5. SHEAVES OF MODULES

So far we have discussed schemes and morphisms between them without mentioning any sheaves other than the structure sheaves. We can increase the flexibility of our technique enormously by considering sheaves of modules on a given scheme. Especially important are quasi-coherent and coherent sheaves, which play the role of modules (resp. f.g.-modules) over a ring. In this section we will develop the basic properties of quasi-coherent and coherent sheaves. In particular, we will introduce the important "twisting sheaf" $\mathcal{O}(1)$ of Serre on a projective scheme.

DEFINITION Let $(X, \mathcal{O}_X)$ be a ringed space. A sheaf of $\mathcal{O}_X$-modules (or simply an $\mathcal{O}_X$-module) is a sheaf $\mathcal{F}$ of abelian groups on X, such that for each open set $U \subseteq X$, the group $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module, and for each inclusion of open sets $V \subseteq U$ the restriction homomorphism $\mathcal{F}(U) \longrightarrow \mathcal{F}(V)$ is compatible with the module structures via the ring homomorphism $\mathcal{O}_X(U) \longrightarrow \mathcal{O}_X(V)$. A morphism $\mathcal{F} \longrightarrow \mathcal{G}$ of sheaves of $\mathcal{O}_X$-modules is a morphism of sheaves, such that for each open set $U \subseteq X$, the map $\mathcal{F}(U) \longrightarrow \mathcal{G}(U)$ is a morphism of $\mathcal{O}_X(U)$-modules.

This defines the category of $\mathcal{O}_X$-modules. This category is additive via $(\psi + \phi)_U = \psi_U + \phi_U$ and the sheaf $U \longmapsto 0$ is a $\mathcal{O}_X$-module with the property that $Hom(\mathcal{F}, 0) = 0$ and $Hom(0, \mathcal{F}) = 0$ for all $\mathcal{O}_X$-modules $\mathcal{F}$. This is a zero object for the category, and the zero morphism $\mathcal{F} \longrightarrow \mathcal{G}$ is $\psi_U = 0 : \mathcal{F}(U) \longrightarrow \mathcal{G}(U)$.

Let $\psi : \mathcal{F} \longrightarrow \mathcal{G}$ be a morphism of $\mathcal{O}_X$-modules. Then the kernel $Ker\psi$ is defined by $(Ker\psi)(U) = Ker\psi_U$. This is a $\mathcal{O}_X$-module and $Ker\psi \longrightarrow \mathcal{F}$ is a morphism of $\mathcal{O}_X$-modules. Note that if $\mathcal{F}$ is a $\mathcal{O}_X$-module and M is a subsheaf of $\mathcal{F}$ (of abelian groups) for which $M(U) \subseteq \mathcal{F}(U)$ is a $\mathcal{O}_X(U)$-submodule for all $U \subseteq X$, then M is a $\mathcal{O}_X$-module and $M \longrightarrow \mathcal{F}$ is a morphism of $\mathcal{O}_X$-modules. It is not hard to check that $Ker\psi \longrightarrow \mathcal{F}$ is a categorical kernel for $\mathcal{O}_X$-modules.
i.e. each $F(U)$, $\mathcal{O}_X(U)$-module and $(r.m)|_V = r|_V \cdot m|_V$

If F is a presheaf of $\mathcal{O}_X$-modules and $x \in X$, then the abelian group F_x becomes a $\mathcal{O}_{X,x}$-module in a canonical way:

$$(U, s) \cdot (V, m) = (U \cap V, s|_{U\cap V} \cdot m|_{U \cap V})$$

Let $\mathcal{F}$ be the sheafification of F. Then $\mathcal{F}$ is a sheaf of abelian groups, and if $U \subseteq X$ is open, $\mathcal{F}(U)$ becomes a $\mathcal{O}_X(U)$-module via

$$r \in \mathcal{O}_X(U)$$

$$s : U \longrightarrow \cup_{x \in U} F_x \qquad s \in \mathcal{F}(U)$$

$$(r \cdot s)(x) = (U, r) \cdot s(x)$$

It is then clear that $\mathcal{F}$ is an $\mathcal{O}_X$-module, and the canonical morphism $F \longrightarrow \mathcal{F}$ is a morphism of presheaves of $\mathcal{O}_X$-modules. If $\psi : \mathcal{F} \longrightarrow \mathcal{G}$ is a morphism of $\mathcal{O}_X$-modules and $\mathcal{G} \longrightarrow \mathcal{G}/Im\psi$ canonical ($\mathcal{G}/Im\psi$ being the presheaf of $\mathcal{O}_X$-modules $U \longmapsto \mathcal{G}(U)/Im\psi_U$) the cokernel of sheaves of abelian groups is the composition $\mathcal{G} \longrightarrow \mathcal{G}/Im\psi \longrightarrow Coker\psi$ where $Coker\psi$ is the sheafification. By the above, $Coker\psi$ is a sheaf of $\mathcal{O}_X$-modules. It is not difficult to check that $\mathcal{G} \longrightarrow Coker\psi$ is also a cokernel for $\mathcal{O}_X$-modules.

To be more explicit, if F is a presheaf of $\mathcal{O}_X$-modules, $\phi : F \longrightarrow \mathcal{H}$ a morphism of presheaves of $\mathcal{O}_X$-modules, where $\mathcal{H}$ is a sheaf, then we claim the induced morphism $\mathcal{F} \xrightarrow{\psi} \mathcal{H}$ (of sheaves of abelian groups) is a morphism of $\mathcal{O}_X$-modules. Let $U \subseteq X$ and $s \in \mathcal{F}(U)$ be given. There is an open cover $U = \cup_i V_i$ and $t_i \in F(V_i)$ s.t. $s|_{V_i} = t_i$. Then by definition $\psi_U(s)|_{V_i} = \phi_{V_i}(t_i)$ $\forall i$. But then for $r \in \mathcal{O}_X(U)$, $(r \cdot s)|_{V_i} = r|_{V_i} \cdot s|_{V_i} = r|_{V_i} \cdot t_i = (r|_{V_i} \cdot t_i)$, so

$$\begin{aligned}\psi_U(r \cdot s)|_{V_i} &= \phi_{V_i}(r|_{V_i} \cdot t_i)\\ &= r|_{V_i} \cdot \phi_{V_i}(t_i)\\ &= r|_{V_i} \cdot \psi_U(s)|_{V_i} = (r \cdot \psi_U(s))|_{V_i}\end{aligned}$$

so ψ is a morphism of $\mathcal{O}_X$-modules.

Let $\psi : \mathcal{F} \longrightarrow \mathcal{G}$ be a morphism of $\mathcal{O}_X$-modules, I the presheaf of $\mathcal{O}_X$-modules $U \longmapsto Im\psi_U$. The image of ψ as a morphism of sheaves of abelian groups is the induced morphism $I^+ \longrightarrow \mathcal{G}$, where I^+ is the sheafification of I. We have shown that $\mathcal{F} \longrightarrow I \longrightarrow I^+$ is an epimorphism and $I^+ \longrightarrow \mathcal{G}$ a monomorphism in the category of sheaves of abelian groups. But it is clear that if a morphism of $\mathcal{O}_X$-modules is epi/mono as a morphism of sheaves of groups, then it is epi/mono as a morphism of $\mathcal{O}_X$-modules. So provided we can show the category $\mathcal{O}_X$-Mod is abelian, $I^+ \longrightarrow \mathcal{G}$ will be a categorical image.

It is shown in our thesis notes that $\mathcal{O}_X$-$\underline{Mod}$ is actually isomorphic to the localisation of a module category over a ringoid. It follows that $\mathcal{O}_X$-$\underline{Mod}$ is a Grothendieck abelian category. We have described above the kernels, cokernels and images. Since the kernels and cokernels are those used for sheaves of abelian groups, it follows that $\psi : \mathcal{F} \longrightarrow \mathcal{G}$ is an epimorphism or monomorphism in $\mathcal{O}_X$-$\underline{Mod}$ if and only if it is in $\underline{Ab}(sh(X))$.

It follows from Mitchell Prop 2.9 and Cor. 2.10 that if we are given a diagram of modules and we take the limit (resp. colimit) as abelian groups, the limit becomes a module (resp the colimit) in a canonical way, so that it is a limit (resp. colimit) in the category of modules. Let (I, M, d) be a diagram scheme and $\{\mathcal{F}_i\}$ a diagram over this scheme in $\mathcal{O}_X$-$\underline{Mod}$. Let L be the limit calculated for sheaves of abelian groups (so L is the pointwise limit, $L(U) = \lim \mathcal{F}_i(U)$). So each $L(U)$ becomes a $\mathcal{O}_X(U)$-module in such a way that $L(U) \longrightarrow \mathcal{F}_i(U)$ is a morphism of $\mathcal{O}_X(U)$-modules $\forall i \in I$. Given $V \subseteq U$ and $r \in \mathcal{O}_X(U)$, $s \in L(U)$ let $\mathcal{O}_X(V) \longrightarrow L(V)$, $\mathcal{O}_X(V) \longrightarrow L(V)$ be determined by $r|_V \cdot s|_V$ and $(r \cdot s)|_V$. Using the limit morphisms it is not hard to check that $r|_V \cdot s|_V = (r \cdot s)|_V$ so L is a sheaf of $\mathcal{O}_X$-modules. It is not hard to check that L is a limit for the diagram in $\mathcal{O}_X$-$\underline{Mod}$. Hence limits in $\mathcal{O}_X$-$\underline{Mod}$ are calculated pointwise.

Let C be the pointwise colimit of the diagram, so for $U \subseteq X$

$$(1) \qquad C(U) = \bigoplus_i \mathcal{F}_i(U) \Big/ \sum_{m \in M} Im(u_k - u_j D(m)) \qquad u_i : \mathcal{F}_i(U) \longrightarrow \bigoplus_i \mathcal{F}_i(U)$$

Then C is a presheaf of $\mathcal{O}_X$-modules and so the sheafification $\underline{a}C$ is a sheaf of $\mathcal{O}_X$-modules and the composites $\mathcal{F}_i \longrightarrow C \longrightarrow \underline{a}C$ are morphisms of $\mathcal{O}_X$-modules. Then $\mathcal{F}_i \longrightarrow \underline{a}C$ are a colimit of sheaves of abelian groups. Let $\mathcal{F}_i \longrightarrow Z$ be morphisms of $\mathcal{O}_X$-modules which form a cocone on the diagram. There is an induced morphism of sheaves of abelian groups $\underline{a}C \longrightarrow Z$ s.t. $\mathcal{F}_i \longrightarrow \underline{a}C \longrightarrow Z = \mathcal{F}_i \longrightarrow Z$ $\forall i$. It suffices to show $\underline{a}C \longrightarrow Z$ is a morphism of $\mathcal{O}_X$-modules. It suffices to show the composite $C \longrightarrow \underline{a}C \longrightarrow Z$ is a morphism of presheaves of $\mathcal{O}_X$-modules. But this follows from (1) and the fact the $\mathcal{F}_i \longrightarrow Z$ are morphism of $\mathcal{O}_X$-modules. In summary:

$\boxed{\mathcal{O}_X-\underline{Mod}}$ Grothendieck abelian

Morphisms Let $\psi : \mathcal{F} \longrightarrow \mathcal{G}$ be a morphism of $\mathcal{O}_X$-modules. The $\underline{\text{kernel}}$ of ψ is the morphism $K \longrightarrow \mathcal{F}$ where K is the sheaf of $\mathcal{O}_X$-modules $U \longmapsto Ker\psi_U$.

The $\underline{\text{cokernel}}$ of ψ is the composite $\mathcal{G} \longrightarrow C \longrightarrow \underline{a}C$ where C is the presheaf of $\mathcal{O}_X$-modules $U \longmapsto \mathcal{G}(U)/Im\psi_U$.

ψ is a $\underline{\text{monomorphism}}$ iff. ψ_U is injective $\forall U \subseteq X$ (iff. ψ_x injective $\forall x$)
ψ is an $\underline{\text{epimorphism}}$ iff. it is locally surjective. That is, for all $V \subseteq X$ and $t \in \mathcal{G}(V)$ there is an open cover $V = \cup_i W_i$ and $s_i \in \mathcal{F}(W_i)$ s.t. $t|_{W_i} = \psi_{W_i}(s_i)$ $\forall i$. (iff ψ_x surjective $\forall x$)
ψ is an $\underline{\text{isomorphism}}$ iff. ψ_U is bijective $\forall U \subseteq X$. (iff. ψ_x bijective $\forall x$)

Any abelian category is balanced, so $\underline{\text{mono + epi = iso}}$

By a $\underline{\text{subsheaf}}$ of $\mathcal{F}$ we mean a $\mathcal{O}_X$-module $\mathcal{G}$ with $\mathcal{G}(V)$ a $\mathcal{O}_X(V)$-submod. of $\mathcal{F}(V)$ $\forall V \subseteq X$.

The $\underline{\text{image}}$ of ψ is the induced morphism $\underline{a}I \longrightarrow \mathcal{G}$ where I is the presheaf of $\mathcal{O}_X$-modules $U \longmapsto Im\psi_U$. So we can identify the image with the subsheaf of $\mathcal{G}$ consisting of $t \in \mathcal{G}(U)$ s.t. there is a cover $U = \cup_i V_i$ and $t|_{V_i} \in Im\psi_{V_i}$ $\forall i$.

So if $\{\mathcal{F}_i\}$ is a diagram and $L \longrightarrow \mathcal{F}_i$ a limit, the $L(U) \longrightarrow \mathcal{F}_i(U)$ are a limit of modules. Conversely if $L \longrightarrow \mathcal{F}_i$ are a limit of modules pointwise $\forall U \subseteq X$ they are a limit in $\mathcal{O}_X$-$\underline{Mod}$

Limits Let $\{\mathcal{F}_i\}$ be a diagram of $\mathcal{O}_X$-modules. For each U let $L(U)$ be the (any limit) limit of the diagram $\{\mathcal{F}_i(U)\}$ (limit as $\mathcal{O}_X(U)$-modules). Include restrictions $L(V) \longrightarrow L(U)$ as usual, using the fact that $L(U)$ is a limit as abelian groups, also. Then L is a sheaf of $\mathcal{O}_X$-modules and is a limit for the $\mathcal{F}_i$.

Colimits Let $\{\mathcal{F}_i\}$ be a diagram of $\mathcal{O}_X$-modules. For each U let $D(U)$ be the colimit of the diagram $\{\mathcal{F}_i(U)\}$ of $\mathcal{O}_X(U)$-modules (choose any colimit, not necc. the canonical one). Then D becomes a presheaf of $\mathcal{O}_X$-modules isomorphic to C (as in (1)), so $\underline{a}D \cong \underline{a}C$ as $\mathcal{O}_X$-modules, and the morphisms $\mathcal{F}_i \longrightarrow D \longrightarrow \underline{a}D$ are a colimit.

- If $\psi : \mathcal{F} \longrightarrow \mathcal{G}$ is monic, then $\mathcal{F}$ is isomorphic to a subsheaf of $\mathcal{G}$, namely $Im\psi$
- If $\mathcal{F}$ is a $\mathcal{O}_X$-module, a collection of submodules $\mathcal{G}(U) \subseteq \mathcal{F}(U)$ defines a $\mathcal{O}_X$-module $\mathcal{G}$ iff. the following conditions hold:
  1. If $V \subseteq U$ and $s \in \mathcal{G}(U)$ then $s|_V \in \mathcal{G}(V)$.
  2. If $U \subseteq X$ and $U = \cup_i V_i$ and $s_i \in \mathcal{G}(V_i)$ are a matching family, then the unique amalgamation of the s_i in $\mathcal{F}(U)$ belongs to $\mathcal{G}(U)$.

A sequence of $\mathcal{O}_X$-modules is exact if it is exact in $\mathcal{O}_X$-Mod, which is iff. the sequence is exact as a sequence of sheaves of abelian groups. If $U \subseteq X$ and if $\mathcal{F}$ is an $\mathcal{O}_X$-module, then $\mathcal{F}|_U$ is an $\mathcal{O}_X|_U$-module. If $\mathcal{F}, \mathcal{G}$ are two $\mathcal{O}_X$-modules, the presheaf

$$U \longmapsto \operatorname{Hom}_{\mathcal{O}_X|_U}(\mathcal{F}|_U, \mathcal{G}|_U) \qquad (1)$$

is a sheaf, which we call the sheaf $\mathcal{H}om$ and denote by $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})$. It is also an $\mathcal{O}_X$-module. To verify these claims, note that (1) is clearly a presheaf of abelian groups, and a suitable adaptation of Ex 1.15 (show f is a morphism of $\mathcal{O}_X$-modules) shows that it is also a sheaf. For $U \subseteq X$, $\operatorname{Hom}_{\mathcal{O}_X|_U}(\mathcal{F}|_U, \mathcal{G}|_U)$ becomes a $\mathcal{O}_X(U)$-module via $(r\cdot\phi)_V(s) = r|_V \cdot \phi_V(s)$. It is now clear that $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})$ is a $\mathcal{O}_X$-module.

Given two $\mathcal{O}_X$-modules $\mathcal{F},\mathcal{G}$, associate to an open set $U \subseteq X$ the $\mathcal{O}_X(U)$-module $\mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$. If $V \subseteq U$ then the map $\mathcal{F}(U) \times \mathcal{G}(U) \longrightarrow \mathcal{F}(V) \otimes_{\mathcal{O}_X(V)} \mathcal{G}(V)$, $(s,t) \longmapsto s|_V \otimes t|_V$ is bilinear, inducing

$$\begin{aligned} \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U) &\longrightarrow \mathcal{F}(V) \otimes_{\mathcal{O}_X(V)} \mathcal{G}(V) \\ s \otimes t &\longmapsto s|_V \otimes t|_V \end{aligned}$$

So this assignment becomes a presheaf of $\mathcal{O}_X$-modules, with $(s\otimes t)|_V = s|_V \otimes t|_V$. We denoted the associated sheaf by $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$, which is a $\mathcal{O}_X$-module. We often simply write $\mathcal{F} \otimes \mathcal{G}$, with $\mathcal{O}_X$ understood.

Note that $\mathcal{O}_X$ is itself a $\mathcal{O}_X$-module. We say an $\mathcal{O}_X$-module is free if it is isomorphic to a direct sum of copies of $\mathcal{O}_X$. It is locally free if X can be covered by open sets U for which $\mathcal{F}|_U$ is a free $\mathcal{O}_X|_U$-module. An $\mathcal{O}_X$-module $\mathcal{F}$ is called an invertible sheaf if X can be covered by open sets U for which $\mathcal{F}|_U \cong \mathcal{O}_X|_U$.

A sheaf of ideals on X is a sheaf of modules $\mathcal{I}$ which is a subsheaf of $\mathcal{O}_X$. In other words, for every open set U, $\mathcal{I}(U)$ is an ideal in $\mathcal{O}_X(U)$.

Let $f:(X,\mathcal{O}_X) \longrightarrow (Y,\mathcal{O}_Y)$ be a morphism of ringed spaces. If $\mathcal{F}$ is an $\mathcal{O}_X$-module, then $f_*\mathcal{F}$ is an $f_*\mathcal{O}_X$-module. Since we have the morphism $f^\# : \mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X$ of sheaves of rings on Y, this gives $f_*\mathcal{F}$ a natural structure of a $\mathcal{O}_Y$-module. This defines the functor $f_* : \mathcal{O}_X\text{-Mod} \longrightarrow \mathcal{O}_Y\text{-Mod}$ on objects, and for a morphism $\phi: \mathcal{F} \longrightarrow \mathcal{G}$ of $\mathcal{O}_X$-modules the pushforward to clearly a morphism of $\mathcal{O}_Y$-modules, giving f_* on morphisms. We call $f_*\mathcal{F}$ the direct image of $\mathcal{F}$ by f. This functor is clearly additive.

LEMMA Let X be a topological space, G a presheaf of abelian groups and R a presheaf of rings. Suppose that further G has the structure of a R-module (i.e. $\forall U \subseteq X$ $G(U)$ is an $R(U)$-module and restriction on G commutes with the action). Then $\mathfrak{a}G$ is a $\mathfrak{a}R$-module.

PROOF Let $U \subseteq X$ and $s \in (\mathfrak{a}G)(U)$, $r \in (\mathfrak{a}R)(U)$ be given. Define $(r\cdot s) : U \longrightarrow \bigcup_{x\in U} G_x$ by

$$(r\cdot s)(x) = r(x)\cdot s(x)$$

where G_x becomes a R_x-module in the canonical way. $(V,a)\cdot(W,g) = (V\cap W, a|_{V\cap W}\cdot g|_{V\cap W})$. To see that $r\cdot s \in (\mathfrak{a}G)(U)$, let $y \in U$ be given and find $y \in V \subseteq U$, $y \in W \subseteq U$, $t \in G(V)$, $q \in R(W)$ s.t. $\forall x \in V$ $s(x) = (V,t)$ $\forall x \in W$ $r(x) = (W,q)$. Then $\forall x \in V\cap W$ $(r\cdot s)(x) = (W\cap V, q|_{V\cap W}\cdot s|_{V\cap W})$. Then it is clear that $(\mathfrak{a}G)(U)$ is a $(\mathfrak{a}R)(U)$-module, and thus that $\mathfrak{a}G$ is an $\mathfrak{a}R$-module. □

Let an $\mathcal{O}_Y$-module $\mathcal{G}$ be given and let G be the presheaf of abelian groups $G(U) = \varinjlim_{V \supseteq f(U)} \mathcal{G}(V)$. Let R be the presheaf of rings $R(U) = \varinjlim_{V\supseteq f(U)} \mathcal{O}_Y(V)$. Then $G(U)$ is an $R(U)$-module via $(V,a)\cdot(W,g) = (V\cap W, a|_{V\cap W}\cdot g|_{V\cap W})$. Then G is an R-module, and so by the Lemma $f^{-1}\mathcal{G}$ becomes a $f^{-1}\mathcal{O}_Y$-module. Because of the adjoint property of f^{-1} (Ex 1.18) we have a morphism $f^{-1}\mathcal{O}_Y \longrightarrow \mathcal{O}_X$ of sheaves of rings on $\mathcal{O}_X$, which makes $\mathcal{O}_X$ into a $f^{-1}\mathcal{O}_Y$-module. We define $f^*\mathcal{G}$ to be the tensor product

$$f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$$

which becomes an $\mathcal{O}_X$-module in a way we will describe shortly. Firstly though we show how the correspondence $\mathcal{G} \longmapsto f^{-1}\mathcal{G}$ defines a functor $\mathcal{O}_Y\text{-Mod} \longrightarrow f^{-1}\mathcal{O}_Y\text{-Mod}$. We need the following:

LEMMA Let $\phi: \mathcal{G} \longrightarrow \mathcal{G}'$ be a morphism of $\mathcal{O}_Y$-modules, G and G' the corresponding presheaves of R-modules on X, where G, G', R sheafify to give $f^{-1}\mathcal{G}, f^{-1}\mathcal{G}', f^{-1}\mathcal{O}_Y$ respectively. Then

$$\begin{aligned} \hat{\phi} : G &\longrightarrow G' \\ (V,g) &\longmapsto (V, \phi_V(g)) \end{aligned}$$

is a morphism of R-modules. □

LEMMA Let G, G' be presheaves of R-modules, where R is a presheaf of rings. If $\Upsilon : G \longrightarrow G'$ is a morphism of R-modules then $\underline{a}\Upsilon : \underline{a}G \longrightarrow \underline{a}G$ is a morphism of $\underline{a}R$-modules.

PROOF It is clear that $\underline{a}\Upsilon$ is a morphism of sheaves of abelian groups. For $x \in X$ we claim $\Upsilon_x : G_x \longrightarrow G'_x$ is a morphism of R_x-module structures. This follows from

$$\begin{aligned}\Upsilon_x((V,a)\cdot(W,g)) &= \Upsilon_x(V\cap W, a|_{V\cap W}\cdot g|_{V\cap W})\\ &= (V\cap W, \Upsilon_{V\cap W}(a|_{V\cap W}\cdot g|_{V\cap W}))\\ &= (V\cap W, a|_{V\cap W}\cdot \Upsilon_{V\cap W}(g|_{V\cap W}))\\ &= (V,a)\cdot \Upsilon_x(W,g)\end{aligned}$$

Now to show $\underline{a}\Upsilon$ is a morphism of $\underline{a}R$-modules, let $s \in (\underline{a}G)(U)$ and $r \in (\underline{a}R)(U)$ be given. Then $\forall x \in X$

$$\begin{aligned}(\underline{a}\Upsilon)_U(r\cdot s)(x) &= \Upsilon_x((r\cdot s)(x))\\ &= \Upsilon_x(r(x)\cdot s(x))\\ &= r(x)\cdot \Upsilon_x(s(x))\\ &= r(x)\cdot(\underline{a}\Upsilon)_U(s)(x)\\ &= (r\cdot(\underline{a}\Upsilon)_U(s))(x)\end{aligned}$$

as required. $\square$

Combining the two Lemmas, if $\mathcal{G}, \mathcal{G}'$ are $\mathcal{O}_Y$-modules then $f^{-1}\phi : f^{-1}\mathcal{G} \longrightarrow f^{-1}\mathcal{G}'$ is a morphism of $f^{-1}\mathcal{O}_Y$-modules for any morphism $\phi : \mathcal{G} \longrightarrow \mathcal{G}'$ of $\mathcal{O}_Y$-modules. It is then easily checked $f^{-1} : \mathcal{O}_Y\text{-}\underline{\mathrm{Mod}} \longrightarrow f^{-1}\mathcal{O}_Y\text{-}\underline{\mathrm{Mod}}$ is a covariant additive functor.

$$f^{-1} : \mathcal{O}_Y\text{-}\underline{\mathrm{Mod}} \longrightarrow f^{-1}\mathcal{O}_Y\text{-}\underline{\mathrm{Mod}} \qquad \begin{array}{l}(f : X \longrightarrow Y \text{ continuous})\\ \mathcal{O}_Y \text{ a sheaf of rings on } Y.\end{array}$$

$\boxed{\text{Objects}}$ Given $\mathcal{G} \in \mathcal{O}_Y\text{-}\underline{\mathrm{Mod}}$ define G by $G(U) = \varinjlim_{V \supseteq f(U)} \mathcal{G}(V)$, then G is a R-module where $R(U) = \varinjlim_{V \supseteq f(U)} \mathcal{O}_Y(V)$. Hence $f^{-1}\mathcal{G} = \underline{a}G$ is a $f^{-1}\mathcal{O}_Y = \underline{a}R$-module.

$\boxed{\text{Morphisms}}$ Given $\phi : \mathcal{G} \longrightarrow \mathcal{G}'$ let $\hat{\phi} : G \longrightarrow G'$ be $(V,g) \longmapsto (V, \phi_V(g))$. Then $f^{-1}\phi : f^{-1}\mathcal{G} \longrightarrow f^{-1}\mathcal{G}'$ is defined by

$$(f^{-1}\phi)_U(s)(x) = \hat{\phi}_x(s(x))$$

Next consider the following scenario: a morphism $\alpha : \mathcal{O} \longrightarrow \mathcal{O}'$ of sheaves of rings on a space X is given. Then $\mathcal{O}'$ becomes a $\mathcal{O}$-module in the obvious way. Tensoring with $\mathcal{O}'$ defines a functor

$$-\otimes_{\mathcal{O}}\mathcal{O}' : \mathcal{O}\text{-}\underline{\mathrm{Mod}} \longrightarrow \mathcal{O}'\text{-}\underline{\mathrm{Mod}}$$

Suppose $\mathcal{F}$ is a $\mathcal{O}$-module. Then $\mathcal{F}\otimes_{\mathcal{O}}\mathcal{O}'$ is the sheafification of the presheaf $U \longmapsto \mathcal{F}(U)\otimes_{\mathcal{O}(U)}\mathcal{O}'(U)$, which is a $\mathcal{O}'(U)$-module in the usual way. It is clear this presheaf is actually a presheaf of $\mathcal{O}'$-modules. So by our earlier comments there is a canonical $\mathcal{O}'$-module structure on $\mathcal{F}\otimes_{\mathcal{O}}\mathcal{O}'$. Let $\phi : \mathcal{F} \longrightarrow \mathcal{F}'$ be a morphism of $\mathcal{O}$-modules. Then for $U \subseteq X$ $\phi_U \otimes 1 : \mathcal{F}(U)\otimes_{\mathcal{O}(U)}\mathcal{O}'(U) \longrightarrow \mathcal{F}'(U)\otimes_{\mathcal{O}(U)}\mathcal{O}'(U)$ is a morphism of $\mathcal{O}'(U)$-modules, which patch together to make a morphism of presheaves of $\mathcal{O}'$-modules. Sheafifying gives a morphism $\mathcal{F}\otimes_{\mathcal{O}}\mathcal{O}' \longrightarrow \mathcal{F}'\otimes_{\mathcal{O}}\mathcal{O}'$ of $\mathcal{O}'$-modules, since

LEMMA Let G, G' be presheaves of $\mathcal{O}$-modules, where $\mathcal{O}$ is a sheaf of rings, and let $\phi : G \longrightarrow G'$ be a morphism of $\mathcal{O}$-modules. Then $\underline{a}\phi : \underline{a}G \longrightarrow \underline{a}G'$ is a morphism of $\mathcal{O}$-modules.

