-1}V$ is actually the diagonal $f^{-1}V \longrightarrow f^{-1}V \times_V f^{-1}V = (fp_1)^{-1}V$ for $f^{-1}V \longrightarrow V$. So we have a pullback

$$\begin{tikzcd}
f^{-1}V \arrow[r, "\Delta"] \arrow[d] & f^{-1}V \times_V f^{-1}V \arrow[d] \\
X \arrow[r, "\Delta"'] & X \times_Y X
\end{tikzcd}$$

Since by assumption Δ is a closed immersion and pullbacks preserve closed immersions, it follows that $f^{-1}V \longrightarrow V$ is separated.

In a similar vein, if $f : X \longrightarrow Y$ is separated, so is $U \longrightarrow X \longrightarrow Y$ for any open $U \subseteq X$. Consider the following diagram:

$$\begin{tikzcd}
U \arrow[r] \arrow[d] & X \arrow[d, bend right, "\Delta"] & \\
p_1^{-1}U \cap p_2^{-1}U \arrow[r] \arrow[d] \arrow[dr] & p_2^{-1}U \arrow[r] \arrow[d] & U \arrow[d] \\
p_1^{-1}U \arrow[r] \arrow[d] & X \times_Y X \arrow[r] \arrow[d] & X \arrow[d] \\
U \arrow[r] & X \arrow[r] & Y
\end{tikzcd}$$

Since $\Delta(U) \subseteq p_1^{-1}U \cap p_2^{-1}U$ we get $U \longrightarrow p_1^{-1}U \cap p_2^{-1}U$. Moreover $\Delta^{-1}(p_1^{-1}U \cap p_2^{-1}U) = U$ so this is a pullback. Since pullbacks of closed immersions are closed immersions and $U \longrightarrow p_1^{-1}U \cap p_2^{-1}U = U \times_Y U$ is easily checked to be the diagonal, it follows that $U \longrightarrow Y$ is separated.

NOTE Local Schemes (EGA I §2.4)

DEFINITION A local scheme is an affine scheme $X = \mathrm{Spec}\, A$ where A is a local ring. Thus there is a closed point $a \in X$ and for all $b \in X$ we have $a \in \overline{\{b\}}$.

If Y is a scheme and $y \in Y$ the local scheme $\mathrm{Spec}(\mathcal{O}_{Y,y})$ is called the local scheme of Y at y. If $V \cong \mathrm{Spec}\, B$ is an open affine neighborhood of y in Y, then there is an isomorphism $\mathcal{O}_{Y,y} \cong B_y$ which gives rise to, via $B \longrightarrow B_y$, a morphism $\mathrm{Spec}(\mathcal{O}_{Y,y}) \longrightarrow \mathrm{Spec}\, B \longrightarrow V$. On composing with the inclusion $V \longrightarrow Y$ we obtain a morphism of schemes $\mathrm{Spec}(\mathcal{O}_{Y,y}) \longrightarrow Y$. We claim that this morphism is independent of V. (Note that $\mathrm{Spec}(\mathcal{O}_{Y,y}) \longrightarrow Y$ is a monomorphism since Spec of the ring epi $B \longrightarrow B_{\mathfrak{p}}$ is a mono of schemes)

If $V' \cong \mathrm{Spec}\, B'$ is another open affine containing y then there is an open affine $y \in W \subseteq V \cap V'$, so we may restrict to the case where $V \subseteq V'$. The morphism $V \longrightarrow V'$ induces $B' \longrightarrow B$ making the following diagram commute

$$\begin{tikzcd} B' \arrow[rr] \arrow[dr] & & B \arrow[dl] \\ & \mathcal{O}_{Y,y} & \end{tikzcd}$$

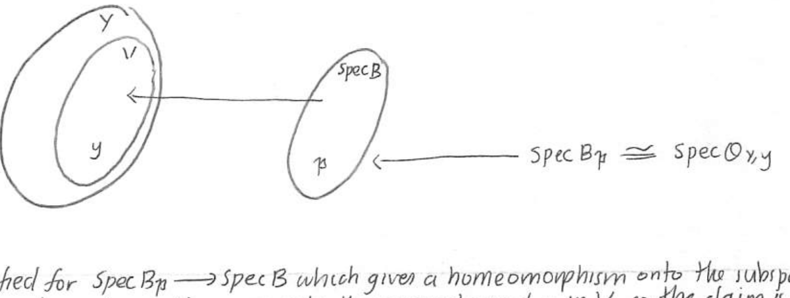
Where of course $B \longrightarrow \mathcal{O}_{Y,y}$ is $B \longrightarrow B_y \cong \mathcal{O}_{Y,y}$. Considering the following commutative diagram

$$\begin{tikzcd} & Y & \\ V' \arrow[ur] & & V \arrow[ul] \arrow[ll] \\ \mathrm{spec}\, B' \arrow[u, Rightarrow] & & \mathrm{Spec}\, B \arrow[u, Rightarrow] \arrow[ll] \\ & \mathrm{Spec}(\mathcal{O}_{Y,y}) \arrow[ul] \arrow[ur] & \end{tikzcd}$$

We see that $\mathrm{Spec}(\mathcal{O}_{Y,y}) \longrightarrow Y$ is independent of the affine set used to define it. Note that the max. ideal of $\mathcal{O}_{Y,y}$ is mapped to $y \in Y$.

PROPOSITION Let Y be a scheme, $y \in Y$, $(f, \psi): \mathrm{Spec}(\mathcal{O}_{Y,y}) \longrightarrow Y$ canonical. Then f gives a homeomorphism with the subspace of Y consisting of all generisations of y (i.e. z with $y \in \overline{\{z\}}$). For every $z \in \mathrm{Spec}(\mathcal{O}_{Y,y})$ $\psi_z : \mathcal{O}_{Y, f(z)} \longrightarrow (\widetilde{\mathcal{O}_{Y,y}})_z$ is an isomorphism.

PROOF Let $y \in V \cong \mathrm{Spec}\, B$ be an open affine neighborhood:



The claim is easily verified for $\mathrm{Spec}\, B_{\mathfrak{p}} \longrightarrow \mathrm{Spec}\, B$ which gives a homeomorphism onto the subspace $\{\mathfrak{q} \mid \mathfrak{q} \subseteq \mathfrak{p}\}$. The generisations of y in Y are precisely the generisations of y in V, so the claim is immediate. The second claim follows from the case $(f, \psi): \mathrm{Spec}\, B_{\mathfrak{p}} \longrightarrow \mathrm{Spec}\, B$, and in this case we have a commutative diagram for $\mathfrak{q} \subseteq \mathfrak{p}$

$$\begin{tikzcd} \mathcal{O}_{B,\mathfrak{q}} \arrow[r] \arrow[d, Rightarrow] & \mathcal{O}_{B_{\mathfrak{p}}, \mathfrak{q}B_{\mathfrak{p}}} \arrow[d, Rightarrow] \\ B_{\mathfrak{q}} \arrow[r] & (B_{\mathfrak{p}})_{\mathfrak{q}B_{\mathfrak{p}}} \\ B \arrow[u] \arrow[r] & (B_{\mathfrak{p}}) \arrow[u] \end{tikzcd}$$

Since the middle row is an isomorphism, we are done. □

NOTE Let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}_{Y,y}$ which is mapped to $y \in Y$ by $f: Spec(\mathcal{O}_{Y,y}) \longrightarrow Y$. We claim the following diagram commutes

$$\begin{tikzcd}
\mathcal{O}_{Y,y} \arrow[r, "f_{\mathfrak{m}}"] \arrow[dr] \arrow[ddr, Rightarrow] & \widetilde{\mathcal{O}_{Y,y}}_{\mathfrak{m}} \arrow[d, phantom, "\cong"] \\
& (\mathcal{O}_{Y,y})_{\mathfrak{m}} \arrow[d, phantom, "\cong"] \\
& \mathcal{O}_{Y,y}
\end{tikzcd}$$

commutes, where the morphisms on the right are intrinsic to $\widetilde{\mathcal{O}_{Y,y}}$. The chain of isomorphisms giving rise to $f_{\mathfrak{m}}$ is as follows: $y \longleftrightarrow \mathfrak{p} \longleftrightarrow \mathfrak{m}$: (the following diagram commutes)

$$\begin{tikzcd}
& & \mathcal{O}_{Y,y} \arrow[d, "\wr"] \\
& & (\mathcal{O}_Y|_V)_y \arrow[d, "\wr"] \\
& & \mathcal{O}_{B,\mathfrak{p}} \arrow[d, "\wr"] \arrow[lld, bend right, "\sim"'] \\
B_{\mathfrak{p}} \arrow[d, "\wr"] & (B_{\mathfrak{p}})_{\mathfrak{p}B_{\mathfrak{p}}} \arrow[l, "\sim"'] \arrow[d, "\wr"] & \mathcal{O}_{B_{\mathfrak{p}}, \mathfrak{p}B_{\mathfrak{p}}} \arrow[l, "\sim"'] \arrow[d, "\wr"] \\
\mathcal{O}_{Y,y} & (\mathcal{O}_{Y,y})_{\mathfrak{m}} \arrow[l, "\sim"'] & \widetilde{\mathcal{O}_{Y,y}}_{\mathfrak{m}} \arrow[l, "\sim"']
\end{tikzcd}$$

It follows from the fact that $B_{\mathfrak{p}} \xrightarrow{\sim} \mathcal{O}_{Y,y}$ is $B_{\mathfrak{p}} \xrightarrow{\sim} \mathcal{O}_{B,\mathfrak{p}} \xrightarrow{\sim} \mathcal{O}_{Y,y}$ that the final composite is $1_{\mathcal{O}_{Y,y}}$.

Now assume X is an integral scheme with generic point ξ, and let $x \in X$ be any point $f: Spec(\mathcal{O}_{X,x}) \longrightarrow X$ canonical. Let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}_{X,x}$. Then $f(\mathfrak{m}) = x$ and since f is a homeomorphism with the generisations of x we also have $f(0) = \xi$.

$$f: Spec(\mathcal{O}_{X,x}) \longrightarrow X$$
$$f(\mathfrak{m}) = x$$
$$f(0) = \xi.$$

Since $\mathcal{O}_{X,x}$ is an integral domain $Spec(\mathcal{O}_{X,x})$ is also an integral scheme, and we have a commutative diagram

$$\begin{tikzcd}
& (\mathcal{O}_{X,x})_0 & \widetilde{\mathcal{O}_{X,x}}_0 \arrow[l, Rightarrow] & \mathcal{O}_{X,\xi} \arrow[l, Rightarrow] \\
\mathcal{O}_{X,x} \arrow[ur] & (\mathcal{O}_{X,x})_{\mathfrak{m}} \arrow[l, Rightarrow] \arrow[u, tail] & \widetilde{\mathcal{O}_{X,x}}_{\mathfrak{m}} \arrow[l, Rightarrow] \arrow[u, tail] & \mathcal{O}_{X,x} \arrow[l] \arrow[u, tail]
\end{tikzcd}$$

Since the bottom row is the identity, it follows that

$$\mathcal{O}_{X,x} \rightarrowtail \mathcal{O}_{X,\xi} \overset{f_0}{\Longrightarrow} \widetilde{\mathcal{O}_{X,x}}_0 \Longrightarrow (\mathcal{O}_{X,x})_0$$

is the canonical inclusion of a domain in its quotient field. We know f_0 is an isomorphism, and $\mathcal{O}_{X,\xi}$ is a quotient field of $\mathcal{O}_{X,\xi}$, so $\mathcal{O}_{X,\xi} \Rightarrow \widetilde{\mathcal{O}_{X,x}}_0 \Rightarrow (\mathcal{O}_{X,x})_0$ is the canonical isomorphism of quotient fields

NOTE Let X be a scheme and $x \in X$, $f: Spec(\mathcal{O}_{X,x}) \longrightarrow X$. On global sections this is just the expected map $\mathcal{O}_X(X) \longrightarrow \mathcal{O}_{X,x} \Longrightarrow \widetilde{\mathcal{O}_{X,x}}(Spec\, \mathcal{O}_{X,x})$.

NOTE FACTORISATION THROUGH LOCAL SCHEMES

Let $f: X \longrightarrow Y$ be a morphism of schemes, with $X = \operatorname{Spec} R$ local and $y \in Y$. Let G denote the set of generisations of y:

$$G = \{ z \in Y \mid y \in \overline{\{z\}} \}$$

We claim that if $f(X) \subseteq G$ then there is a unique factorisation of f through $\operatorname{Spec}(\mathcal{O}_{Y,y}) \longrightarrow Y$

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Y \\ & \operatorname{Spec}(\mathcal{O}_{Y,y}) \arrow[ur] & \end{tikzcd}$$

Let $V \subseteq \operatorname{Spec} A$ be an affine open neighborhood of y. Since open sets are closed under generisation, $G \subseteq V$. Let $\mathfrak{p} \in \operatorname{Spec} A$ correspond to y, so $G \cong \{ \mathfrak{q} \mid \mathfrak{q} \subseteq \mathfrak{p} \}$. If $A \longrightarrow A_{\mathfrak{p}}$ and $\operatorname{Spec}(\mathcal{O}_{Y,y}) \cong \operatorname{Spec} A_{\mathfrak{p}} \longrightarrow \operatorname{Spec} A$ is canonical), we can reduce to showing that if $Y = \operatorname{Spec} A$ is affine, $\mathfrak{p} \subseteq A$ prime, and $f: X \longrightarrow Y$ s.t. $f(X) \subseteq \{ \mathfrak{q} \mid \mathfrak{q} \subseteq \mathfrak{p} \}$ then f factors uniquely through $\operatorname{Spec} A_{\mathfrak{p}} \longrightarrow \operatorname{Spec} A$.

Let $\varphi: A \longrightarrow R$ determine f. Then $f(X) \subseteq \{ \mathfrak{q} \mid \mathfrak{q} \subseteq \mathfrak{p} \}$ means that if $s \notin \mathfrak{p}$ then $\varphi(s)$ is a unit, since otherwise $\varphi(s) \in \mathfrak{m}$ for some maximal $\mathfrak{m}$, contradicting $\varphi^{-1}\mathfrak{m} \subseteq \mathfrak{p}$. Hence there is a unique $A_{\mathfrak{p}} \longrightarrow R$ making the following diagram commute:

$$\begin{tikzcd} A \arrow[rr] \arrow[dr] & & A_{\mathfrak{p}} \arrow[dl] \\ & R & \end{tikzcd} \qquad \begin{tikzcd} \operatorname{Spec} A & & \operatorname{Spec} A_{\mathfrak{p}} \arrow[ll] \\ & \operatorname{Spec} R \arrow[ul] \arrow[ur] & \end{tikzcd}$$

as required.

NOTE : SPECIALISATION AND MORPHISMS OF LOCAL RINGS

Let X be a scheme, $x_0, x_1 \in X$ with x_0 a specialisation of x_1 : $x_0 \in \{x_1\}^-$. Since open sets are closed under generisation, any open set containing x_0 also contains x_1, so there is a morphism of rings:

$$\begin{aligned} \mathcal{O}_{X,x_0} &\longrightarrow \mathcal{O}_{X,x_1} \\ (U,s) &\longmapsto (U,s) \end{aligned} \qquad (1)$$

If X is integral then this morphism is injective : let $U \cong \operatorname{Spec} A$ be an affine open neighborhood of x_0 (hence also x_1). Then A is a domain and x_0, x_1 correspond to primes $\mathfrak{p}_0, \mathfrak{p}_1$ with $\mathfrak{p}_1 \subseteq \mathfrak{p}_0$, and $\mathcal{O}_{X,x_0} \longrightarrow \mathcal{O}_{X,x_1}$ corresponds to the canonical map $A_{\mathfrak{p}_0} \longrightarrow A_{\mathfrak{p}_1}$, which is injective. In particular if ξ is the generic point of X, $K(X) = \mathcal{O}_{X,\xi}$ the function field of X (see Ex 3.6) then we can consider every $\mathcal{O}_{X,x}$ as a <u>subring</u> of $K(X)$.

<u>PROPOSITION</u> Let X be a scheme and $x, y \in X$. Then $\operatorname{Spec}(\mathcal{O}_{X,x}) \longrightarrow X$ factors through $\operatorname{Spec}(\mathcal{O}_{X,y}) \longrightarrow X$ if and only if x is a generisation of y (i.e. $y \in \{x\}^-$), and in this case the unique factorisation $\operatorname{Spec}(\mathcal{O}_{X,x}) \longrightarrow \operatorname{Spec}(\mathcal{O}_{X,y})$ is the morphism of schemes corresponding to the morphism of rings defined in (1) above.

<u>PROOF</u> Both morphisms are monomorphisms, so in terms of subobjects in the category <u>Sch</u> we are claiming

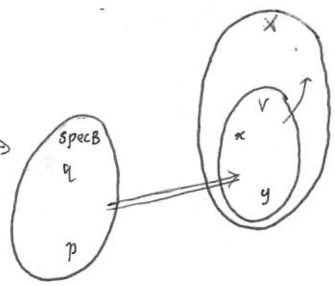
$$\operatorname{Spec}(\mathcal{O}_{X,x}) \leq \operatorname{Spec}(\mathcal{O}_{X,y}) \quad \text{iff.} \quad x \rightsquigarrow y$$

From earlier notes we know that $\operatorname{Spec}(\mathcal{O}_{X,y}) \longrightarrow X$ is a homeomorphism of $\operatorname{Spec}(\mathcal{O}_{X,y})$ with the subspace of all generisations of y, so if $\operatorname{Spec}(\mathcal{O}_{X,x}) \leq \operatorname{Spec}(\mathcal{O}_{X,y})$ clearly $x \rightsquigarrow y$. Conversely if $x \rightsquigarrow y$ then, every point in the image of $\operatorname{Spec}(\mathcal{O}_{X,x}) \longrightarrow X$ is a generisation of y, so by the previous note (Fact. Local schemes) $\operatorname{Spec}(\mathcal{O}_{X,x}) \leq \operatorname{Spec}(\mathcal{O}_{X,y})$. Let $\phi : \mathcal{O}_{X,y} \longrightarrow \mathcal{O}_{X,x}$ be as defined in (1). To complete the proof we need only show that the following diagram commutes:

$$\begin{tikzcd} \operatorname{Spec}(\mathcal{O}_{X,x}) \arrow[rr] \arrow[dr, "\operatorname{Spec}\phi"'] & & X \\ & \operatorname{Spec}(\mathcal{O}_{X,y}) \arrow[ur] & \end{tikzcd}$$

Let $V \cong \operatorname{Spec} B$ be any affine open neighborhood of y (which is therefore a neighborhood of x as well). The two morphisms $\longrightarrow X$ are then the composites in the following diagram:

$$\begin{tikzcd} \operatorname{Spec}(\mathcal{O}_{X,x}) \arrow[r, "\sim"] \arrow[d, "\operatorname{Spec}\phi"'] \arrow[dr, phantom, "ⓐ"] & \operatorname{Spec}(B_{\mathfrak{q}}) \arrow[d] \arrow[r] \arrow[dr, phantom, "ⓑ"] & \operatorname{Spec} B \\ \operatorname{Spec}(\mathcal{O}_{X,y}) \arrow[r, "\sim"] & \operatorname{Spec}(B_{\mathfrak{p}}) \arrow[ur] & \end{tikzcd}$$



We need to show that ⓐ, ⓑ commute, where $B_{\mathfrak{p}} \longrightarrow B_{\mathfrak{q}}$ is $a/s \longmapsto a/s$ (since $\mathfrak{q} \subseteq \mathfrak{p}$ by assumption). But this is straightforward. $\square$

In the case of an integral scheme with generic point ξ and quotient field $K = \mathcal{O}_{X,\xi}$ then every point is a specialisation of ξ, so for $x \in X$, $\operatorname{Spec}(\mathcal{O}_{X,x}) \longrightarrow X$ and $\operatorname{Spec} K \longrightarrow X$ fit into a commutative diagram

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[dr] & X \\ & \operatorname{Spec} \mathcal{O}_{X,x} \arrow[u] \end{tikzcd}$$

NOTE FIELDS AND DIAGONALS

Let $f: X \longrightarrow Y$ be a morphism of schemes, K a field and $U = \operatorname{Spec} K$. Denote the maximal ideal of K by $t \in U$. Let $\Delta: X \longrightarrow X \times_Y X$ be the diagonal. Then a morphism $g: U \longrightarrow X \times_Y X$ factors through the diagonal if and only if $g(x) \in \Delta(X)$; and this factorisation is unique:

$$\begin{tikzcd}
U \arrow[rr, dashed, "m"] \arrow[dr, "g"'] & & X \arrow[dl, "\Delta"'] \arrow[dr, Rightarrow, bend left] \arrow[dl, Rightarrow, shift left] & \\
 & X \times_Y X \arrow[rr, "p_1"] \arrow[d, "p_2"] & & X \arrow[d, "f"] \\
 & X \arrow[rr, "f"] & & Y
\end{tikzcd}$$

Suppose $g(t) \in \Delta(X)$, say $g(x) = \xi = \Delta(x)$. Then we have ring morphisms

$$\begin{tikzcd}
\mathcal{O}_{X,x} \arrow[r, "p_{1,x}"] \arrow[r, bend left, "p_{2,x}"] & \mathcal{O}_{X \times_Y X, g(t)} \arrow[l, bend left, "\Delta_x"]
\end{tikzcd}$$

By definition $\Delta_x p_{1,x} = \Delta_x p_{2,x} = 1$ and all three morphisms are local. That is, $\Delta_x^{-1} m_x = m_{g(t)}$. Let $a \in \mathcal{O}_{X,x}$ be given. Then $\Delta_x(p_{1,x}(a) - p_{2,x}(a)) = a - a = 0 \in m_x$ so $p_{1,x}(a) - p_{2,x}(a) \in m_{g(t)}$. But $g_t: \mathcal{O}_{X \times_Y X, g(t)} \longrightarrow \mathcal{O}_{U,t}$ is also local, so $\operatorname{Ker} g_t = m_{g(t)}$. Hence $g_t p_{1,x} = g_t p_{2,x}$. It follows from the Lemmas on the previous page that $p_2 g = p_1 g$. Putting $m = p_2 g$ and using the pullback, we see that $g = \Delta m$, as required. The morphism m is clearly unique.

NOTE Fields and Closed Subschemes.

Let us reprove Ex 2.4. ($\operatorname{Spec} k(y) \longrightarrow Y$ being $\operatorname{Spec} k(y) \longrightarrow \operatorname{Spec} \mathcal{O}_{Y,y} \longrightarrow Y$) †Note $\operatorname{Spec} k(y) \longrightarrow Y$ is a monomorphism of schemes.

LEMMA Let Y be a scheme and $y \in Y$. If K is a field and $U = \operatorname{Spec} K$ then any morphism $f: U \longrightarrow Y$ factors uniquely through $\operatorname{Spec} k(y) \longrightarrow Y$, assuming $f(m) = y$.

PROOF Let $\operatorname{Spec} K \longrightarrow Y$ be given. The image of this morphism is $\{y\} \subseteq \overline{\{y\}}$ so there is a unique factorisation of $\operatorname{Spec} K \longrightarrow Y$ through $\operatorname{Spec} \mathcal{O}_{Y,y} \longrightarrow Y$. The ring morphism $\mathcal{O}_{Y,y} \longrightarrow K$ must have kernel m_y, so we get $k(y) \longrightarrow K$ fitting into a commutative diagram

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[rr] \arrow[dr] \arrow[ddrr, bend right] & & Y \\
 & \operatorname{Spec} \mathcal{O}_{Y,y} \arrow[ur] & \\
 & & \operatorname{Spec} k(y) \arrow[ul] \arrow[uu]
\end{tikzcd}$$

Since $\mathcal{O}_{Y,y} \longrightarrow k(y)$ is an epimorphism, $\operatorname{Spec} k(y) \longrightarrow \operatorname{Spec} \mathcal{O}_{Y,y}$ is a monomorphism, so the factorisation is unique. $\square$

<u>LEMMA</u> The product of two nonempty schemes over a field is nonempty.

<u>PROOF</u> Say X, Y nonempty and we have a pullback

$$\begin{tikzcd} P \arrow[r, "g"] \arrow[d, "f"'] & X \arrow[d] \\ Y \arrow[r] & \operatorname{Spec} k \end{tikzcd}$$

Let $U \subseteq X$, $V \subseteq Y$ be nonempty affine subsets, say $U \cong \operatorname{Spec} A$, $V \cong \operatorname{Spec} B$. Then $f^{-1}V \cap g^{-1}U \cong \operatorname{Spec}(A \otimes_k B)$, so it suffices to show $A \otimes_k B \neq 0$. But k is a field, so A, B are free and $- \otimes -$ preserves coproducts, so this is trivial. $\square$

<u>PROPOSITION</u> Let $f: X \longrightarrow Y$ be a closed immersion with $y \in Y$ belonging to the image of f. Suppose $g: \operatorname{Spec} K \longrightarrow Y$ is a morphism with K a field and $g(m) = y$. Then g factors uniquely through f.

<u>PROOF</u> Uniqueness is clear since a closed immersion is a monomorphism. To show existence, factor g uniquely through $\operatorname{Spec} k(y) \longrightarrow Y$. Form the pullbacks $X_y = \operatorname{Spec} k(y) \times_Y X$ and $P = \operatorname{Spec} K \times_{k(y)} X_y$:

$$\begin{tikzcd} P \arrow[rr] \arrow[dr] \arrow[dd] & & X \arrow[dd] \\ & X_y \arrow[ur] \arrow[dd] & \\ \operatorname{Spec} K \arrow[rr] \arrow[dr] & & Y \\ & \operatorname{Spec} k(y) \arrow[ur] & \end{tikzcd}$$

All three squares are therefore pullbacks. From earlier notes we know that $X_y \cong f^{-1}y$, which is a singleton since f is injective, and $y \in f(X)$. So by the previous Lemma $P \neq \emptyset$. But pullbacks preserve closed immersions, so $P \longrightarrow \operatorname{Spec} K$ is a closed immersion. Since $P \neq \emptyset$ we must have $P \cong \operatorname{Spec} K/0 = \operatorname{Spec} K$, so $P \longrightarrow \operatorname{Spec} K$ is an isomorphism. Hence $\operatorname{Spec} K \longrightarrow Y$ factors through f. $\square$

Next we will discuss the valuative criterion of separatedness. The rough idea is that in order for a scheme X to be separated, it should not contain any subscheme which looks like a curve with a doubled point, as in the example above. Another way of saying this is that if C is a curve, and P a point of C, then given any morphism of $C-P$ into X, it should admit at most one extension to a morphism of all of C into X. (cf. (I, 6.8) where we showed that a projective variety has this property.