PROOF Let $U \subseteq X$, $s \in (\underline{a}G)(U)$, $r \in \mathcal{O}(U)$ and $x \in U$ be given. Then

$$\begin{aligned}(\underline{a}\phi)_U(r\cdot s)(x) &= \phi_x((r\cdot s)(x)) = \phi_x((U,r)\cdot s(x))\\ &= (U,r)\cdot \phi_x(s(x))\\ &= (U,r)\cdot(\underline{a}\phi)_U(s)(x)\\ &= (r\cdot(\underline{a}\phi)_U(s))(x). \ \square\end{aligned}$$

Given $\phi : \mathcal{F} \longrightarrow \mathcal{F}'$ let P, P' be the presheaves of $\mathcal{O}'$-modules which sheafify to give $\mathcal{F}\otimes_{\mathcal{O}}\mathcal{O}'$, $\mathcal{F}'\otimes_{\mathcal{O}}\mathcal{O}'$ resp. and let $\hat{\phi} : P \longrightarrow P'$ sheafify to give $\phi\otimes_{\mathcal{O}}\mathcal{O}' : \mathcal{F}\otimes_{\mathcal{O}}\mathcal{O}' \longrightarrow \mathcal{F}'\otimes_{\mathcal{O}}\mathcal{O}'$. If $\psi : \mathcal{F}' \longrightarrow \mathcal{F}''$ is another morphism of $\mathcal{O}$-modules and P'', $\hat{\psi}$ defined similarly, it is clear that $\hat{\psi}\hat{\phi} = \widehat{(\psi\phi)}$, since $\hat{\phi}_U = \phi_U \otimes 1$. For $U \subseteq X$,

$$(\phi\otimes_{\mathcal{O}}\mathcal{O}')_U(s)(x) = \hat{\phi}_x(s(x))$$

which makes it straight forward to see that $-\otimes_{\mathcal{O}}\mathcal{O}'$ is an additive functor, since $\widehat{\psi+\phi} = \hat{\psi}+\hat{\phi}$ also.

$- \otimes_{\mathcal{O}} \mathcal{O}' : \mathcal{O}\text{-}\underline{\mathrm{Mod}} \longrightarrow \mathcal{O}'\text{-}\underline{\mathrm{Mod}}$ $\mathcal{O}, \mathcal{O}'$ sheaves of rings on X
$\alpha : \mathcal{O} \to \mathcal{O}'$ makes $\mathcal{O}'$ into a $\mathcal{O}$-module

$\boxed{\text{Objects}}$ Given $\mathcal{F} \in \mathcal{O}\text{-}\underline{\mathrm{Mod}}$, $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{O}'$ is the sheafification of the presheaf
$U \longmapsto \mathcal{F}(U) \otimes_{\mathcal{O}(U)} \mathcal{O}'(U)$.

$\boxed{\text{Morphisms}}$ Given a morphism $\phi : \mathcal{F} \longrightarrow \mathcal{F}'$, $\phi \otimes_{\mathcal{O}} \mathcal{O}' : \mathcal{F} \otimes_{\mathcal{O}} \mathcal{O}' \longrightarrow \mathcal{F} \otimes_{\mathcal{O}} \mathcal{O}'$ is defined by

$$(\phi \otimes_{\mathcal{O}} \mathcal{O}')_U(s)(x) = \hat{\phi}_x(s(x))$$

where $\hat{\phi}_U : \mathcal{F}(U) \otimes_{\mathcal{O}(U)} \mathcal{O}'(U) \longrightarrow \mathcal{F}'(U) \otimes_{\mathcal{O}(U)} \mathcal{O}'(U)$ is $\phi_U \otimes 1$.

Now let us return to our morphism of ringed spaces $f : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$. By adjointness we have a morphism $f^{-1}\mathcal{O}_Y \longrightarrow \mathcal{O}_X$ of sheaves of rings, and hence two additive functors

$$\mathcal{O}_Y\text{-}\underline{\mathrm{Mod}} \xrightarrow{f^{-1}} f^{-1}\mathcal{O}_Y\text{-}\underline{\mathrm{Mod}} \xrightarrow{- \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X} \mathcal{O}_X\text{-}\underline{\mathrm{Mod}}$$

We denote the composite by $f^* : \mathcal{O}_Y\text{-}\underline{\mathrm{Mod}} \longrightarrow \mathcal{O}_X\text{-}\underline{\mathrm{Mod}}$. We claim that $f^* \dashv f_*$.

$$\begin{tikzcd} \mathcal{O}_X\text{-}\underline{\mathrm{Mod}} \arrow[r, bend left, "f_*"] & \mathcal{O}_Y\text{-}\underline{\mathrm{Mod}} \arrow[l, bend left, "f^*"] \end{tikzcd}$$

First we define a unit $\eta : 1 \longrightarrow f_* f^*$. Let $\mathcal{G}$ be a $\mathcal{O}_Y$-module. Then

$$(f_* f^* \mathcal{G})(V) = (f^*\mathcal{G})(f^{-1}V)$$
$$f^*\mathcal{G} = f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$$

We define $\eta \; (= \eta_{\mathcal{G}}) : \mathcal{G} \longrightarrow f_* f^* \mathcal{G}$ by $\eta_V(s) = \dot{(\dot{(V,s)} \otimes 1)}$ where dots denote canonical morphisms into sheafifications. It is easily checked that η is a morphism of sheaves of abelian groups. To see that it preserves the module structure, we need to understand $f^\# : \mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X$ and its adjoint partner $f^{-1}\mathcal{O}_Y \longrightarrow \mathcal{O}_X$ better:

$$\frac{f^\# : \mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X}{\xi : f^{-1}\mathcal{O}_Y \longrightarrow \mathcal{O}_X}$$

is the composite

induced by $\psi : \mathcal{A}' \longrightarrow \mathcal{O}_X$
$\psi_U((V,t)) = t|_U$.

$$\begin{tikzcd} f^{-1}\mathcal{O}_Y \arrow[r, "f^{-1}f^\#"] \arrow[d, "\text{sheaf of}"'] & f^{-1}f_*\mathcal{O}_X \arrow[r, "\varepsilon"] \arrow[d, "\text{sheaf of}"] & \mathcal{O}_X \\ \mathcal{A} \arrow[r, "\hat{f}^\#"] & \mathcal{A}' & \end{tikzcd}$$

$\mathcal{A}(V) = \varinjlim_{W \supseteq f(V)} \mathcal{O}_Y(W)$ $\mathcal{A}'(V) = \varinjlim_{W \supseteq f(V)} \mathcal{O}_X(f^{-1}W)$

$$\hat{f}^\#_V(W, m) = (W, f^\#_W(m))$$

Let $W \supseteq f(V)$ and $m \in \mathcal{O}_Y(W)$ be given. Then this composite $\xi : f^{-1}\mathcal{O}_Y \longrightarrow \mathcal{O}_X$ maps $\dot{(W,m)} \in (f^{-1}\mathcal{O}_Y)(V)$ to $\hat{f}^\#_V(\dot{W}, m) = (W, \dot{f_W^\#}(m))$ in $(f^{-1}f_*\mathcal{O}_X)(V)$ and then to $f^\#_W(m)|_V$ in $\mathcal{O}_X(V)$. In particular for $r \in \mathcal{O}_Y(Q)$, putting $W = Q$, $V = f^{-1}Q$ gives $\xi_{f^{-1}Q}(\dot{(Q,r)}) = f^\#_Q(r)$.

Now we are ready to show that $\eta : \mathcal{G} \longrightarrow f_* f^* \mathcal{G}$ is a morphism of $\mathcal{O}_Y$-modules. Let $r \in \mathcal{O}_Y(Q)$ be given and $s \in \mathcal{G}(Q)$. Then by definition the $\mathcal{O}_Y$-module structure on $f_* f^* \mathcal{G}$ arises from $\mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X$ and the canonical $f_*\mathcal{O}_X$-module structure on $f^*\mathcal{G}$, arising from the $\mathcal{O}_X$-module structure on $f^*\mathcal{G} = f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$. So

$$r \cdot \eta_Q(s) = f^\#_Q(r) \cdot \eta_Q(s)$$

Let $y \in V$. Then

$$
\begin{aligned}
(r \cdot \eta_Q(s))(y) &= (f^{\#}_Q(r) \cdot \eta_Q(s))(y) \\
&= (f^{-1}Q,\ f^{\#}_Q(r)) \cdot \eta_Q(s)(y) \\
&= (f^{-1}Q,\ f^{\#}_Q(r)) \cdot (f^{-1}Q,\ (Q,s) \otimes 1) \\
&= (f^{-1}Q,\ f^{\#}_Q(r) \cdot ((Q,s) \otimes 1)) \\
&= (f^{-1}Q,\ (Q,s) \otimes f^{\#}_Q(r)) \\
&= (f^{-1}Q,\ (Q,s) \otimes \xi_{f^{-1}Q}((Q,r))) \\
&= (f^{-1}Q,\ (Q,r) \cdot (Q,s) \otimes 1) \\
&= (f^{-1}Q,\ (Q, r \cdot s) \otimes 1) \\
&= ((Q, r \cdot s) \otimes 1)(y) \\
&= \eta_Q(r \cdot s)(y)
\end{aligned}
$$

Hence $r \cdot \eta_Q(s) = \eta_Q(r \cdot s)$ and η_Q is a morphism of $\mathcal{O}_Y(Q)$-modules as required. It is easily checked that for any $\phi : \mathcal{G} \longrightarrow \mathcal{H}$ that η is natural:

$$
\begin{tikzcd}
\mathcal{G} \arrow[r, "\phi"] \arrow[d, "\eta_G"'] & \mathcal{H} \arrow[d, "\eta_H"] \\
f_* f^* \mathcal{G} \arrow[r, "f_* f^* \phi"'] & f_* f^* \mathcal{H}
\end{tikzcd}
$$

We now check that $(f^*\mathcal{G}, \eta_G)$ is a reflection of $\mathcal{G}$ along f_*. Let $\mathcal{F}$ be a $\mathcal{O}_X$-module and $\phi : \mathcal{G} \longrightarrow f_*\mathcal{F}$ a morphism of $\mathcal{O}_Y$-modules. We define $\psi : f^*\mathcal{G} \longrightarrow \mathcal{F}$ as follows. Firstly, we use the adjunction between $f_* : \underline{Ab}(X) \longrightarrow \underline{Ab}(Y)$ and $f^{-1} : \underline{Ab}(Y) \longrightarrow \underline{Ab}(Y)$ to produce $\psi' : f^{-1}\mathcal{G} \longrightarrow \mathcal{F}$, a morphism of sheaves of abelian groups defined by

$$
\begin{aligned}
G(U) &= \varinjlim_{V \supseteq f(U)} \mathcal{G}(V) \\
\hat{\psi} &: G \longrightarrow \mathcal{F} \\
\hat{\psi}_U((V,s)) &= \phi_V(s)|_U
\end{aligned}
$$

$$
\begin{aligned}
\psi'_U &: (f^{-1}\mathcal{G})(U) \longrightarrow \mathcal{F}(U) \\
\operatorname{germ}_y \psi'_U(s) &= \hat{\psi}_y(s(y))
\end{aligned}
$$

Let R be the presheaf $R(V) = \varinjlim_{V \supseteq f(U)} \mathcal{O}_Y(V)$, so that $f^{-1}\mathcal{O}_Y = \underline{a}R$. Let $\alpha \in (f^{-1}\mathcal{O}_Y)(U)$ and $s \in (f^{-1}\mathcal{G})(U)$ be given. Then for $y \in U$

$$
\begin{aligned}
\operatorname{germ}_y \psi'_U(\alpha \cdot s) = \hat{\psi}_y(\alpha(y) \cdot s(y)) = \hat{\psi}_y((W,a) \cdot (V,t)) &= \hat{\psi}_y((W \cap V,\ a|_{W \cap V} \cdot t|_{W \cap V})) \\
&= (W \cap V,\ \hat{\psi}_{W \cap V}(a|_{W \cap V} \cdot t|_{W \cap V}))
\end{aligned}
$$

where $\alpha(y) = (W,a) \in R_y$ and $s(y) = (V,t) \in G_y$. Say $a = (Z, a')$, $a' \in \mathcal{O}_Y(Z)$ and $t = (T, t')$, $t' \in \mathcal{G}(T)$. Then

$$
\begin{aligned}
\hat{\psi}_{W \cap V}(a|_{W \cap V} \cdot t|_{W \cap V}) &= \hat{\psi}_{W \cap V}((Z,a')|_{W \cap V} \cdot (T,t')|_{W \cap V}) \\
&= \hat{\psi}_{W \cap V}((Z \cap T,\ a'|_{Z \cap T} \cdot t'|_{Z \cap T})) \\
&= \phi_{Z \cap T}(a'|_{Z \cap T} \cdot t'|_{Z \cap T})|_{W \cap V} \\
&= (a'|_{Z \cap T} \cdot \phi_{Z \cap T}(t'|_{Z \cap T}))|_{W \cap V}
\end{aligned}
$$

But by definition $f_*\mathcal{F}$ becomes a $\mathcal{O}_Y$-module via $f^{\#} : \mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X$, so continuing

$$
\begin{aligned}
&= \{ f^{\#}_{Z \cap T}(a'|_{Z \cap T}) \cdot \phi_{Z \cap T}(t'|_{Z \cap T}) \}|_{W \cap V} \\
&= f^{\#}_{Z \cap T}(a'|_{Z \cap T})|_{W \cap V} \cdot \phi_{Z \cap T}(t'|_{Z \cap T})|_{W \cap V}
\end{aligned}
$$

But $f^{\#}_{Z\cap T}(a'|_{Z\cap T})|_{W\cap V} = f^{\#}_{Z}(a')|_{W\cap V}$ and so

$$\hat{\psi}_{W\cap V}(a|_{W\cap V}\cdot t|_{W\cap V}) = f^{\#}_{Z}(a')|_{W\cap V}\cdot \phi_T(t')|_{W\cap V}$$

Hence

$$\begin{aligned} \operatorname{germ}_y \psi'_U(\alpha\cdot s) &= (W\cap V,\ f^{\#}_{Z}(a')|_{W\cap V}\cdot \phi_T(t')|_{W\cap V}) \\ &= (W,\ f^{\#}_{Z}(a')|_{W})\cdot (V,\ \phi_T(t')|_V) \\ &= (W,\ f^{\#}_{Z}(a')|_{W})\cdot \hat{\psi}_y(s(y)) \end{aligned}$$

But $f^{\#}_{Z}(a')|_W = \xi_W((\dot{Z}, a')) = \xi_W(\dot{a})$. Hence in $\mathcal{O}_{X,y}$ $(W, f^{\#}_{Z}(a')|_W) = (W, \xi_W(\dot{a})) = \xi_y(W, \dot{a})$.
$= \xi_y \operatorname{germ}_y \alpha = \operatorname{germ}_y \xi_U(\alpha)$. Hence

$$\begin{aligned} \operatorname{germ}_y \psi'_U(\alpha\cdot s) &= \operatorname{germ}_y \xi_U(\alpha)\cdot \operatorname{germ}_y \psi'_U(s) \\ &= \operatorname{germ}_y (\xi_U(\alpha)\cdot \psi'_U(s)) \qquad (1) \\ &= \operatorname{germ}_y (\xi_U(\alpha)\cdot \psi'_U(s)) \end{aligned}$$

Hence $\psi'_U(\alpha\cdot s) = \xi_U(\alpha)\cdot\psi'_U(s)$, which we will use below.

Let $P(U) = (f^{-1}\mathcal{G})(U) \otimes_{(f^{-1}\mathcal{O}_Y)(U)} \mathcal{O}_X(U)$ be the presheaf of $\mathcal{O}_X$-modules which sheafifies to give $f^*\mathcal{G}$. Let $U \subseteq X$ be fixed and consider the map

$$\begin{aligned} (f^{-1}\mathcal{G})(U)\times \mathcal{O}_X(U) &\longrightarrow \mathcal{F}(U) \\ (s, r) &\longmapsto r\cdot \psi'_U(s) = \psi'_U(r\cdot s) \end{aligned}$$

This is clearly bilinear in $(f^{-1}\mathcal{O}_Y)(U)$, thus inducing a morphism of abelian groups $\psi''_U : (f^{-1}\mathcal{G})(U)\otimes \mathcal{O}_X(U) \longrightarrow \mathcal{F}(U)$ defined by $\psi''_U(s\otimes r) = r\cdot\psi'_U(s)$ (Bilinearity follows from (1)). Then ψ'' is readily seen to be a morphism of presheaves of $\mathcal{O}_X$-modules. We define ψ to be the induced morphism of $\mathcal{O}_X$-modules $\psi : f^{-1}\mathcal{G}\otimes \mathcal{O}_X \longrightarrow \mathcal{F}$.

Next we have to show that ψ is unique making the following diagram commute

$$\begin{tikzcd} & \mathcal{G} \arrow[dl, "\eta_a"'] \arrow[dr, "\phi"] & \\ f_*f^*\mathcal{G} \arrow[rr, "f_*\psi"'] & & f_*\mathcal{F} \end{tikzcd} \qquad (2)$$

It is easy enough to check the diagram commutes, for $s \in \mathcal{G}(V)$

$$s \longmapsto ((\dot{V}, s)\otimes 1) \longmapsto \psi'_{f^{-1}V}((\dot{V}, s)) = \phi_V(s)|_{f^{-1}V} = \phi_V(s)$$

Next we check that ψ is unique. Suppose $\psi' : f^*\mathcal{G} \longrightarrow \mathcal{F}$ is another morphism of $\mathcal{O}_X$-modules making (2) commute.

$$\begin{tikzcd} & \mathcal{G}(U) \arrow[dl] \arrow[dr, "\phi_U"] & \\ f^*\mathcal{G}(f^{-1}U) \arrow[rr, bend left, "\psi_{f^{-1}U}"] \arrow[rr, "\psi'_{f^{-1}U}"'] & & \mathcal{F}(f^{-1}U) \end{tikzcd} \qquad (3)$$

To show $\psi = \psi'$ it suffices to show that for all $V \subseteq X$, ψ_V and ψ'_V agree on $(f^{-1}\mathcal{G})(V)\otimes \mathcal{O}_X(V)$.
For $s \in (f^{-1}\mathcal{G})(V)$ and $r \in \mathcal{O}_X(V)$ let $V = \bigcup_i V_i$ be a cover and $(W_i, s_i) \in G(V_i)$ s.t. $s|_{V_i} = (\dot{W_i}, s_i)$ $\forall i$. Then

$$\begin{aligned} \psi'_V(s\dot{\otimes} r)|_{V_i} &= r|_{V_i}\cdot \psi'_V(s\dot{\otimes} 1)|_{V_i} \\ &= r|_{V_i}\cdot \psi'_{V_i}(s|_{V_i}\dot{\otimes} 1) \\ &= r|_{V_i}\cdot \psi'_{V_i}((\dot{W_i}, s_i)\otimes 1) \end{aligned}$$

Note that $(W_i, s_i) \in G(V_i)$, but $(W_i, s_i) \in G(f^{-1}W_i)$ also. Moreover $V_i \subseteq f^{-1}W_i$ and $(W_i, s_i)|_{V_i} = (W_i, s_i)$. So

$$\begin{aligned}
\psi'_V(s \dot\otimes r)|_{V_i} &= r|_{V_i} \cdot \psi'_{V_i}(\{(W_i,s_i) \dot\otimes 1\}|_{V_i}) \\
&= r|_{V_i} \cdot \psi'_{f^{-1}W_i}((W_i,s_i) \dot\otimes 1)|_{V_i} \\
&= r|_{V_i} \cdot \psi'_{f^{-1}W_i}(\eta_{W_i}(s_i))|_{V_i} \\
&= r|_{V_i} \cdot \psi_{f^{-1}W_i}(\eta_{W_i}(s))|_{V_i} \\
&= \psi_V(s \dot\otimes r)|_{V_i}
\end{aligned}$$

⌜See (3)⌟

Since the V_i cover V, $\psi'_V(s \dot\otimes r) = \psi_V(s \dot\otimes r)$, as required. Hence $\psi = \psi'$, so ψ is unique and the proof that f^* is left adjoint to f_* is complete.

__NOTE__ Let $f : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Then we claim $f^*\mathcal{O}_Y \cong \mathcal{O}_X$. By definition $f^*\mathcal{O}_Y = f^{-1}\mathcal{O}_Y \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$, so $f^*\mathcal{O}_Y$ is the sheafification of the presheaf of $\mathcal{O}_X$-modules $P(U) = (f^{-1}\mathcal{O}_Y)(U) \otimes_{f^{-1}\mathcal{O}_Y(U)} \mathcal{O}_X(U) \cong \mathcal{O}_X(U)$ (iso of $\mathcal{O}_X(U)$-modules). This is natural in U since the $f^{-1}\mathcal{O}_Y(U)$-module structure on $\mathcal{O}_X(U)$ comes from $\xi : f^{-1}\mathcal{O}_Y \longrightarrow \mathcal{O}_X$. This sheafifies to give an isomorphism $f^*\mathcal{O}_Y \cong \mathcal{O}_X$ of $\mathcal{O}_X$-modules.

__NOTE__ A sequence $\mathcal{F}_{i-1} \longrightarrow \mathcal{F}_i \longrightarrow \mathcal{F}_{i+1}$ of $\mathcal{O}_X$-modules for a ringed space $\mathcal{O}_X$ is exact iff. it is exact as a sequence of sheaves of abelian groups hence iff. $\forall x \in X$ the sequence of abelian groups $\mathcal{F}_{i-1,x} \longrightarrow \mathcal{F}_{i,x} \longrightarrow \mathcal{F}_{i+1,x}$ is exact (Ex 1.2) hence:

⌜Sequence of $\mathcal{O}_X$-modules $\mathcal{F}_{i-1} \longrightarrow \mathcal{F}_i \longrightarrow \mathcal{F}_{i+1}$ exact iff. $\forall x \in X$ the sequence $\mathcal{F}_{i-1,x} \longrightarrow \mathcal{F}_{i,x} \longrightarrow \mathcal{F}_{i+1,x}$ of $\mathcal{O}_{X,x}$-modules is exact.⌟

__NOTE__ Let $f : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. For $x \in X$ there is a morphism of abelian groups compatible with the ring morphism $\mathcal{O}_{Y,f(x)} \longrightarrow \mathcal{O}_{X,x}$

$$\theta_x : (f_*\mathcal{F})_{f(x)} \longrightarrow \mathcal{F}_x$$

NOTE Locally Free Sheaves

Let X be a topological space, $x \in X$ and $i : \{x\} \longrightarrow X$ the inclusion. Put $1 = \{x\}$. Then we have functors

$$\begin{tikzcd} Sh(1) \arrow[r, bend left, "i_*"] & Sh(X) \arrow[l, bend left, "i^{-1}"] \end{tikzcd} \qquad \begin{array}{l} i^{-1} \dashv i_* \\ i^{-1} \cong \text{stalk}_x \end{array}$$

For sheaves of sets, groups or rings. But $\underline{Sets} = Sh(1)$, $\underline{Ab} = \underline{Ab}(1)$, $\underline{Rng} = \underline{Rng}(1)$ (category isomorphisms). An isomorphism is a left and right adjoint for its inverse, so the stalk functors $Sh(X) \longrightarrow \underline{Sets}$, $\underline{Ab}(X) \longrightarrow \underline{Ab}$, $\underline{Rng}(X) \longrightarrow \underline{Rng}$ all have right adjoints, hence preserve all colimits. (To be careful, one checks that for sets, groups, rings i^{-1} is actually isomorphic to stalk_x, considering sheafification etc.)

Let $\mathcal{O}_X$ be a sheaf of rings, $\mathcal{F}$ an $\mathcal{O}_X$-module. Then for $x \in X$ the stalk $\mathcal{F}_x$ becomes a $\mathcal{O}_{X,x}$-module in a natural way, and this defines a covariant additive functor $\mathcal{O}_X\text{-}\underline{Mod} \longrightarrow \mathcal{O}_{X,x}\text{-}\underline{Mod}$, where $\psi : \mathcal{F} \longrightarrow \mathcal{F}'$ induces $\psi_x : \mathcal{F}_x \longrightarrow \mathcal{F}'_x$ in the usual way. Call this functor Stalk_x. Define

$$\text{Sky}_x : \mathcal{O}_{X,x}\text{-}\underline{Mod} \longrightarrow \mathcal{O}_X\text{-}\underline{Mod}$$

$$\text{Sky}_x(M)(U) = \begin{cases} M & x \in U \\ 0 & \text{otherwise} \end{cases}$$

This is a sheaf of $\mathcal{O}_X$-modules, where $\text{Sky}_x(M)(U)$ becomes a $\mathcal{O}_X|U)$-module via $\mathcal{O}_X(U) \longrightarrow \mathcal{O}_{X,x}$. One defines Sky_x on morphisms in the obvious way. Clearly Sky_x is additive. To see $\text{Stalk}_x \dashv \text{Sky}_x$, define for an $\mathcal{O}_X$-module $\mathcal{F}$ the following morphism of $\mathcal{O}_X$-modules

$$\eta : \mathcal{F} \longrightarrow \text{Sky}_x \mathcal{F}_x$$

$$\text{If } x \in U \quad \eta_U(m) = (U, m) \in \mathcal{F}_x$$

If $\phi : \mathcal{F} \longrightarrow \text{Sky}_x M$ is another morphism of $\mathcal{O}_X$-modules for a $\mathcal{O}_{X,x}$-module M, then $\psi : \mathcal{F}_x \longrightarrow M$ defined by $(U, m) \longmapsto \phi_U(m)$ is a unique morphism of $\mathcal{O}_{X,x}$-modules s.t.

$$\begin{tikzcd} \mathcal{F} \arrow[rr, "\phi"] \arrow[dr, "\eta"'] & & \text{Sky}_x M \\ & \text{Sky}_x \mathcal{F}_x \arrow[ur, "\text{Sky}_x \psi"'] & \end{tikzcd}$$

commutes. Hence $\text{Stalk}_x \dashv \text{Sky}_x$ and so taking stalks preserves all colimits. It follows from Ex 1.2 that the functor Stalk_x is actually exact.

<u>DEFINITION</u> An $\mathcal{O}_X$-module $\mathcal{F}$ is <u>free</u> if it is a coproduct $\bigoplus_{i \in I} \mathcal{O}_X$ for some index set I (possibly empty or infinite). The <u>rank</u> of $\mathcal{F}$ is the cardinality of I (just ∞ if I is infinite. We do not distinguish between cardinals). Any module over $\mathcal{O}_X$ has rank 0 if $X = \emptyset$.

Suppose there are morphisms $u_i : \mathcal{O}_X \longrightarrow \mathcal{F}$ making $\mathcal{F}$ a coproduct $\bigoplus_{i \in I} \mathcal{O}_X$. Then the morphisms $u_{i,x} : \mathcal{O}_{X,x} \longrightarrow \mathcal{F}_x$ are a coproduct, so $\mathcal{F}_x$ is a free $\mathcal{O}_{X,x}$-module of rank $|I|$. Since rank is well-defined for modules over a commutative ring, the rank of $\mathcal{F}$ is well-defined. That is, if $\mathcal{F}$ can be written as a coproduct of $\mathcal{O}_X$ over two sets I, J then either I, J are both infinite or they are both finite and have the same number of elements. So $\text{rank}_{\mathcal{O}_X} \mathcal{F} = \text{rank}_{\mathcal{O}_{X,x}} \mathcal{F}_x$ for all $x \in X$.

<u>DEFINITION</u> An $\mathcal{O}_X$-module $\mathcal{F}$ is <u>locally free</u> if X can be covered by open sets U for which $\mathcal{F}|_U$ is a free $\mathcal{O}_X|_U$-module. If $\mathcal{F}|_U$ is a free $\mathcal{O}_X|_U$-module for $U \subseteq X$, the <u>rank</u> of $\mathcal{F}$ on U is the rank of $\mathcal{F}|_U$ as an $\mathcal{O}_X|_U$-module. In this case if $x \in U$ then

$$\text{rank}_{\mathcal{O}_{X,x}} \mathcal{F}_x = \text{rank}_{(\mathcal{O}_X|_U)_x} (\mathcal{F}|_U)_x = \text{rank}_U \mathcal{F}|_U \qquad (1)$$

If $X = \emptyset$ then 0 is locally free of rank 0. Any $\mathcal{F}$ has rank 0 on $\emptyset \subseteq X$.

So if $\mathcal{F}$ is locally free, $\mathcal{F}_x$ is a free $\mathcal{O}_{X,x}$-module for all $x \in X$. We call $\operatorname{rank}_{\mathcal{O}_{X,x}} \mathcal{F}_x$ the rank of $\mathcal{F}$ at x. If $\mathcal{F}$ is locally free and has the same rank $n \in \{0, 1, \dots\} \cup \{\infty\}$ at every point, we say $\mathcal{F}$ is locally free of rank n. From (1) it follows that if X is connected, $\mathcal{F}$ has the same rank everywhere.

NOTE In (1) we have $\mathcal{F}_x \cong (\mathcal{F}|_U)_x$ as modules via $\mathcal{O}_{X,x} \cong (\mathcal{O}_X|_U)_x$.

An invertible sheaf is a locally free sheaf of rank 1. So every sheaf on $\emptyset$ is invertible, and $\forall x \in X$ $\operatorname{rank}_{\mathcal{O}_{X,x}} \mathcal{F}_x = 1$. That is, $\mathcal{F}_x \cong \mathcal{O}_{X,x}$ as $\mathcal{O}_{X,x}$-modules.

At this point you should consult our typed notes “Locally Free Sheaves”.

<u>NOTE</u> The functors $- \otimes_{\mathcal{O}_X} \mathscr{G}$ and $\mathscr{F} \otimes_{\mathcal{O}_X} -$

Let $\mathcal{O}_X$ be a sheaf of rings, $\mathscr{F}$ an $\mathcal{O}_X$-module. Let $\phi : \mathscr{G} \longrightarrow \mathscr{G}'$ be a morphism of $\mathcal{O}_X$-modules, $P(U) = \mathscr{F}(U) \otimes_{\mathcal{O}_X(U)} \mathscr{G}(U)$ and $P'(U) = \mathscr{F}(U) \otimes_{\mathcal{O}_X(U)} \mathscr{G}'(U)$ the presheaves that give rise to $\mathscr{G}, \mathscr{G}'$. Let $\hat{\phi} : P \longrightarrow P'$ be defined by $\hat{\phi}_U = 1 \otimes \phi_U$. Then $\hat{\phi}$ is a morphism of presheaves of $\mathcal{O}_X$-modules. We let $\mathscr{F} \otimes \phi$ be the unique morphism of $\mathcal{O}_X$-modules making

$$\begin{tikzcd}
P \arrow[r, "\hat{\phi}"] \arrow[d] & P' \arrow[d] \\
\mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{G} \arrow[r, "\mathscr{F} \otimes \phi"'] & \mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{G}'
\end{tikzcd}$$

$$(\mathscr{F} \otimes \phi)_U(s)(x) = \hat{\phi}_x(s(x))$$

It is clear that $\mathscr{F} \otimes 1_{\mathscr{G}} = 1_{\mathscr{F} \otimes \mathscr{G}}$, and if $\psi : \mathscr{G}' \longrightarrow \mathscr{G}''$ then $\widehat{\psi\phi} = \hat{\psi}\hat{\phi}$, so $\mathscr{F} \otimes \psi\phi = (\mathscr{F} \otimes \psi)(\mathscr{F} \otimes \phi)$. Since $(\phi + \phi')^{\wedge} = \hat{\phi} + \hat{\phi}'$ it also follows that $\mathscr{F} \otimes (\phi + \phi') = \mathscr{F} \otimes \phi + \mathscr{F} \otimes \phi'$, so $\mathscr{F} \otimes -$ is an additive functor.

Similarly given an $\mathcal{O}_X$-module $\mathscr{G}$ and $\phi : \mathscr{F} \longrightarrow \mathscr{F}'$ let $\hat{\phi} : P \longrightarrow P'$ be $\hat{\phi}_U = \phi_U \otimes 1$. Then there is a morphism $\phi \otimes \mathscr{G} : \mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{G} \longrightarrow \mathscr{F}' \otimes_{\mathcal{O}_X} \mathscr{G}$ of $\mathcal{O}_X$-modules defined by

$$(\phi \otimes \mathscr{G})_U(s)(x) = \hat{\phi}_x(s(x))$$

This makes $- \otimes_{\mathcal{O}_X} \mathscr{G}$ into an additive functor. One checks easily that if $\phi : \mathscr{F} \longrightarrow \mathscr{F}'$, $\psi : \mathscr{G} \longrightarrow \mathscr{G}'$ then $(\phi \otimes \mathscr{G}')(\mathscr{F} \otimes \psi) = (\mathscr{F}' \otimes \psi)(\phi \otimes \mathscr{G})$. We call this $\phi \otimes \psi : \mathscr{F} \otimes \mathscr{G} \longrightarrow \mathscr{F}' \otimes \mathscr{G}'$. It is defined by $(\phi \otimes \psi)_U(s)(x) = \widehat{(\phi \otimes \psi)}_x(s(x))$ where $\phi \otimes \psi_U : \mathscr{F}(U) \otimes \mathscr{G}(U) \longrightarrow \mathscr{F}'(U) \otimes \mathscr{G}'(U)$ is defined in the obvious way. Clearly $(\phi' \otimes \psi')(\phi \otimes \psi) = (\phi'\phi) \otimes (\psi'\psi)$.

<u>NOTE</u> $(\mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{G})|_U \cong \mathscr{F}|_U \otimes_{\mathcal{O}_X|_U} \mathscr{G}|_U$.

Let $\mathcal{O}_X$ be a sheaf of rings, $U \subseteq X$ open, $\mathscr{F}, \mathscr{G}$ two $\mathcal{O}_X$-modules. Then $\mathscr{F} \otimes \mathscr{G}$ is the sheafification of the presheaf of $\mathcal{O}_X$-modules $V \overset{P}{\longmapsto} \mathscr{F}(V) \otimes_{\mathcal{O}_X(V)} \mathscr{G}(V)$, and $\mathscr{F}|_U \otimes_{\mathcal{O}_X|_U} \mathscr{G}|_U$ is the sheafification of the presheaf $P|_U$. So there is an isomorphism of sheaves of abelian groups on U:

$$\eta : \mathscr{F}|_U \otimes_{\mathcal{O}_X|_U} \mathscr{G}|_U \longrightarrow (\mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{G})|_U$$

$$\eta_V(s)(x) = K_x(s(x)) \qquad K_x : (P|_U)_x \Rightarrow P_x$$

It is easily checked that η is a morphism of $\mathcal{O}_X|_U$-modules. It is also easily checked that these isomorphisms are natural in both $\mathscr{F}$ and $\mathscr{G}$. That is, for $\phi : \mathscr{F} \longrightarrow \mathscr{F}'$ and $\psi : \mathscr{G} \longrightarrow \mathscr{G}'$ the following diagrams commute:

$$\begin{tikzcd}
(\mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{G})|_U \arrow[r, Rightarrow] \arrow[d, "(\phi \otimes 1)|_U"'] & \mathscr{F}|_U \otimes_{\mathcal{O}_X|_U} \mathscr{G}|_U \arrow[d, "\phi|_U \otimes 1"] \\
(\mathscr{F}' \otimes_{\mathcal{O}_X} \mathscr{G})|_U \arrow[r, Rightarrow] & \mathscr{F}'|_U \otimes_{\mathcal{O}_X|_U} \mathscr{G}|_U
\end{tikzcd}$$

$$\begin{tikzcd}
(\mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{G})|_U \arrow[r, Rightarrow] \arrow[d, "(1 \otimes \psi)|_U"'] & \mathscr{F}|_U \otimes_{\mathcal{O}_X|_U} \mathscr{G}|_U \arrow[d, "1 \otimes \psi|_U"] \\
(\mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{G}')|_U \arrow[r, Rightarrow] & \mathscr{F}|_U \otimes_{\mathcal{O}_X|_U} \mathscr{G}'|_U
\end{tikzcd}$$

If $V \subseteq U$, $a \in \mathscr{F}(V)$, $b \in \mathscr{G}(V)$ then η_V identifies $a \dot{\otimes} b \in (\mathscr{F}|_U \otimes \mathscr{G}|_U)(V)$ and $a \dot{\otimes} b \in (\mathscr{F} \otimes \mathscr{G})(V)$.

NOTE $(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})_x \cong \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{G}_x$

Let $\mathcal{O}_X$ be a sheaf of rings, $\mathcal{F}, \mathcal{G}$ two $\mathcal{O}_X$-modules. Then for $x \in X$ $\mathcal{G}_x, \mathcal{F}_x$ are naturally $\mathcal{O}_{X,x}$-modules, and we claim $(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})_x \cong \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{G}_x$ as $\mathcal{O}_{X,x}$-modules.

Let P be the presheaf of $\mathcal{O}_X$-modules $P(U) = \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$. Define

$$\varepsilon : \mathcal{F}_x \times \mathcal{G}_x \longrightarrow P_x$$

$$\varepsilon((U,s),(V,t)) = (U \cap V, s|_{U\cap V} \otimes t|_{U \cap V})$$

It is easily checked that ε is $\mathcal{O}_{X,x}$-bilinear, and we claim ε is actually a tensor product. Suppose M is an abelian group and $\phi : \mathcal{F}_x \times \mathcal{G}_x \longrightarrow M$ is $\mathcal{O}_{X,x}$-bilinear. For $x \in U$ the map $\tilde{\theta}_U : \mathcal{F}(U) \times \mathcal{G}(U) \longrightarrow M$ defined by $\tilde{\theta}_U(s,t) = \phi((U,s),(U,t))$ is $\mathcal{O}_X(U)$-bilinear and hence induces $\theta_U : \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U) \longrightarrow M$, $\theta_U(s \otimes t) = \phi((U,s),(U,t))$. Since θ commutes with restriction, we induce a morphism of abelian groups $\psi : P_x \longrightarrow M$ given by

$$\psi((U, s\otimes t)) = \phi((U,s),(U,t))$$

Clearly $\psi\varepsilon = \phi$ and ψ is unique, so ε is a tensor product, and there is an isomorphism $\alpha_x : P_x \longrightarrow \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{G}_x$ given by

$$\begin{tikzcd} & \mathcal{F}_x \times \mathcal{G}_x \arrow[dl, "\varepsilon"'] \arrow[dr] & \\ P_x \arrow[rr, "\alpha_x"'] & & \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{G}_x \end{tikzcd}$$

$$\alpha_x(U, s\otimes t) = (U,s) \otimes (U,t)$$

⌜α_x is an isomorphism of $\mathcal{O}_{X,x}$-modules⌟

Now, $(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})_x \cong P_x$ via $(U,q) \longmapsto q(x)$, so there is an isomorphism of abelian groups

$$\alpha'_x : (\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})_x \longrightarrow \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{G}_x$$

To see this is an isomorphism of $\mathcal{O}_{X,x}$-modules let $U \subseteq X$ and $q \in (\mathcal{F} \otimes \mathcal{G})(U)$. Say $r \in \mathcal{O}_X(U)$, and let $x \in V \subseteq U$ and $t \in P(V)$ be s.t. $q|_V = \dot{t}$. Then if $t = a \otimes b$ we have

$$\begin{aligned} \alpha'_x((U,r)\cdot(U,q)) &= \alpha'_x((U,r)\cdot(V,\dot{t})) \\ &= \alpha'_x(U\cap V, r\cdot \dot{t}) \\ &= \alpha_x((r\cdot\dot{t})(x)) \\ &= \alpha_x((U,r)\cdot(V, a\otimes b)) \\ &= \alpha_x(U\cap V, a|_{U\cap V} \otimes r|_{U\cap V}\cdot b|_{U\cap V}) \\ &= (U\cap V, a|_{U\cap V}) \otimes (U\cap V, r|_{U\cap V}\cdot b|_{U\cap V}) \\ &= (U,r)\cdot \alpha'_x(U,q) \end{aligned}$$

⌜This is actually trivial, since $(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}_x) \cong P_x$ is an iso of $\mathcal{O}_{X,x}$-mods⌟

⌜Should also use $t = \sum_i a_i \otimes b_i$ to be completely correct⌟

as required. Moreover, this isomorphism is <u>natural</u> in $\mathcal{F}$ and $\mathcal{G}$. Let $\phi : \mathcal{F} \longrightarrow \mathcal{F}'$ be a morphism of $\mathcal{O}_X$-modules. We claim the following diagram commutes:

$$\begin{tikzcd} (\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})_x \arrow[r, Rightarrow, "\alpha'_x"] \arrow[d, "(\phi\otimes\mathcal{G})_x"] & \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{G}_x \arrow[d, "\phi_x \otimes \mathcal{G}_x"] \\ (\mathcal{F}' \otimes_{\mathcal{O}_X} \mathcal{G})_x \arrow[r, Rightarrow, "\alpha'_x"'] & \mathcal{F}'_x \otimes_{\mathcal{O}_X} \mathcal{G}_x \end{tikzcd}$$

For $s \in (\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})(U)$ let $x \in V$ and $a \otimes b \in P(V)$ be s.t. $s|_V = (a\dot{\otimes} b)$. Then

⌜Technically should use $t = a_i \otimes b^i$ (sum over i) But it doesn't make a difference⌟

$$\begin{aligned} \alpha'_x(\phi\otimes\mathcal{G})_x(U,s) &= \alpha'_x(U, (\phi\otimes\mathcal{G})_U(s)) = \alpha_x((\phi\otimes\mathcal{G})_U(s)(x)) \\ &= \alpha_x(\hat{\phi}_x(s(x))) = \alpha_x(\hat{\phi}_x(V, a\otimes b)) \\ &= \alpha_x(V, \phi_V(a)\otimes b) = (V, \phi_V(a)) \otimes (V,b) = (\phi_x \otimes \mathcal{G}_x)\alpha'_x(U,s) \end{aligned}$$

Similarly if $\phi : \mathcal{G} \longrightarrow \mathcal{G}'$ is a morphism of $\mathcal{O}_X$-modules the following diagram commutes

$$\begin{tikzcd}
(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})_x \arrow[r, Rightarrow] \arrow[d, "(\mathcal{F} \otimes \phi)_x"] & \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{G}_x \arrow[d, "\mathcal{F}_x \otimes \phi_x"] \\
(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}')_x \arrow[r, Rightarrow] & \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{G}'_x
\end{tikzcd}$$

NOTE $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X \cong \mathcal{F}$

Let $\mathcal{O}_X$ be a sheaf of rings, $\mathcal{F}$ an $\mathcal{O}_X$-module, P the presheaf $P(U) = \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(U)$. Then there is an isomorphism of $\mathcal{O}_X(U)$-modules

$$\lambda_U : P(U) \longrightarrow \mathcal{F}(U)$$
$$m \otimes r \longmapsto r \cdot m$$

This defines an isomorphism of presheaves of $\mathcal{O}_X$-modules $\lambda : P \longrightarrow \mathcal{F}$, so P is a sheaf and so there is an isomorphism of $\mathcal{O}_X$-modules $\mathcal{F} \Longrightarrow \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X$, which maps $m \in \mathcal{F}(U)$ to $(m \dot\otimes 1) \in (\mathcal{F} \otimes \mathcal{O}_X)(U)$. This isomorphism is clearly natural in $\mathcal{F}$: for a morphism $\phi : \mathcal{F} \longrightarrow \mathcal{F}'$ we have a commutative diagram

$$\begin{tikzcd}
\mathcal{F} \arrow[r, Rightarrow] \arrow[d, "\phi"'] & \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X \arrow[d, "\phi \otimes \mathcal{O}_X"] \\
\mathcal{F}' \arrow[r, Rightarrow] & \mathcal{F}' \otimes_{\mathcal{O}_X} \mathcal{O}_X
\end{tikzcd}$$

The inverse $\mathcal{F} \otimes \mathcal{O}_X \longrightarrow \mathcal{F}$ maps $m \dot\otimes r$ to $r.m$.

NOTE $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \cong \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{F}$

Let $\mathcal{O}_X$ be a sheaf of rings, $\mathcal{F}, \mathcal{G}$ $\mathcal{O}_X$-modules. Let $P(U) = \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$, $Q(U) = \mathcal{G}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{F}(U)$. Then there is an isomorphism of presheaves of $\mathcal{O}_X$-modules

$$\lambda : P \longrightarrow Q$$
$$\lambda_U (a \otimes b) = b \otimes a$$

The sheafification of λ gives an isomorphism of $\mathcal{O}_X$-modules $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \Longrightarrow \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{F}$, which is natural in both variables. That is, for $\phi : \mathcal{F} \longrightarrow \mathcal{F}'$ and $\psi : \mathcal{G} \longrightarrow \mathcal{G}'$ the following commute:

$$\begin{tikzcd}
\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \arrow[r, Rightarrow] \arrow[d, "\phi \otimes \mathcal{G}"'] & \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{F} \arrow[d, "\mathcal{G} \otimes \phi"] \\
\mathcal{F}' \otimes_{\mathcal{O}_X} \mathcal{G} \arrow[r, Rightarrow] & \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{F}'
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \arrow[r, Rightarrow] \arrow[d, "\mathcal{F} \otimes \psi"'] & \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{F} \arrow[d, "\psi \otimes \mathcal{F}"] \\
\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}' \arrow[r, Rightarrow] & \mathcal{G}' \otimes_{\mathcal{O}_X} \mathcal{F}
\end{tikzcd}$$

Note that $(\mathcal{F} \otimes \mathcal{G})(U) \Longrightarrow (\mathcal{G} \otimes \mathcal{F})(U)$ maps $a \dot\otimes b$ to $b \dot\otimes a$ for $a \in \mathcal{F}(U)$, $b \in \mathcal{G}(U)$.

NOTE $(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}) \otimes_{\mathcal{O}_X} \mathcal{H} \cong \mathcal{F} \otimes_{\mathcal{O}_X} (\mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{H})$

Let $P(U) = (\mathcal{F} \otimes \mathcal{G})(U) \otimes_{\mathcal{O}_X(U)} \mathcal{H}(U)$, and $Q(U) = \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} (\mathcal{G} \otimes \mathcal{H})(U)$ be the corresponding presheaves of $\mathcal{O}_X$-modules ($\mathcal{O}_X$ a sheaf of rings, $\mathcal{F}, \mathcal{G}, \mathcal{H}$ $\mathcal{O}_X$-modules). Then there is an isomorphism of $\mathcal{O}_{X,x}$-modules

$$\begin{aligned}
\lambda_x : P_x &\cong (\mathcal{F} \otimes \mathcal{G})_x \otimes_{\mathcal{O}_{X,x}} \mathcal{H}_x \cong (\mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{G}_x) \otimes_{\mathcal{O}_{X,x}} \mathcal{H}_x \\
&\cong \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} (\mathcal{G}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{H}_x) \cong \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} (\mathcal{G} \otimes \mathcal{H})_x \\
&\cong Q_x
\end{aligned}$$

and we define $\lambda : (\mathcal{F} \otimes \mathcal{G}) \otimes \mathcal{H} \longrightarrow \mathcal{F} \otimes (\mathcal{G} \otimes \mathcal{H})$ by $\lambda_U(s)(x) = \lambda_x(s(x))$. This is easily checked to be a well-defined morphism of $\mathcal{O}_X$-modules, which is an isomorphism since $\forall x \in X$ λ_x is iso. Here

$$\lambda_x(U, a \otimes b) = \textstyle\sum_j (V \cap U,\ c_j|_{V \cap U} \otimes (d_j|_{V \cap W} \dot\otimes b|_{V \cap W})) \quad \text{where} \quad a(x) = (V, \textstyle\sum_j c_j \otimes d_j)$$

This isomorphism is natural, in the sense that for $\phi : \mathcal{F} \longrightarrow \mathcal{F}'$, $\psi : \mathcal{G} \longrightarrow \mathcal{G}'$ and $\xi : \mathcal{H} \longrightarrow \mathcal{H}'$ the following diagrams commute:

$$\begin{tikzcd}
(\mathcal{F} \otimes \mathcal{G}) \otimes \mathcal{H} \arrow[r, Rightarrow] \arrow[d, "(\phi \otimes \mathcal{G}) \otimes \mathcal{H}"'] & \mathcal{F} \otimes (\mathcal{G} \otimes \mathcal{H}) \arrow[d, "\phi \otimes (\mathcal{G} \otimes \mathcal{H})"] \\
(\mathcal{F}' \otimes \mathcal{G}) \otimes \mathcal{H} \arrow[r, Rightarrow] & \mathcal{F}' \otimes (\mathcal{G} \otimes \mathcal{H})
\end{tikzcd}$$

$$\begin{tikzcd}
(\mathcal{F} \otimes \mathcal{G}) \otimes \mathcal{H} \arrow[r, Rightarrow] \arrow[d, "(\mathcal{F} \otimes \psi) \otimes \mathcal{H}"'] & \mathcal{F} \otimes (\mathcal{G} \otimes \mathcal{H}) \arrow[d, "\mathcal{F} \otimes (\psi \otimes \mathcal{H})"] \\
(\mathcal{F} \otimes \mathcal{G}') \otimes \mathcal{H} \arrow[r, Rightarrow] & \mathcal{F} \otimes (\mathcal{G}' \otimes \mathcal{H})
\end{tikzcd}$$

$$\begin{tikzcd}
(\mathcal{F} \otimes \mathcal{G}) \otimes \mathcal{H} \arrow[r, Rightarrow] \arrow[d, "(\mathcal{F} \otimes \mathcal{G}) \otimes \xi"'] & \mathcal{F} \otimes (\mathcal{G} \otimes \mathcal{H}) \arrow[d, "\mathcal{F} \otimes (\mathcal{G} \otimes \xi)"] \\
(\mathcal{F} \otimes \mathcal{G}) \otimes \mathcal{H}' \arrow[r, Rightarrow] & \mathcal{F} \otimes (\mathcal{G} \otimes \mathcal{H}')
\end{tikzcd}$$

Of course for any $x \in X$ by definition of λ the following diagram commutes

$$\begin{tikzcd}
\{(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}) \otimes_{\mathcal{O}_X} \mathcal{H}\}_x \arrow[r, Rightarrow, "\lambda"] \arrow[d, Rightarrow] & \{\mathcal{F} \otimes_{\mathcal{O}_X} (\mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{H})\}_x \arrow[d, Rightarrow] \\
(\mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{G}_x) \otimes_{\mathcal{O}_{X,x}} \mathcal{H}_x \arrow[r, Rightarrow] & \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} (\mathcal{G}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{H}_x)
\end{tikzcd}$$

Where the verticals are the canonical isomorphisms and the bottom map the associator for $\mathcal{O}_{X,x}$-modules. Also note that if $a \in \mathcal{F}(U)$, $b \in \mathcal{G}(U)$, $c \in \mathcal{H}(U)$ then

$$\lambda_U((a \dot{\otimes} b) \dot{\otimes} c) = a \dot{\otimes} (b \dot{\otimes} c)$$

NOTE $\mathcal{F}\otimes -$ and $-\otimes\mathcal{F}$ are right exact

Let $(X,\mathcal{O}_X)$ be a ringed space. We claim that for a $\mathcal{O}_X$-module $\mathcal{F}$ the additive functor $\mathcal{F}\otimes - : \mathcal{O}_X\text{-}\underline{\mathrm{Mod}} \longrightarrow \mathcal{O}_X\text{-}\underline{\mathrm{Mod}}$ is right exact (a.k.a cokernel preserving). Suppose we are given an exact sequence

$$\mathcal{G}' \longrightarrow \mathcal{G} \longrightarrow \mathcal{G}'' \longrightarrow 0 \qquad (1)$$

We need to show that the sequence

$$\mathcal{F}\otimes\mathcal{G}' \longrightarrow \mathcal{F}\otimes\mathcal{G} \longrightarrow \mathcal{F}\otimes\mathcal{G}'' \longrightarrow 0 \qquad (2)$$

is exact. It suffices to show that it is exact on stalks $\forall x\in X$. But commutativity of the following diagram:

$$\begin{tikzcd}
(\mathcal{F}\otimes\mathcal{G}')_x \arrow[r] \arrow[d, Rightarrow] & (\mathcal{F}\otimes\mathcal{G})_x \arrow[r] \arrow[d, Rightarrow] & (\mathcal{F}\otimes\mathcal{G}'')_x \arrow[r] \arrow[d, Rightarrow] & 0 \\
\mathcal{F}_x\otimes\mathcal{G}'_x \arrow[r] & \mathcal{F}_x\otimes\mathcal{G}_x \arrow[r] & \mathcal{F}_x\otimes\mathcal{G}''_x \arrow[r] & 0
\end{tikzcd}$$

Together with exactness of $\mathcal{G}'_x \longrightarrow \mathcal{G}_x \longrightarrow \mathcal{G}''_x \longrightarrow 0$ and the fact that $\mathcal{F}_x\otimes - : \mathcal{O}_{X,x}\text{-}\underline{\mathrm{Mod}} \longrightarrow \mathcal{O}_{X,x}\text{-}\underline{\mathrm{Mod}}$ is right exact implies that (2) is exact, as required. Similarly the functor $-\otimes\mathcal{F}$ is also right exact. So both $\mathcal{F}\otimes -$ and $-\otimes\mathcal{F}$ are additive and finite colimit preserving.

<u>DEFINITION</u> Let $(X,\mathcal{O}_X)$ be a ringed space. A $\mathcal{O}_X$-module $\mathcal{F}$ is <u>flat</u> if the functor $-\otimes\mathcal{F}$ is exact. (equiv. $\mathcal{F}\otimes -$ is exact). That is, $-\otimes\mathcal{F}$ preserves monics.

<u>LEMMA</u> If $\mathcal{F}$ is a $\mathcal{O}_X$-module and $\mathcal{F}_x$ is a flat $\mathcal{O}_{X,x}$-module $\forall x\in X$ then $\mathcal{F}$ is flat. In particular any locally free module is flat.

<u>PROOF</u> Use the above argument. $\square$

NOTE $f_*\mathcal{F}\otimes_{\mathcal{O}_Y} f_*\mathcal{G} \longrightarrow f_*(\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{G})$ $\qquad f:(X,\mathcal{O}_X)\longrightarrow(Y,\mathcal{O}_Y)$

Let $f:(X,\mathcal{O}_X)\longrightarrow(Y,\mathcal{O}_Y)$ be a ringed space morphism, $\mathcal{F},\mathcal{G}$ $\mathcal{O}_X$-modules. For $V\subseteq Y$ open define

$$\mathcal{F}(f^{-1}V)\times\mathcal{G}(f^{-1}V) \longrightarrow (\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{G})(f^{-1}V)$$
$$(a,b) \longmapsto a\,\dot{\otimes}\,b$$

This is $\mathcal{O}_Y(V)$-bilinear and induces $\mathcal{F}(f^{-1}V)\otimes_{\mathcal{O}_Y(V)}\mathcal{G}(f^{-1}V) \longrightarrow (\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{G})(f^{-1}V)$ and then after some checking a morphism of $\mathcal{O}_Y$-modules

$$\alpha : f_*\mathcal{F}\otimes_{\mathcal{O}_Y} f_*\mathcal{G} \longrightarrow f_*(\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{G})$$
$$\alpha_V(a\,\dot{\otimes}\,b) = a\,\dot{\otimes}\,b$$

This morphism is natural in both variables, but is not necessarily an isomorphism (although this is the case if f is an isomorphism).

NOTE Stalks under direct and inverse image.

$$\ulcorner (f^*\mathcal{G})_x \cong \mathcal{G}_{f(x)} \otimes_{\mathcal{O}_{Y,f(x)}} \mathcal{O}_{X,x} \lrcorner$$

Let $f: X \longrightarrow Y$ be a continuous map, $f^{-1}: \underline{Ab}(Y) \longrightarrow \underline{Ab}(X)$ and $f_*: \underline{Ab}(X) \longrightarrow \underline{Ab}(Y)$ the usual adjoint pair. We claim that there is a canonical isomorphism $(f^{-1}G)_x \cong G_{f(x)}$. (This also holds for sheaves of sets or rings, in which case we have isos of sets / rings)

Let P be the presheaf on X defined by $P(U) = \varinjlim_{V \supseteq f(U)} G(V)$. Then $(f^{-1}G)_x \cong P_x$ via $(U,s) \mapsto s(x)$, as abelian groups. We define $\phi: P_x \longrightarrow G_{f(x)}$ by

$$\phi(V,(W,t)) = (W,t) \qquad (W,t) \in P(V) = \varinjlim_{W \supseteq f(V)} G(W)$$
$$t \in G(W)$$

To show this is well-defined, suppose $(V',(W',t')) \in P_x$ with $(V,(W,t)) = (V',(W',t'))$. This means there is an open neighborhood Q of x s.t. $(W',t')|_Q = (W,t)|_Q$ in $P(Q) = \varinjlim_{D \supseteq f(Q)} G(D)$. By definition this means there is an open set Z in Y with $f(Q) \subseteq Z \subseteq W \cap W'$ and $t|_Z = t'|_Z$. Hence $(W,t) = (W',t')$ in $G_{f(x)}$.

To see that ϕ is injective, suppose $\phi(V,(W,t)) = \phi(V',(W',t'))$, say $f(x) \in Q$ is s.t. $t|_Q = t'|_Q$, $Q \subseteq W \cap W'$. Then $f^{-1}Q \cap V \cap V'$ is an open neighborhood of x in X and $(W,t)|_{f^{-1}Q \cap V \cap V'} = (W',t')|_{f^{-1}Q \cap V \cap V'}$ in $P(f^{-1}Q \cap V \cap V')$, so $(V,(W,t)) = (V',(W',t'))$. To see that ϕ is surjective let $(W,t) \in G_{f(x)}$ be given. Then $(W,t) \in P(f^{-1}W)$ and $\phi(f^{-1}W,(W,t)) = (W,t)$, as required. It is easy to see that ϕ is a morphism of groups or rings. It is easily checked that $(f^{-1}G)_x \Longrightarrow G_{f(x)}$ is natural in G.

Now let $f: \mathcal{O}_X \longrightarrow \mathcal{O}_Y$ be a morphism of ringed spaces.

$$\begin{tikzcd} \mathcal{O}_X\text{-}\underline{Mod} \arrow[r, bend left, "f_*"] & \mathcal{O}_Y\text{-}\underline{Mod} \arrow[l, bend left, "f^*"] \end{tikzcd}$$

We claim that $(f^*\mathcal{G})_x \cong \mathcal{G}_{f(x)} \otimes_{\mathcal{O}_{Y,f(x)}} \mathcal{O}_{X,x}$ where $\mathcal{O}_{X,x}$ becomes a $\mathcal{O}_{Y,f(x)}$ module via $f_x: \mathcal{O}_{Y,f(x)} \longrightarrow \mathcal{O}_{X,x}$. By the above we know there is an isomorphism of rings $(f^{-1}\mathcal{O}_Y)_x \cong \mathcal{O}_{Y,f(x)}$. Thus $(f^{-1}\mathcal{G})_x$ becomes a $\mathcal{O}_{Y,f(x)}$-module, and first we show $(f^{-1}\mathcal{G})_x \cong \mathcal{G}_{f(x)}$ is an isomorphism of $\mathcal{O}_{Y,f(x)}$-modules.

$$\alpha: (f^{-1}\mathcal{O}_Y)_x \xrightarrow{\sim} R_x \xrightarrow{\sim} \mathcal{O}_{Y,f(x)}$$
$$\beta: (f^{-1}\mathcal{G})_x \xrightarrow{\sim} P_x \xrightarrow[T]{\sim} \mathcal{G}_{f(x)} \qquad \ulcorner P \text{ sheafifies to give } f^{-1}\mathcal{G} \lrcorner$$

For $m \in \mathcal{O}_Y(Q)$, $x \in Q$ and $s \in (f^{-1}\mathcal{G})(T)$, $x \in T$. Then if $s(x) = (V,(W,t))$ with $W \supseteq f(V)$, $t \in \mathcal{G}(W)$, then

$$\begin{aligned}
\beta((Q,m)\cdot(T,s)) &= \beta(\alpha^{-1}(Q,m)\cdot(T,s)) \\
&= \beta((f^{-1}Q,(Q,m))\cdot(T,s)) \\
&= \beta(f^{-1}Q \cap T, (Q,m)|_{f^{-1}Q\cap T}\cdot s|_{f^{-1}Q\cap T}) \\
&= T((f^{-1}Q,(Q,m))\cdot(V,(W,t))) && \ulcorner\text{action of } R_x \text{ on } P_x \lrcorner \\
&= T(f^{-1}Q\cap V, (Q,m)\cdot(W,t)) && \ulcorner\text{action of } \mathcal{O}_Y \text{ on } \mathcal{G} \\
&= T(f^{-1}Q\cap V, (Q\cap W, m|_{Q\cap W}\cdot t|_{Q\cap W})) \\
&= (Q\cap W, m|_{Q\cap W}\cdot t|_{Q\cap W}) \\
&= (Q,m)\cdot(W,t) \\
&= (Q,m)\cdot\beta(T,s)
\end{aligned}$$

as required. Let Z be the presheaf $Z(U) = (f^{-1}\mathcal{G})(U) \otimes_{(f^{-1}\mathcal{O}_Y)(U)} \mathcal{O}_X(U)$ which sheafifies to give $f^*\mathcal{G}$. Each $Z(U)$ is both a $(f^{-1}\mathcal{O}_Y)(U)$-module and a $\mathcal{O}_X(U)$-module, so Z is both a $f^{-1}\mathcal{O}_Y$-module and an $\mathcal{O}_X$-module.