In practice, this rough idea has to be modified. The question is local, so we replace the curve by its local ring at a point, which is a discrete valuation ring. Then since our schemes may be quite general, we must consider arbitrary (not necessarily discrete) valuation rings. Finally, we make the criterion relative over the image scheme Y of a morphism.

<u>THEOREM 4.3</u> (Valuative Criterion of Separatedness) Let $f: X \longrightarrow Y$ be a morphism of schemes, and assume that X is noetherian. Then f is separated if and only if the following condition holds: for any valuation ring R with quotient field K, let $T = \operatorname{Spec} R$, $U = \operatorname{Spec} K$ and let $i: U \longrightarrow T$ be the morphism induced by the inclusion $R \longrightarrow K$. Given a morphism of T to Y, and given a morphism of U to X which makes a commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d, "i"'] & X \arrow[d, "f"] \\ T \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

there is at most one morphism of T to X making the whole diagram commutative.
(NOTE: Same result is true if we take K any field and $R \subseteq K$ a valuation ring of K)

We will need two lemmas.

<u>LEMMA 4.4</u> Let R be a valuation ring of field K. Let $T = \operatorname{Spec} R$ and $U = \operatorname{Spec} K$. To give a morphism of U to a scheme X is equivalent to giving a point $x_1 \in X$ and an inclusion of fields $k(x_1) \to K$. To give a morphism of T to X is equivalent to giving two points x_0, x_1 in X, with x_0 a specialisation of x_1 (that is, $x_0 \in \{x_1\}^-$ see Ex 3.17) and an inclusion of fields $k(x_1) \longrightarrow K$, such that R dominates the local ring $\mathcal{O}$ of x_0 on the subscheme $Z = \{x_1\}^-$ of X with its induced reduced structure.

<u>PROOF</u> U is a one point scheme, with structure sheaf K. To give a local homomorphism $\mathcal{O}_{X,x_1} \longrightarrow K$ is the same as giving a ring morphism $k(x_1) \longrightarrow K$, so the first part is obvious. The spectrum of R con two points $t_0 = \mathfrak{m}_R$ and $t_1 = (0)$, with $\{t_1\}^- = \operatorname{Spec} R$. If $f: T \longrightarrow X$ is a morphism, let $x_0 = f(t_0)$, $x_1 = f(t_1)$. Then it is clear that $x_0 \in \{x_1\}^-$. Let $Z = \{x_1\}^-$. Then if $\mathcal{O}_Z$ is the induced reduced scheme structure on Z, then since Z is irreducible, Z is an integral scheme (3.1) with generic point x_1. Thus $\mathcal{O}_{Z,x_1}$ is a field and there is an injective morphism $\mathcal{O}_{Z,x_0} \longrightarrow \mathcal{O}_{Z,x_1} \cong k(x_1)$. Since T is reduced and $f(T) \subseteq Z$ there is a unique morphism $T \longrightarrow Z$ factoring f. (Ex 3.11) This gives local ring morphisms $\mathcal{O}_{Z,x_0} \longrightarrow \mathcal{O}_{T,t_0}$ and $\mathcal{O}_{Z,x_1} \longrightarrow \mathcal{O}_{T,t_1}$ which fit into the following commutative diagram

$$\begin{tikzcd} (U,s) \arrow[d] & \mathcal{O}_{Z,x_0} \arrow[r] \arrow[d] & \mathcal{O}_{T,t_0} \arrow[d] \arrow[r, phantom, "\cong"] & R \arrow[d] \\ (U,s) & \mathcal{O}_{Z,x_1} \arrow[r] \arrow[d, phantom, "\cong"] & \mathcal{O}_{T,t_1} \arrow[r, phantom, "\cong"] & K \\ & k(x_1) \arrow[urr, bend right] & & \end{tikzcd}$$

So we have our inclusion of fields $k(x_1) \longrightarrow K$. Note that $k(x_1) = \mathcal{O}_{X,x_1}/\mathfrak{m}_{x_1}$, and there is an iso of rings $\mathcal{O}_{Z,x_1} \Rightarrow \mathcal{O}_{X,x_1}/\mathcal{I}_{Z,x_1}$. We need only show $\mathcal{I}_{Z,x_1} = \mathfrak{m}_{x_1}$. Clearly $\subseteq$. For $\supseteq$, notice that if $s \in \mathcal{O}_X(U)$ then $\{x \in U \mid \operatorname{germ}_x s \in \mathfrak{m}_x\}$ is closed, so if $\operatorname{germ}_{x_1} s \in \mathfrak{m}_{x_1}$ then $s \in \mathcal{I}_Z(U)$ since $Z = \{x_1\}^-$. Note that then if $k(x_1) \to K$ is induced by $f_{x_1}: \mathcal{O}_{X,x_1} \longrightarrow \mathcal{O}_{T,t_1}$ the above diagram commutes. (since $k(x_1) \Rightarrow \mathcal{O}_{Z,x_1}$ arises from $\mathcal{O}_{X,x_1} \longrightarrow \mathcal{O}_{Z,x_1}$ and $T \to Z \to X = T \longrightarrow X$). It is clear that $k(x_1) \longrightarrow K$ maps the subring $\mathcal{O}_{Z,x_0}$ to a local subring of K dominated by R...... <u>NOTE</u> Nicer proof overleaf.

LEMMA Let R be a local ring, $T = \operatorname{Spec} R$. To give a morphism from T to a scheme X is equivalent to giving a point $x_0 \in X$ and a local morphism $\mathcal{O}_{X,x_0} \longrightarrow R$.

PROOF Let a morphism $f: T \longrightarrow X$ be given. Denote by $\mathfrak{m}$ the maximal ideal of R, and let $x_0 = f(\mathfrak{m})$. Every point $y \in f(T)$ is a generisation of x_0 since $\overline{f^{-1}\{y\}}$ is closed, contains some $\mathfrak{p} \subseteq R$ and hence contains $\mathfrak{m}$. So the image of f is contained in $G = \{y \in X \mid y \text{ is a generisation of } x_0\}$. By our earlier notes, f factors uniquely through $\operatorname{Spec}(\mathcal{O}_{X,x_0}) \longrightarrow X$:

$$\begin{tikzcd} T \arrow[rr] \arrow[dr, "\operatorname{Spec}\alpha"'] & & X \\ & \operatorname{Spec}(\mathcal{O}_{X,x_0}) \arrow[ur] & \end{tikzcd}$$

Let $\alpha: \mathcal{O}_{X,x_0} \longrightarrow R$ correspond to $T \longrightarrow \operatorname{Spec}(\mathcal{O}_{X,x_0})$. We get a commutative diagram (see earlier note)

$$\begin{tikzcd} & \mathcal{O}_{T,\mathfrak{m}} & & \mathcal{O}_{X,x_0} \arrow[ll] \arrow[dl, "\wr"] \arrow[ddll, bend left, "1"] \\ R \arrow[ur, phantom, sloped, "\cong"] & & & \\ & & \widetilde{\mathcal{O}_{X,x_0}}_{\mathfrak{n}} \arrow[uul] & \\ & \mathcal{O}_{X,x_0} \arrow[uul, "\alpha"] \arrow[ur, phantom, sloped, "\cong"] & & \end{tikzcd}$$

$\ulcorner \mathfrak{n}$ the maximum ideal $\lrcorner$

From which it follows that α is local. So associated to f is (x_0, α). Given $x_0 \in X$ and $\alpha: \mathcal{O}_{X,x_0} \longrightarrow R$ local, let f be the composite of $\operatorname{Spec}\alpha$ and $\operatorname{Spec}(\mathcal{O}_{X,x_0}) \longrightarrow X$. These are mutually inverse since the above factorisation is unique and since α is local the f induced by (x_0, α) maps $\mathfrak{m}$ to x_0. $\square$

Hence if morphisms out of the spectrum of a local ring agree as functions and have the same local morphism on the max. ideal, they are equal.

LEMMA Let R be a local domain, $T = \operatorname{Spec} R$. Let $t_0 = \mathfrak{m}$ be the maximal ideal of R and $t_0 = (0)$ its generic point. If $f, g: T \longrightarrow X$ are two morphisms to a scheme X, and if $f(t_0) = g(t_0)$ and $f(t_1) = g(t_1)$ and f, g induce the same ring morphism $\mathcal{O}_{X,x_1} \longrightarrow \mathcal{O}_{T,t_1}$ then $f = g$.

PROOF There are canonical morphisms $\mathcal{O}_{X,x_0} \longrightarrow \mathcal{O}_{X,x_1}$ and $\mathcal{O}_{T,t_0} \longrightarrow \mathcal{O}_{T,t_1}$, with the latter injective ($x_1 = f(t_1)$, $x_0 = f(t_0)$), fitting into commutative diagrams

$$\begin{tikzcd} \mathcal{O}_{T,t_0} \arrow[d, tail] & \mathcal{O}_{X,x_0} \arrow[l, "f_{t_0}"'] \arrow[d] \\ \mathcal{O}_{T,t_1} & \mathcal{O}_{X,x_1} \arrow[l, "f_{t_1}"] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{O}_{T,t_0} \arrow[d, tail] & \mathcal{O}_{X,x_0} \arrow[l, "g_{t_0}"'] \arrow[d] \\ \mathcal{O}_{T,t_1} & \mathcal{O}_{X,x_1} \arrow[l, "g_{t_1}"] \end{tikzcd}$$

Since $f_{t_1} = g_{t_1}$ it follows that $f_{t_0} = g_{t_0}$ and hence $f = g$ by the previous Lemma. $\square$

LEMMA Let X be a scheme, x_0, x_1 points of X with x_0 a specialisation of x_1. Then there is a valuation ring R and a morphism $f: T \longrightarrow X$ from $T = \operatorname{Spec} R$ to X, which maps the generic point t_1 of R to x_1 and the closed point t_0 to x_0.

PROOF Let $Z = \overline{\{x_1\}}$ be the induced reduced scheme structure. By (3.1) Z is an integral scheme, so $\mathcal{O}_{Z,x_0}$ is a local domain. Let $\operatorname{Spec}(\mathcal{O}_{Z,x_0}) \longrightarrow Z$ be canonical. Since the image of this morphism is the set of generisations of x_0, there is a prime $\mathfrak{p}$ of $\mathcal{O}_{Z,x_0}$ mapping to x_1. Clearly $\mathfrak{p} = (0)$.

Let K be the quotient field of $\mathcal{O}_{Z,x_0}$ and R a valuation ring of K dominating $\mathcal{O}_{Z,x_0}$. So there is a morphism $\alpha: \mathcal{O}_{Z,x_0} \longrightarrow R$ with $\alpha^{-1}\mathfrak{m} = x_0$ and $\alpha^{-1} 0 = x_1$. The morphism $\operatorname{Spec}\alpha: \operatorname{Spec} R \longrightarrow \operatorname{Spec}(\mathcal{O}_{Z,x_0})$ composed with the canonical map $\operatorname{Spec}(\mathcal{O}_{Z,x_0}) \longrightarrow Z \longrightarrow X$ gives the desired morphism. $\square$

LEMMA 4.5 Let $f: X \longrightarrow Y$ be a quasi-compact morphism of schemes (see Ex 3.2). Then the subset $f(X)$ of Y is closed if and only if it is stable under specialisation (Ex 3.17e).

PROOF One implication is obvious, so we only have to show that if $f(X)$ is stable under specialisation, then it is closed. Let $Z = f(X)^-$. Since f factors through the induced reduced scheme structure on Z, and we proved in our notes on induced reduced schemes that if $U \subseteq X$ is open and affine, then $Z \cap U$ is affine in Z, it follows that $X \longrightarrow Z$ is quasi-compact, since $f^{-1}U = f^{-1}(Z \cap U)$. Let $X_{red} \longrightarrow X$ be canonical — then the underlying morphism of spaces is the identity, so $X_{red} \longrightarrow X \to Z$ is quasi-compact and has image $f(X)$. Now $f(X)$ is closed under specialisation in Z iff. it is in Y, and closed iff. it is closed in Z, so we can reduce to the case where X, Y are both reduced, and $Y = f(X)^-$.

Let $y \in Y$ be a point. We wish to show that $y \in f(X)$. Let V be an open affine neighborhood of y. Then by Ex 3.2 $f^{-1}V$ is quasi-compact, and it suffices to show y is in the image of the induced morphism $f^{-1}V \longrightarrow V$. So we can assume X is (nonempty, since $f(X)^- = Y$), covered by a finite number of open affines X_i, and that Y is affine, $f: X \longrightarrow Y$ quasi-compact. Note that even with these replacements, we still have $Y = f(X)^-$, and $f(X)$ is still closed under specialisation. For some i we have $y \in f(X_i)^-$, since otherwise for all i there is $y \in U_i \subseteq (f(X_i)^-)^c$, whence (finite no. i) $U = \bigcap_i U_i$ is an open neighborhood of y not meeting $f(X)$, which is a contradiction. Let $Y_i = f(X_i)^-$ with the induced reduced structure. Since Y is affine, $Y \cap Y_i = Y_i$ is affine in Y_i, and $f|_{X_i}$ factors uniquely

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ X_i \arrow[u] \arrow[r] & Y_i \arrow[u] \\ \operatorname{Spec} A \arrow[u, "\cong"] \arrow[r] & \operatorname{Spec} B \arrow[u, "\cong"'] \end{tikzcd}$$

The ring morphism $B \xrightarrow{\varphi} A$ corresponding to $X_i \longrightarrow Y_i$ is injective, since $X_i \to Y_i$ is a dominant morphism of reduced affine schemes (see our solution to Ex 2.18). The point $y \in Y_i$ corresponds to a prime ideal $\mathfrak{p} \subseteq B$. Let $\mathfrak{p}' \subseteq \mathfrak{p}$ be a minimal prime ideal of B contained in $\mathfrak{p}$, and let $y' \in Y_i$ correspond to $\mathfrak{p}'$. Since y' specialises to y and $f(X)$ is closed under specialisation, to complete the proof, it suffices to show that $\mathfrak{p}' = \varphi^{-1}\mathfrak{q}$ for some prime $\mathfrak{q} \subseteq A$.