The isomorphism $(f^*\mathcal{G})_x \longrightarrow Z_x$, $(V,s) \longmapsto s(x)$ is an isomorphism of $(f^{-1}\mathcal{O}_Y)_x$-modules, and hence of $\mathcal{O}_{Y,f(x)}$-modules (via α). From our earlier notes we know there is an isomorphism of $(f^{-1}\mathcal{O}_Y)_x$-modules

$$Z_x \xrightarrow{\ \sim\ } (f^{-1}\mathcal{G})_x \otimes_{(f^{-1}\mathcal{O}_Y)_x} \mathcal{O}_{X,x}$$

If we construct the tensor product in the usual way, we can use α to replace $(f^{-1}\mathcal{O}_Y)_x$ by $\mathcal{O}_{Y,f(x)}$, making $(f^{-1}\mathcal{G})_x$ and $\mathcal{O}_{X,x}$ into $\mathcal{O}_{Y,f(x)}$-modules via α. Then upon composition with the isomorphism of $\mathcal{O}_{Y,f(x)}$-modules $\beta \otimes 1$ we have an isomorphism of $\mathcal{O}_{Y,f(x)}$-modules (also of $\mathcal{O}_{X,x}$-modules)

$$\eta : (f^*\mathcal{G})_x \xrightarrow{\ \sim\ } Z_x \xrightarrow{\ \sim\ } \mathcal{G}_{f(x)} \otimes_{\mathcal{O}_{Y,f(x)}} \mathcal{O}_{X,x}$$

Let $(V,s) \in (f^*\mathcal{G})_x$ with $s \in (f^*\mathcal{G})(V)$ and $s(x) = (U, a_i \otimes b^i) \in Z_x$, $a_i \in (f^{-1}\mathcal{G})(U)$, $b^i \in \mathcal{O}_X(U)$ and suppose $a_i(x) = (Q_i, (W_i, t_i)) \in P_x$, $(W_i, t_i) \in P(Q_i) = \varinjlim_{W \supseteq f(Q_i)} \mathcal{G}(W)$, $t \in \mathcal{G}(W_i)$. Then [sum over repeated indices]

$$\eta(V,s) = (W_i, t_i) \otimes (U, b^i)$$

We claim η is natural in $\mathcal{G}$. If $\phi : \mathcal{G} \longrightarrow \mathcal{G}'$ is a morphism of $\mathcal{O}_Y$-modules, we need to show

$$\begin{tikzcd}
(f^*\mathcal{G})_x \arrow[r, "\eta"] \arrow[d, "(f^*\phi)_x"'] & \mathcal{G}_{f(x)} \otimes_{\mathcal{O}_{Y,f(x)}} \mathcal{O}_{X,x} \arrow[d, "\phi_{f(x)} \otimes \mathcal{O}_{X,x}"] \\
(f^*\mathcal{G}')_x \arrow[r, "\eta"'] & \mathcal{G}'_{f(x)} \otimes_{\mathcal{O}_{Y,f(x)}} \mathcal{O}_{X,x}
\end{tikzcd}$$

commutes. But if (V,s) is as above, and if $\phi' : P \longrightarrow P'$, $\phi'' : f^{-1}\mathcal{G} \longrightarrow f^{-1}\mathcal{G}'$, $\phi''' : Z \longrightarrow Z'$ are canonical,

$$\begin{aligned}
(f^*\phi)_x(V,s) &= (V, (f^*\phi)_V(s)) \\
(f^*\phi)_V(s)(x) &= \phi'''_x(s(x)) \\
&= \phi'''_x(U, a_i \otimes b^i) \\
&= (U, \phi'''_U(a_i \otimes b^i)) \\
&= (U, \phi''_U(a_i) \otimes b^i) \\
\phi''_U(a_i)(x) &= \phi'_x(a_i(x)) \\
&= \phi'_x(Q_i, (W_i, t_i)) \\
&= (Q_i, \phi'_{Q_i}(W_i, t_i)) \\
&= (Q_i, (W_i, \phi_{W_i}(t_i)))
\end{aligned}$$

So finally, $\eta (f^*\phi)_x(V,s) = (W_i, \phi_{W_i}(t_i)) \otimes (U, b^i) = (\phi_{f(x)} \otimes \mathcal{O}_{X,x})\,\eta(V,s)$, as required.

<u>EXAMPLE</u> Let $f : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces, $\mathcal{F}$ a $\mathcal{O}_Y$-module. With all morphisms canonical, the following diagram commutes $(x \in X)$:

$$\begin{tikzcd}
\mathcal{F}_{f(x)} \arrow[r, "\eta_{f(x)}"] \arrow[d] & (f_* f^* \mathcal{F})_{f(x)} \arrow[d] \\
\mathcal{F}_{f(x)} \otimes_{\mathcal{O}_{Y,f(x)}} \mathcal{O}_{X,x} & (f^*\mathcal{F})_x \arrow[l, Rightarrow]
\end{tikzcd}
\qquad \text{(unit of } f^* \dashv f_*)$$

Beginning with (V,s), $s \in \mathcal{F}(V)$ we have $(V, ((V,s) \otimes 1))$ and then $(f^{-1}V, ((V,s) \otimes 1))$, which maps to $(V,s) \otimes 1 \in (\mathcal{F}_{f(x)} \otimes_{\mathcal{O}_{Y,f(x)}} \mathcal{O}_{X,x})$

At this point you should consult our typed notes “Inverse Image Preserves Tensor Products”.

NOTE $(M \otimes_A B)_\mathfrak{p} \cong M_{\varphi^{-1}\mathfrak{p}} \otimes_{A_{\varphi^{-1}\mathfrak{p}}} B_\mathfrak{p}$ $\quad \ulcorner \varphi : A \longrightarrow B,\ AM \lrcorner$

Let $\varphi : A \longrightarrow B$ be a morphism of rings, M an A-module, and $\mathfrak{p} \subseteq B$ a prime. Define

$$\varepsilon : M_{\varphi^{-1}\mathfrak{p}} \times B_\mathfrak{p} \longrightarrow (M \otimes_A B)_\mathfrak{p}$$
$$\varepsilon(m/s, b/t) = \frac{m \otimes b}{\varphi(s)t}$$

Suppose $m/s = m'/s'$ in $M_{\varphi^{-1}\mathfrak{p}}$, so there is $q \notin \varphi^{-1}\mathfrak{p}$ with $q(s'.m - s.m') = 0$. Then $\varphi(q) \notin \mathfrak{p}$ and

$$\begin{aligned} &\varphi(q)(\{\varphi(s')t\}(m \otimes b) - \{\varphi(s)t\}(m' \otimes b)\} \\ &= \varphi(q)(m \otimes \varphi(s')tb - m' \otimes \varphi(s)tb) \\ &= \varphi(q)((s'm - sm') \otimes tb) = q(s'm - sm') \otimes tb = 0 \end{aligned}$$

Hence $m \otimes b/\varphi(s)t = m' \otimes b/\varphi(s')t$ in $(M \otimes_A B)_\mathfrak{p}$. Similarly one shows that if $b/t = b'/t'$ in $B_\mathfrak{p}$ then $m \otimes b/\varphi(s)t = m \otimes b'/\varphi(s)t'$ in $(M \otimes_A B)_\mathfrak{p}$. It follows that ε is well-defined. It is easily checked that ε is $A_{\varphi^{-1}\mathfrak{p}}$-bilinear, where $B_\mathfrak{p}$ becomes an $A_{\varphi^{-1}\mathfrak{p}}$-module via $\varphi_\mathfrak{p} : A_{\varphi^{-1}\mathfrak{p}} \longrightarrow B_\mathfrak{p}$. Let Z be another abelian group and $\psi : M_{\varphi^{-1}\mathfrak{p}} \times B_\mathfrak{p} \longrightarrow Z$ a $A_{\varphi^{-1}\mathfrak{p}}$-bilinear map.

$$\begin{tikzcd} & M_{\varphi^{-1}\mathfrak{p}} \times B_\mathfrak{p} \arrow[dl, "\psi"'] \arrow[dr, "\varepsilon"] & \\ Z & & (M \otimes_A B)_\mathfrak{p} \arrow[ll, "\phi"] \end{tikzcd}$$

We define ϕ as follows. Firstly, for $s \notin \mathfrak{p}$ define $\phi'_s : M \times B \longrightarrow Z$ by $\phi'_s(m,b) = \psi(m/1, b/s)$. Then ϕ'_s is A-bilinear, inducing a morphism of abelian groups

$$\phi''_s : M \otimes_A B \longrightarrow Z$$
$$\phi''_s(m \otimes b) = \psi(m/1, b/s)$$

Define ϕ by $\phi(a/s) = \phi''_s(a)$. To see this is well-defined, suppose $a/s = a'/s'$ with $a = \sum_i m_i \otimes b_i$ and $a' = \sum_i m_i' \otimes b_i'$. Let $t \notin \mathfrak{p}$ be s.t. $ts'a = tsa'$. That is, $\sum_i m_i \otimes b_i ts' = \sum_i m_i' \otimes tsb_i'$. Applying $\phi''_{tss'}$ to both sides gives $\phi''_s(a) = \phi''_{s'}(a')$, so ϕ is well-defined. It is easily checked that

$$\phi''_s(a) + \phi''_{s'}(a') = \phi''_{ss'}(s'a + sa')$$

so ϕ is a morphism of groups. Since $\phi(m \otimes b/s) = \psi(m/1, b/s)$ and $\psi(m/s, b/t) = \psi(1/s . m/1, b/t) = \psi(m/1, b/\varphi(s)t)$ it follows that $\phi\varepsilon = \psi$. Uniqueness of ϕ is clear. Hence we have isomorphisms of abelian groups

$$\begin{tikzcd} M_{\varphi^{-1}\mathfrak{p}} \otimes_{A_{\varphi^{-1}\mathfrak{p}}} B_\mathfrak{p} \arrow[r, bend left, "K"] & (M \otimes_A B)_\mathfrak{p} \arrow[l, bend left, "\phi"] \end{tikzcd}$$

$$\phi(m \otimes b/s) = m/1 \otimes b/s$$
$$K(m/s \otimes b/t) = \frac{m \otimes b}{\varphi(s)t}$$

It is easily checked that K is an isomorphism of $B_\mathfrak{p}$-modules and that these isomorphisms are natural in M. That is, for a morphism $\alpha : M \longrightarrow N$ of A-modules, the following diagram commutes:

$$\begin{tikzcd} (M \otimes_A B)_\mathfrak{p} \arrow[r, Rightarrow, "\phi"] \arrow[d, "(\alpha \otimes B)_\mathfrak{p}"'] & M_{\varphi^{-1}\mathfrak{p}} \otimes_{A_{\varphi^{-1}\mathfrak{p}}} B_\mathfrak{p} \arrow[d, "\alpha_{\varphi^{-1}\mathfrak{p}} \otimes B_\mathfrak{p}"] \\ (N \otimes_A B)_\mathfrak{p} \arrow[r, Rightarrow, "\phi"'] & N_{\varphi^{-1}\mathfrak{p}} \otimes_{A_{\varphi^{-1}\mathfrak{p}}} B_\mathfrak{p} \end{tikzcd}$$

$\underline{\text{NOTE}}$ Extending isomorphisms from a basis

Let X be a topological space with basis $\{U_i\}_{i\in I}$ (i.e. a nonempty collection of open U_i s.t. $x \in \overset{\text{open}}{U} \subseteq X \Rightarrow \exists i\ x \in U_i \subseteq U$) and let $\mathcal{F}, \mathcal{G}$ be two sheaves on X (of sets, groups, rings or even $\mathcal{O}_X$-modules for some sheaf of rings $\mathcal{O}_X$). Suppose for all $i \in I$ we are given a morphism $\varphi_i : \mathcal{F}(U_i) \longrightarrow \mathcal{G}(U_i)$ (of sets, groups, rings or modules). We claim there exists a unique morphism $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ (of sheaves of sets, rings, groups, modules) with $\varphi|_{U_i} = \varphi_i$, provided the φ_i are <u>natural</u>: that is, if $U_i \subseteq U_j$ then $\varphi_j(x)|_{U_i} = \varphi_i(x|_{U_i})$.

Let $U \subseteq X$ and $s \in \mathcal{F}(U)$ be given. Then $\{\varphi_i(s|_{U_i}) \mid U_i \subseteq U\}$ is a matching family since the φ_i are natural. Hence there is unique $\varphi_U(s) \in \mathcal{G}(U)$ with $\varphi_U(s)|_{U_i} = \varphi_i(s|_{U_i})\ \forall U_i \subseteq U$. It is clear that $\forall U \subseteq V\ \varphi_V(s)|_U = \varphi_U(s|_U)$. It is also clear that if the φ_i are morphisms of sets, rings, groups or modules then so is φ. The uniqueness of φ is clear.

Finally, if each φ_i is bijective, then so is φ_U for all $U \subseteq X$. So an isomorphism on a basis extends to an isomorphism on the whole space. Uniqueness means that if $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ is a morphism and φ_U bijective $\forall U$ in a basis, then φ is an isomorphism.

In particular, let $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ be a morphism of sheaves, $X = \dot{\cup}_i U_i$ an open cover. If $\varphi|_{U_i} : \mathcal{F}|_{U_i} \longrightarrow \mathcal{G}|_{U_i}$ is an isomorphism of sheaves on U_i for all i, then φ is an isomorphism since the set of open sets which are contained in some U_i is a basis.

Now that we have the general notion of a sheaf of modules on a ringed space, we specialise to the case of schemes. We start by defining the sheaf of modules $\tilde{M}$ on $\operatorname{Spec} A$ associated to a module M over a ring A.

<u>DEFINITION</u> Let A be a ring and let M be an A-module. We define the <u>sheaf associated</u> to M on $\operatorname{Spec} A$, denoted by $\tilde{M}$, as follows. For each prime ideal $\mathfrak{p} \cdot A$, let $M_\mathfrak{p}$ be the localisation of M at $\mathfrak{p}$. For any open set $U \subseteq \operatorname{Spec} A$ we define the group $\tilde{M}(U)$ to be the set of functions $s : U \longrightarrow \coprod_{\mathfrak{p} \in U} M_\mathfrak{p}$ such that for each $\mathfrak{p} \in U$, $s(\mathfrak{p}) \in M_\mathfrak{p}$ and such that s is locally a fraction m/f with $m \in M$ and $f \in A$. To be precise, we require that for each $\mathfrak{p} \in U$ there is a neighborhood V of $\mathfrak{p}$ in U, and there are elements $m \in M$ and $f \in A$, such that for each $\mathfrak{q} \in V$, $f \notin \mathfrak{q}$, and $s(\mathfrak{q}) = m/f$ in $M_\mathfrak{q}$. We make $\tilde{M}$ into a sheaf of abelian groups using the obvious structure maps.

<u>PROPOSITION 5.1</u> Let A be a ring, let M be an A-module, and let $\tilde{M}$ be the sheaf on $X = \operatorname{Spec} A$ associated to M. Then

(a) $\tilde{M}$ is a $\mathcal{O}_X$-module

(b) For each $\mathfrak{p} \in X$ the stalk $(\tilde{M})_\mathfrak{p}$ of the sheaf $\tilde{M}$ at $\mathfrak{p}$ is isomorphic as an abelian group to $M_\mathfrak{p}$. (and as an $A_\mathfrak{p}$-module)

(c) For any $f \in A$, the A_f-module $\tilde{M}(D(f))$ is isomorphic as an A_f-module to M_f.

(d) In particular $\Gamma(X, \tilde{M}) \cong M$ as an A-module.

<u>PROOF</u> Let $U \subseteq X$ be open and let $m \in \tilde{M}(U)$, $r \in \mathcal{O}_X(U)$ be given. For $\mathfrak{p} \in X$, $M_\mathfrak{p}$ is an $A_\mathfrak{p}$-module in the canonical way, and we define

$$(r \cdot m)(\mathfrak{p}) = r(\mathfrak{p}) \cdot m(\mathfrak{p})$$

It is easily checked that this definition makes $\tilde{M}$ into an $\mathcal{O}_X$-module.

(b) For $\mathfrak{p} \in X$ define $(\tilde{M})_\mathfrak{p} \longrightarrow M_\mathfrak{p}$ by mapping (V, m) to $m(\mathfrak{p})$. This is a morphism of abelian groups, which is an isomorphism by the same arguments used in Prop 2.2a). This isomorphism identifies the $\mathcal{O}_{X,\mathfrak{p}}$-module structure on $(\tilde{M})_\mathfrak{p}$ with the $A_\mathfrak{p}$-module structure on $M_\mathfrak{p}$ in a way compatible with $\mathcal{O}_{X,\mathfrak{p}} \cong A_\mathfrak{p}$. That is, if we make $(\tilde{M})_\mathfrak{p}$ into an $A_\mathfrak{p}$-module, $(\tilde{M})_\mathfrak{p} \longrightarrow M_\mathfrak{p}$ is an isomorphism of $A_\mathfrak{p}$-modules.

(c) Define $\psi : M_f \longrightarrow \tilde{M}(D(f))$ by mapping a/f^n to the section $\mathfrak{p} \longmapsto a/f^n$. This is clearly a morphism of abelian groups. The proof of Prop 2.2 shows that it is a bijection. When we make $\tilde{M}(D(f))$ into an A_f-module via $A_f \cong \mathcal{O}_X(D(f))$ it is clear ψ is an isomorphism of A_f-modules. Note that the diagram commutes:

$$\begin{array}{ccc} M & \Longrightarrow & \tilde{M}(X) \\ \downarrow & & \downarrow \\ M_f & \Longrightarrow & \tilde{M}(D(f)) \end{array}$$

(d) Clear from (c). $\square$

<u>NOTE</u> (b) shows that if two maps $s, t \in \tilde{M}(U)$ agree at $\mathfrak{p} \in U$, they agree on a neighborhood of $\mathfrak{p}$.

<u>PROPOSITION 5.2</u> Let A be a ring and $X = \operatorname{Spec} A$. Also let $f : \operatorname{Spec} B \longrightarrow \operatorname{Spec} A$ be induced from a ring morphism $\varphi : A \longrightarrow B$. Then:

(a) The map $M \longmapsto \tilde{M}$ gives an exact, fully faithful functor from the category of A-modules to the category of $\mathcal{O}_X$-modules.

(b) If M, N are A-modules then $(M \otimes_A N)^\sim \cong \tilde{M} \otimes_{\mathcal{O}_X} \tilde{N}$

(c) If $\{M_i\}$ is any family of A-modules, then $(\bigoplus_i M_i)^\sim \cong \bigoplus_i \tilde{M}_i$

(d) For any B-module N, we have $f_*(\tilde{N}) \cong ({}_A N)^\sim$ where ${}_A N$ means N considered as an A-module

(e) For any A-module M we have $f^*(\tilde{M}) \cong (M \otimes_A B)^\sim$

<u>PROOF</u> If $\alpha : M \longrightarrow N$ is a morphism of A-modules, then $\tilde{\alpha} : \tilde{M} \longrightarrow \tilde{N}$ defined by $\tilde{\alpha}_U(s)(\mathfrak{p}) = \alpha_\mathfrak{p}(s(\mathfrak{p}))$, where $\alpha_\mathfrak{p} : M_\mathfrak{p} \longrightarrow N_\mathfrak{p}$ is canonical, is a morphism of $\mathcal{O}_X$-modules. Clearly $\tilde{1} = 1$ and $\widetilde{\alpha\beta} = \tilde{\alpha}\tilde{\beta}$ so $\sim$ is an additive functor A-<u>Mod</u> $\longrightarrow$ $\mathcal{O}_X$-<u>Mod</u>. Since $\tilde{\alpha}_X(\dot{m})(\mathfrak{p}) = \alpha(m)/1$, it follows that $\sim$ is faithful (since if $m, n \in M$ agree in $M_\mathfrak{p}$ $\forall \mathfrak{p}$, then $\operatorname{Ann}(m-n)$ is improper and so $m = n$). To see that $\sim$ is full, let $\kappa : \tilde{M} \longrightarrow \tilde{N}$ be a morphism of $\mathcal{O}_X$-modules. Induce $\alpha : M \longrightarrow N$ via

$$\begin{array}{ccc} M & \xrightarrow{\alpha} & N \\ \cong & & \cong \\ \tilde{M}(X) & \xrightarrow[f_X]{} & \tilde{N}(X) \end{array}$$

For each $\mathfrak{p}$ it is clear that $K_\mathfrak{p} : (\tilde{M})_\mathfrak{p} \longrightarrow (\tilde{N})_\mathfrak{p}$ is a morphism of $\mathcal{O}_{X,\mathfrak{p}}$-modules, hence of $A_\mathfrak{p}$-modules. All sides in the following diagram commute, save possibly the bottom face:

$$\begin{tikzcd}
M \arrow[r, "\alpha"] \arrow[d, "t"'] \arrow[drr, Rightarrow] & N \arrow[d] \arrow[drr, Rightarrow] & & \\
M_\mathfrak{p} \arrow[r, "\alpha_\mathfrak{p}"] \arrow[drr, Rightarrow] & N_\mathfrak{p} \arrow[drr, Rightarrow] & \tilde{M}(X) \arrow[r, Rightarrow, "K_X"] \arrow[d] & \tilde{N}(X) \arrow[d] \\
& & \tilde{M}_\mathfrak{p} \arrow[r, "K_\mathfrak{p}"'] & \tilde{N}_\mathfrak{p}
\end{tikzcd}$$

Since the bottom diagram consists of morphisms of $A_\mathfrak{p}$-modules and both legs agree on $\operatorname{Im}(M \longrightarrow M_\mathfrak{p})$, it follows (since $m/f = 1/f \cdot m/1$) that the bottom diagram commutes also. Hence for $U \subseteq X$ and $s \in (\tilde{M})(U)$ we have $K_U(s)(\mathfrak{p}) = \alpha_\mathfrak{p}(s(\mathfrak{p}))$, so $K = \tilde{\alpha}$ as required. Hence $\sim$ is full.

To see that $\tilde{\ }$ is exact, we use the fact that a sequence of $\mathcal{O}_X$-modules is exact iff. it is exact as a sequence of sheaves of abelian groups, which by Ex 1.2 is iff. the sequences of stalks are exact. But if $M \xrightarrow{\phi} N \xrightarrow{\psi} L$ is an exact sequence of A-modules, for any $\mathfrak{p} \in \operatorname{Spec} A$ $M_\mathfrak{p} \xrightarrow{\phi_\mathfrak{p}} N_\mathfrak{p} \xrightarrow{\psi_\mathfrak{p}} L_\mathfrak{p}$ is exact and

$$\begin{tikzcd}
(\tilde{M})_\mathfrak{p} \arrow[r, "\tilde{\phi}_\mathfrak{p}"] \arrow[d, phantom, "\cong" rotate=90] & (\tilde{N})_\mathfrak{p} \arrow[r, "\tilde{\psi}_\mathfrak{p}"] \arrow[d, phantom, "\cong" rotate=90] & (\tilde{L})_\mathfrak{p} \arrow[d, phantom, "\cong" rotate=90] \\
M_\mathfrak{p} \arrow[r, "\phi_\mathfrak{p}"'] & N_\mathfrak{p} \arrow[r, "\psi_\mathfrak{p}"'] & L_\mathfrak{p}
\end{tikzcd}$$

commutes, so $\tilde{M} \xrightarrow{\tilde{\phi}} \tilde{N} \xrightarrow{\tilde{\psi}} \tilde{L}$ is exact on stalks and thus exact. Hence $\sim$ is an exact functor, and thus preserves limits and colimits of all finite diagrams.

Next we show that $\sim$ preserves coproducts. Let modules $\{M_i\}_{i \in I}$ be given, and let $u_i : M_i \longrightarrow \oplus_i M_i$ be the coproduct. Let $v_i : \tilde{M}_i \longrightarrow \oplus_i \tilde{M}_i$ be the coproduct in $\mathcal{O}_X$-Mod, so $\oplus_i \tilde{M}_i$ is the sheafification of the presheaf $P(U) = \oplus_i \tilde{M}_i(U)$. Note that $P_\mathfrak{p} \cong \oplus_i (M_i)_\mathfrak{p}$ via $(U, (s_i)) \longmapsto (s_i(\mathfrak{p}))$ as abelian groups. There is a morphism Y of $\mathcal{O}_X$-modules unique making the following diagram commute for all i:

$$\begin{tikzcd}
\tilde{M}_i \arrow[r, "\tilde{u}_i"] \arrow[dr, "v_i"'] & \widetilde{\oplus_i M_i} \\
& \oplus_i \tilde{M}_i \arrow[u, "Y"']
\end{tikzcd}$$

To show that $\{\tilde{u}_i\}$ is a coproduct we need only show Y is an isomorphism, and by Prop 1.1 it suffices to show $Y_\mathfrak{p}$ is an isomorphism of abelian groups for all $\mathfrak{p}$. But $(\oplus_i \tilde{M}_i)_\mathfrak{p} \cong P_\mathfrak{p} \cong \oplus_i M_{i\mathfrak{p}}$ and $(\widetilde{\oplus_i M_i})_\mathfrak{p} \cong (\oplus_i M_i)_\mathfrak{p}$ and the following diagram commutes

$$\begin{tikzcd}
\widetilde{\oplus_i M_i}_\mathfrak{p} \arrow[r, Rightarrow] & (\oplus_i M_i)_\mathfrak{p} & \sum_i \frac{u_i(m_i)}{s_i} \\
(\oplus_i \tilde{M}_i)_\mathfrak{p} \arrow[r, Rightarrow] \arrow[u, "Y_\mathfrak{p}"] & \oplus_i M_{i\mathfrak{p}} \arrow[u, Rightarrow] & (m_i/s_i) \arrow[u, mapsto]
\end{tikzcd}$$

Since if $s : U \longrightarrow \bigcup_{\mathfrak{p} \in U} P_\mathfrak{p}$ belongs to $(\oplus_i \tilde{M}_i)(U)$, let $\mathfrak{p} \in V \subseteq U$ and $(t_i) \in P(V)$ be s.t. $s|_V = \dot{(t_i)}$. Then the bottom maps (U, s) to $(t_i(\mathfrak{p}))$ and then up to $\sum_i u_i(m_i)/s_i$ where $t_i(\mathfrak{p}) = m_i/s_i$. Going around the other way, $Y_\mathfrak{p}(U, s) = Y_\mathfrak{p}(V, \dot{(t_i)}) = (V, Y_V(\dot{(t_i)}))$. But $\dot{(t_i)} = \sum_i v_i(\dot{t_i})$, so

$$\begin{aligned}
Y_\mathfrak{p}(U, s) &= (V, \textstyle\sum_i Y_V(v_{iV}(t_i))) \\
&= (V, \textstyle\sum_i \tilde{u}_{iV}(t_i))
\end{aligned}$$

which is mapped by the top row to

$$\begin{aligned}
(\textstyle\sum_i \tilde{u}_{iV}(t_i))(\mathfrak{p}) &= \textstyle\sum_i \tilde{u}_{iV}(t_i)(\mathfrak{p}) \\
&= \textstyle\sum_i (u_i)_\mathfrak{p}(t_i(\mathfrak{p})) \\
&= \textstyle\sum_i (u_i)_\mathfrak{p}(m_i/s_i) = \sum_i u_i(m_i)/s_i
\end{aligned}$$

As required. Hence for all $\mathfrak{p}$, $Y_\mathfrak{p}$ is an isomorphism, and hence Y is an isomorphism and $\sim$ preserves coproducts. Hence $\sim$ is colimit preserving.

Next we show that $(M \otimes_A N)^\sim \cong \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}$. For this we need to establish a pair of isomorphisms:

$$\widetilde{M}_{\mathfrak{p}} \otimes_{\mathcal{O}_{X,\mathfrak{p}}} \widetilde{N}_{\mathfrak{p}} \xrightarrow{\sim} M_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} N_{\mathfrak{p}} \xrightarrow{\sim} (M \otimes_A N)_{\mathfrak{p}}$$

The second isomorphism is elementary, given by $m/s \otimes n/t \longmapsto m \otimes n/st$. To define the first map, consider

$$\varepsilon : \widetilde{M}_{\mathfrak{p}} \times \widetilde{N}_{\mathfrak{p}} \longrightarrow M_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} N_{\mathfrak{p}}$$
$$\varepsilon((U,s),(V,t)) = s(\mathfrak{p}) \otimes t(\mathfrak{p})$$

This is easily checked to be $\mathcal{O}_{X,\mathfrak{p}}$-bilinear. If K is an abelian group and $\Upsilon : \widetilde{M}_{\mathfrak{p}} \times \widetilde{N}_{\mathfrak{p}} \longrightarrow K$ is $\mathcal{O}_{X,\mathfrak{p}}$-bilinear, then in the usual way we define $\phi : M_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} N_{\mathfrak{p}} \longrightarrow K$ by

$$\phi(m/s \otimes n/t) = \Upsilon((D(s), (\dot{m/s})), (D(t), (\dot{n/t})))$$

It is clear that $\phi\varepsilon = \Upsilon$ and ϕ is unique, so ε is a tensor product and there is an isomorphism of groups

$$\begin{tikzcd}
 & \widetilde{M}_{\mathfrak{p}} \times \widetilde{N}_{\mathfrak{p}} \arrow[dl, "\varepsilon"'] \arrow[dr] & \\
M_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} N_{\mathfrak{p}} & & \widetilde{M}_{\mathfrak{p}} \otimes_{\mathcal{O}_{X,\mathfrak{p}}} \widetilde{N}_{\mathfrak{p}} \arrow[ll, "Y_{\mathfrak{p}}"]
\end{tikzcd}$$

$$Y_{\mathfrak{p}}((U,s) \otimes (V,t)) = s(\mathfrak{p}) \otimes t(\mathfrak{p})$$

Note that for $r \in \mathcal{O}_X(Q)$, $Y_{\mathfrak{p}}((Q,r)\cdot(U,s) \otimes (V,t)) = r(\mathfrak{p}) \cdot (s(\mathfrak{p}) \otimes t(\mathfrak{p})) = r(\mathfrak{p}) \cdot Y_{\mathfrak{p}}((U,s) \otimes (V,t))$.
For $\mathfrak{p} \in \operatorname{Spec} A$ let $\delta_{\mathfrak{p}} : \widetilde{M}_{\mathfrak{p}} \otimes_{\mathcal{O}_{X,\mathfrak{p}}} \widetilde{N}_{\mathfrak{p}} \longrightarrow (M \otimes_A N)_{\mathfrak{p}}$ be the isomorphism just constructed. This isomorphism maps the action of $\mathcal{O}_{X,\mathfrak{p}}$ to the action of $A_{\mathfrak{p}}$.