So we have reduced to the following algebra problem: $\varphi: B \longrightarrow A$ is an injective morphism of reduced rings, $\mathfrak{p}' \subseteq B$ is minimal and we wish to find $\mathfrak{q} \subseteq A$ prime with $\varphi^{-1}\mathfrak{q} = \mathfrak{p}'$. Consider A as a B-module and localise at $\mathfrak{p}'$:

$$\begin{tikzcd} B \arrow[r] \arrow[d] & A \arrow[d, "a \mapsto a \otimes 1"] \\ B_{\mathfrak{p}'} \arrow[r, "b/s \mapsto 1 \otimes b/s"'] & A \otimes B_{\mathfrak{p}'} \end{tikzcd}$$

Since localisation is exact, the bottom row is injective, and $B_{\mathfrak{p}'}$ has only prime, which must be 0 since localisation preserves nilradicals. Hence $B_{\mathfrak{p}'}$ is a field, and $A \otimes B_{\mathfrak{p}'}$ a nonzero ring. Let $\mathfrak{q}_0'$ be any prime of $A \otimes B_{\mathfrak{p}'}$. Then $\mathfrak{q}_0' \cap B_{\mathfrak{p}'} = (0)$. Let $\mathfrak{q}'$ be the inverse image of $\mathfrak{q}_0'$ under $A \longrightarrow A \otimes B_{\mathfrak{p}'}$. Then $\varphi^{-1}\mathfrak{q}' = \mathfrak{p}' B_{\mathfrak{p}'} \cap B = \mathfrak{p}'$, which completes the proof. □

PROOF OF THEOREM 4.3 First suppose that f is separated, and suppose given a diagram as above where there are two morphisms $h, h': T \longrightarrow X$ making the whole diagram commutative.

$$\begin{tikzcd} U \arrow[r] \arrow[d, "i"'] & X \arrow[d, "f"] \\ T \arrow[r] \arrow[ur, shift left, "h"] \arrow[ur, shift right, "h'"'] & Y \end{tikzcd} \qquad (1)$$

Then we obtain a morphism $h'': T \longrightarrow X \times_Y X$. If t_1 is the generic point of T, the fact that $hi = h'i$ means that we get a commutative diagram

$$
\begin{tikzcd}
U \arrow[dr] \arrow[dd, "i"'] & & \\
& X \arrow[d, "\Delta"] \arrow[dr, bend left, Rightarrow] \arrow[ddl, bend right, Rightarrow] & \\
T \arrow[rr, bend left, "h'"] \arrow[r, dashed, "h''"] \arrow[d, "h"'] & X\times_Y X \arrow[r] \arrow[dl] & X \arrow[d] \\
X \arrow[rr] & & Y
\end{tikzcd}
$$

Since $\Delta(X)$ is closed, and t_1 has its image in $\Delta(X)$, it follows that the image of t_0 also belongs to $\Delta(X)$ hence $h(t_0) = h'(t_0)$. Commutativity of (1) on stalks shows that $h_{x_1} = h'_{x_1} : \mathcal{O}_{X,x_1} \longrightarrow \mathcal{O}_{T,t_1}$, so by the preceeding Lemmas $h = h'$, as required. (This part does not need X noethenan, and only requires R to be a local domain with quotient field K).

Conversely, let us suppose the condition of the theorem is satisfied. To show that f is separated, it is sufficient by (4.2) to show that $\Delta(X)$ is a closed subset of $X \times_Y X$. Since we have assumed X is noethenan (hence $sp(X)$ is a noethenan space, so every open subset is quasi-compact Ex 2.13) the morphism Δ is quasi-compact, so by (4.5) it will be sufficient to show that $\Delta(X)$ is stable under specialisation. So let $\xi_1 \in \Delta(X)$ be a point and $\xi_1 \rightsquigarrow \xi_0$ a specialisation. Then there is a valuation ring R, and a morphism $T = \operatorname{Spec} R \longrightarrow X \times_Y X$ mapping the closed point t_0 of $\operatorname{Spec} R$ to ξ_0 and the generic point t_1 of $\operatorname{Spec} R$ to ξ_1. Let K be the quotient field of R, $U = \operatorname{Spec} K$ and $U \longrightarrow T$ canonical, $t \longmapsto t_1$. We have a diagram

$$
\begin{tikzcd}
T \arrow[ddr] & & U \arrow[ll] \arrow[ddr, dashed] & \\
& & X \arrow[dl, "\Delta"'] \arrow[dr, Rightarrow] \arrow[ddl, Rightarrow] & \\
& X\times_Y X \arrow[rr] \arrow[d] & & X \arrow[d, "f"] \\
& X \arrow[rr, "f"'] & & Y
\end{tikzcd}
$$

In our notes on fields and diagonals we showed that $U \longrightarrow T \longrightarrow X \times_Y X \longrightarrow X$ is well-defined (i.e. it doesn't matter which projection you use) since $\xi_1 \in \Delta(X)$. By the condition, the two morphisms $T \longrightarrow X$ must be the same, and therefore $T \longrightarrow X \times_Y X$ factors through Δ, showing that $\xi_0 \in \Delta(X)$, which completes the proof. $\square$

<u>COROLLARY 4.6</u> Assume that all schemes are noethenan in the following statements:

(a) Open and closed immersions are separated
(b) A composition of two separated morphisms is separated
(c) Separated morphisms are closed under base extension
(d) If $f : X \longrightarrow Y$ and $f' : X' \longrightarrow Y'$ are separated morphisms of schemes over a base scheme S, then the product morphism $f \times f' : X \times_S X' \longrightarrow Y \times_S Y'$ is also separated.
(e) If $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are two morphisms, and if $g \circ f$ is separated, then f is separated
(f) A morphism $f : X \longrightarrow Y$ is separated if and only if Y can be covered by open subsets V_i such that $f^{-1}(V_i) \longrightarrow V_i$ is separated for each i, and if f is separated and $V \subseteq Y$ open then $f^{-1}V \longrightarrow V$ is separated.

<u>PROOF</u> (a) An open immersion is a monomorphism of schemes, hence by (4.3) clearly separated. Similarly a closed immersion is a monomorphism, hence separated. (Need domains noethenan)
(we actually proved this earlier without any noethenan assumptions)

Note a morphism $\emptyset \longrightarrow Y$ is always separated.

(b) Consider the following diagram:

$$\begin{tikzcd}
U \arrow[r] \arrow[dd] & X \arrow[d, "f"] \\
 & Y \arrow[d, "g"] \\
T \arrow[r] \arrow[uur, bend left, "h"] \arrow[uur, bend right, "h'"'] & Z
\end{tikzcd}$$

Separatedness of g means $fh = fh'$, and then separatedness of f means $h = h'$. (Need X noethenan)

(c) Easy: if $f: X \longrightarrow Y$ is separated, $Y' \longrightarrow Y$ base change $\begin{tikzcd}[cramped] X' \arrow[r] \arrow[d] & Y' \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$ pullback and X' noethenan, $X' \longrightarrow Y'$ separated.

(d) Consider the following diagram and use the pullback $X \times_S X'$: (We need $X \times_S X'$ noethenan - don't nec. need X, X' to be).

$$\begin{tikzcd}
U \arrow[r] \arrow[dd] & X\times_S X' \arrow[rr] \arrow[d] \arrow[dd] & & X' \arrow[d] \arrow[dd, "f'"] \\
 & X \arrow[rr] \arrow[dd, "f"] & & S \arrow[dd] \\
T \arrow[r] \arrow[uur, bend left] \arrow[uur, bend right] & Y\times_S Y' \arrow[rr] \arrow[d] & & Y' \arrow[d] \\
 & Y \arrow[rr] & & S
\end{tikzcd}$$

(e) Easy. (Need X noethenan)

(f) Let f be separated, $V \subseteq X$ open and assume V noethenan. Then $f|_V = V \longrightarrow X \longrightarrow Y$ is separated by (a), (b). Now assume X noethenan and f separated, $V \subseteq Y$ open. Then considering the following diagram we see that $f^{-1}V \longrightarrow V$ is separated: (One can also show this without the noethenan condition - see an earlier note)

$$\begin{tikzcd}
X \arrow[d, "f"] & f^{-1}V \arrow[l] \arrow[d] & U \arrow[l] \arrow[d] \\
Y & V \arrow[l] \arrow[r] & T \arrow[ul, bend left] \arrow[ul, bend right]
\end{tikzcd}$$

Now suppose there is an open cover $Y = \bigcup_i V_i$ s.t. $f|_{f^{-1}V_i}: f^{-1}V_i \longrightarrow V_i$ is separated $\forall i$. Start with a commutative diagram

$$\begin{tikzcd}
U \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
T \arrow[r, "j"'] & Y
\end{tikzcd}$$

Now every point of $j(T)$ is a generisation of $j(t_0)$, where t_0 is the closed point, so there is some V_i with $j(T) \subseteq V_i$. Using the fact that $V_i \longrightarrow Y$, $f^{-1}V_i \longrightarrow X$ are monomorphisms we get a commutative diagram

$$\begin{tikzcd}
U \arrow[r] \arrow[rr, bend right=10] \arrow[dd] & f^{-1}V_i \arrow[r] \arrow[d] & X \arrow[dd, "f"] \\
 & V_i \arrow[dr] & \\
T \arrow[ur] \arrow[rr] \arrow[uurr, bend left, "h"] \arrow[uurr, bend right, "h'"'] & & Y
\end{tikzcd}$$

⌜Need X noetherian for the second part⌟

Given h, h' as indicated, both factor through $f^{-1}V_i \longrightarrow X$, and the fact that $f^{-1}V_i \longrightarrow V_i$ is separated completes the proof. $\square$

DEFINITION A morphism $f: X \longrightarrow Y$ is proper if it is separated, of finite type, and universally closed. Here we say that a morphism is closed if the image of any closed subset is closed. A morphism $f: X \longrightarrow Y$ is universally closed if it is closed, and for any morphism $Y' \longrightarrow Y$, the corresponding morphism $f': X' \longrightarrow Y'$ obtained by base extension is also closed. (This is independent of the chosen pullback). That is, any pullback of f is closed.

EXAMPLE 4.6.1 Let k be a field and let $X = \operatorname{Spec} k[x]$ be the affine line over k. Then X is separated and of finite type over k (any morphism of aff. schemes is separated), but it is not proper over k. Indeed, take the base extension $X \longrightarrow k$. The morphism $X \times_k X \longrightarrow X$ is $\operatorname{Spec}(k[x] \otimes_k k[y]) \longrightarrow \operatorname{Spec}(k[x])$ which is the same as $\operatorname{Spec}(k[x,y]) \longrightarrow \operatorname{Spec}(k[x])$. Let $\varphi: k[x] \longrightarrow k[x,y]$. Then (assume k alg. closed) $\varphi^{-1}(x-a, xy-b) = (x-a)$ so the projection of the hyperbola $xy-1$ is the x-axis minus the origin, which is not closed. (if $x \in \mathfrak{p}$ and $xy - 1 \in \mathfrak{p}$ then $\mathfrak{p}$ is not proper).

$\boxed{\operatorname{Spec}(k[x,y])}$		$\boxed{\operatorname{Spec} k[x]}$
Maximal ideals $(x-a, y-b)$	$\longmapsto$	Maximal ideal $(x-a)$
Prime ideals $(x-a)$	$\longmapsto$	Maximal ideal $(x-a)$
All other primes	$\longmapsto$	Prime ideal 0

Since if $\mathfrak{p} \in \operatorname{Spec}(k[x,y])$ is not 0 and not maximal, it is $(f(x))$ for an irreducible monic polynomial f. But then $\varphi^{-1}\mathfrak{p} = 0$ or $(x-c)$ for some $c \in k$, and if $x - c \in \mathfrak{p}$ then $f(x) = x - c$. So the primes containing $(xy-1)$ are mapped to the whole spectrum minus (x). Since $\operatorname{Spec} k[x]$ is irreducible this set cannot be closed.

Of course it is clear that what is missing in this example is the point at infinity on the hyperbola. This suggests that the projective line would be proper over k. In fact, we will see later (4.9) that any projective variety over a field is proper.

NOTE It is easily checked that if $f: X \longrightarrow Y$ is proper so are $Z \Rightarrow X \longrightarrow Y$, $X \longrightarrow Y \Rightarrow Z$ for isomorphisms $\Rightarrow$ Any morphism out of or into $(\emptyset, 0)$ is proper.

THEOREM 4.7 (Valuative Criterion of Properness) Let $f: X \longrightarrow Y$ be a morphism of finite type, with X noetherian. Then f is proper if and only if for every valuation ring R and for every morphism of U to X and T to Y forming a commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ T \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

(using the notation of (4.3)) there exists a unique morphism $T \longrightarrow X$ making the whole diagram commutative. (NOTE The same result is true if we let K be a field, $R \subseteq K$ a valuation ring of K).

PROOF First assume that f is proper. Then by definition f is separated, so the uniqueness of the morphism $T \longrightarrow X$ will follow from (4.3), once we know it exists. For the existence, we consider the base extension $T \longrightarrow Y$, and let $X_T = X \times_Y T$. We get a map $U \longrightarrow X_T$ from the given maps $U \longrightarrow X$ and $U \longrightarrow T$.

$$\begin{tikzcd} U \arrow[r] \arrow[d, dashed] & X_T \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Z \arrow[ur] & T \arrow[r] & Y \end{tikzcd}$$

Let $\xi_1 \in X_T$ be the image of the unique point t_1 of U. Let $Z = \overline{\{\xi_1\}}$. Then Z is a closed, irreducible subset of X_T which we give the induced reduced scheme structure, so that $U \longrightarrow X_T$ factors uniquely through the closed immersion $Z \longrightarrow X_T$. Since f is proper, it is universally closed, so the morphism $f': X_T \longrightarrow T$ must be closed, so $f'(Z)$ is a closed subset of T. But $f'(\xi_1) = t_1$, so in fact $f'(Z) = T$. Hence there is a point $\xi_0 \in Z$ with $f'(\xi_0) = t_0$. Consider the following commutative diagram (see Ex 3.6 for some details) where $\mathcal{O}_{T,t_0} \longrightarrow \mathcal{O}_{Z,\xi_0}$ is injective since $\mathcal{O}_{T,t_1} \longrightarrow \mathcal{O}_{Z,\xi_1}$ is, which follows since $\mathcal{O}_{T,t_1} \longrightarrow \mathcal{O}_{U,t_1}$ is an isomorphism.

$$\begin{tikzcd}
 & \mathcal{O}_{Z,\xi_0} \arrow[d] & & \mathcal{O}_{T,t_0} \arrow[ll] \arrow[d] \arrow[r, phantom, "\cong"] & R \arrow[d] \\
\text{field} \arrow[r] & \mathcal{O}_{Z,\xi_1} \arrow[d] & & \mathcal{O}_{T,t_1} \arrow[ll] \arrow[r, phantom, "\cong"] \arrow[dll, Rightarrow] & K \\
 & \mathcal{O}_{U,t_1} \cong K & & &
\end{tikzcd}$$

Since $\mathcal{O}_{T,t_0} \longrightarrow \mathcal{O}_{Z,\xi_0}$ is local, $\mathcal{O}_{Z,\xi_0}$ maps to a local subring of K dominating R. Since R is a valuation ring, by (I, 6.1A) R is maximal w.r.t. domination, so $\mathcal{O}_{T,t_0} \longrightarrow \mathcal{O}_{Z,\xi_0}$ is an isomorphism. Let $\alpha : \mathcal{O}_{Z,\xi_0} \longrightarrow \mathcal{O}_{T,t_0}$ be the inverse. Applying an earlier Lemma we obtain

$$\begin{array}{ccccccc}
T & \Longrightarrow & \operatorname{Spec}(\mathcal{O}_{Z,\xi_0}) & \longrightarrow & Z & \longrightarrow & X_T \\
t_0 & \cdots\cdots & m & \cdots\cdots & \xi & \cdots\cdots & \xi \\
t_1 & \cdots\cdots & 0 & \cdots\cdots & \xi & \cdots\cdots & \xi
\end{array}$$

The local morphism of $T \longrightarrow Z$ on ξ_1 must fit into the above diagram, hence must be inverse to $\mathcal{O}_{T,t_1} \longrightarrow \mathcal{O}_{X_T,\xi_1} \longrightarrow \mathcal{O}_{Z,\xi_1}$. It follows that $U \longrightarrow T \longrightarrow Z = U \longrightarrow Z$. Considering

$$\begin{tikzcd}
 & \mathcal{O}_{X_T,\xi_1} \arrow[d] & & \mathcal{O}_{X,q} \arrow[ll] \arrow[dl] \\
\mathcal{O}_{U,t_1} & \mathcal{O}_{Z,\xi_1} \arrow[l] \arrow[r, Rightarrow] & \mathcal{O}_{T,t_1} \arrow[ul] \arrow[ur, phantom, "\mathrm{I}"] & \mathcal{O}_{Y,n} \arrow[l] \arrow[u]
\end{tikzcd}$$

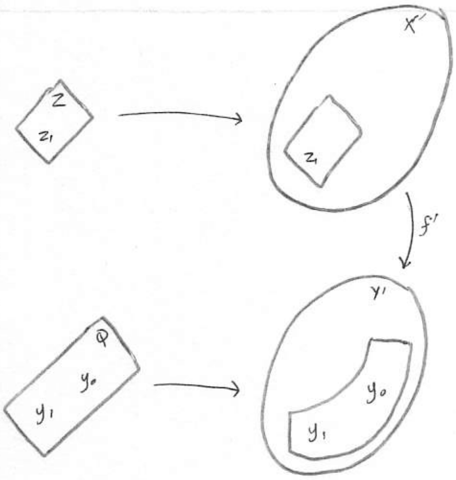
$f(q) = n$
$t_1 \longmapsto n$

We see that the two maps $T \longrightarrow Y$ and $T \longrightarrow X \dashrightarrow Y$ agree on t_0, t_1 and have the same local morphism on the image of t_1, hence are equal, which completes the first part of the proof (We have not used X noetherian or f of finite type - we need f separated and universally closed)

Conversely, suppose the condition of the theorem holds. To show f is proper, we only have to show it is universally closed, since it is of finite type by hypothesis, and it is separated by (4.3). So let $Y' \longrightarrow Y$ be any morphism (choose 1_Y to get f closed) and let $f' : X' \longrightarrow Y'$ be the morphism obtained from f by base extension. Let Z be a closed subset of X', and give it the reduced induced structure.

$$\begin{tikzcd}
Z \arrow[r] & X' \arrow[r] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
 & Y' \arrow[r] & Y
\end{tikzcd}$$

We need to show that $f'(Z)$ is closed in Y'. Since f is of finite type, so is f' and so is the restriction of f' to Z (Ex 3.13). In particular $Z \longrightarrow Y'$ is quasi-compact, so by (4.5) we have only to show that $f'(Z)$ is closed under specialisation. So let $z_1 \in Z$ be a point, let $y_1 = f'(z_1)$ and let $y_1 \rightsquigarrow y_0$ be a specialisation of y_1. Let $Q = \overline{\{y_1\}}$ with the induced reduced structure. There are isomorphisms $k(y_1) \cong \mathcal{O}_{Q,y_1}$ and $k(z_1) \cong \mathcal{O}_{Z,z_1}$ and a commutative diagram:



$$\begin{tikzcd}
R \arrow[r, hook] & \mathcal{O}_{Z,z_1} \arrow[r, equal, "\sim"] & k(z_1) & \mathcal{O}_{X',z_1} \arrow[l] \arrow[ll, bend right] \\
\mathcal{O}_{Q,y_0} \arrow[r, tail] \arrow[u, dashed, "\alpha"] & \mathcal{O}_{Q,y_1} \arrow[u, tail] \arrow[r, equal, "\sim"] & k(y_1) \arrow[u, tail] & \mathcal{O}_{Y',y_1} \arrow[l] \arrow[u] \arrow[ll, bend left]
\end{tikzcd} \qquad (1)$$

Let R be a valuation ring of the field $\mathcal{O}_{Z,z_1}$ dominating the image of $\mathcal{O}_{Q,y_0}$. So we get a local morphism $\mathcal{O}_{Q,y_0} \longrightarrow R$ making the above diagram commute. Put $U = \operatorname{Spec}(\mathcal{O}_{Z,z_1})$ (since $\mathcal{O}_{Z,z_1}$ is isomorphic to the quotient field of R, there is no harm) and let $i : U \longrightarrow T = \operatorname{Spec} R$ be canonical. Let $U \longrightarrow Z$ be the canonical morphism, and $T \longrightarrow Y'$ the composite $\operatorname{Spec} R \longrightarrow \operatorname{Spec}(\mathcal{O}_{Q,y_0}) \longrightarrow Q \longrightarrow Y'$. We claim the following diagram commutes:

$$\begin{tikzcd}
U \arrow[r] \arrow[d, "i"'] & Z \arrow[d] \\
T \arrow[r] & Y'
\end{tikzcd} \qquad (2)$$

It is not difficult to check the underlying maps of spaces commute, and it then suffices to show the square of local maps corresponding to the single point $t_0 \in U$ commutes:

$$\begin{tikzcd}
\mathcal{O}_{Z,z_1} & (\mathcal{O}_{Z,z_1})_0 \arrow[l, Rightarrow] & \mathcal{O}_{U,t_0} \arrow[l, Rightarrow] \arrow[dr, phantom, "\mathrm{I}"] & \mathcal{O}_{Z,z_1} \arrow[l, Rightarrow] & \\
 & & & & \mathcal{O}_{X',z_1} \arrow[ul] \\
 & & \mathcal{O}_{T,t_0} \arrow[uu] \arrow[dl, Rightarrow] & \mathcal{O}_{Y',y_1} \arrow[l] \arrow[uu] \arrow[ur] \arrow[dd] & \\
 & R_0 \arrow[uuu] & & & \\
 & & \widetilde{\mathcal{O}_{Q,y_0}}_0 \arrow[uu] \arrow[d, Rightarrow] & \mathcal{O}_{Q,y_1} \arrow[l, Rightarrow] & \\
 & & (\mathcal{O}_{Q,y_0})_0 \arrow[uul, bend left] & \mathcal{O}_{Q,y_0} \arrow[l] \arrow[u, tail] &
\end{tikzcd}$$

To show I commutes we use the fact that (1) commutes, $\mathcal{O}_{Q,y_0} \longrightarrow \mathcal{O}_{Y',y_1}$ is an epimorphism (since $\mathcal{O}_{Q,y_1} \cong (\mathcal{O}_{Q,y_0})_0$) and all other squares commute. This shows that (2) commutes. Composing with morphisms $Z \longrightarrow X' \longrightarrow X$ and $Y' \longrightarrow Y$, we get morphisms $U \longrightarrow X$ and $T \longrightarrow Y$ to which we can apply the condition of the theorem. So there is a morphism $T \longrightarrow X$ making the diagram commute. Since X' is a fibred product, it lifts to give a morphism $T \longrightarrow X'$, and using the pullback it is not hard to check that $U \longrightarrow T \longrightarrow X' = U \longrightarrow Z \longrightarrow X'$. Hence the generic point of T goes to $z_1 \in Z$, and since Z is closed this morphism $T \longrightarrow X'$ factors through $Z \longrightarrow X'$. Now let z_0 be the image of t_0. Then $f'(z_0) = y_0$, so $y_0 \in f'(Z)$. This completes the proof. $\square$

<u>COROLLARY 4.8</u> In the following statements, we take all schemes to be noetherian:

(a) A closed immersion is proper
(b) A composition of proper morphisms is proper
(c) Proper morphisms are stable under base extension
(d) If $f: X \longrightarrow Y$ and $f': X' \longrightarrow Y'$ are proper morphisms of schemes over a base scheme S, then the product morphism $f \times f': X \times_S X' \longrightarrow Y \times_S Y'$ is also proper
(e) If $f: X \longrightarrow Y$ and $g: Y \longrightarrow Z$ are two morphisms, if $g \circ f$ is proper, and if g is separated, then f is proper.
(f) If $f: X \longrightarrow Y$ is proper and $V \subseteq Y$ open then $f^{-1}V \longrightarrow V$ is proper. If Y can be covered by open subsets V_i such that $f^{-1}V_i \longrightarrow V_i$ is proper for each i, then f is proper.

<u>PROOF</u> A closed immersion is proper without any noetherian assumptions, since any closed immersion is separated, of finite type (Ex 3.13) closed (clearly) and universally closed by (Ex 3.11a).

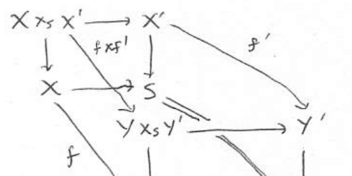
(b) If $f: X \longrightarrow Y$, $g: Y \longrightarrow Z$ proper and X is noetherian, $g \circ f$ is proper

(c) Consider a pullback

$$\begin{tikzcd} X' \arrow[r, "g"] \arrow[d] & Y' \arrow[d] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

If f is proper then by (Ex 3.13) g is of finite type and is universally closed by inspection. Provided X' is noetherian, g is separated as well, and hence proper.

(d) Consider the following diagram, assume $X \times_S X'$ is noetherian, $f: X \longrightarrow Y$ and $f': X' \longrightarrow Y'$ proper morphisms of schemes over S:



For any choice of the pullbacks,

Products of morphisms of finite type are of finite type (Ex 3.13) so we simply apply the Theorem.

(e) If $f: X \longrightarrow Y$ and $g: Y \longrightarrow Z$ are given, with $g \circ f$ proper and g separated. We assume X is noetherian. Then f is quasi-compact and hence by Ex 3.13 f is of finite type. Also f is separated by (4.6). So we need only show existence in the valuative criterion, given a commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ T \arrow[r] \arrow[ur, dashed] & Y \arrow[d, "g"] \\ & Z \end{tikzcd}$$

Use $g \circ f$ proper to get $T \longrightarrow X$ and g separated to show $T \longrightarrow X \longrightarrow Y = T \longrightarrow Y$.