Let P be the presheaf giving rise to $\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}$ and for $\mathfrak{p} \in \operatorname{Spec} A$, $\alpha_{\mathfrak{p}} : P_{\mathfrak{p}} \longrightarrow \widetilde{M}_{\mathfrak{p}} \otimes_{\mathcal{O}_{X,\mathfrak{p}}} \widetilde{N}_{\mathfrak{p}}$ the isomorphism established in an earlier note. We define

$$\eta : \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N} \longrightarrow (M \otimes_A N)^\sim$$
$$\eta_U : (\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N})(U) \longrightarrow (M \otimes_A N)^\sim(U)$$
$$\eta_U(s)(\mathfrak{p}) = \delta_{\mathfrak{p}} \alpha_{\mathfrak{p}}(s(\mathfrak{p}))$$

⌜If $s(\mathfrak{p}) = (U, m \otimes n)$ and $m(\mathfrak{p}) = m/f$ $n(\mathfrak{p}) = n/g$ then $\eta_U(s)(\mathfrak{p}) = \frac{m \otimes n}{fg}$⌟

It is not difficult to check η is a morphism of $\mathcal{O}_X$-modules. For each $\mathfrak{p}$ $\delta_{\mathfrak{p}}\alpha_{\mathfrak{p}} : P_{\mathfrak{p}} \longrightarrow (M \otimes_A N)_{\mathfrak{p}}$ is an isomorphism, hence η_U is injective. We define the inverse by

$$\eta'_U(t)(\mathfrak{p}) = (D(t), (\dot{\tfrac{m}{1}}) \otimes (\dot{\tfrac{n}{t}}))$$

$t \in (M \otimes_A N)^\sim(U)$, $t(\mathfrak{p}) = \frac{m \otimes n}{t}$

Then $\eta'_U(t)$ is clearly regular and $\eta_U(\eta'_U(t)) = t$, so η is an isomorphism of $\mathcal{O}_X$-modules. We claim that this isomorphism is natural in M and N. Let $\phi : M \longrightarrow M'$ be a morphism of A-modules. Then the following diagram commutes:

$$\begin{tikzcd}
\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N} \arrow[r, Rightarrow, "\eta"] \arrow[d, "\widetilde{\phi} \otimes \widetilde{N}"'] & (M \otimes_A N)^\sim \arrow[d, "(\phi \otimes N)^\sim"] \\
\widetilde{M'} \otimes_{\mathcal{O}_X} \widetilde{N} \arrow[r, Rightarrow, "\eta'"] & (M' \otimes_A N)^\sim
\end{tikzcd}$$

For $U \subseteq X$ and $s \in (\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N})(U)$ and $\mathfrak{p} \in U$ suppose $s(\mathfrak{p}) = (U, m_i \otimes n^i)$ and $m_i(\mathfrak{p}) = m_i/f_i$, $n^i(\mathfrak{p}) = n^i/g_i$ Then

sum over repeated indices,

$$\begin{aligned}
(\phi \otimes N)^{\sim}_U \eta_U(s)(\mathfrak{p}) &= (\phi \otimes N)^{\sim}_U (\eta_U(s))(\mathfrak{p}) \\
&= (\phi \otimes N)_{\mathfrak{p}} \delta_{\mathfrak{p}} \alpha_{\mathfrak{p}} (s(\mathfrak{p})) \\
&= (\phi \otimes N)_{\mathfrak{p}} (m_i \otimes n^i / f_i g_i) \\
&= \phi(m_i) \otimes n^i / f_i g_i
\end{aligned}$$

$$\begin{aligned}
\eta'_U (\widetilde{\phi} \otimes \widetilde{N})_U (s)(\mathfrak{p}) &= \delta_{\mathfrak{p}} \alpha_{\mathfrak{p}} (\hat{K}_{\mathfrak{p}} (s(\mathfrak{p}))) \\
&= \delta_{\mathfrak{p}} \alpha_{\mathfrak{p}} (U, \hat{K}_U (m_i \otimes n^i)) \\
&= \delta_{\mathfrak{p}} \alpha_{\mathfrak{p}} (U, \widetilde{\phi}_U (m_i) \otimes n^i) \\
&= \delta_{\mathfrak{p}} ((U, \widetilde{\phi}_U(m)) \otimes (U, n)) \\
&= \phi(m_i) \otimes n^i / f_i g_i
\end{aligned}$$

$\hat{K}_U : \widetilde{M}(U) \otimes \widetilde{N}(U) \longrightarrow \widetilde{M}'(U) \otimes \widetilde{N}(U) = \widetilde{\phi} \otimes 1$

$\widetilde{\phi}_U(m)(\mathfrak{p}) = \phi_{\mathfrak{p}}(m(\mathfrak{p}))$

Similarly if $\psi : N \longrightarrow N'$ is a morphism of A-modules,

$$\begin{tikzcd}
\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N} \arrow[r, Rightarrow, "\eta"] \arrow[d, "\widetilde{M} \otimes \widetilde{\psi}"'] & (M \otimes_A N)^{\sim} \arrow[d, "(M \otimes \psi)^{\sim}"] \\
\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N'} \arrow[r, Rightarrow, "\eta'"'] & (M \otimes_A N')^{\sim}
\end{tikzcd}$$

(d) The ring morphism $\varphi : A \longrightarrow B$ and the induced $f : \operatorname{Spec} B \longrightarrow \operatorname{Spec} A$ induce functors; where $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$:

$$\begin{tikzcd}
\mathcal{O}_Y\text{-}\underline{\mathrm{Mod}} \arrow[r, bend left, "f_*"] & \mathcal{O}_X\text{-}\underline{\mathrm{Mod}} \arrow[l, bend left, "f^*"] \\
\underline{\mathrm{Mod}\,B} \arrow[u, "\sim"] \arrow[r, bend left, "{}_A-"] & \underline{\mathrm{Mod}\,A} \arrow[u] \arrow[l, bend left, "- \otimes_A B"]
\end{tikzcd} \qquad (1)$$

where ${}_A-$ is restriction of scalars and $- \otimes_A B$ is extension of scalars. We claim both squares in (1) commute (up to natural equivalence). Firstly we define an isomorphism of $\mathcal{O}_X$-modules $({}_AN)^{\sim} \longrightarrow f_*(\widetilde{N})$ for any B-module N. For a prime $\mathfrak{q} \subseteq B$ we define

$$\begin{aligned}
Z_{\mathfrak{q}} : ({}_AN)_{\varphi^{-1}\mathfrak{q}} &\longrightarrow N_{\mathfrak{q}} \\
n/s &\longmapsto n/\varphi(s)
\end{aligned}$$

This is a morphism of abelian groups and for $a/s \in A_{\varphi^{-1}\mathfrak{q}}$ we have

$$Z_{\mathfrak{q}}(a/s \cdot n/t) = \varphi_{\mathfrak{q}}(a/s) \cdot Z_{\mathfrak{q}}(n/t)$$

Define $\eta : ({}_AN)^{\sim} \longrightarrow f_*(\widetilde{N})$ as follows: for $V \subseteq X$ and $s \in ({}_AN)^{\sim}(V)$ we define for $\mathfrak{q} \in f^{-1}V$

$$\eta_V(s)(\mathfrak{q}) = Z_{\mathfrak{q}}(s(\varphi^{-1}\mathfrak{q}))$$

This is regular, so $\eta_V(s) \in f_*(\widetilde{N})(V)$. η_V is clearly a morphism of groups, and to see η is a morphism of $\mathcal{O}_X$-modules let $r \in \mathcal{O}_X(V)$ be given and suppose $r(\varphi^{-1}\mathfrak{q}) = a/s \in A_{\varphi^{-1}\mathfrak{q}}$, $s(\varphi^{-1}\mathfrak{q}) = n/t$. Then

$$\begin{aligned}
\eta_V(r \cdot s)(\mathfrak{q}) &= Z_{\mathfrak{q}}((r \cdot s)(\varphi^{-1}\mathfrak{q})) = Z_{\mathfrak{q}}(r(\varphi^{-1}\mathfrak{q}) \cdot s(\varphi^{-1}\mathfrak{q})) \\
&= \varphi_{\mathfrak{q}}(r(\varphi^{-1}\mathfrak{q})) \cdot Z_{\mathfrak{q}}(s(\varphi^{-1}\mathfrak{q})) = f^{\#}_V(r)(\mathfrak{q}) \cdot \eta_V(s)(\mathfrak{q}) \\
&= (f^{\#}_V(r) \cdot \eta_V(s))(\mathfrak{q}) = (r \cdot \eta_V(s))(\mathfrak{q}).
\end{aligned}$$

Since the $\mathcal{O}_X$-module structure on $f_*(\tilde{N})$ comes via $f^\#: \mathcal{O}_X \to f_*\mathcal{O}_Y$. It only remains to show that η is an isomorphism. Since the open sets $D(f)$, $f \in A$ are a basis for $\operatorname{Spec} A$, it suffices to show that $\eta_{D(f)}$ is an isomorphism for all $f \in A$. But for $f \in A$, $({}_AN)_f \to N_{\varphi(f)}$ defined by $n/f^m \mapsto n/\varphi(f)^m$ is an isomorphism of groups, and the following diagram commutes: (by (5.1)c, the vertical maps are iso)

$$\begin{array}{ccc}
({}_AN)^\sim(D(f)) & \xrightarrow{\eta_{D(f)}} & (f_*\tilde{N})(D(f)) \\
\| & & \| \\
({}_AN)_f & \Longrightarrow & N_{\varphi(f)}
\end{array}$$

This completes the proof that $f_*(\tilde{N}) \cong ({}_AN)^\sim$.

(e) Let M be an A-module. We claim that $f^*(\tilde{M}) \cong (M \otimes_A B)^\sim$. Let Z be the presheaf $Z(U) = (f^{-1}\tilde{M})(U) \otimes_{(f^{-1}\mathcal{O}_X)(U)} \mathcal{O}_Y(U)$. We showed in an earlier note that there is an isomorphism of groups

$$\begin{array}{ccc}
Z_{\mathfrak{p}} & \longrightarrow & \tilde{M}_{\varphi^{-1}\mathfrak{p}} \otimes_{\mathcal{O}_{X,\varphi^{-1}\mathfrak{p}}} \mathcal{O}_{Y,\mathfrak{p}} \\
(U, a \otimes b) & & \\
a \in (f^{-1}\tilde{M})(U) & \longmapsto & (W,t) \otimes (U,b) \\
a(\mathfrak{p}) = (Q, (W,t)) & & \\
t \in \tilde{M}(W),\ W \supseteq f(Q) & &
\end{array}$$

Using the isomorphism of rings $\mathcal{O}_{X,\varphi^{-1}\mathfrak{p}} \cong A_{\varphi^{-1}\mathfrak{p}}$ we turn $\tilde{M}_{\varphi^{-1}\mathfrak{p}}$, $\mathcal{O}_{Y,\mathfrak{p}}$ into $A_{\varphi^{-1}\mathfrak{p}}$ modules. Then $\tilde{M}_{\varphi^{-1}\mathfrak{p}} \cong M_{\varphi^{-1}\mathfrak{p}}$ and $\mathcal{O}_{Y,\mathfrak{p}} \cong B_{\mathfrak{p}}$ as $A_{\varphi^{-1}\mathfrak{p}}$-modules ($B_{\mathfrak{p}}$ becomes an $A_{\varphi^{-1}\mathfrak{p}}$-module via $\varphi_{\mathfrak{p}}: A_{\varphi^{-1}\mathfrak{p}} \to B_{\mathfrak{p}}$, and $\mathcal{O}_{Y,\mathfrak{p}} \to B_{\mathfrak{p}}$ is an isomorphism of $A_{\varphi^{-1}\mathfrak{p}}$-modules since $\mathcal{O}_{X,\varphi^{-1}\mathfrak{p}} \to \mathcal{O}_{Y,\mathfrak{p}} \xrightarrow{\sim} B_{\mathfrak{p}} = \mathcal{O}_{X,\varphi^{-1}\mathfrak{p}} \xrightarrow{\sim} A_{\varphi^{-1}\mathfrak{p}} \to B_{\mathfrak{p}}$). So there is an iso of groups

$$\begin{array}{ccc}
\tilde{M}_{\varphi^{-1}\mathfrak{p}} \otimes_{\mathcal{O}_{X,\varphi^{-1}\mathfrak{p}}} \mathcal{O}_{Y,\mathfrak{p}} & \longrightarrow & M_{\varphi^{-1}\mathfrak{p}} \otimes_{A_{\varphi^{-1}\mathfrak{p}}} B_{\mathfrak{p}} \\
(W,t) \otimes (U,b) & \longmapsto & t(\varphi^{-1}\mathfrak{p}) \otimes b(\mathfrak{p})
\end{array}$$

There is another note which shows that $M_{\varphi^{-1}\mathfrak{p}} \otimes_{A_{\varphi^{-1}\mathfrak{p}}} B_{\mathfrak{p}} \xrightarrow{\sim} (M \otimes_A B)_{\mathfrak{p}}$ via $m/s \otimes b/t \mapsto m \otimes b/\varphi(s)t$. So finally there is an isomorphism of groups

$$\begin{array}{ccc}
T_{\mathfrak{p}} : Z_{\mathfrak{p}} & \longrightarrow & (M \otimes_A B)_{\mathfrak{p}} \\
(U, a \otimes b) & & \\
a \in (f^{-1}\tilde{M})(U) & & \\
b \in \mathcal{O}_Y(U) & \longmapsto & \dfrac{m \otimes b}{\varphi(s)t} \\
a(\mathfrak{p}) = (Q, (W,t)) & & \\
t \in \tilde{M}(W),\ t(\varphi^{-1}\mathfrak{p}) = m/s & & \\
b(\mathfrak{p}) = b/t & &
\end{array}$$

of course this gives only the values on generators $a \otimes b$. It is easy enough to extend.

We define

$$\eta: f^*(\tilde{M}) \longrightarrow (M \otimes_A B)^\sim$$
$$\eta_V(s)(\mathfrak{p}) = T_{\mathfrak{p}}(s(\mathfrak{p}))$$

Given $s \in f^*(\tilde{M})(V)$ our first task is to show that $\eta_V(s)$ is regular. Let $\mathfrak{p} \in V$ be given and let U be an open neighborhood of $\mathfrak{p}$ and $a_i \otimes b^i \in Z(U)$ s.t. $s|_U = (a_i \otimes b^i)$ (sum over i). Say $a_i \in (f^{-1}\tilde{M})(U)$ and $b^i \in \mathcal{O}_Y(U)$ admit neighborhoods M_i and M'_i of $\mathfrak{p}$ in U and $(W_i, t_i) \in P(M_i)$, $c_i, q_i \in B$ such that $q_i \notin \mathfrak{q}$ $\forall \mathfrak{q} \in M'_i$ and $a_i|_{M_i} = (W_i, t_i)$ and $b^i|_{M'_i} = (c_i/q_i)$ for all i. Here $P(M_i) = \varinjlim_{W \supseteq f(M_i)} \tilde{M}(W)$ so $t_i \in \tilde{M}(W_i)$. Finally let Z_i be an open neighborhood of $f(\mathfrak{p})$ in W_i and $m_i \in M$, $s_i \in A$ s.t. $\forall \mathfrak{q} \in Z_i$ $s_i \notin \mathfrak{q}$ and $t_i|_{Z_i} = (m_i/s_i)$. Since there are a finite number of indices, form the open neighborhood of $\mathfrak{p}$: $T = \bigcap_i M_i \cap \bigcap_i M'_i \cap \bigcap_i f^{-1}Z_i$.

Then for $q \in T$ we have

$$\begin{aligned} \eta_V(s)(q) &= T_q(s(q)) \\ &= T_q(U, a_i \otimes b^i) \\ &= \sum_i \frac{m_i \otimes c_i}{\varphi(s_i) q_i} \end{aligned}$$

This shows that $\eta_V(s)$ is regular. It is easy to see η is a morphism of sheaves of abelian groups. To see it is a morphism of $\mathcal{O}_Y$-modules let $s \in f^*\tilde{M}(V)$, $r \in \mathcal{O}_Y(V)$ be given, with $p \in V$. Say

$$\begin{aligned} r(p) &= d/y \in B_p \\ s(p) &= (U, a_i \otimes b^i) \\ a_i(p) &= (Q_i, (W_i, t_i)) \\ &\qquad t_i(\varphi^{-1}p) = m_i/s_i \in M_{\varphi^{-1}p} \\ b^i(p) &= c_i/q_i \in B_p \end{aligned}$$

Then

$$\begin{aligned} \eta_V(r \cdot s)(p) &= T_p((r \cdot s)(p)) \\ &= T_p((V, r) \cdot s(p)) \\ &= T_p(V \cap U, r|_{V \cap U} \cdot (a_i|_{V\cap U} \otimes b^i|_{V \cap U})) \\ &= T_p(V \cap U, a_i|_{V \cap U} \otimes b^i|_{V\cap U} r|_{V \cap U}) \\ &= \sum_i \frac{m_i \otimes d c_i}{\varphi(s_i) y q_i} \\ &= d/y \cdot \sum_i \frac{m_i \otimes c_i}{\varphi(s_i) q_i} \\ &= r(p) \cdot \eta_V(s)(p) = (r \cdot \eta_V(s))(p) \end{aligned}$$

Since each T_p is an isomorphism it is clear that η_V is injective. To see that it is surjective, notice that T_p^{-1} is defined by linear extension of:

$$T_p^{-1}\left(\frac{m \otimes b}{s}\right) = \left(D(s), (X, \tfrac{\dot{m}}{1}) \otimes \tfrac{\dot{b}}{s}\right) \qquad \begin{aligned} &\tfrac{\dot{m}}{1} \in \tilde{M}(X) \quad \tfrac{\dot{b}}{s} \in \mathcal{O}_Y(D(s)) \\ &(X, \tfrac{\dot{m}}{1}) \in P(D(s)) \\ &(X, \tfrac{\dot{m}}{1}) \in (f^{-1}\tilde{M})(D(s)) \end{aligned}$$

Given $e \in (M \otimes_A B)^\sim(V)$ define $e' \in f^*\tilde{M}(V)$ by $e'(p) = T_p^{-1}(e(p))$. Since it is clear that $\eta_V(e') = e$ it only remains to show e' is regular. But if $p \in V$ let $p \in U \subseteq V$ and $m_i \otimes b^i \in M \otimes_A B$, and $s \in B$ be s.t. $s \notin q$ $\forall q \in U$ and $e(q) = m_i \otimes b^i/s$ $\forall q \in U$. Then for $q \in U$

$$\begin{aligned} e'(q) &= T_p^{-1}(m_i \otimes b^i/s) \\ &= \left(D(s), \textstyle\sum_i (X, \tfrac{\dot{m}}{1}) \otimes \tfrac{\dot{b}_i}{s}\right) \\ &= \left(U, \textstyle\sum_i (X, \tfrac{\dot{m}}{1})|_U \otimes \tfrac{\dot{b}_i}{s}|_U\right) \end{aligned}$$

This completes the proof that η is an isomorphism of $\mathcal{O}_Y$-modules.

<u>NATURALITY</u> Finally we show the isomorphisms of (d) and (e) are natural. Let $\phi : N \longrightarrow N'$ be a morphism of B-modules. We claim the following diagram commutes:

$$\begin{array}{ccc} (AN)^\sim & \xrightarrow{\ \eta\ } & f_*(\tilde{N}) \\ {\scriptstyle\tilde{\phi}}\downarrow & & \downarrow{\scriptstyle f_*\tilde{\phi}} \\ (AN')^\sim & \xrightarrow[\ \eta\ ]{} & f_*(\tilde{N'}) \end{array}$$

Say $V \subseteq X$, $s \in (AN)^\sim(V)$, $q \in f^{-1}V$ and $s(\varphi^{-1}q) = n/t$. Then both ways 'round produce $\phi(n)/\varphi(t)$, as required.

Let $\psi : M \longrightarrow M'$ be a morphism of A-modules. We claim the following diagram commutes:

$$\begin{tikzcd} f^*(\widetilde{M}) \arrow[r, Rightarrow, "\eta"] \arrow[d, "f^*\widetilde{\psi}"'] & (M \otimes_A B)^\sim \arrow[d, "\widetilde{\psi \otimes B}"] \\ f^*(\widetilde{M'}) \arrow[r, Rightarrow, "\eta"'] & (M' \otimes_A B)^\sim \end{tikzcd}$$

Let $\widetilde{\psi} : \widetilde{M} \longrightarrow \widetilde{M'}$, $f^{-1}\widetilde{\psi} : f^{-1}\widetilde{M} \longrightarrow f^{-1}\widetilde{M'}$ be canonical, and let $V \subseteq Y$, $s \in f^*\widetilde{M}(V)$, $q \in V$ and suppose $s(q) = (U, a_i \otimes b^i)$, $b^i(q) = c_i/q_i$, $a_i(q) = (Q_i, (W_i, t_i))$, $t_i(q) = m_i/s_i$. Then

$$\begin{aligned} \eta_V (f^*\widetilde{\psi})_V(s)(q) &= \tau_q((f^*\widetilde{\psi})_U(s)(q)) \\ &= \tau_q(U, (f^{-1}\widetilde{\psi})_U(a_i) \otimes b^i) \\ &= \sum_i \frac{\psi(m_i) \otimes c_i}{\varphi(s_i) q_i} \\ &= (\psi \otimes B)_q(\tau_q(s(q))) \\ &= (\widetilde{\psi \otimes B})_q \, \eta_V(s)(q) \end{aligned}$$

as required. $\square$

<u>EXAMPLES</u> Let X be a topological space, $U \subseteq X$ open, $i : U \longrightarrow X$ the inclusion,

$$\begin{tikzcd} Sh(U) \arrow[r, bend left, "i_*"] & Sh(X) \arrow[l, bend left, "i^{-1}"] \end{tikzcd} \qquad i^{-1} \dashv i_*$$

Note that for a sheaf F on X, $i^{-1}F$ is the sheafification of the presheaf $P(V) = \varinjlim_{W \supseteq V} F(W)$ which is naturally isomorphic to $F|_U$ via $P(V) \xrightarrow{\sim} F(V)$, $(W, s) \longmapsto s|_V$. It follows that $F|_U$ is isomorphic to $i^{-1}F$ via

$$\alpha : F|_U \longrightarrow i^{-1}F$$
$$\alpha_V : F(V) \longrightarrow (i^{-1}F)(V)$$
$$\alpha_V(s)(x) = (V, (V, s))$$

If we consider instead sheaves of abelian groups or rings, α gives an isomorphism of these sheaves. For a morphism of sheaves $\phi : F \longrightarrow F'$ it is not hard to check that the following diagram commutes:

$$\begin{tikzcd} i^{-1}F \arrow[r, "i^{-1}\phi"] & i^{-1}F' \\ F|_U \arrow[u] \arrow[r, "\phi|_U"'] & F'|_U \arrow[u] \end{tikzcd}$$

So there is actually a natural equivalence of functors i^{-1}, and $-|_U$ for sheaves of sets, groups and rings. It is also easy to check that i_* is fully faithful, so the counit $\varepsilon : i^{-1} i_* \longrightarrow 1$ is an isomorphism.

<u>EXAMPLE</u> Let $\mathcal{O}_X$ be a sheaf of rings on a space X, and let $U \subseteq X$ be open. Then $\mathcal{O}_X|_U$ is a sheaf of rings and $f : \mathcal{O}_X|_U \longrightarrow \mathcal{O}_X$ given by $U \hookrightarrow X$ and $f^\# : \mathcal{O}_X \longrightarrow i_* \mathcal{O}_X|_U$, defined by restriction $\mathcal{O}_X(V) \longrightarrow \mathcal{O}_X(V \cap U)$, is a morphism of ringed spaces. Let $-|_U : \mathcal{O}_X\text{-}\underline{\text{Mod}} \longrightarrow \mathcal{O}_X|_U\text{-}\underline{\text{Mod}}$ be defined by restriction.

$$\begin{tikzcd} \mathcal{O}_X|_U\text{-}\underline{\text{Mod}} \arrow[r, bend left, "f_*"] & \mathcal{O}_X\text{-}\underline{\text{Mod}} \arrow[l, "f^*"'] \arrow[l, bend left, "-|_U"] \end{tikzcd}$$

$-|_U$ is an additive functor,

We claim that $f^* \cong -|_U$. Firstly, let $f^{-1}\mathcal{O}_X \xrightarrow{\xi} \mathcal{O}_X|_U$ be the adjoint partner of $f^\#$. We claim that the following diagram commutes:

$$\begin{tikzcd} f^{-1}\mathcal{O}_X \arrow[r, "\xi"] & \mathcal{O}_X|_U \\ \mathcal{O}_X|_U \arrow[u, "\alpha"] \arrow[ur, equal] & \end{tikzcd}$$

Let $V \subseteq U$ and $s \in \mathcal{O}_X(V)$. By definition ξ is the composite $f^{-1}\mathcal{O}_X \to f^{-1}f_*\mathcal{O}_X|_U \to \mathcal{O}_X|_U$ and it is not difficult to check (see earlier notes where we give the counit explicitly) that $\xi\alpha$ maps s to itself. But in forming the presheaf $Z(V) = (f^{-1}\mathcal{F})(V) \otimes_{(f^{-1}\mathcal{O}_X)(V)} \mathcal{O}_X|_U(V)$ which sheafifies to give $f^*\mathcal{F}$ for a $\mathcal{O}_X$-module $\mathcal{F}$, $\mathcal{O}_X|_U(V)$ becomes a $(f^{-1}\mathcal{O}_X)(V)$-module via ξ_V. Hence when we replace $(f^{-1}\mathcal{O}_X)(V)$ by $\mathcal{O}_X(V)$ we get an isomorphism of abelian groups

$$\beta_V : \mathcal{F}(V) \xrightarrow{\alpha_V} (f^{-1}\mathcal{F})(V) \longrightarrow (f^{-1}\mathcal{F})(V) \otimes_{\mathcal{O}_X(V)} \mathcal{O}_X(V)$$

$$= (f^{-1}\mathcal{F})(V) \otimes_{(f^{-1}\mathcal{O}_X)(V)} \mathcal{O}_X|_U(V) = Z(V)$$

$$\beta_V(s) = \alpha_V(s) \otimes 1$$

To see that β is an isomorphism of $\mathcal{O}_X|_U$-modules, let $r \in \mathcal{O}_X(V)$ be given. Then $\alpha_V(r\cdot s) = \alpha_V(r)\cdot\alpha_V(s)$ and

$$\begin{aligned} \beta_V(r\cdot s) &= \alpha_V(r\cdot s) \otimes 1 \\ &= \alpha_V(r)\cdot\alpha_V(s) \otimes 1 \\ &= \alpha_V(s) \otimes \alpha_V(r)\cdot 1 \\ &= \alpha_V(s) \otimes \xi_V(\alpha_V(r)) = \alpha_V(s) \otimes r = r \cdot \beta_V(s) \end{aligned}$$

Hence $\beta : \mathcal{F}|_U \to Z$ is an isomorphism of presheaves of $\mathcal{O}_X|_U$-modules, hence Z is a sheaf and $Z \to aZ = f^*\mathcal{F}$ is an isomorphism of $\mathcal{O}_X|_U$-modules. So finally we have an isomorphism of $\mathcal{O}_X|_U$-modules

$$\gamma : \mathcal{F}|_U \longrightarrow f^*\mathcal{F}$$

$$\gamma_V(s) = \overset{\bullet}{(\alpha_V(s) \otimes 1)}$$

$$\gamma_V(s)(x) = (V, \alpha_V(s) \otimes 1)$$

Given a morphism $\phi : \mathcal{F} \to \mathcal{F}'$ of $\mathcal{O}_X$-modules it is not hard to check that the following diagram commutes

$$\begin{tikzcd} \mathcal{F}|_U \arrow[r, "\gamma"] \arrow[d, "\phi|_U"'] & f^*\mathcal{F} \arrow[d, "f^*\phi"] \\ \mathcal{F}'|_U \arrow[r, "\gamma"'] & f^*\mathcal{F}' \end{tikzcd}$$

That is, the functors f^* and $-|_U$ are naturally equivalent.