(f) Let $f: X \longrightarrow Y$ be proper and $V \subseteq Y$ open. Since being of finite type and being separated is local on the base (assuming X noetherian) it follows that $f^{-1}V \longrightarrow V$ is of finite type and separated. Checking an appropriate diagram shows that $f^{-1}V \longrightarrow V$ is proper by the Theorem.

Now suppose $f: X \longrightarrow Y$ is a morphism, X noethenan and there is an open cover $\{V_i\}$ of Y such that $f^{-1}V_i \longrightarrow V_i$ is proper for all i. Then since finiteness and separatedness are local, f is a separated morphism of finite type. Applying the argument of (4.6f) we see that f is proper. $\square$

┌ Here and elsewhere graded ring = +ve graded and graded module = $\mathbb{Z}$-graded. See our Graded Rings and Modules notes ┘

Let S be a graded ring, M a graded S-module. A <u>graded submodule</u> of M is a submodule $N \subseteq M$ which is a graded module in such a way that $N \longrightarrow M$ preserves degrees. Put differently, $N \subseteq M$ is a submodule with the property that if $m \in N$ and $m = \sum_{n\in\mathbb{Z}} m_n$ in M then $\forall n \in \mathbb{Z}$ $m_n \in N$.

<u>LEMMA</u> A submodule $N \subseteq M$ is a graded submodule if and only if it is generated as an S-module by a set of homogenous elements of M.

<u>PROOF</u> If $N \subseteq M$ is graded, it is generated by all its homogenous elements. For the converse, let $\{m_i\}_{i\in I}$ be a set of homogenous elements of M with $N = (\{m_i\})$. Let $n \in N$ be given, say (0 trivially graded, so assume $I \neq \emptyset$)

$$n = s_1 m_1 + \cdots + s_n m_n$$

Then for $d \in \mathbb{Z}$

$$\begin{aligned} n_d &= (s_1 m_1)_d + \cdots + (s_n m_n)_d \\ &= s_{1\,d-d_1} m_1 + \cdots + s_{n\,d-d_n} m_n \qquad m_i \in M_{d_i} \end{aligned}$$

Hence $n_d \in N$, as required. $\square$

<u>NOTE</u> If $N \subseteq M$ is a graded submodule, M/N becomes a graded S-module, $(M/N)_d = \{a + N \mid a \in M_d\}$.

Let M, N be graded S-modules, $M \otimes_S N$ their tensor product. We show that $M \otimes_S N$ becomes a graded S-module with

$$(M \otimes_S N)_n = \{ \textstyle\sum_i m_i \otimes n_i \mid \deg m_i + \deg n_i = n \ \forall i \} \qquad (1)$$

┌ m_i, n_i homogenous ┘

First consider M, N as groups. Let B be the subgroup of the free abelian group A on the set $M \times N$ consisting of the relations

$$\begin{aligned} &① \quad (x+x', y) - (x,y) - (x',y) \\ & \quad (x, y+y') - (x,y) - (x,y') \\ &② \quad (x\cdot s, y) - (x, s\cdot y) \end{aligned}$$

Then $B = P + L$ where P is the subgroup generated by the relations in ①, L by the relations in ②. By definition $M \otimes_{\mathbb{Z}} N = A/P$. Since the tensor product preserves direct sums, we have

$$M \otimes_{\mathbb{Z}} N \cong \bigoplus_{q\in\mathbb{Z}} \bigoplus_{m+n=q} M_m \otimes_{\mathbb{Z}} N_n$$

where a sequence $(m_{m,n,q} \otimes n_{m,n,q})_{m,n,q}$ is mapped to the sum of all the entries. In other words, $M \otimes_{\mathbb{Z}} N$ is the direct sum of the following subgroups:

$$(M \otimes_{\mathbb{Z}} N)_q = \{ \textstyle\sum_i m_i \otimes n_i \mid \deg m_i + \deg n_i = q \} \qquad (2)$$

There is a canonical morphism of groups $M \otimes_{\mathbb{Z}} N \xrightarrow{\alpha} M \otimes_S N$ associated to $A/P \longrightarrow A/B$, the kernel of which is $L + P/P$ — the subgroup of $M \otimes_{\mathbb{Z}} N$ generated by elements of the form $x \cdot s \otimes y - x \otimes s \cdot y$. Call this subgroup P'. Considering $M \otimes_{\mathbb{Z}} N$ as a graded $\mathbb{Z}$-module ($\mathbb{Z}_0 = \mathbb{Z}$, $\mathbb{Z}_n = 0$ $n>0$) with the grading in (2), P' is a graded submodule, since it is generated by homogenous elements:

$$\begin{aligned} x\cdot s \otimes y - x \otimes s \cdot y &= (\textstyle\sum_i x_i)\cdot(\sum_d s_d) \otimes (\sum_j y_j) \\ &\quad - (\textstyle\sum_i x_i) \otimes (\sum_d s_d)\cdot(\sum_j y_j) \\ &= \textstyle\sum_{i,d,j} (x_i \cdot s_d \otimes y_j - x_i \otimes s_d \cdot y_j) \end{aligned}$$

where each of these summands is homogenous, of degree $i+d+j$. With the notation of (2), α is epi and $(M \otimes_S N)_n = \alpha((M \otimes_{\mathbb{Z}} N)_n)$, and the fact that $M \otimes_S N = \bigoplus_n (M \otimes_S N)_n$ follows from the fact that $M \otimes_{\mathbb{Z}} N / P' \cong M \otimes_S N$ as abelian groups, using the quotient grading.

At this point you should consult our typed notes "The Proj Construction".

Originally this note was titled "Notes on *Proj*", but it was combined with "Projective Space over a Scheme" and together they are now named "The Proj Construction". In our written notes there may be references to these notes separately.

Our next objective is to define projective morphisms and to show that any projective morphism is proper. Recall that in Section 2 we defined projective n-space $\mathbb{P}^n_A$ over any ring A to be $\operatorname{Proj} A[x_0,\dots,x_n]$. As we have just shown, if $A \longrightarrow B$ is a morphism of rings then $\mathbb{P}^n_B \cong \mathbb{P}^n_A \times_{\operatorname{Spec} A} \operatorname{Spec} B$. This motivates the following definition for any scheme Y.

<u>DEFINITION</u> If Y is any scheme, we define <u>projective n-space</u> over Y, denoted $\mathbb{P}^n_Y$, to be $\mathbb{P}^n_{\mathbb{Z}} \times_{\operatorname{Spec}\mathbb{Z}} Y$. That is, $\mathbb{P}^n_Y$ is the product of Y and $\mathbb{P}^n_{\mathbb{Z}}$, hence only determined up to canonical isomorphism. Since projective space over a ring satisfies $\mathbb{P}^n_A \cong \mathbb{P}^n_{\mathbb{Z}} \times \operatorname{Spec} A$ we have $\mathbb{P}^n_{\operatorname{Spec} A} \cong \mathbb{P}^n_A$ for any ring A. A morphism $f : X \longrightarrow Y$ of schemes is <u>projective</u> if it factors into a closed immersion $i : X \longrightarrow \mathbb{P}^n_Y$ for some $n \geq 1$, followed by the projection $\mathbb{P}^n_Y \longrightarrow Y$. This is independent of the choice of scheme to call $\mathbb{P}^n_Y$, in the sense that if f factors through one projection via a closed immersion then it factors through all of them by a closed immersion.

A morphism is <u>quasi-projective</u> if it factors into an open immersion $j : X \longrightarrow X'$ followed by a projective morphism $g : X' \longrightarrow Y$.

<u>EXAMPLE 4.8.1</u> Let A be a ring, let S be a graded ring with $S_0 \cong A$ as rings, which is finitely generated as an A-algebra by S_1. Then the natural map $\operatorname{Proj} S \longrightarrow \operatorname{Spec} A$ is a projective morphism. Indeed, by hypothesis there is a surjective morphism of graded rings $A[x_0,\dots,x_n] \longrightarrow S$. Hence by Ex 3.12 the induced morphism $\operatorname{Proj} S \longrightarrow \mathbb{P}^n_A$ is a closed immersion. It is clear that the following diagram commutes:

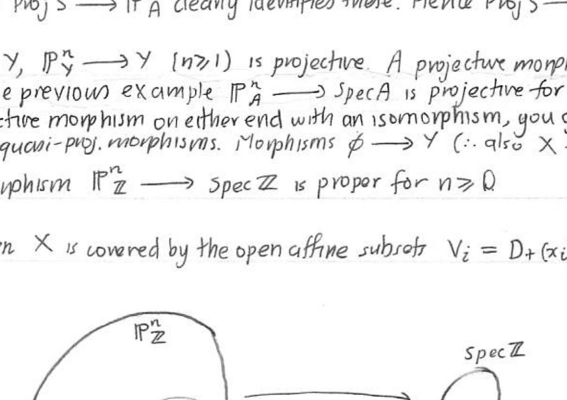
$$\begin{tikzcd} & \mathbb{P}^n_A \arrow[dr] & \\ \operatorname{Proj} S \arrow[ur] \arrow[rr] & & \operatorname{Spec} A \end{tikzcd}$$

Since on global sections $a \in A$ becomes $a \mapsto a/1 \in \Gamma(\mathbb{P}^n_A)$ and $a \mapsto a/1 \in \Gamma(\operatorname{Proj} S)$. The morphism $\operatorname{Proj} S \longrightarrow \mathbb{P}^n_A$ clearly identifies these. Hence $\operatorname{Proj} S \longrightarrow \operatorname{Spec} A$ is projective.

<u>EXAMPLE</u> For any scheme Y, $\mathbb{P}^n_Y \longrightarrow Y$ $(n \geq 1)$ is projective. A projective morphism is quasi-projective. As shown by the previous example $\mathbb{P}^n_A \longrightarrow \operatorname{Spec} A$ is projective for $n \geq 1$, any ring A. If you compose a projective morphism on either end with an isomorphism, you get a projective morphism. The same is true of quasi-proj. morphisms. Morphisms $\emptyset \longrightarrow Y$ ($\therefore$ also $X \longrightarrow \emptyset$) are always projective.

<u>THEOREM 4.9</u> The morphism $\mathbb{P}^n_{\mathbb{Z}} \longrightarrow \operatorname{Spec}\mathbb{Z}$ is proper for $n \geq 0$

<u>PROOF</u> Let $X = \mathbb{P}^n_{\mathbb{Z}}$. Then X is covered by the open affine subsets $V_i = D_+(x_i)$ $0 \leq i \leq n$



So to show $\mathbb{P}^n_{\mathbb{Z}} \longrightarrow \mathbb{Z}$ is of finite type it suffices to show $\mathbb{Z}[x_0,\dots,x_n]_{(x_i)}$ is a f.g. $\mathbb{Z}$-algebra. But in our Ch 1 §3 notes we showed that $\mathbb{Z}[x_0,\dots,x_n]_{(x_i)} \cong \mathbb{Z}[y_1,\dots,y_n]$ as $\mathbb{Z}$-algebras, so $\mathbb{P}^n_{\mathbb{Z}} \longrightarrow \mathbb{Z}$ is of finite type. Since $\operatorname{Spec}\mathbb{Z}$ is noetherian it follows from Ex. 3.13 that $\mathbb{P}^n_{\mathbb{Z}}$ is noetherian, so we are in a position to use (4.7), imitating the proof of (I, 6.8). The proof is by induction on n.

$\boxed{n=0}$ In this case the canonical morphism $\mathbb{P}^0_{\mathbb{Z}} \longrightarrow \mathbb{Z}$ is trivially proper, since it is an isomorphism.

$\boxed{n \geq 1}$ Suppose that $\mathbb{P}^{n-1}_{\mathbb{Z}} \longrightarrow \operatorname{Spec}\mathbb{Z}$ is proper and that we are given a valuation ring R and a commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & \mathbb{P}^n_{\mathbb{Z}} \arrow[d] \\ R \arrow[r] & \operatorname{Spec}\mathbb{Z} \end{tikzcd} \qquad (1)$$

Let ξ_1 be the image in $\mathbb{P}^n_{\mathbb{Z}}$ of the single point of U. For $0 \le i \le n$ let $V_i = D_+(x_i)$. Suppose that for some i $\xi_1 \notin V_i$. There is a closed immersion $\mathbb{P}^{n-1}_{\mathbb{Z}} \longrightarrow \mathbb{P}^n_{\mathbb{Z}}$ with image $\mathbb{P}^n_{\mathbb{Z}} - V_i$, and by assumption ξ_1 is in the image of this closed immersion, so by our earlier notes on "Factorisation through Closed Immersions" $U \longrightarrow \mathbb{P}^n_{\mathbb{Z}}$ factors uniquely through $\mathbb{P}^{n-1}_{\mathbb{Z}} \longrightarrow \mathbb{P}^n_{\mathbb{Z}}$. Since by assumption this morphism is proper, there is a unique morphism $T \longrightarrow \mathbb{P}^{n-1}_{\mathbb{Z}}$ making the outside of the following diagram commute:

$$\begin{tikzcd}
 & & \mathbb{P}^{n-1}_{\mathbb{Z}} \arrow[dl] \arrow[ddl] \\
U \arrow[urr] \arrow[r] \arrow[d] & \mathbb{P}^n_{\mathbb{Z}} \arrow[d] & \\
T \arrow[r] \arrow[uurr] \arrow[ur] & \operatorname{Spec}\mathbb{Z} &
\end{tikzcd}$$

(Since $\operatorname{Spec}\mathbb{Z}$ is terminal, we can avoid some comm. checks). Let $T \longrightarrow \mathbb{P}^n_{\mathbb{Z}}$ be the composite $T \longrightarrow \mathbb{P}^{n-1}_{\mathbb{Z}} \longrightarrow \mathbb{P}^n_{\mathbb{Z}}$. It is clear that $U \longrightarrow T \longrightarrow \mathbb{P}^n_{\mathbb{Z}} = T \longrightarrow \mathbb{P}^n_{\mathbb{Z}}$. To show uniqueness, suppose $\eta : T \longrightarrow \mathbb{P}^n_{\mathbb{Z}}$ is another morphism making the diagram commute. Then $\operatorname{Im}\eta \subseteq V_i^c$ so again η factors uniquely through $\mathbb{P}^{n-1}_{\mathbb{Z}}$, since R is a reduced ring. Using uniqueness of $T \longrightarrow \mathbb{P}^{n-1}_{\mathbb{Z}}$ and the fact that $\mathbb{P}^{n-1}_{\mathbb{Z}} \longrightarrow \mathbb{P}^n_{\mathbb{Z}}$ is a monomorphism, we conclude that $T \longrightarrow \mathbb{P}^n_{\mathbb{Z}}$ is unique, so $\mathbb{P}^n_{\mathbb{Z}} \longrightarrow \operatorname{Spec}\mathbb{Z}$ is proper.

We are left with the case where $\xi_1 \in \bigcap_{i=0}^n V_i$. For each i there is an isomorphism

$$D_+(x_i) \cong \operatorname{Spec}\mathbb{Z}[x_0,\dots,x_n]_{(x_i)} \cong \operatorname{Spec}\mathbb{Z}\left[\frac{x_0}{x_i},\dots,\frac{x_n}{x_i}\right]$$

($\mathbb{Z}[x_0/x_i,\dots,x_n/x_i]$ denoting $\mathbb{Z}[y_0,\dots,y_{n-1}]$ with nice labels). Under this isomorphism x_j/x_i is identified with the section $\mathfrak{p} \longmapsto x_j/x_i \in \mathbb{Z}[x_0,\dots,x_n]_{(\mathfrak{p})}$ of $\mathbb{P}^n_{\mathbb{Z}}$ over $D_+(x_i)$. The morphism $U \longrightarrow \mathbb{P}^n_{\mathbb{Z}}$ gives a morphism of rings $\mathcal{O}_{\xi_1} \longrightarrow K$. Since $\xi_1 \in \bigcap V_i$ the sections $x_j/x_i \in D_+(x_i)$ become units in $\mathcal{O}_{\xi_1}$. (Let x_i/x_i denote the identity on $D_+(x_i)$). Let f_{ij} be the image of $x_i/x_j \in \mathcal{O}_{\xi_1}$ in K for all $0 \le i,j \le n$. These are all nonzero and

$$f_{ik} = f_{ij} f_{jk} \qquad 0 \le i,j,k \le n$$

Let $v : K \longrightarrow G$ be the valuation on K determined by the valuation ring R. Let $g_i = v(f_{i0})$ for $i = 0,\dots,n$. Since $f_{ij} = f_{ji}^{-1}$ if we choose k s.t. g_k is minimal among $\{g_0,\dots,g_n\}$ (possible since G totally ordered) then for each i we have

$$v(f_{ik}) = g_i - g_k \geqslant 0$$

Hence $f_{ik} \in R$ for $0 \le i \le n$. Define a morphism of rings $\mathbb{Z}[x_0/x_k,\dots,x_n/x_k] \longrightarrow R$ by $x_i/x_k \longmapsto f_{ik}$. This induces a morphism of schemes $T \longrightarrow \mathbb{P}^n_{\mathbb{Z}}$ given by $T \longrightarrow \operatorname{Spec}\mathbb{Z}[x_0/x_k,\dots,x_n/x_k] \cong D_+(x_k) \longrightarrow \mathbb{P}^n_{\mathbb{Z}}$. To show that $T \longrightarrow \mathbb{P}^n_{\mathbb{Z}}$ makes (1) commute, we have to show that the outside in the following diagram commutes:

$$\begin{tikzcd}
U \arrow[rr] \arrow[dr] \arrow[ddr] \arrow[ddd] & & \mathbb{P}^n_{\mathbb{Z}} \\
 & D_+(x_k) \arrow[ur] & \\
 & \operatorname{Spec}\mathbb{Z}[x_0/x_k,\dots,x_n/x_k] \arrow[u, "\cong"] & \\
T \arrow[ur] & &
\end{tikzcd} \qquad (2)$$

I

Here $\xi_1 \in D_+(x_k)$ so $U \longrightarrow \mathbb{P}^n_{\mathbb{Z}}$ factorises through $D_+(x_k)$, and $U \longrightarrow \operatorname{Spec}\mathbb{Z}[x_0/x_k,\dots,x_n/x_k]$ is the map $\mathbb{Z}[x_0/x_k,\dots,x_n/x_k] \longrightarrow K$, $x_i/x_k \longmapsto f_{ik}$. So it suffices to show the triangle I commutes. But morphisms into $\operatorname{Spec}\mathbb{Z}[x_0/x_k,\dots]$ are in bijection with ring morphisms $\mathbb{Z}[x_0/x_k,\dots] \longrightarrow$ so this is easily checked. It only remains to show that $T \longrightarrow \mathbb{P}^n_{\mathbb{Z}}$ is unique making this diagram commute.

Suppose $T \xrightarrow{\gamma} \mathbb{P}^n_{\mathbb{Z}}$ were another morphism s.t. $U \longrightarrow T \longrightarrow \mathbb{P}^n_{\mathbb{Z}} = U \longrightarrow \mathbb{P}^n_{\mathbb{Z}}$. The image of the maximal ideal of R is contained in some $D_+(x_i)$, hence the image of T is contained in $D_+(x_i)$.

It follows that Υ factors uniquely through the open immersion $\operatorname{Spec}\mathbb{Z}[x_0/x_i, \dots, x_n/x_i] \longrightarrow \mathbb{P}^n_{\mathbb{Z}}$. Let $\mathbb{Z}[x_0/x_i, \dots, x_n/x_i] \longrightarrow K$ be $x_j/x_i \longmapsto f_{ji}$. Then as we checked for k, $\operatorname{Spec} K \longrightarrow \operatorname{Spec}\mathbb{Z}[x_0/x_i, \dots, x_n/x_i] \longrightarrow \mathbb{P}^n_{\mathbb{Z}} = U \longrightarrow \mathbb{P}^n_{\mathbb{Z}}$ and by assumption $U \longrightarrow T \longrightarrow \mathbb{P}^n_{\mathbb{Z}} = U \longrightarrow T$:

$$\begin{tikzcd}
U \arrow[r] \arrow[d] \arrow[dr] & \mathbb{P}^n_{\mathbb{Z}} \\
T \arrow[r] \arrow[ur, "\Upsilon"] & \operatorname{Spec}\mathbb{Z}[x_0/x_i, \dots, x_n/x_i] \arrow[u]
\end{tikzcd}$$

[and that $T \longrightarrow \operatorname{Spec}\mathbb{Z}[x_0/x_i, \dots, x_n/x_i]$ is $\operatorname{Spec}\phi$, where ϕ is defined below]

Since an open immersion is a monomorphism, the bottom left triangle commutes, and it follows that $f_{ji} \in R$ for all $0 \le j \le n$. But then $g_j - g_i = v(f_{j0}) + v(f_{0i}) = v(f_{j0} f_{0i}) = v(f_{ji}) \geqslant 0$ and in particular $g_k \geqslant g_i$. Minimality of g_k means that $g_i = g_k$, and hence f_{ki}, f_{ik} are units in R. If $i = k$ then we are done, since $\mathbb{Z}[x_0/x_i, \dots, x_n/x_i] \longrightarrow R$ is the morphism used to define $T \longrightarrow \mathbb{P}^n_{\mathbb{Z}}$ in (2). So assume $i \neq k$. Since f_{ki}, f_{ik} are units in R, the morphisms

$$\varphi : \mathbb{Z}[x_0/x_k, \dots, x_n/x_k] \longrightarrow R \qquad x_j/x_k \longmapsto f_{jk}$$
$$\phi : \mathbb{Z}[x_0/x_i, \dots, x_n/x_i] \longrightarrow R \qquad x_j/x_i \longmapsto f_{ji}$$

map x_i/x_k and x_k/x_i resp. to units, thus inducing morphisms out of the localisations at x_i/x_k, x_k/x_i resp. But $\mathbb{Z}[x_0/x_k, \dots, x_n/x_k]_{x_i/x_k} \cong \mathbb{Z}[x_0, \dots, x_n]_{(x_i x_k)} \cong \mathbb{Z}[x_0/x_i, \dots, x_n/x_i]_{x_k/x_i}$ and using notes at the end of §2 there is a commutative diagram ($b_j = f_{jk}$ for $0 \le j \le n$)

$$\begin{tikzcd}
 & & R & & \\
\mathbb{Z}[x_0/x_i, \dots, x_n/x_i] \arrow[r] \arrow[urr, "\phi"] & \mathbb{Z}[x_0/x_i, \dots, x_n/x_i]_{x_k/x_i} \arrow[ur] & & \mathbb{Z}[x_0/x_k, \dots, x_n/x_k]_{x_i/x_k} \arrow[ul] & \mathbb{Z}[x_0/x_k, \dots, x_n/x_k] \arrow[l] \arrow[ull, "\varphi"'] \\
 & & \mathbb{Z}[x_0, \dots, x_n]_{(x_i x_k)} \arrow[ul, Rightarrow] \arrow[ur, Rightarrow] & &
\end{tikzcd} \tag{3}$$

This gives a morphism $\operatorname{Spec}\mathbb{Z}[x_0, \dots, x_n]_{(x_i x_k)} \longrightarrow R$. Consider the following diagram:

$$\begin{tikzcd}
 & & \mathbb{P}^n_{\mathbb{Z}} & & \\
 & D_+(x_i) \arrow[ur] \arrow[rr, phantom, "\mathrm{I}"] & & D_+(x_k) \arrow[ul] & \\
\operatorname{Spec}\mathbb{Z}[x_0/x_i, \dots, x_n/x_i] \arrow[ur, Rightarrow] \arrow[rr, phantom, "a"] & & D_+(x_i x_k) \arrow[ul] \arrow[ur] \arrow[rr, phantom, "b"] & & \operatorname{Spec}\mathbb{Z}[x_0/x_k, \dots, x_n/x_k] \arrow[ul, Rightarrow] \\
 & & \operatorname{Spec}\mathbb{Z}[x_0, \dots, x_n]_{(x_i x_k)} \arrow[u, Rightarrow] \arrow[ull] \arrow[urr] \arrow[ll, phantom, "c"] \arrow[rr, phantom, "d"] & & \\
 & & T \arrow[u] \arrow[uull, bend left] \arrow[uurr, bend right] & &
\end{tikzcd}$$

Squares c, d commute by (3), I trivially and a, b by our notes at the end of §2. It follows that the two ways around the outside are equal. But one is $\Upsilon : T \longrightarrow \mathbb{P}^n_{\mathbb{Z}}$ and the other is our original map, completing the proof of uniqueness, and thus the proof that $\mathbb{P}^n_{\mathbb{Z}} \longrightarrow \operatorname{Spec}\mathbb{Z}$ is proper. $\square$

COROLLARY If A is noetherian and $n \geq 0$ then $\mathbb{P}^n_A \longrightarrow \operatorname{spec} A$ is proper ($\mathbb{P}^n_A$ is also noetherian)

PROOF For any ring A and $n \geq 0$ there is a pullback diagram

$$\begin{tikzcd} \mathbb{P}^n_A \arrow[r] \arrow[d] & \mathbb{P}^n_{\mathbb{Z}} \arrow[d] \\ \operatorname{spec} A \arrow[r] & \operatorname{Spec} \mathbb{Z} \end{tikzcd}$$

Since $\mathbb{P}^n_{\mathbb{Z}} \longrightarrow \operatorname{spec}\mathbb{Z}$ is of finite type, so is $\mathbb{P}^n_A \longrightarrow \operatorname{spec} A$, by Ex 3.13. So if A is noetherian, so is $\mathbb{P}^n_A$, again by Ex 3.13. Since $\mathbb{P}^n_{\mathbb{Z}} \longrightarrow \operatorname{Spec}\mathbb{Z}$ is proper, $\mathbb{P}^n_A \longrightarrow \operatorname{spec} A$ is proper by (4.8c). □

COROLLARY A projective morphism of noetherian schemes is proper.