<u>NOTE</u> If $\mathcal{O}_X$ is a sheaf of rings on X and $U \subseteq V$ are open subsets, $\mathcal{O}_X|_U \to \mathcal{O}_X$, $\mathcal{O}_X|_V \to \mathcal{O}_X$ and $\mathcal{O}_X|_U = (\mathcal{O}_X|_V)|_U \to \mathcal{O}_X|_V$ canonical, it is clear that the following two diagrams commute:

$$\begin{tikzcd} & \mathcal{O}_X & \\ \mathcal{O}_X|_U \arrow[ur] \arrow[rr] & & \mathcal{O}_X|_V \arrow[ul] \end{tikzcd} \qquad \begin{tikzcd} & \mathcal{O}_X\text{-}\underline{\mathrm{Mod}} \arrow[dl, "-|_U"'] \arrow[dr, "-|_V"] & \\ \mathcal{O}_X|_U\text{-}\underline{\mathrm{Mod}} & & \mathcal{O}_X|_V\text{-}\underline{\mathrm{Mod}} \arrow[ll, "-|_U"] \end{tikzcd}$$

NOTE $(gf)_* = g_* f_*$ and $f^{-1}g^{-1} \cong (gf)^{-1}$

Let $f: X \longrightarrow Y$ and $g: Y \longrightarrow Z$ be continuous maps. We have functors

$$\begin{tikzcd} Sh(X) \arrow[r, bend left, "f_*"] & Sh(Y) \arrow[l, bend left, "f^{-1}"] \arrow[r, bend left, "g_*"] & Sh(Z) \arrow[l, bend left, "g^{-1}"] \end{tikzcd}$$

It is easy enough to check that $g_* f_* = (gf)_*$. We also claim that $f^{-1}g^{-1} \cong (gf)^{-1}$. Let $\mathcal{G}$ be a sheaf on Z. Then $(gf)^{-1}\mathcal{G}$ is the sheafification of the presheaf

$$Z'(U) = \varinjlim_{W \supseteq gf(U)} \mathcal{G}(W)$$

whereas $f^{-1}g^{-1}\mathcal{G}$ is the sheafification of

$$Z(U) = \varinjlim_{V \supseteq f(U)} g^{-1}\mathcal{G}(V)$$

Given $U \subseteq X$ and $W \supseteq gf(U)$, $s \in \mathcal{G}(W)$ we have $(W,s) \in P(g^{-1}W)$ where P is the presheaf which sheafifies to give $g^{-1}\mathcal{G}$, $P(g^{-1}W) = \varinjlim_{T \supseteq gg^{-1}W} \mathcal{G}(T)$. Then $(\dot{W,s}) \in (g^{-1}\mathcal{G})(g^{-1}W)$ and thus $(g^{-1}W, (\dot{W,s}))$ belongs to $Z(U)$. This defines a map $\mathcal{G}(W) \longrightarrow Z(U)$, which is a morphism of groups or rings if $\mathcal{G}$ is a sheaf of groups or rings. Moreover these maps are compatible with the diagram defining $Z'(U)$, hence induce

$$\phi_U : Z'(U) \longrightarrow Z(U)$$
$$\phi_U(W,s) = (g^{-1}W, (\dot{W,s}))$$

which is a morphism of groups or rings as appropriate. Since ϕ_U is clearly natural in U, this gives a morphism $\phi: Z' \longrightarrow Z$. Let $x \in X$ be given. Then by an earlier note there are isomorphisms of groups

$$Y: Z'_x \Longrightarrow \mathcal{G}_{gf(x)} \qquad Y(U,(W,s)) = (W,s)$$
$$Y': Z_x \Longrightarrow (g^{-1}\mathcal{G})_{f(x)} \Longrightarrow \mathcal{G}_{gf(x)}$$
$$Y'(U,(V,m)) = (T,n) \text{ where } m(f(x)) = (Q,(T,n)) \in P_{f(x)}$$
$$n \in \mathcal{G}(T).$$

We claim that the following diagram commutes:

$$\begin{tikzcd} Z'_x \arrow[r, "\phi_x"] \arrow[d, Rightarrow] & Z_x \arrow[d, Rightarrow] \\ \mathcal{G}_{gf(x)} \arrow[dr, equal] & (g^{-1}\mathcal{G})_{f(x)} \arrow[d, Rightarrow] \\ & \mathcal{G}_{gf(x)} \end{tikzcd}$$

Since $Y'\phi_x(U,(W,s)) = Y'(U,(g^{-1}W,(\dot{W,s}))) = (W,s) = Y(U,(W,s))$. Hence ϕ_x is an isomorphism.

But for all x the following diagram commutes:

$$\begin{tikzcd} ((gf)^{-1}\mathcal{G})_x \arrow[r, "\hat{\phi}_x"] \arrow[d, Rightarrow] & (f^{-1}g^{-1}\mathcal{G})_x \arrow[d, Rightarrow] \\ Z'_x \arrow[r, "\phi_x"] & Z_x \end{tikzcd} \qquad (1)$$

where $\hat{\phi}: (gf)^{-1}\mathcal{G} \longrightarrow (f^{-1}g^{-1})\mathcal{G}$ is the sheafification of ϕ. By Prop 1.1 $\hat{\phi}$ is an isomorphism of sheaves (of sets, groups, or rings).

To see that this isomorphism is natural in $\mathcal{G}$, let $\psi:\mathcal{G}\longrightarrow\mathcal{G}'$ be a morphism of sheaves. If $Z'_{\mathcal{G}}, Z'_{\mathcal{G}'}$ sheafify to give $(gf)^{-1}\mathcal{G}$, $(gf)^{-1}\mathcal{G}'$ resp. and $Z_{\mathcal{G}}, Z_{\mathcal{G}'}$ sheafify to give $f^{-1}g^{-1}\mathcal{G}$, $f^{-1}g^{-1}\mathcal{G}'$ we have

$$\psi:\mathcal{G}\longrightarrow\mathcal{G}'$$
$$\bar{\psi}: Z'_{\mathcal{G}}\longrightarrow Z'_{\mathcal{G}'}$$
$$g^{-1}\psi: g^{-1}\mathcal{G}\longrightarrow g^{-1}\mathcal{G}'$$
$$\overline{g^{-1}\psi}: Z_{\mathcal{G}}\longrightarrow Z_{\mathcal{G}'}$$

we must show that the following diagram commutes:

$$\begin{tikzcd}
(gf)^{-1}\mathcal{G} \arrow[r, Rightarrow, "\hat{\phi}"] \arrow[d, "(gf)^{-1}\psi"'] & (f^{-1}g^{-1})\mathcal{G} \arrow[d, "f^{-1}g^{-1}\psi"] \\
(gf)^{-1}\mathcal{G}' \arrow[r, Rightarrow, "\hat{\phi}"'] & (f^{-1}g^{-1})\mathcal{G}'
\end{tikzcd}$$

But for $U\subseteq X$, $q\in(gf)^{-1}\mathcal{G}(U)$ and $x\in U$, say $q(x)\in Z'_x$ is $(U,(W,s))$. Then

$$\begin{aligned}
\hat{\phi}_U\,(gf)^{-1}\psi_U(q)(x) &= \phi_x\left((gf)^{-1}\psi_U(q)(x)\right)\\
&= \phi_x\left(\bar{\psi}_x(q(x))\right)\\
&= \phi_x\bar{\psi}_x(U,(W,s))\\
&= \phi_x(U,(W,\psi_W(s)))\\
&= (U,(g^{-1}W,(W,\dot{\psi}_W(s))))
\end{aligned}$$

$$\begin{aligned}
(f^{-1}g^{-1}\psi)_U\,\hat{\phi}_U(q)(x) &= \overline{g^{-1}\psi}_x\left(\hat{\phi}_U(q)(x)\right)\\
&= \overline{g^{-1}\psi}_x\left(\phi_x(U,(W,s))\right)\\
&= \overline{g^{-1}\psi}_x(U,(g^{-1}W,(W,s)))\\
&= (U,(g^{-1}W,(g^{-1}\psi)_{g^{-1}W}(\dot{W},s)))\\
&= (U,(g^{-1}W,(W,\dot{\psi}_W(s))))
\end{aligned}$$

as required. Hence $f^{-1}g^{-1}\cong(gf)^{-1}$ for sheaves of sets, groups or rings. The equality $g_*f_*=(gf)_*$ also holds for sheaves of sets, groups or rings. Clearly $1_*=1$.

<u>COROLLARY</u> If $f:X\longrightarrow Y$ is a homeomorphism then $Sh(X)$ is isomorphic to $Sh(Y)$ (resp. for <u>Ab</u>, <u>Rng</u>)

<u>PROOF</u> Let $g:Y\longrightarrow X$ be inverse to f. Then $(fg)_*=1_*=1$ so $f_*g_*=1$, $g_*f_*=1$. □

NOTE f_* and f^{-1} for a homeomorphism $f : X \longrightarrow Y$

Let $f : X \longrightarrow Y$ be a homeomorphism with inverse $g : Y \longrightarrow X$. We claim that

$$f^{-1} \cong g_* : Sh(Y) \longrightarrow Sh(X)$$

for sheaves of abelian groups, sets or rings. Let $\mathcal{G}$ be a sheaf on Y and let $P(U) = \varinjlim_{V \supseteq f(U)} \mathcal{G}(V)$ be the presheaf which sheafifies to give $f^{-1}\mathcal{G}$. There is an isomorphism of presheaves

$$\psi : g_*\mathcal{G} \longrightarrow P$$
$$\psi_U(s) = (f(U), s)$$

It follows that P is a sheaf and $\psi' : g_*\mathcal{G} \longrightarrow f^{-1}\mathcal{G}$, $\psi'_U(s) = (f(U), s)$ is an isomorphism of sheaves (of sets, groups, rings). It is easily checked that for a morphism $\phi : \mathcal{G} \longrightarrow \mathcal{G}'$ the following diagram commutes:

$$\begin{tikzcd}
g_*\mathcal{G} \arrow[r] \arrow[d, "g_*\phi"'] & f^{-1}\mathcal{G} \arrow[d, "f^{-1}\phi"] \\
g_*\mathcal{G}' \arrow[r] & f^{-1}\mathcal{G}'
\end{tikzcd}$$

so there is a natural equivalence $g_* \cong f^{-1}$.

Let P be a presheaf (of sets, groups, rings) on X. Then there is an isomorphism of sheaves (of sets, groups or rings)

$$\omega : \underline{a}(f_*P) \longrightarrow f_*(\underline{a}P)$$
$$\omega_V(s)(x) = \alpha_y(s(f(x)))$$

$$\alpha_x : (f_*P)_{f(x)} \longrightarrow P_x$$
$$\alpha_x(W, s) = (f^{-1}W, s)$$

So direct image commutes with sheafification.

At this point you should consult our typed notes “Inverse and Direct Images of Sheaves”.

NOTE f_* and f^* are local

Let $f : X \longrightarrow Y$ be a morphism of ringed spaces, $U \subseteq X$ open and suppose $V \subseteq Y$ is open with $f(U) \subseteq V$. Let $g : U \longrightarrow V$ be unique morphism of ringed spaces making the following diagram commute:

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ U \arrow[u, "i"] \arrow[r, "g"'] & V \arrow[u, "j"'] \end{tikzcd}$$

where $U \longrightarrow X$, $V \longrightarrow Y$ are the canonical inclusions, also morphisms of ringed spaces. Then up to natural equivalence the following two diagrams commute (the first only commutes for $U = f^{-1}V$)

$$\begin{tikzcd} \mathcal{O}_X\text{-}\underline{\mathrm{Mod}} \arrow[r, "f_*"] \arrow[d, "-|_U"'] \arrow[dr, phantom, "\boxed{U = f^{-1}V \text{ only}}"] & \mathcal{O}_Y\text{-}\underline{\mathrm{Mod}} \arrow[d, "-|_V"] \\ \mathcal{O}_X|_U\text{-}\underline{\mathrm{Mod}} \arrow[r, "g_*"'] & \mathcal{O}_Y|_V\text{-}\underline{\mathrm{Mod}} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{O}_X\text{-}\underline{\mathrm{Mod}} \arrow[d, "-|_U"'] & \mathcal{O}_Y\text{-}\underline{\mathrm{Mod}} \arrow[l, "f^*"'] \arrow[d, "-|_V"] \\ \mathcal{O}_X|_U\text{-}\underline{\mathrm{Mod}} & \mathcal{O}_Y|_V\text{-}\underline{\mathrm{Mod}} \arrow[l, "g^*"] \end{tikzcd}$$

It is easy to check that $g_*(-|_U) = (-|_V) f_*$ in the case $U = f^{-1}V$. The second diagram commutes (up to iso) for general U, V with $f(U) \subseteq V$ since $-|_U \cong i^*$, $-|_V \cong j^*$ and hence

$$\begin{aligned}(-|_U) f^* &\cong i^* f^* \\ &\cong (fi)^* \\ &= (jg)^* \\ &\cong g^* j^* \\ &\cong g^*(-|_V)\end{aligned}$$

COROLLARY Let $f : X \longrightarrow Y$ be a morphism of ringed spaces. Then

(i) If $\mathcal{F}$ is a free $\mathcal{O}_Y$-module, then $f^*\mathcal{F}$ is free of the same rank

(ii) If $\mathcal{F}$ is a locally free $\mathcal{O}_Y$-module, $f^*\mathcal{F}$ is locally free

(iii) If $\mathcal{F}$ is a locally free $\mathcal{O}_Y$-module of rank $n \in \{0,1,2,\ldots,\infty\}$ then $f^*\mathcal{F}$ is locally free of the same rank.

PROOF Since f^* has a right adjoint f_*, (i) is obvious (since $f^*\mathcal{O}_Y \cong \mathcal{O}_X$). (ii) For $x \in X$ let $f(x) \in V$ be an open neighborhood with $\mathcal{F}|_V$ free. Let $g : f^{-1}V \longrightarrow V$ be the induced map. Then

$$(f^*\mathcal{F})|_{f^{-1}V} \cong g^* \mathcal{F}|_V$$

So $f^*\mathcal{F}$ is locally free by (i). (iii) follows by the same argument, where we choose V so that $\mathcal{F}|_V$ is free of rank n. $\square$

In particular the inverse image of an invertible sheaf is invertible.

NOTE f_* and f^* are local on the base (Old, for reference only)

Let $f : X \longrightarrow Y$ be a morphism of schemes, $V \subseteq Y$ open, $f' : f^{-1}V \longrightarrow V$ the induced map. We claim the following diagram (both squares) commutes up to natural equivalence :

$$\begin{tikzcd} \mathcal{O}_X\text{-}\underline{\mathrm{Mod}} \arrow[r, bend left, "f_*"] \arrow[d, "-|_{f^{-1}V}"'] & \mathcal{O}_Y\text{-}\underline{\mathrm{Mod}} \arrow[l, bend left, "f^*"] \arrow[d, "-|_V"] \\ \mathcal{O}_X|_{f^{-1}V}\text{-}\underline{\mathrm{Mod}} \arrow[r, bend left, "f'_*"] & \mathcal{O}_Y|_V\text{-}\underline{\mathrm{Mod}} \arrow[l, bend left, "f'^*"] \end{tikzcd}$$

It is easily seen that $(-|_V) f_* = f'_* (-|_{f^{-1}V})$. Next notice that for any type of sheaves the following diagram commutes up to isomorphism

$$\begin{tikzcd} sh(X) \arrow[d] & sh(Y) \arrow[l, "f^{-1}"'] \arrow[d] \\ sh(f^{-1}V) & sh(V) \arrow[l, "f'^{-1}"] \end{tikzcd}$$

Given a sheaf G on Y, let P sheafify to give $f^{-1}G$, P' sheafify to give $f'^{-1}(G|_V)$. There is an isomorphism $\phi_x : P'_x \Rightarrow (P|_{f^{-1}V})_x \Rightarrow P_x$ given by $(Q, (W, t)) \longmapsto (Q, (W, t))$, and we define

$$\eta : f'^{-1}(G|_V) \longrightarrow (f^{-1}G)|_{f^{-1}V}$$
$$\eta_U(s)(x) = \phi_x(s(x))$$

This is natural isomorphism of sheaves (of sets, groups or rings).

Now let $\mathcal{G}$ be a sheaf of $\mathcal{O}_Y$-modules, and let

$$f^{-1}\mathcal{O}_Y = \text{sheafification of } R(U) = \varinjlim_{W \supseteq f(U)} \mathcal{O}_Y(W)$$
$$f'^{-1}(\mathcal{O}_Y|_V) = \text{sheafification of } R'(U) = \varinjlim_{V \supseteq W \supseteq f(U)} \mathcal{O}_Y(W)$$

$$f^{-1}\mathcal{G} = \text{sheafification of } P(U) = \varinjlim_{W \supseteq f(U)} \mathcal{G}(W)$$
$$f'^{-1}(\mathcal{G}|_V) = \text{sheafification of } P'(U) = \varinjlim_{V \supseteq W \supseteq f(U)} \mathcal{G}(W)$$

Let $\eta : f'^{-1}(\mathcal{G}|_V) \longrightarrow (f^{-1}\mathcal{G})|_{f^{-1}V}$, $\gamma : f'^{-1}(\mathcal{O}_Y|_V) \longrightarrow f^{-1}(\mathcal{O}_Y)|_{f^{-1}V}$ be isomorphisms of sheaves of abelian groups and rings respectively. Of course $f'^{-1}(\mathcal{G}|_V)$ is a $f'^{-1}(\mathcal{O}_Y|_V)$-module and $(f^{-1}\mathcal{G})|_{f^{-1}V}$ is a $f^{-1}\mathcal{O}_Y|_{f^{-1}V}$-module, and we claim η, γ are compatible in the obvious sense. Let $\phi_x : P'_x \longrightarrow P_x$ and $\psi_x : R'_x \longrightarrow R_x$ be as above, $s \in f'^{-1}(\mathcal{G}|_V)(U)$, $r \in f'^{-1}(\mathcal{O}_Y|_V)(U)$ and say $s(x) = (Q', (W', m))$, $r(x) = (Q, (W, t))$. Then

$$\begin{aligned} \eta_U(r \cdot s)(x) &= \phi_x(r(x) \cdot s(x)) \\ &= (Q \cap Q', (W \cap W', t|_{W \cap W'} \cdot m|_{W \cap W'})) \\ &= \psi_x(r(x)) \cdot \phi_x(s(x)) \\ &= (\gamma_U(r) \cdot \eta_U(s))(x). \end{aligned}$$

as required. Now, let

$$f^*\mathcal{G} = \text{sheafification of } Q(U) = (f^{-1}\mathcal{G})(U) \otimes_{(f^{-1}\mathcal{O}_Y)(U)} \mathcal{O}_X(U)$$
$$f'^*(\mathcal{G}|_V) = \text{sheafification of } Q'(U) = (f'^{-1}\mathcal{G}|_V)(U) \otimes_{(f'^{-1}\mathcal{O}_Y|_V)(U)} (\mathcal{O}_X|_{f^{-1}V})(U)$$

Let $\xi : f^{-1}\mathcal{O}_Y \longrightarrow \mathcal{O}_X$ be the adjoint partner of $f^\#$, $\xi' : f'^{-1}(\mathcal{O}_Y|_V) \longrightarrow \mathcal{O}_X|_{f^{-1}V}$ the adjoint partner of $f'^\#$. We claim the following diagram commutes:

$$\begin{tikzcd}
(f^{-1}\mathcal{O}_Y)|_{f^{-1}V} \arrow[rr, "\xi|_{f^{-1}V}"] & & \mathcal{O}_X|_{f^{-1}V} \\
f'^{-1}(\mathcal{O}_Y|_V) \arrow[u] \arrow[urr, "\xi'"'] & &
\end{tikzcd}$$

The morphisms ξ, ξ' are induced by morphisms out of R', R resp. and since $f^\#_W(t) = f'^\#_W(t)$ for $W \subseteq V$ and $t \in \mathcal{O}_Y(W)$ it is not hard to check commutativity. Hence when we make $(\mathcal{O}_X|_{f^{-1}V})(U)$ into a $(f'^{-1}\mathcal{O}_Y|_V)(U)$ module via ξ' and then compose with $(f^{-1}\mathcal{O}_Y)(U) \longrightarrow (f'^{-1}\mathcal{O}_Y|_V)(U)$, the $(f^{-1}\mathcal{O}_Y)(U)$-module structure on $(\mathcal{O}_X|_{f^{-1}V})(U)$ is the canonical one given by ξ. Hence for $U \subseteq f^{-1}V$ open there is an isomorphism of $\mathcal{O}_X(U)$-modules

$$Q'(U) = (f'^{-1}\mathcal{G}|_V)(U) \otimes_{(f'^{-1}\mathcal{O}_Y|_V)(U)} (\mathcal{O}_X|_{f^{-1}V})(U)$$

$$\Big\downarrow \text{equality}$$

$$(f'^{-1}\mathcal{G}|_V)(U) \otimes_{(f^{-1}\mathcal{O}_Y)(U)} \mathcal{O}_X(U)$$

$$\Big\downarrow \cong$$

$$(f^{-1}\mathcal{G})(U) \otimes_{(f^{-1}\mathcal{O}_Y)(U)} \mathcal{O}_X(U) = Q(U)$$

$$\beta_U : Q'(U) \longrightarrow Q(U)$$
$$m \otimes r \longmapsto \eta_U(m) \otimes r$$

Then β is an isomorphism of presheaves of $\mathcal{O}_X|_{f^{-1}V}$-modules.
The following diagram of sheaves of groups on $f^{-1}V$ thus commutes

$$\begin{tikzcd}
aQ' = f'^*(\mathcal{G}|_V) \arrow[r, Rightarrow] & a(Q|_{f^{-1}V}) \Rightarrow (aQ)|_{f^{-1}V} = (f^*\mathcal{G})|_{f^{-1}V} \\
Q' \arrow[u] \arrow[r, Rightarrow, "\beta"'] & Q \arrow[u]
\end{tikzcd}$$

Let $\varepsilon_x : (Q|_{f^{-1}V})_x \Rightarrow Q_x$ be canonical. Then the isomorphism of sheaves of abelian groups

$$\Phi : f'^*(\mathcal{G}|_V) \longrightarrow (f^*\mathcal{G})|_{f^{-1}V}$$

$$\Phi_U(s)(x) = \varepsilon_x \beta_x(s(x))$$

$\ulcorner$ If $s(x) = (W, \sum_i m_i \otimes r_i)$ with $W \subseteq U$, $m_i \in (f'^{-1}\mathcal{G}|_V)(W)$, $r_i \in \mathcal{O}_X(W)$ then $\Phi_U(s)(x) = (W, \sum_i \eta_W(m_i) \otimes r_i)$ $\lrcorner$

This is an isomorphism of $\mathcal{O}_X|_{f^{-1}V}$-modules since $\Phi_U(r\cdot s)(x) = \varepsilon_x\beta_x((U,r)\cdot s(x)) = (U,r)\cdot \varepsilon_x\beta_x(s(x)) = (r\cdot \Phi_U(s))(x)$. Naturality of Φ in $\mathcal{G}$ follows from naturality of η on the previous page. Hence we have proven that

$$f'^*(-|_V) \cong (-|_{f^{-1}V})\, f^*$$

NOTE f_* and f^* for an isomorphism $f: \mathcal{O}_X \longrightarrow \mathcal{O}_Y$

Let $f: \mathcal{O}_X \longrightarrow \mathcal{O}_Y$ be an isomorphism of ringed spaces with inverse $g: \mathcal{O}_Y \longrightarrow \mathcal{O}_X$. We have already seen that f_* is an isomorphism $\mathcal{O}_X$-$\underline{Mod} \longrightarrow \mathcal{O}_Y$-$\underline{Mod}$. We claim that

$$f^* \cong g_* : \mathcal{O}_Y\text{-}\underline{Mod} \longrightarrow \mathcal{O}_X\text{-}\underline{Mod}$$

Firstly let $P(U) = \varinjlim_{V \supseteq f(U)} \mathcal{F}(V)$ and $R(U) = \varinjlim_{V \supseteq f(U)} \mathcal{O}_Y(V)$ be the presheaves of abelian groups which sheafify to give $f^{-1}\mathcal{F}$ and $f^{-1}\mathcal{O}_Y$ for a $\mathcal{O}_Y$-module $\mathcal{F}$. For $U \subseteq X$, $f(U) = g^{-1}U$, and there is an isomorphism of presheaves of abelian groups

$$\psi' : g_*\mathcal{F} \longrightarrow P$$
$$\psi'_U(s) = (f(U), s)$$

It follows that $\psi: g_*\mathcal{F} \longrightarrow f^{-1}\mathcal{F}$, $\psi_U(s) = \overset{\bullet}{(f(U),s)}$ is an isomorphism of sheaves of abelian groups.

Since the functor $f_*: \underline{Rng}(X) \longrightarrow \underline{Rng}(Y)$ is fully faithful, the counit $f^{-1}f_* \longrightarrow 1$ is an isomorphism. Hence the composite $f^{-1}\mathcal{O}_Y \longrightarrow f^{-1}f_*\mathcal{O}_X \longrightarrow \mathcal{O}_X$ is an isomorphism, since $f^\#: \mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X$ is. Hence the adjoint partner $\xi: f^{-1}\mathcal{O}_Y \longrightarrow \mathcal{O}_X$ of $f^\#$ is an isomorphism of sheaves of rings. It follows that if

$$Z(U) = (f^{-1}\mathcal{F})(U) \otimes_{(f^{-1}\mathcal{O}_Y)(U)} \mathcal{O}_X(U)$$

is the presheaf of $\mathcal{O}_X$-modules sheafifying to give $f^*\mathcal{F}$ then there is an isomorphism of groups

$$\kappa_U : (g_*\mathcal{F})(U) \xrightarrow[\sim]{\psi_U} (f^{-1}\mathcal{F})_U \xrightarrow{\sim} (f^{-1}\mathcal{F})(U) \otimes_{(f^{-1}\mathcal{O}_Y)(U)} \mathcal{O}_X(U) = Z(U)$$
$$\kappa_U(s) = \overset{\bullet}{(f(U),s)} \otimes 1.$$

It is easily checked that κ is an isomorphism of sheaves of abelian groups. We must show it is a morphism of sheaves of $\mathcal{O}_X$-modules. The sheaf $g_*\mathcal{F}$ becomes an $\mathcal{O}_X$-module via $g^\#: \mathcal{O}_X \longrightarrow g_*\mathcal{O}_Y$. So let $r \in \mathcal{O}_X(U)$ and $s \in (g_*\mathcal{F})(U)$ be given. Note that $\mathcal{O}_X(U)$ becomes a $(f^{-1}\mathcal{O}_Y)(U)$-module via ξ_U above. Hence

$$\begin{aligned} \kappa_U(r\cdot s) &= \overset{\bullet}{(f(U), g^\#_U(r)\cdot s)} \otimes 1 \qquad\qquad U \subseteq X \\ r\cdot\kappa_U(s) &= \overset{\bullet}{(f(U),s)} \otimes r \\ &= \overset{\bullet}{(f(U),s)} \otimes \xi_U^{-1}(r)\cdot 1 \\ &= \xi_U^{-1}(r)\cdot \overset{\bullet}{(f(U),s)} \otimes 1 \end{aligned}$$

So it is sufficient to show that $\xi_U^{-1}(r)\cdot\overset{\bullet}{(f(U),s)} = \overset{\bullet}{(f(U), g^\#_U(r)\cdot s)}$. Let $x \in U$ be given, and note that from earlier notes (section 1) $\xi_U^{-1}(r)(x) = (U_x, (V_x, t_x))$ where $x \in U_x$, $V_x \supseteq f(U_x)$ and $t_x \in \mathcal{O}_Y(V_x)$ so that for $y \in U_x$

⌜We can assume that $\xi_U^{-1}(r)(y) = (U_x,(V_x,t_x))$ for all $y \in U_x$⌟

$$\mathrm{germ}_{f(y)}\, t_x = f^{\#\,-1}_{f(y)}\, \mathrm{germ}_y\, r \quad \text{in } \mathcal{O}_X, \qquad (1)$$

Then

⌜This is probably simpler if we get ξ directly, without using ε⌟

$$\begin{aligned} \{\xi_U^{-1}(r)\cdot\overset{\bullet}{(f(U),s)}\}(x) &= \xi_U^{-1}(r)(x)\cdot(U, (f(U),s)) \\ &= (U_x, (V_x \cap f(U), t_x|_{V_x\cap f(U)}\cdot s|_{V_x\cap f(U)})) \end{aligned}$$

⌜We may assume $U_x \subseteq U$⌟

$$\overset{\bullet}{(f(U), g^\#_U(r)\cdot s)}(x) = (U, (f(U), g^\#_U(r)\cdot s)) \quad \text{in } P_x$$

So it suffices to show $(V_x \cap f(U), t_x|_{V_x\cap f(U)}\cdot s|_{V_x\cap f(U)}) = (f(U), g^\#_U(r)\cdot s)$ in $P(U_x)$. Since $(g^\#_U(r)\cdot s)|_{V_x\cap f(U)} = g^\#_U(r)|_{V_x\cap f(U)}\cdot s|_{V_x\cap f(U)}$ it suffices to show that $t_x|_{V_x\cap f(U)}$ and $g^\#_U(r)|_{V_x\cap f(U)}$ agree on a neighborhood of every point of $f(U_x)$. Since for $W \subseteq Y$, $g^\#_{f^{-1}W} = f^{\#\,-1}_W$ (1) implies that for $y \in U_x$,

$$\mathrm{germ}_{f(y)}\, t_x = g^\#_y\, \mathrm{germ}_y\, r = \mathrm{germ}_{f(y)}\, g^\#_U(r)$$

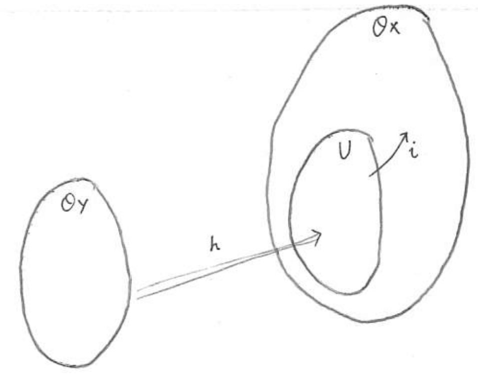
as required. Hence κ is an isomorphism of sheaves of $\mathcal{O}_X$-modules.