PROOF Suppose $f: X \longrightarrow Y$ is projective with X, Y noetherian. Then there is a closed immersion $i: X \longrightarrow \mathbb{P}^n_Y$ for some $n \geq 1$ so that the following diagram commutes:

$$\begin{tikzcd} & \mathbb{P}^n_Y \arrow[dr] & \\ X \arrow[ur] \arrow[rr] & & Y \end{tikzcd} \qquad (1)$$

But closed immersions are proper, so the result follows from (4.8) b and the following Corollary. □

COROLLARY If Y is a noetherian scheme, $\mathbb{P}^n_Y \longrightarrow Y$ projective n-space over Y for some $n \geq 1$, then $\mathbb{P}^n_Y \longrightarrow Y$ is proper. (and $\mathbb{P}^n_Y$ is noetherian)

PROOF Recall "projective n-space over Y" means "product of $\mathbb{P}^n$ and Y". So there is a pullback

$$\begin{tikzcd} \mathbb{P}^n_Y \arrow[r] \arrow[d] & Y \arrow[d] \\ \mathbb{P}^n_{\mathbb{Z}} \arrow[r] & \operatorname{Spec}\mathbb{Z} \end{tikzcd}$$

By 3.13 $\mathbb{P}^n_Y \longrightarrow Y$ is of finite type, so $\mathbb{P}^n_Y$ is noetherian and so by (4.8c) $\mathbb{P}^n_Y \longrightarrow Y$ is proper. □

NOTE If $f: X \longrightarrow Y$ is projective, Y noetherian then X is noetherian (Consider (1) and Ex 3.13)

COROLLARY A quasi-projective morphism of noetherian schemes is of finite type and separated.

PROOF Let $f: X \longrightarrow Y$ be quasi-projective with X, Y noetherian and suppose f factors as $X \xrightarrow{h} X' \xrightarrow{g} Y$ with h an open immersion and g projective. Since Y is noetherian, so is X', so g is proper, hence of finite type and separated. But X is noetherian, so h is a quasi-compact open immersion, hence by Ex 3.13 also of finite type and separated. By Ex 3.13 and (4.6) it follows that f is of finite type and separated. □

LEMMA If $f: X \longrightarrow Y$ is projective, Y noetherian, then X is noetherian and f is proper.

PROOF Combine the note and Corollary. □

LEMMA If $f: X \longrightarrow Y$ is quasi-projective, Y noetherian, then X is noetherian and f is of finite type and separated.

PROOF Suppose f factorises as an open immersion $X \longrightarrow X'$ followed by a projective morphism $X' \longrightarrow Y$. Then X' is noetherian, and X is isomorphic to an open subset of X', so X is also noetherian. □

At this point you should consult our typed notes “The Proj Construction”. Look for the Section “Projective space over a Scheme”.

This was originally a separate note, which was combined with “Notes on *Proj*” and together they are now named “The Proj Construction”. In our written notes there may be references to these notes separately.

NOTE Projectivity is Local

LEMMA If $f : X \longrightarrow Y$ is a projective morphism of schemes then so is $f^{-1}V \longrightarrow V$ for any open $V \subseteq Y$.

PROOF Suppose $i : X \longrightarrow \mathbb{P}^n_Y$ $(n \geq 1)$ is a closed immersion s.t. $X \xrightarrow{i} \mathbb{P}^n_Y \xrightarrow{g} Y = f$. Then every face in the following diagram commutes, and all square faces are pullbacks:

$$\begin{tikzcd}
& \mathbb{P}^n_Y \arrow[dr, "g"] & \\
X \arrow[ur, "i"] \arrow[rr] & & Y \\
& g^{-1}V \arrow[uu] \arrow[dr] & \\
f^{-1}V \arrow[uu] \arrow[ur] \arrow[rr] & & V \arrow[uu]
\end{tikzcd}$$

By earlier notes $f^{-1}V \longrightarrow g^{-1}V$ is a closed immersion, and $g^{-1}V = V \times_{\operatorname{spec} \mathbb{Z}} \mathbb{P}^n_{\mathbb{Z}}$ with projection $g^{-1}V \longrightarrow V$, so $f^{-1}V \longrightarrow V$ is projective. $\square$

NOTE Localisation and Change of Rings (EGA0, 1.5)

Let A, B be rings $\varphi: A \longrightarrow B$ a morphism of rings, $S \subseteq A$ and $T \subseteq B$ multiplicative sets, such that $\varphi(S) \subseteq T$. Then we get a morphism $\varphi^T: S^{-1}A \longrightarrow T^{-1}B$ of rings making the following commute:

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d] & B \arrow[d] \\ S^{-1}A \arrow[r, "\varphi^T"'] & T^{-1}B \end{tikzcd} \qquad \varphi^T(a/s) = \varphi(a)/\varphi(s)$$

Let M be an B-module. Then let $M_{[\varphi]}$ denote M considered as an A-module. There is an morphism of $S^{-1}A$-modules natural in M:

$$\phi: S^{-1}(M_{[\varphi]}) \longrightarrow (T^{-1}M)_{[\varphi^T]}$$
$$\phi(m/s) = m/\varphi(s)$$

If $T = \varphi(S)$ then ϕ is an isomorphism. If N is another B-module then composing the morphism $S^{-1}((M \otimes_B N)_{[\varphi]}) \longrightarrow (T^{-1}(M \otimes_B N))_{[\varphi^T]}$ just defined with $T^{-1}(M \otimes_B N) \longrightarrow T^{-1}M \otimes_{T^{-1}B} T^{-1}N$ we get a morphism of $S^{-1}A$-modules

$$S^{-1}((M \otimes_B N)_{[\varphi]}) \longrightarrow (T^{-1}M \otimes_{T^{-1}B} T^{-1}N)_{[\varphi^T]}$$
$$m \otimes n/s \longmapsto m/1 \otimes n/\varphi(s)$$

Now let L be an A-module, and form the tensor product $L \otimes_A B$, which is a B-module. We claim there is an isomorphism of $T^{-1}B$-modules natural in L:

$$\tau: (S^{-1}L) \otimes_{S^{-1}A} T^{-1}B \longrightarrow T^{-1}(L \otimes_A B)$$

we define $\tau(e/s \otimes b/t) = e \otimes b/\varphi(s)t$, in the usual way starting with $S^{-1}L \times T^{-1}B \longrightarrow T^{-1}(L \otimes B)$. There is a canonical isomorphism of abelian groups $T^{-1}(L \otimes_A B) \Rightarrow (L \otimes_A B) \otimes_B T^{-1}B \Rightarrow L \otimes_A (B \otimes_B T^{-1}B) \Rightarrow L \otimes_A T^{-1}B$ defined by $n \otimes b/s \longmapsto n \otimes b/s$. (Here $T^{-1}B$ is an A-module via $A \longrightarrow B \longrightarrow T^{-1}B$). One checks that $L \times T^{-1}B \longrightarrow (S^{-1}L) \otimes_{S^{-1}A} T^{-1}B$, $(e, b/t) \longmapsto \frac{e}{1} \otimes b/t$ is well-defined and A-linear, giving a morphism of abelian groups

$$\tau': T^{-1}(L \otimes_A B) \longrightarrow (S^{-1}L) \otimes_{S^{-1}A} T^{-1}B$$
$$\tau'(n \otimes b/s) = n/1 \otimes b/s$$

Since τ is clearly inverse to τ', τ is an isomorphism of $T^{-1}B$-modules. Naturality in L is easily checked. We proved this result elsewhere in the case of prime localisation (the proof here is more elegant, but the other proof is also educational).

At this point you should consult our typed notes “Varieties as Schemes”.

$\boxed{Q4.3}$ First, let $f : X \longrightarrow Y$ be a closed immersion, $U \subseteq Y$ an affine open subset. We claim $f^{-1}U$ is an affine open subset of X. If $f^{-1}U$ is empty this is trivial, and in any case $f^{-1}U \longrightarrow U \Rightarrow \operatorname{Spec} A$ is a closed immersion. Hence $f^{-1}U \cong \operatorname{Spec}(A/\mathfrak{a})$ for some ideal $\mathfrak{a}$, by Ex 3.11.

Now suppose $f : X \longrightarrow S$ is separated, $U, V \subseteq X$ open affine subsets, and S affine.

$$\begin{tikzcd}
X \arrow[dr, "\Delta"] & & \\
& X \times_S X \arrow[r, "p_1"] \arrow[d, "p_2"'] & X \arrow[d] \\
& X \arrow[r] & S
\end{tikzcd}$$

By earlier notes, $p_1^{-1}U \cap p_2^{-1}U \cong U \times_S V$, which is affine. But combining the previous paragraph with the fact that $\Delta^{-1}(p_1^{-1}U \cap p_2^{-1}U) = U \cap V$ we see that $U \cap V$ is affine.

$\boxed{Q4.4}$ We use some results from §5. See our notes on the scheme theoretic image there, in particular. Don't worry, this exercise is not used until Remark 5.16.1, so there is no logical problem! First we prove

LEMMA Let $f : X \longrightarrow Y$ be (over S) morphism of schemes which are both of finite type and separated over a noetherian scheme S. If X is proper over S, then f is a proper morphism.

PROOF Consider the following pullback:

$$\begin{tikzcd}
X \times_S Y \arrow[r] \arrow[d, "q"'] & X \arrow[d] \\
Y \arrow[r] & S
\end{tikzcd}$$

Since $X \longrightarrow S$ is proper it follows from (4.8) that q is proper, provided we can show $X \times_S Y$ is noetherian. But $Y \longrightarrow S$, $X \longrightarrow S$ are of finite type, so by Ex 3.13 (c), (d) it follows that $X \times_S Y \longrightarrow S$ is of finite type, so $X \times_S Y$ is noetherian by Ex 3.13 (g). Let $\Gamma_f : X \longrightarrow X \times_S Y$ be the graph morphism. Then $q\Gamma_f = f$ and X is noetherian, so by (4.8b) it suffices to show that Γ_f is proper. We show that Γ_f is a closed immersion. From our earlier notes on the graph we know there is a pullback

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\Gamma_f"'] & Y \arrow[d, "\Delta"] \\
X \times_S Y \arrow[r, "f \times 1"'] & Y \times_S Y
\end{tikzcd}$$

By assumption Δ is a closed immersion, hence so is Γ_f, completing the proof. $\square$

⌜see LaTeX notes for rest of proof⌟

$\boxed{\text{Q4.5}}$ Let X be an integral scheme over a field k, with generic point ξ and function field K. A <u>valuation of</u> K/k is a valuation v on K with $v(x) = 0$ for all $x \in k - \{0\}$ (equivalently $k \subseteq R$ where R is the valuation ring of v). Here we consider k as a subfield of K via $k \longrightarrow \mathcal{O}_X(X) \longrightarrow \mathcal{O}_{X,\xi} = K$.
Let v be a valuation of K/k with valuation ring R, with all morphisms in the following commutative diagram canonical (so $\operatorname{Spec} K \longrightarrow X$ is $\operatorname{Spec}(\mathcal{O}_{X,\xi}) \longrightarrow X$):

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d] \\
\operatorname{Spec} R \arrow[r] & \operatorname{Spec} k
\end{tikzcd} \qquad (1)$$

Morphisms $\operatorname{Spec} R \xrightarrow{f} X$ are in bijection with pairs (x, α) consisting of a point $x \in X$ and a local morphism $\alpha : \mathcal{O}_{X,x} \longrightarrow R$. Suppose f makes (1) into a commutative diagram, and let $x = f(\mathfrak{m})$ and α be such that $\operatorname{Spec} R \longrightarrow X = \operatorname{Spec} R \xrightarrow{\operatorname{Spec}\alpha} \operatorname{Spec}(\mathcal{O}_{X,x}) \longrightarrow X$. Of course $\operatorname{Spec} K \longrightarrow X$ factors uniquely through $\operatorname{Spec}(\mathcal{O}_{X,x}) \longrightarrow X$ via the canonical inclusion $\mathcal{O}_{X,x} \longrightarrow K$, so we get a diagram

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[rr] \arrow[dr] \arrow[dd] & & X \arrow[dd] \\
& \operatorname{Spec}(\mathcal{O}_{X,x}) \arrow[ur] & \\
\operatorname{Spec} R \arrow[ur] \arrow[uurr, "f"'] \arrow[rr] \arrow[u, phantom, "@"] & & \operatorname{Spec} k
\end{tikzcd}$$

The triangle @ commutes since $\operatorname{Spec}(\mathcal{O}_{X,x}) \longrightarrow X$ is monic. So as subrings of K, R dominates $\mathcal{O}_{X,x}$. Conversely, given a point $x \in X$ with R dominating $\mathcal{O}_{X,x}$ in K, the inclusion $\mathcal{O}_{X,x} \xrightarrow{\alpha} R$ is a local morphism and the corresponding morphism $\operatorname{Spec} R \longrightarrow X = \operatorname{Spec} R \longrightarrow \operatorname{Spec}(\mathcal{O}_{X,x}) \longrightarrow X$ makes both triangles in (1) commute. So the bijection above gives rise to a bijection: (for any valuation v of K/k with val. ring R)

$$(2) \qquad \left[\begin{array}{c}\text{Morphisms } \operatorname{Spec} R \longrightarrow X \text{ making} \\ \text{(1) commute}\end{array}\right] \cong \left[\begin{array}{c}\text{Points } x \in X \text{ with } R \text{ dominating} \\ \mathcal{O}_{X,x} \text{ in } K\end{array}\right]$$

If X is any integral scheme over a field k, we call $x \in X$ a <u>center</u> of a valuation v of K/k if R dominates $\mathcal{O}_{X,x}$.

(a) Let X be an integral scheme which is separated over a field k. (ie $X \longrightarrow \operatorname{Spec} k$ is separated). The uniqueness part of (4.3) does not need X noetherian, and applied to (2) it shows that for a valuation v of K/k the center of v is unique (if it exists).

(b) Let X be an integral scheme which is proper over a field k. It follows immediately from (4.7) and (2) that every valuation of K/k has a unique center.