NOTE f_* and f^* for an open immersion $f : \mathcal{O}_Y \longrightarrow \mathcal{O}_X$

Let $f : \mathcal{O}_Y \longrightarrow \mathcal{O}_X$ be an open immersion. So there is an open subset $U \subseteq X$ s.t. $f(Y) = U$ and the induced morphism $h : \mathcal{O}_Y \longrightarrow \mathcal{O}_X|_U$ is an isomorphism.

$$\begin{tikzcd} \mathcal{O}_Y \arrow[rr, "f"] \arrow[dr, shift right, "h"'] & & \mathcal{O}_X \\ & \mathcal{O}_X|_U \arrow[ul, shift right, "g"'] \arrow[ur, "i"'] & \end{tikzcd}$$

$g : \mathcal{O}_X|_U \longrightarrow \mathcal{O}_Y$ inverse to h



Note that $f_* = i_* h_*$. We have already noted that i_* and h_* are fully faithful, hence so is f_* (as functors on sheaves of sets, abelian groups or rings). Hence the counit $f^{-1}f_*\mathcal{O}_Y \longrightarrow \mathcal{O}_Y$ is an isomorphism. Beginning with $f^\#: \mathcal{O}_X \longrightarrow f_*\mathcal{O}_Y$ and applying previous notes, we have

$$\begin{tikzcd} \xi : f^{-1}\mathcal{O}_X \arrow[r, "f^{-1}f^\#"] \arrow[d, equal] & f^{-1}f_*\mathcal{O}_Y \arrow[r, Rightarrow, "\varepsilon"] \arrow[d, equal] & \mathcal{O}_Y \\ (ih)^{-1}\mathcal{O}_X \arrow[r, "(ih)^{-1}f^\#"] \arrow[d, phantom, "\cong" rotate=90] & (ih)^{-1}f_*\mathcal{O}_Y \arrow[d, phantom, "\cong" rotate=90] & \\ h^{-1}i^{-1}\mathcal{O}_X \arrow[r, "h^{-1}i^{-1}f^\#"] \arrow[d, phantom, "\cong" rotate=90] & h^{-1}i^{-1}f_*\mathcal{O}_Y \arrow[d, phantom, "\cong" rotate=90] & \\ h^{-1}\mathcal{O}_X|_U \arrow[r, "h^{-1}f^\#|_U"] \arrow[d, phantom, "\cong" rotate=90] & h^{-1}(f_*\mathcal{O}_Y)|_U \arrow[d, phantom, "\cong" rotate=90] & \\ g_*\mathcal{O}_X|_U \arrow[r, "g_* f^\#|_U"] & g_*(f_*\mathcal{O}_Y)|_U & \end{tikzcd}$$

⌜$(ih)^{-1} \cong h^{-1}i^{-1}$⌟

⌜$i^{-1} \cong -|_U$⌟

⌜$g : U \longrightarrow Y$ inverse to h, $h^{-1} \cong g_*$⌟

It is easily verified that the bottom row is an isomorphism – hence so is $\xi : f^{-1}\mathcal{O}_X \longrightarrow \mathcal{O}_Y$. It follows that there is an isomorphism of sheaves of abelian groups for any $\mathcal{F} \in \mathcal{O}_X\text{-}\underline{\mathrm{Mod}}$

$$\Psi : (ih)^{-1}\mathcal{F} \longrightarrow Z \qquad \ulcorner Z(V) = (ih)^{-1}\mathcal{F}(V) \otimes_{(f^{-1}\mathcal{O}_X)} \mathcal{O}_Y(V) \lrcorner$$

$$\Psi_V(a) = a \otimes 1$$

And hence an isomorphism $\Psi' : (ih)^{-1}\mathcal{F} \longrightarrow f^*\mathcal{F}$. It is not difficult to check that Ψ' is natural in $\mathcal{F}$ and combined with the natural isomorphisms

$$g_*\mathcal{F}|_U \Longrightarrow h^{-1}\mathcal{F}|_U \Longrightarrow h^{-1}i^{-1}\mathcal{F} \Longrightarrow (ih)^{-1}\mathcal{F} \Longrightarrow f^*\mathcal{F} \qquad (2)$$

We have an isomorphism $K : g_*\mathcal{F}|_U \longrightarrow f^*\mathcal{F}$ of sheaves of abelian groups, natural in $\mathcal{F}$. To show that $f^* \cong g_*(-)|_U : \mathcal{O}_X\text{-}\underline{\mathrm{Mod}} \longrightarrow \mathcal{O}_Y\text{-}\underline{\mathrm{Mod}}$ it only remains to show that $K_V : \mathcal{F}(g^{-1}V) \longrightarrow (f^*\mathcal{F})(V)$ is a morphism of $\mathcal{O}_Y(V)$-modules for all $V \subseteq Y$. The sheaf $g_*\mathcal{F}|_U$ becomes a $\mathcal{O}_Y$-module via $\mathcal{O}_Y \xrightarrow{g^\#} g_*\mathcal{O}_X|_U$. First we have to unravel the isomorphisms in (2). Note that since $Z \longrightarrow f^*\mathcal{F}$ is a morphism of $\mathcal{O}_Y$-modules, it suffices to show that $g_*\mathcal{F}|_U \longrightarrow Z$ is a morphism of modules.

The isomorphism $\alpha : \mathcal{F}|_U \longrightarrow i^{-1}\mathcal{F}$ is given by $\alpha_V(s)(x) = (V, (V,s))$ for $V \subseteq U$ and $x \in V$. The isomorphism $\hat{\phi} : (ih)^{-1}\mathcal{F} \longrightarrow h^{-1}i^{-1}\mathcal{F}$ is described earlier in these notes. Putting these together, we have for $a \in \mathcal{F}(g^{-1}V)$, $V \subseteq Y$

$$
\begin{array}{ccc}
(g_* \mathcal{F}_U)(V) & & a \\
\big\downarrow & & \\
(h^{-1}\mathcal{F}|_U)(V) & & \dot{(h(V), a)} \qquad \ulcorner (h(V), a) \text{ belongs to } P(V) = \varinjlim_{W \supseteq h(V)} (\mathcal{F}|_U)(W) \lrcorner \\
h^{-1}\alpha \big\downarrow & & \\
(h^{-1}i^{-1}\mathcal{F})(V) & & \dot{(h(V), \alpha_{h(V)}(a))} \\
\hat{\phi} \big\downarrow & & \\
(ih)^{-1}\mathcal{F}(V) & & \hat{\phi}_V^{-1}(\dot{h(V), \alpha_{h(V)}(a)}) \\
\big\downarrow & & \\
Z(V) & & \hat{\phi}_V^{-1}(\dot{h(V), \alpha_{h(V)}(a)}) \otimes 1 \\
= (ih)^{-1}\mathcal{F}(V) \otimes_{(f^{-1}\mathcal{O}_X)(Y)} \mathcal{O}_Y(V) & &
\end{array}
$$

Let $r \in \mathcal{O}_Y(V)$ be given. It suffices to show that in $(ih)^{-1}\mathcal{F}(V)$

$$\xi_V^{-1}(r) \cdot \hat{\phi}_V^{-1}(\dot{h(V), \alpha_{h(V)}(a)}) = \hat{\phi}_V(\dot{h(V), \alpha_{h(V)}(g_V^{\#}(r) \cdot a)}) \qquad (3)$$

Let $M(C) = \varinjlim_{W \supseteq (ih)(C)} \mathcal{F}(W)$ be the presheaf which sheafifies to give $(ih)^{-1}\mathcal{F}$. Then it suffices to show the two functions in (3) agree in M_x for all $x \in V$. Considering Eq (1) in our Note: $(gf)_* = g_* f_*$ and $f^{-1}g^{-1} = (gf)^{-1}$ we see that

$$\hat{\phi}_V^{-1}(\dot{h(V), \alpha_{h(V)}(a)})(x) = \phi_x^{-1}(V, (h(V), \alpha_{h(V)}(a))) = (V, (h(V), a))$$

Since $\alpha_{h(V)}(a)(h(x)) = (h(V), (h(V), a))$. So (3) reduces to showing that for all $x \in V$,

$$\xi_V^{-1}(r)(x) \cdot (V, (h(V), a)) = (V, (h(V), g_V^{\#}(r) \cdot a)) \qquad \text{in } M_x$$

Note that for $x \in V$, $f_{f(x)}^{\#} : \mathcal{O}_{X, f(x)} \longrightarrow \mathcal{O}_{Y,x}$ is an isomorphism of rings, with inverse $g_x^{\#}$ (technically, $f_{f(x)}^{\# \, -1}$ is $g_x^{\#} : \mathcal{O}_{Y,x} \longrightarrow (\mathcal{O}_X|_U)_{h(x)}$ followed by $(\mathcal{O}_X|_U)_{h(x)} \cong \mathcal{O}_{X, f(x)}$). By our earlier discussion of the counit ε we know that there is an open neighborhood $x \in O \subseteq V$ and $(W, t) \in N(O)$ $\ulcorner$here N sheafifies to $f^{-1}\mathcal{O}_X$, so $X \supseteq W \supseteq f(\ )$ and $t \in \mathcal{O}_X(W)\lrcorner$, so that $\forall y \in O$ $\xi_V^{-1}(r)(y) = (O, (W, t))$ and $\operatorname{germ}_{f(y)} t = f_{f(y)}^{\# \, -1} \operatorname{germ}_y r$. Hence for all $y \in O$

$$
\begin{aligned}
\operatorname{germ}_{f(y)} t &= g_y^{\#} \operatorname{germ}_y r \qquad (4) \\
&= \operatorname{germ}_{f(y)} g_V^{\#}(r)
\end{aligned}
$$

It follows that there is an open set $W \cap f(V) \supseteq Q \supseteq f(O)$ with $t|_Q = g_V^{\#}(r)|_Q$. But then

$$
\begin{aligned}
\xi_V^{-1}(r)(x) \cdot (V, (h(V), a)) &= (O, (W, t) \cdot (h(V), a)) \\
&= (O, (W \cap h(V), t|_{W \cap h(V)} \cdot a|_{W \cap h(V)}))
\end{aligned}
$$

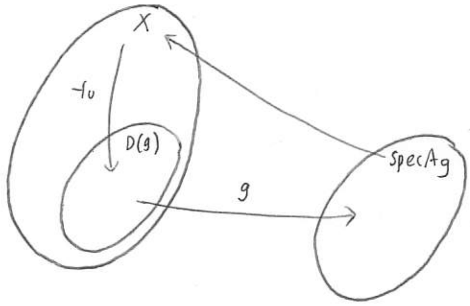
But then (4) implies that $(W \cap f(V), t|_{W \cap f(V)} \cdot a|_{W \cap f(V)}) = (f(V), g_V^{\#}(r) \cdot a)$ in $M(O)$, completing the proof.
Hence there is a natural equivalence

$$
\begin{aligned}
& g_*(-)|_U \Longrightarrow f^* \\
& \kappa : g_* \mathcal{F}|_U \longrightarrow f^* \mathcal{F} \\
& \kappa_V(a) = \dot{A} \otimes 1 \quad \text{where } A \in (f^{-1}\mathcal{F})(V) \text{ is defined by} \\
& \qquad A(x) = (V, (f(V), a)) \quad x \in V
\end{aligned}
$$

<u>NOTE</u> $\tilde{M}|_{D(g)} \cong (M \otimes_A A_g)^\sim$

Let $X = \operatorname{Spec} A$ be an affine scheme and M an A-module. Let $g \in A$ and $f: \operatorname{Spec} A_g \to \operatorname{Spec} A$ be the canonical open immersion. If $g: \mathcal{O}_X|_{D(g)} \longrightarrow \mathcal{O}_{\operatorname{Spec} A_g}$ is inverse to $h: \mathcal{O}_{\operatorname{Spec} A_g} \to \mathcal{O}_X|_{D(g)}$, then the previous note shows that

$$f^* \cong g_* (-)|_U$$



Hence since $f^* \tilde{M} \cong (M \otimes_A A_g)^\sim$ we see that $g_* \tilde{M}|_{D(g)} \cong (M \otimes_A A_g)^\sim \cong \widetilde{M_g}$

These sheaves of the form $\widetilde{M}$ on affine schemes are our models for quasi-coherent sheaves. A quasi-coherent sheaf on a scheme X will be an $\mathcal{O}_X$-module which is locally of the form $\widetilde{M}$. In the next few lemmas and propositions, we will show that this is a local property, and we will establish some facts about quasi-coherent and coherent sheaves.

<u>DEFINITION</u> Let $(X, \mathcal{O}_X)$ be a scheme. A sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is <u>quasi-coherent</u> if X can be covered by open affine subsets $U_i = \operatorname{Spec} A_i$ such that for each i there is an A_i-module M_i with $\mathcal{F}|_{U_i} \cong \widetilde{M_i}$. We say that $\mathcal{F}$ is <u>coherent</u> if furthermore each M_i can be taken to be a finitely generated A_i-module. (the isomorphism $\mathcal{F}|_{U_i} \longrightarrow \widetilde{M_i}$ being of sheaves of abelian groups, and of modules, compatible with $\mathcal{O}_X(V) \cong \mathcal{O}_{\operatorname{Spec} A_i}(V)$ for $V \subseteq U_i$)

The zero module over $(\emptyset, 0)$ is coherent. The definition above is equivalent to every point having a suitable open affine neighborhood (this avoids thinking about covers involving $(\emptyset, 0)$). Although we have just defined the notion of quasi-coherent and coherent sheaves on an arbitrary scheme, we will normally not mention coherent sheaves unless the scheme is noetherian. This is because the notion of coherence is not at all well-behaved on a nonnoetherian scheme. Clearly any module isomorphic to a quasi-coherent (coherent) module is quasi-coherent (coherent).

<u>EXAMPLE 5.2.1</u> On any scheme X, the structure sheaf $\mathcal{O}_X$ is quasi-coherent (and in fact coherent).

<u>EXAMPLE 5.2.2</u> Clearly if M is an A-module, $\widetilde{M}$ is a quasi-coherent $\operatorname{Spec} A$-module, and $\widetilde{M}$ is coherent if M is finitely generated. Let $\varphi : A \longrightarrow B$ be a ring morphism and $f : \mathcal{O}_Y \longrightarrow \mathcal{O}_X$ the associated morphism of schemes, where $\mathcal{O}_X = \operatorname{Spec} A$ and $\mathcal{O}_Y = \operatorname{Spec} B$. Then considering B as a left module over itself, $\widetilde{B} = \mathcal{O}_Y$ as $\mathcal{O}_Y$-modules, so

$$f_*(\mathcal{O}_Y) = f_*(\widetilde{B}) \cong \widetilde{({}_A B)} \qquad \text{(by Prop 5.2)}$$

Hence $f_* \mathcal{O}_Y$ is quasi-coherent (coherent if φ makes B into a f.g. module). In particular, if $X = \operatorname{Spec} A$ is an affine scheme and $Y \subseteq X$ is the closed subscheme defined by $\mathfrak{a} \subseteq A$, and if $i : Y \longrightarrow X$ is the inclusion morphism, then $i_* \mathcal{O}_Y$ is quasi-coherent (in fact, coherent) as a $\mathcal{O}_X$-module. Indeed, it is isomorphic to $(A/\mathfrak{a})^\sim$.

<u>EXAMPLE 5.2.3</u> If U is an open subscheme of a scheme X, with inclusion map $j : U \longrightarrow X$, then the sheaf $j_!(\mathcal{O}_U)$ obtained by extending $\mathcal{O}_U$ by U outside of U (Ex 1.19) is an $\mathcal{O}_X$-module, but it is not in general quasi-coherent.

<u>EXAMPLE 5.2.5</u> Let X be an integral noetherian scheme, and let $\mathcal{K}$ be the constant sheaf with group K equal to the function field $K(X)$ of X (Ex 3.6). Then $\mathcal{K}$ is the sheafification of the presheaf $U \longmapsto K$ $(U \neq \emptyset)$ which is clearly an $\mathcal{O}_X$-module via $\mathcal{O}_X(U) \longrightarrow \mathcal{O}_{X,\xi} = K(X)$. Hence $\mathcal{K}$ is an $\mathcal{O}_X$-module, isomorphic as an $\mathcal{O}_X$-module to the version where $s \in \mathcal{K}(V)$ are maps $s : V \longrightarrow K$ which are locally constant, and $(r \cdot s)(x) = r \cdot s(x) = (V, r)s(x)$. We claim that $\mathcal{K}$ is quasi-coherent.

Let $x \in X$ be given and $x \in U \subseteq X$ an affine open neighborhood with $: U \cong \operatorname{Spec} A$. Of course A is a domain. Let Then ξ (the generic point) is in U and corresponds to $(0) \in \operatorname{Spec} A$. $i : A \longrightarrow Q(A)$ be canonical. This makes $Q(A)$ into an A-module, and hence $\widetilde{Q(A)}$ is a quasi-coherent $\operatorname{Spec} A$-module. For each $\mathfrak{p} \in \operatorname{Spec} A$ there is an isomorphism of rings:

$$\phi_{\mathfrak{p}} : K \longrightarrow \mathcal{O}_{\operatorname{Spec} A, (0)} \longrightarrow Q(A) \longrightarrow Q(A)_{\mathfrak{p}}$$

and $\eta : \mathcal{K} \longrightarrow \widetilde{Q(A)}$ (taking certain liberties to identify U with $\operatorname{Spec} A$) by

$$\eta_V(s)(\mathfrak{p}) = \phi_{\mathfrak{p}}(s(\mathfrak{p}))$$

It is easily checked that this is an isomorphism of sheaves of abelian groups. Given $r \in \mathcal{O}_X(V)$, let $r' \in \mathcal{O}_{\operatorname{Spec} A}(V)$ correspond to r. Then for $t \in \widetilde{Q(A)}(V)$, $(r' \cdot t)(\mathfrak{p}) = r'(\mathfrak{p}) \cdot t(\mathfrak{p}) = i_{\mathfrak{p}}(r'(\mathfrak{p}))t(\mathfrak{p})$ where $i_{\mathfrak{p}} : A_{\mathfrak{p}} \longrightarrow Q(A)_{\mathfrak{p}}$ is canonical. Note that $\phi_{\mathfrak{p}}(V, r) = r'(0)/1 \in Q(A)_{\mathfrak{p}}$. Since A is an integral domain, and r' is regular, there are $a, s \in A$ and open $\mathfrak{p} \in Q \subseteq V$ s.t. $\forall \mathfrak{q} \in Q$ $r'(\mathfrak{q}) = a/s$ in $A_{\mathfrak{q}}$. But Q is open, hence contains (0), so it follows that $\forall \mathfrak{p} \in V$ $r'(0)/1 = i_{\mathfrak{p}}(r'(\mathfrak{p}))$ in $Q(A)_{\mathfrak{p}}$. Hence

$$\begin{aligned} \eta_V(r \cdot s)(\mathfrak{p}) &= \phi_{\mathfrak{p}}((V, r)s(\mathfrak{p})) = \phi_{\mathfrak{p}}(V, r)\phi_{\mathfrak{p}}(s(\mathfrak{p})) = r'(0)/1 \cdot \eta_V(s)(\mathfrak{p}) \\ &= (r' \cdot \eta_V(s))(\mathfrak{p}) \end{aligned}$$

so η is an isomorphism of modules, and so $\mathcal{K}$ is quasi-coherent.

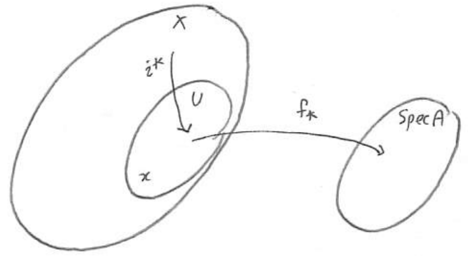
NOTE The definition of quasi-coherent modules could be restated more precisely as follows:

DEFINITION Let $(X, \mathcal{O}_X)$ be a scheme. A sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is quasi-coherent if every point $x \in X$ has an affine open neighborhood U s.t. if $i : U \longrightarrow X$ is the inclusion and $f : U \longrightarrow \operatorname{Spec} A$ the isomorphism, there is an A-module M s.t.

$$f_* i^* \mathcal{F} \cong \widetilde{M} \quad \text{as } \mathcal{O}_{\operatorname{Spec} A}\text{-modules}$$

$i^* : \mathcal{O}_X\text{-}\underline{\mathrm{Mod}} \longrightarrow \mathcal{O}_X|_U\text{-}\underline{\mathrm{Mod}}$

$f_* : \mathcal{O}_X|_U\text{-}\underline{\mathrm{Mod}} \longrightarrow \mathcal{O}_{\operatorname{Spec} A}\text{-}\underline{\mathrm{Mod}}$



Equivalently, $f_* \mathcal{F}|_U \cong \widetilde{M}$. Then $\mathcal{F}$ is coherent if for each x, M can be taken to be finitely-generated.

NOTE If $f : \mathcal{O}_X \longrightarrow \mathcal{O}_Y$ is an isomorphism of ringed spaces and $U \subseteq X$ is open, $V = f(U)$, then it is easily checked that the following diagram of functors commutes: ($f|_U : \mathcal{O}_X|_U \longrightarrow \mathcal{O}_Y|_V$ is clearly iso)

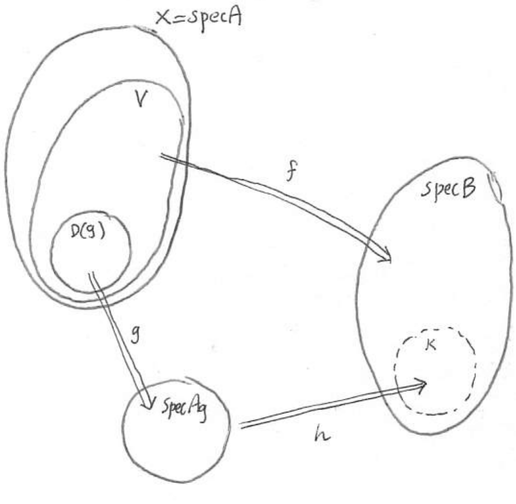
$$\begin{tikzcd}
\mathcal{O}_X\text{-}\underline{\mathrm{Mod}} \arrow[r, "f_*"] \arrow[d, "-|_U"'] & \mathcal{O}_Y\text{-}\underline{\mathrm{Mod}} \arrow[d, "-|_V"] \\
\mathcal{O}_X|_U\text{-}\underline{\mathrm{Mod}} \arrow[r, "f|_{U *}"'] & \mathcal{O}_Y|_V\text{-}\underline{\mathrm{Mod}}
\end{tikzcd}$$

LEMMA 5.3 Let $X = \operatorname{Spec} A$ be an affine scheme, let $f \in A$, $D(f) \subseteq X$ be the corresponding open set, and let $\mathcal{F}$ be a quasi-coherent sheaf on X.

(a) If $s \in \Gamma(X, \mathcal{F})$ is a global section of $\mathcal{F}$ whose restriction to $D(f)$ is 0, then for some n, $f^n s = 0$.

(b) Given a section $t \in \mathcal{F}(D(f))$ of $\mathcal{F}$ over the open set $D(f)$, then for some $n > 0$, $f^n t$ extends to a global section of $\mathcal{F}$ over X. (To be precise $(f|_{D(f)})^n t$ extends)

PROOF If $D(f)$ is empty (so f is nilpotent), both results are trivial. So we may assume $D(f) \neq \emptyset$. Since $\mathcal{F}$ is quasi-coherent, X can be covered by open affine subsets of the form $V = \operatorname{Spec} B$, such that $\mathcal{F}|_V \cong \widetilde{M}$. for some B-module M. Now the open subsets of the form $D(g)$ form a base for the topology of X, so we can cover V by open sets of the form $D(g)$ for various $g \in A$.



$h = (f|_{D(f)})\, g^{-1}$

$\therefore\ g = h^{-1} f|_{D(f)}$

$g_* \cong h^* (f|_{D(f)})_*$

The morphism $\operatorname{Spec} A_g \longrightarrow \operatorname{Spec} B$ arises from some $\varphi : B \longrightarrow A_g$. Hence by (5.2) if we map $\mathcal{F}|_V$ to the $\mathcal{O}_{\operatorname{Spec} B}$-module $f_* \mathcal{F}|_V \cong \widetilde{M}$ then restrict to give $(f|_{D(g)})_*(\mathcal{F}|_{D(g)}) \cong \widetilde{M}|_K$ and then apply $(h^{-1})_*$ to find $(M \otimes_B A_g)^\sim \cong (h^{-1})_*(f|_{D(g)})_*(\mathcal{F}|_{D(g)}) = g_*(\mathcal{F}|_{D(g)})$ (see an earlier note which shows what inverse image is for open immersions). Thus we have shown that if $\mathcal{F}$ is quasi-coherent on X, then X can be covered by open sets of the form $D(g_i)$ where for each i, $\mathcal{F}|_{D(g_i)} \cong \widetilde{M}_i$ for some module M_i over the ring A_{g_i}. Since X is quasi-compact, a finite number of these open sets will do.

<u>NOTE</u> If $\mathcal{F}$ is coherent then we can M is a f.g. B-module. Then $M \otimes_B A_g$ is a f.g. A_g-module, so we see that X can be covered by open sets $D(g_i)$ where for each i, $\mathcal{F}|_{D(g_i)} \cong \widetilde{M}_i$ for a f.g. A_{g_i}-module M_i.

(a) Now suppose we are given $s \in \mathcal{F}(X)$ with $s|_{D(f)} = 0$. For each i let $s_i \in M_i$ correspond to $s|_{D(g_i)} \in \mathcal{F}(D(g_i))$. Then $D(f) \cap D(g_i) = D(fg_i)$ corresponds to $D(f/1) \subseteq \operatorname{Spec} A_{g_i}$ and the fact that $s|_{D(f)} = 0$ means that $s_i/1 = 0$ in $(M_i)_{f/1}$ by (5.1c). Hence $(f/1)^{n_i} s_i = 0$ for some $n_i > 0$. The fact that $\mathcal{F}|_{D(g_i)} \cong \widetilde{M}_i$ means that $f|_{D(\,)} \cdot s|_{D(\,)}$ is identified with $(f/1) \cdot s_i \in \widetilde{M}_i(\operatorname{Spec} A_{g_i})$, which corresponds to $f/1 \cdot s_i \in M$. Similarly we see that $(f^{n_i} \cdot s)_{D(\,)} = 0$. Since there are a finite number of n_i, we can find N large enough so that $(f^N \cdot s)|_{D(g_i)} = 0 \ \forall i$. Hence since the g_i cover X, $f^N \cdot s = 0$.

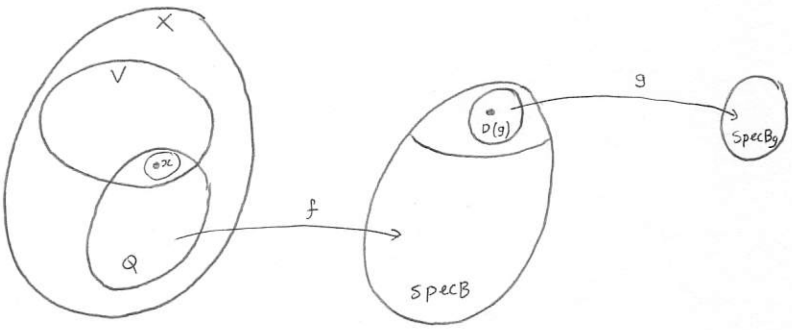
(b) Let $t \in \mathcal{F}(D(f))$ be given and restrict it for each i to get an element of $\mathcal{F}(D(fg_i))$. Let $t_i \in \widetilde{M}_i(D(f))$ correspond to this restriction. Then for some $n_i > 0$ t_i corresponds to q_i/f^{n_i} in $(M_i)_f$. It follows that in $\widetilde{M}_i(D(f))$, $\bar{f}^{n_i}|_{D(f)} \cdot t_i = \bar{q}_i|_{D(f)}$. Translating this back to $\mathcal{F}$, and letting $m_i \in \mathcal{F}(D(g_i))$ correspond to q_i, we have $m_i|_{D(fg_i)} = (\bar{f}^{n_i} \cdot t)|_{D(fg_i)}$ where $\bar{f} = f|_{D(f)}$. Once again by making N sufficiently large and modifying the m_i, we end up with $N > 0$ and $m_i \in \mathcal{F}(D(g_i))$ with the property that for all i

$$(\bar{f}^N \cdot t)|_{D(fg_i)} = m_i|_{D(fg_i)} \qquad (1)$$

We claim that for any i, j there exists an integer $n > 0$ with $(f|_{D(g_ig_j)})^n (m_i|_{D(g_ig_j)} - m_j|_{D(g_ig_j)}) = 0$. But this follows from (1) and (a), since $D(f) \cap D(g_ig_j)$ corresponds to $D(f/1)$ in $\operatorname{Spec} A_{g_ig_j}$. To apply (a), we must first note that by the comments in the first paragraph, $\mathcal{F}|_{D(g_ig_j)}$ does correspond to $\widetilde{M}$ on $\mathcal{O}_{\operatorname{Spec} A_{g_ig_j}}$ for some $A_{g_ig_j}$-module M. This n depends on i and j, but we can take one $N' > 0$ large enough for all. Now the local sections $(f|_{D(g_i)})^{N'} m_i$ of $\mathcal{F}$ on $D(g_i)$ glue together to give a global section s of $\mathcal{F}$, whose restriction to $D(f)$ is $\bar{f}^{N'+N} t$, as required. □

<u>PROPOSITION 5.4</u> Let X be a scheme. Then an $\mathcal{O}_X$-module $\mathcal{F}$ is quasi-coherent if and only if for every open affine subset $U \cong \operatorname{Spec} A$ of X, there is an A-module M such that $\mathcal{F}|_U \cong \widetilde{M}$. If X is noetherian, then $\mathcal{F}$ is coherent if and only if the same is true, with the extra condition that M be a finitely-generated A-module.

<u>PROOF</u> If $X = \emptyset$ then this is trivial. So assume $X \neq \emptyset$. Let $\mathcal{F}$ be quasi-coherent on X and let $V \subseteq X$ be open. If $x \in V$ then let $x \in Q \subseteq X$ be an affine open subset, $f : \mathcal{O}_X|_Q \longrightarrow \operatorname{Spec} B$ an isomorphism and M a B-module s.t. $f_* \mathcal{F}|_Q \cong \widetilde{M}$. Then $f(Q \cap V)$ is an open neighborhood of $f(x)$ in $\operatorname{Spec} B$, so there must be $g \in B$ s.t. $f(x) \in D(g) \subseteq f(Q \cap V)$. If $g : \mathcal{O}_{\operatorname{Spec} B}|_{D(g)} \longrightarrow \operatorname{Spec} B_g$ is the canonical isomorphism then we have already shown that $g_* \widetilde{M}|_{D(g)} \cong (M \otimes_B B_g)^\sim$.



Then $f|_{f^{-1}D(g)}$ gives an isomorphism of schemes $\mathcal{O}_X|_{f^{-1}D(g)} \xrightarrow{h} \operatorname{Spec} B_g$ and $h_* = g_*(f|_{f^{-1}D(g)})_*$ so

$$\begin{aligned} h_* \mathcal{F}|_{f^{-1}D(g)} &= g_*(f|_{f^{-1}D(g)})_* \mathcal{F}|_{f^{-1}D(g)} \\ &= g_*(f_* \mathcal{F}|_Q)|_{D(g)} \end{aligned}$$

But $f_*\mathcal{F}|_Q \cong \widetilde{M}$ so we have $g_*(f_*\mathcal{F}|_Q)|_{D(g)} \cong g_*\widetilde{M}|_{D(g)} \cong (M\otimes_B B_g)^\sim$. So we conclude that there is a basis for the topology of X consisting of open affines for which the restriction of $\mathcal{F}$ is the sheaf associated to a module. In particular, $\mathcal{F}|_V$ is a quasi-coherent $\mathcal{O}_X|_V$-module for any open $V\subseteq X$.

If $U\cong \operatorname{Spec}A$ is affine, $U\subseteq X$, then $\mathcal{F}|_U$ is quasi-coherent, so we can reduce to the case of $X=\operatorname{Spec}A$ being affine, and showing that any quasi-coherent sheaf $\mathcal{F}$ on X is isomorphic to $\widetilde{M}$ for some A-module M. Let $\mathcal{F}$ be given and put $M=\Gamma(X,\mathcal{F})$. For $f\in A$ define

$$\varphi : M_f \longrightarrow \mathcal{F}(D(f))$$
$$\varphi(m/f^n) = \bar{f}^{-n}\cdot m|_{D(f)} \qquad\qquad \bar{f} = f|_{D(f)}$$

It is easily checked that this map is well-defined, and is a morphism of A_f-modules (via $A_f\cong \mathcal{O}_X(D(f))$). If $\varphi(m/f^n)=0$ then $m|_{D(f)}=0$ so by the Lemma, for some $n>0$ $f^n m = 0$ in $M=\mathcal{F}(X)$. Hence $m/f^n=0$ in A_f. If $c\in\mathcal{F}(D(f))$ then by the Lemma, for some $n>0$ there is $m\in\mathcal{F}(X)$ s.t. $m|_{D(f)}=\bar{f}^n c$. Then $c=\varphi(m/f^n)$, so φ is an isomorphism.

There is a canonical morphism of $\mathcal{O}_X$-modules $\alpha:\widetilde{M}\longrightarrow\mathcal{F}$ defined by $\operatorname{germ}_{\mathfrak{p}}\alpha_U(s)=\kappa_{\mathfrak{p}}(s(\mathfrak{p}))$ where $\kappa_{\mathfrak{p}}(a/s)=(D(s), \bar{1/s}\cdot a|_{D(s)})$. For $f\in A$ it is not difficult to check that the following diagram commutes

$$\begin{tikzcd}
\widetilde{M}(D(f)) \arrow[r, "\alpha_{D(f)}"] & \mathcal{F}(D(f)) \\
M_f \arrow[u, Rightarrow] \arrow[ur, "\varphi"'] &
\end{tikzcd}$$

Hence $\alpha_{D(f)}$ is an isomorphism. Since the $D(f)$ form a basis for X, it follows that α is an isomorphism of modules, as required.

Now suppose that X is noetherian, and $\mathcal{F}$ coherent. If $U\cong\operatorname{Spec}A$ is an open affine subset, then by (3.2) A is noetherian. If $V\subseteq X$ is any open subset and $x\in V$ then there is an open affine $Q\cong\operatorname{Spec}B$ with $x\in Q$ s.t. $\mathcal{F}|_Q\cong\widetilde{M}$ where M is a f.g. B-module. Then B is noetherian, and $M\otimes_B B_g$ is generated by $m_1\otimes 1,\dots,m_n\otimes 1$ if the m_i generate M, so $\mathcal{F}|_{D(g)}$ is isomorphic to $\widetilde{M}$ for a f.g. B_g-module M. Hence $\mathcal{F}|_V$ is coherent for any open $V\subseteq X$. (In fact, this does not require X to be noetherian). So we reduce to showing that if $X=\operatorname{Spec}A$ with A noetherian and if $\mathcal{F}$ is coherent, there is f.g. M with $\mathcal{F}\cong\widetilde{M}$.

Let $M=\mathcal{F}(X)$ be as above. Then we can cover X by open sets $D(g_i)$ so that $\mathcal{F}|_{D(g_i)}\cong\widetilde{M}_i$ for a f.g. A_{g_i}-module M_i. We can assume the list of g_i's is finite: $g_1,\dots,g_r$. Notice that $\mathcal{F}(D(g_i))\Longrightarrow\widetilde{M}_i(\operatorname{Spec}A_{g_i})$ is an isomorphism of A_{g_i}-modules, where we use $A_{g_i}\cong\mathcal{O}_X(D_{g_i})$ and $A_{g_i}\cong\mathcal{O}_{\operatorname{Spec}A_{g_i}}(\operatorname{Spec}A_{g_i})$. But by definition the isomorphism $\mathcal{F}|_{D(g_i)}\Rightarrow\widetilde{M}_i$ maps the action of $\mathcal{O}_X$ to the action of $\mathcal{O}_{\operatorname{Spec}A_{g_i}}$, and

$$\begin{tikzcd}
& A_{g_i} \arrow[dl] \arrow[dr, Rightarrow] & \\
\mathcal{O}_X(D(g_i)) \arrow[rr, Rightarrow] & & \mathcal{O}_{\operatorname{Spec}A_{g_i}}(\operatorname{Spec}A_{g_i})
\end{tikzcd}$$

commutes, so $\mathcal{F}(D(g_i))\Rightarrow\widetilde{M}_i(\operatorname{Spec}A_{g_i})$ is an iso of A_{g_i}-modules. It follows that M_{g_i} and M_i must be isomorphic as A_{g_i}-modules. Hence M_{g_i} is finitely generated. So we are reduced to the following algebra problem: A is noetherian, $g_1,\dots,g_r$ generate the unit ideal and each M_{g_i} is finitely-generated, and we want to show M is f.g. But A, A_{g_i} are noetherian, so it suffices to show that if the M_{g_i} are noetherian, so is M. For this we just use the proof of (3.2) with A replaced by M in appropriate places. $\square$

<u>NOTE</u> As we showed directly in the proof, (5.4) implies that if X is a scheme, $\mathcal{F}$ a quasi-coherent sheaf (coherent sheaf) and $V\subseteq X$ open, then $\mathcal{F}|_V$ is a quasi-coherent (coherent) sheaf on V.

<u>COROLLARY 5.5</u> Let A be a ring and $X=\operatorname{Spec}A$. The functor $M\longmapsto\widetilde{M}$ gives an equivalence of categories between the category of A-modules and the category of quasi-coherent $\mathcal{O}_X$-modules. Its inverse is the functor $\mathcal{F}\longmapsto\Gamma(X,\mathcal{F})$. If A is noetherian, the same functor also gives an equivalence between the category of finitely-generated A-modules and the category of coherent $\mathcal{O}_X$-modules.

<u>PROOF</u> The only new information here is that if $\mathcal{F}$ is quasi-coherent then $\mathcal{F}\cong\widetilde{\mathcal{F}(X)}$, which follows from (5.4). $\square$

<u>NOTE</u> To summarise (5.4) if $X=\operatorname{Spec}A$ and $\mathcal{F}$ is quasi-coherent, the morphism $\alpha:\widetilde{M}\longrightarrow\mathcal{F}$, $M=\mathcal{F}(X)$, is an isomorphism. If X is noetherian and $\mathcal{F}$ coherent, M is f.g.

$\underline{\text{NOTE}}$ X scheme, $U \subseteq X$, $-|_U : \mathcal{O}_X\text{-}\underline{\mathrm{Mod}} \longrightarrow \mathcal{O}_X|_U\text{-}\underline{\mathrm{Mod}}$

Let X be a scheme, $U \subseteq X$ open, $f : \mathcal{O}_X|_U \longrightarrow \mathcal{O}_X$ the canonical inclusion. We have additive functors

$$\begin{tikzcd} \mathcal{O}_X|_U\text{-}\underline{\mathrm{Mod}} \arrow[r, bend left, "f_*"] & \mathcal{O}_X\text{-}\underline{\mathrm{Mod}} \arrow[l, "f^*"'] \arrow[l, bend left, "-|_U"] \end{tikzcd}$$

Since $f^* \dashv f_*$ and $f^* \cong -|_U$, $-|_U \dashv f_*$. Hence $-|_U$ preserves epimorphisms and all colimits. For a morphism $\psi : \mathcal{F} \longrightarrow \mathcal{F}'$ of $\mathcal{O}_X$-modules, it is not hard to check that $(\mathrm{Ker}\,\psi)|_U = \mathrm{Ker}\,\psi|_U$ so $-|_U$ also preserves kernels. Hence $-|_U$ is an exact functor, so $-|_U$ preserves all finite limits and all colimits, preserves monos and epis. Since $f^* \cong -|_U$ the same is true of f^*. Up to this point $\mathcal{O}_X$ could have been any ringed space.

We know from (5.4) that $-|_U$ actually preserves quasi-coherent and coherent modules, when X is a scheme.

$\underline{\text{LEMMA}}$ If X is a scheme, $X = \bigcup_i V_i$ any open cover, then a $\mathcal{O}_X$-module $\mathcal{F}$ is quasi-coherent (coherent) iff. for all i, $\mathcal{F}|_{V_i}$ is a quasi-coherent (coherent) $\mathcal{O}_X|_{V_i}$-module.

$\underline{\text{LEMMA}}$ Let $\mathcal{O}_X$ be a ringed space, $X = \bigcup_i V_i$ an open cover. A sequence of $\mathcal{O}_X$-modules

$$\cdots \longrightarrow \mathcal{F}_{j-1} \longrightarrow \mathcal{F}_j \longrightarrow \mathcal{F}_{j+1} \longrightarrow \cdots \qquad (1)$$

is exact if and only if for each i, the sequence of $\mathcal{O}_X|_{V_i}$-modules $\cdots \longrightarrow \mathcal{F}_{j-1}|_{V_i} \longrightarrow \mathcal{F}_j|_{V_i} \longrightarrow \mathcal{F}_{j+1}|_{V_i} \longrightarrow \cdots$ is exact.

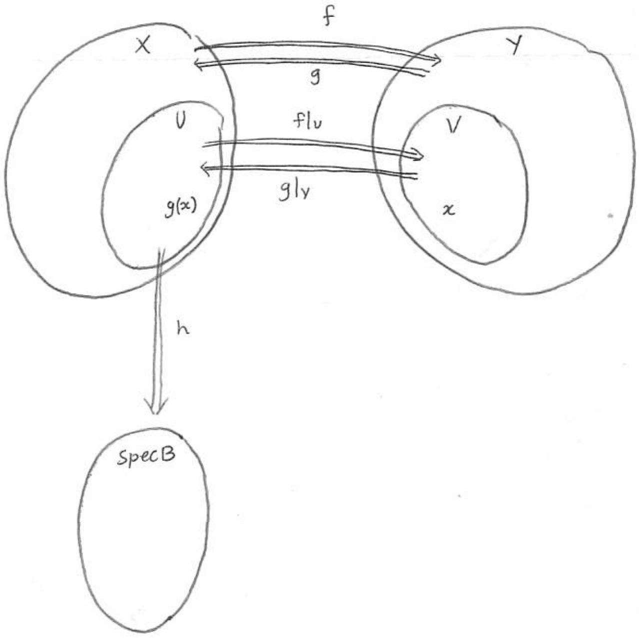
$\underline{\text{PROOF}}$ The condition is clearly necessary. To show sufficiency, it suffices to consider a sequence of the form $\mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}''$. We know this sequence of $\mathcal{O}_X$-modules is exact iff. it is exact as a sequence of sheaves of abelian groups, which by Ex1.2 is iff. $\mathcal{F}'_x \longrightarrow \mathcal{F}_x \longrightarrow \mathcal{F}''_x$ is an exact sequence of abelian groups for all $x \in X$. Suppose $\forall i$ the sequence $\mathcal{F}'|_{V_i} \longrightarrow \mathcal{F}|_{V_i} \longrightarrow \mathcal{F}''|_{V_i}$ is exact. Then for $x \in V_i$ we have a commutative diagram

$$\begin{tikzcd} \mathcal{F}'_x \arrow[r] \arrow[d, phantom, "\cong" rotate] & \mathcal{F}_x \arrow[r] \arrow[d, phantom, "\cong"] & \mathcal{F}''_x \arrow[d, phantom, "\cong"] \\ (\mathcal{F}'|_{V_i})_x \arrow[r] & (\mathcal{F}|_{V_i})_x \arrow[r] & (\mathcal{F}''|_{V_i})_x \end{tikzcd}$$

By assumption the bottom row is exact, hence so is the top row. Since the V_i cover X, this completes the proof. $\square$

NOTE Isomorphism $f: X \longrightarrow Y$ then f_* preserves quasi-coherent

Let $f: X \longrightarrow Y$ be an isomorphism of schemes. Then $f_*: \mathcal{O}_X-\underline{\mathrm{Mod}} \longrightarrow \mathcal{O}_Y-\underline{\mathrm{Mod}}$ is an isomorphism. We claim that f identifies quasi-coherent $\mathcal{O}_X$-modules with quasi-coherent $\mathcal{O}_Y$-modules. If $g: Y \longrightarrow X$ is inverse to f then $g_* = f_*^{-1}$, so it suffices to show that if $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module, then $f_* \mathcal{F}$ is a quasi-coherent $\mathcal{O}_Y$-module.



Let $x \in Y$ be given and let U be an affine open neighborhood of $g(x)$ and h an isomorphism such that there is a B-module M and $h_* \mathcal{F}|_U \cong \widetilde{M}$. Then if $V = f(U)$ we get isomorphisms of schemes $f|_U$, $g|_V$. Then

$$\begin{aligned}(h g|_V)_* (f_* \mathcal{F})|_V &= h_* (g|_V)_* (f|_U)_* \mathcal{F}|_U \\ &= h_* \mathcal{F}|_U \cong \widetilde{M}\end{aligned}$$

So $f_* \mathcal{F}$ is quasi-coherent. Similarly if $\mathcal{F}$ is coherent, $f_* \mathcal{F}$ is coherent, so isomorphisms of schemes identify quasi-coherent and coherent sheaves.

NOTE Tensor Products preserve Quasi-Coherent

Let X be a scheme, $\mathcal{F}, \mathcal{G}$ quasi-coherent. We claim that $\mathcal{F} \otimes \mathcal{G}$ is quasi-coherent. Let $x \in X$ be given, and let $x \in U \subseteq X$, $g: U \Longrightarrow \operatorname{Spec} A$ be an affine open neighborhood of x. By (5.4) we have $g_* \mathcal{F} \cong \widetilde{M}$ and $g_* \mathcal{G} \cong \widetilde{N}$ for some A-modules M, N. Then

$$\begin{aligned} g_*(\mathcal{F} \otimes \mathcal{G}) &\cong g_* \mathcal{F} \otimes g_* \mathcal{G} \\ &\cong \widetilde{M} \otimes \widetilde{N} \\ &\cong \widetilde{M \otimes N}\end{aligned}$$

This implies that $\mathcal{F} \otimes \mathcal{G}$ is quasi-coherent. If X is Noetherian, $\mathcal{F}, \mathcal{G}$ both coherent, then M, N are f.g. hence so is $M \otimes N$ and so $\mathcal{F} \otimes \mathcal{G}$ is coherent.

<u>PROPOSITION 5.6</u> Let X be an affine scheme, let $0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$ be an exact sequence of $\mathcal{O}_X$-modules, and assume that $\mathcal{F}'$ is quasi-coherent. Then the sequence

$$0 \longrightarrow \Gamma(X, \mathcal{F}') \longrightarrow \Gamma(X, \mathcal{F}) \longrightarrow \Gamma(X, \mathcal{F}'') \longrightarrow 0$$

is exact.

<u>PROOF</u> We already know that Γ is a left exact functor (Ex 1.8), so we have only to show that the last map is surjective. Let $s \in \Gamma(X, \mathcal{F}'')$ be a global section of $\mathcal{F}''$. Since the map of sheaves $\mathcal{F} \longrightarrow \mathcal{F}''$ is surjective, for any $x \in X$ there is an open neighborhood $D(f)$ of x, such that $s|_{D(f)}$ lifts to a section $t \in \mathcal{F}(D(f))$. I claim that for some $n > 0$, $f^n s$ lifts to a global section of $\mathcal{F}$.

Indeed, we can cover X with a finite number of open sets $D(g_i)$ such that for each i, $s|_{D(g_i)}$ lifts to a section $t_i \in \mathcal{F}(D(g_i))$. On $D(f) \cap D(g_i) = D(fg_i)$ we have two sections $t, t_i \in \mathcal{F}(D(fg_i))$ both lifting s. Since we can assume wlog that $\mathcal{F}'$ is a subsheaf of $\mathcal{F}$, this means $t - t_i \in \mathcal{F}'(D(fg_i))$. Since $\mathcal{F}'$ is quasi-coherent, by (5.3)b for some $n > 0$, $f^n(t - t_i)$ extends to a section $u_i \in \mathcal{F}'(D(g_i))$ (g_i is a unit on $D(g_i)$). As usual, we pick one n to work for all i. Let $t'_i = f^n t_i + u_i$. Then t'_i is a lifting of $f^n s$ on $D(g_i)$, and furthermore t'_i and $f^n t$ agree on $D(fg_i)$. Now on $D(g_ig_j)$ we have two sections t'_i and t'_j of $\mathcal{F}$, both of which lift $f^n s$, so $t'_i - t'_j \in \mathcal{F}'(D(g_ig_j))$. Furthermore, t'_i and t'_j are equal on $D(fg_ig_j)$, so by (5.3a) applied to $D(g_ig_j) \cong \operatorname{Spec} A_{g_ig_j}$ we have $f^m(t'_i - t'_j) = 0$ for some $m > 0$, which we may take independent of i and j. Now the sections $f^m t'_i$ of $\mathcal{F}$ glue to give a global section t'' of $\mathcal{F}$ over X, which lifts $f^{n+m} s$. This proves the claim.

Now cover X by a finite number of open sets $D(f_i)$, $i = 1, \dots, r$ such that $s|_{D(f_i)}$ lifts to a section of $\mathcal{F}$ over $D(f_i)$ for each i. Then by the claim, we can find an integer n (one for all i) and global sections $t_i \in \Gamma(X, \mathcal{F})$ such that t_i is a lifting of $f_i^n s$. Now the open sets $D(f_i)$ cover X, so the ideal $(f_1^n, \dots, f_r^n)$ is the unit ideal of A, and we can write $1 = \sum_{i=1}^r a_i f_i^n$ with $a_i \in A$. Let $t = \sum a_i t_i$. Then t is a global section of $\mathcal{F}$ whose image in $\Gamma(X, \mathcal{F}'')$ is $\sum a_i f_i^n s = s$. This completes the proof. □

<u>NOTE</u> If X is a scheme and $X = \bigcup_i V_i$ an open cover of X, then a $\mathcal{O}_X$-module $\mathcal{F}$ is quasi-coherent if and only if $\mathcal{F}|_{V_i}$ is a quasi-coherent $\mathcal{O}_X|_{V_i}$-module $\forall i$. (This is also true for coherent modules)

<u>PROPOSITION 5.7</u> Let X be a scheme. The kernel, cokernel and image of any morphism of quasi-coherent sheaves are quasi-coherent. Any extension of quasi-coherent sheaves is quasi-coherent. If X is noetherian, the same is true for coherent sheaves.

<u>PROOF</u> Let $X = \bigcup_i V_i$ be an affine open cover of X, $V_i \cong \operatorname{Spec} A_i$. Then for any morphism $\varphi : \mathcal{F} \longrightarrow \mathcal{F}'$ to show the kernel, cokernel and image are quasi-coherent, it suffices to show the <u>canonical</u> choices are quasi-coherent. Since the kernel of $\operatorname{Coker} \varphi$ is an image for φ, it suffices to show the result for kernels and cokernels. Since for each i $-|_{V_i} : \mathcal{O}_X\text{-}\underline{\mathrm{Mod}} \longrightarrow \mathcal{O}_X|_{V_i}\text{-}\underline{\mathrm{Mod}}$ preserves kernels and cokernels, and isomorphisms $\mathcal{O}_X|_{V_i}\text{-}\underline{\mathrm{Mod}} \cong \mathcal{O}_{\operatorname{Spec} A_i}\text{-}\underline{\mathrm{Mod}}$ preserve quasi-coherent (and coherent) modules, we can reduce to the case where $X = \operatorname{Spec} A$ is affine. (For both quasi- and coherent sheaves).

The case of quasi-coherent sheaves now follows from (5.5) and the fact that $\sim$ is exact. If X is noetherian, we can use (5.5) once again to show the result (noting that A is noetherian, so submodules of f.g.-modules are f.g.).

Next we show that an extension of quasi-coherent sheaves is quasi-coherent. Let $0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$ be an exact sequence of $\mathcal{O}_X$-modules with $\mathcal{F}', \mathcal{F}''$ quasi-coherent. Then $0 \to \mathcal{F}'|_{V_i} \to \mathcal{F}|_{V_i} \to \mathcal{F}''|_{V_i} \to 0$ is an exact sequence of $\mathcal{O}_X|_{V_i}$-modules, so once again we reduce to the case where $X = \operatorname{Spec} A$ is affine. By (5.6) the corresponding sequence of global sections over X is exact, say $0 \to M' \to M \to M'' \to 0$. Applying $\sim$, using naturality of the counit $\widetilde{\mathcal{F}(X)} \to \mathcal{F}$ of the adjunction $\sim \dashv \Gamma$ we have an exact commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widetilde{M'} & \longrightarrow & \widetilde{M} & \longrightarrow & \widetilde{M''} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{F}' & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{F}'' & \longrightarrow & 0 \end{array}$$

The two outside arrows are isomorphisms, since $\mathcal{F}'$ and $\mathcal{F}''$ are quasi-coherent. Since $\mathcal{O}_X$-$\underline{\mathrm{Mod}}$ is abelian the 5-Lemma implies that $\widetilde{M} \longrightarrow \mathcal{F}$ is also an isomorphism, so $\mathcal{F}$ is quasi-coherent.

In the noetherian case, if $\mathcal{F}'$ and $\mathcal{F}''$ are coherent then M' and M'' are f.g., so M is f.g. and hence $\mathcal{F}$ is coherent. □

$\underline{\text{NOTE}}$ Sheaves of Ideals of Affine Schemes

Let A be a ring, $\mathfrak{a} \subseteq A$ an ideal. The inclusion $\mathfrak{a} \xrightarrow{i} A$ becomes a morphism of $\mathcal{O}_X$-modules $\tilde{\mathfrak{a}} \xrightarrow{\tilde{i}} \tilde{A} = \mathcal{O}_X$ $(X = \operatorname{Spec} A)$ which is injective since $\sim$ is exact. For $U \subseteq X$, we claim that if $\mathfrak{a}$ is radical then

$$\operatorname{Im} \tilde{i}_U = \{ s \in \mathcal{O}_X(U) \mid s(\mathfrak{p}) \in \mathfrak{a}A_\mathfrak{p} \ \ \forall \mathfrak{p} \in U \} \qquad (1)$$

One inclusion is clear, since $\tilde{\mathfrak{a}} \longrightarrow \tilde{A}$ is defined by $\mathfrak{a}_\mathfrak{p} \xrightarrow{\phi_\mathfrak{p}} A_\mathfrak{p}$ $a/s \longmapsto a/s$. In fact, this map is bijective with $\mathfrak{a}A_\mathfrak{p}$ and makes the following diagram commute

$\ulcorner \Gamma\mathfrak{a} = A$ is trivial$\lrcorner$

$$\begin{tikzcd}
\tilde{\mathfrak{a}} \arrow[r] \arrow[d, Rightarrow] & \tilde{A} \arrow[d, Rightarrow] \\
\mathfrak{a}_\mathfrak{p} \arrow[r] \arrow[dr, Rightarrow, "\phi"'] & A_\mathfrak{p} \\
 & \mathfrak{a}A_\mathfrak{p} \arrow[u]
\end{tikzcd}$$

So let $s \in \mathcal{O}_X(U)$ be given and assume $s(\mathfrak{p}) \in \mathfrak{a}A_\mathfrak{p}$ $\forall \mathfrak{p} \in U$. Define $t : U \longrightarrow \bigcup_{\mathfrak{p} \in U} \mathfrak{a}_\mathfrak{p}$ by $\mathfrak{p} \longmapsto \phi^{-1}(s(\mathfrak{p}))$. To show that (1) is true, we need only show t is regular. Let $\mathfrak{p} \in U$ be given, find $\mathfrak{p} \in V \subseteq U$ $b, s \in A$ s.t. $s \notin \mathfrak{q}$ $\forall \mathfrak{q} \in V$ and $s(\mathfrak{q}) = b/s \in A_\mathfrak{q}$ $\forall \mathfrak{q} \in V$. Let $f \in A$ be s.t. $\mathfrak{p} \in D(f) \subseteq V$. Then f belongs to every prime in $V(\mathfrak{a}) - V \cap V(\mathfrak{a})$ and b belongs to every prime in $V(\mathfrak{a}) \cap V$, since $s(\mathfrak{q}) \in \mathfrak{a}A_\mathfrak{q} \Rightarrow tb \in \mathfrak{a}$ for some $t \notin \mathfrak{q}$, hence $b \in \mathfrak{q}$. It follows that fb belongs to every prime containing $\mathfrak{a}$, and hence to $\mathfrak{a}$ itself, since $\mathfrak{a}$ is radical. Say $bf = a \in \mathfrak{a}$, and put $t = sf$. Then $sf \notin \mathfrak{q}$ $\forall \mathfrak{q} \in D(f)$ and for $\mathfrak{q} \in D(f)$ $t(\mathfrak{q}) = a/t$, as required. Hence t is regular and (1) holds.

Now let $\mathscr{I}$ be a quasi-coherent sheaf of ideals on X. Let $\mathfrak{a} \subseteq A$ be the ideal corresponding to $\mathscr{I}(X)$ under the isomorphism $\mathcal{O}_X(X) \cong A$. Then by (5.4) the following is an isomorphism of $\mathcal{O}_X$-modules:

$$\beta : \tilde{\mathfrak{a}} \longrightarrow \mathscr{I}$$
$$\beta_U(s)(\mathfrak{p}) = \phi_\mathfrak{p}(s(\mathfrak{p}))$$

$\ulcorner \phi_\mathfrak{p} : \mathfrak{a}_\mathfrak{p} \longrightarrow A_\mathfrak{p}$ canonical$\lrcorner$

Hence there is a commutative diagram

$$\begin{tikzcd}
\mathscr{I}_\mathfrak{p} \arrow[r, tail] \arrow[d, Rightarrow] & \mathcal{O}_{X,\mathfrak{p}} \arrow[d, Rightarrow] \\
\mathfrak{a}_\mathfrak{p} \arrow[r, tail] & A_\mathfrak{p}
\end{tikzcd}$$

which shows that $\mathcal{O}_{X,\mathfrak{p}} \cong A_\mathfrak{p}$ identifies $\mathscr{I}_\mathfrak{p}$ with $\mathfrak{a}A_\mathfrak{p}$. Hence

$$(\mathcal{O}_X/\mathscr{I})_\mathfrak{p} \cong \mathcal{O}_{X,\mathfrak{p}}/\mathscr{I}_\mathfrak{p} \cong A_\mathfrak{p}/\mathfrak{a}A_\mathfrak{p} \qquad (2)$$

Clearly the following diagram commutes:

$$\begin{tikzcd}
\tilde{\mathfrak{a}} \arrow[rr, Rightarrow] \arrow[dr] & & \mathscr{I} \arrow[dl] \\
 & \mathcal{O}_X &
\end{tikzcd}$$

If we associate with every ideal $\mathfrak{a}$ the quasi-coherent sheaf of ideals $\operatorname{Im}(\tilde{\mathfrak{a}} \longrightarrow \mathcal{O}_X)$ then we have

$\underline{\text{LEMMA}}$ There is a bijection between ideals $\mathfrak{a} \subseteq A$ and quasi-coherent sheaves of ideals of $\mathcal{O}_X$, defined by

$$\mathfrak{a} \longmapsto \operatorname{Im}(\tilde{\mathfrak{a}} \longrightarrow \tilde{A})$$
$$\mathscr{I} \longmapsto \mathfrak{a} \text{ where } \mathscr{I}, \mathfrak{a} \text{ correspond under } \mathcal{O}_X(X) \cong A.$$

We denote the quasi-coherent sheaf associated to $\mathfrak{a}$ by $\mathscr{I}_\mathfrak{a}$. Note that $\mathscr{I}_A = \mathcal{O}_X$ and $\mathscr{I}_0 = 0$.

PROPOSITION Let $\mathfrak{a} \subseteq A$ be an ideal, $\mathcal{I}_{\mathfrak{a}}$ the associated quasi-coherent sheaf of ideals. Then

$$\operatorname{Supp} \mathcal{O}_X/\mathcal{I}_{\mathfrak{a}} = V(\mathfrak{a})$$

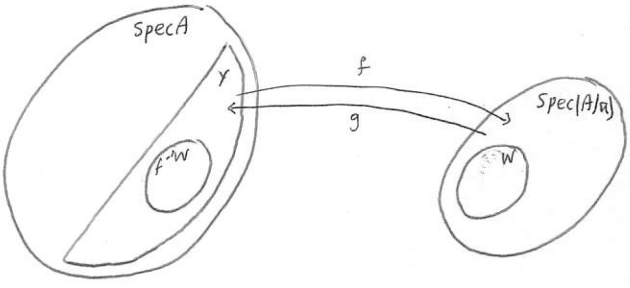
Let $Y = V(\mathfrak{a})$ and $i: Y \longrightarrow X$ be the inclusion. Then if $\mathcal{O}_Y = i^{-1}(\mathcal{O}_X/\mathcal{I}_{\mathfrak{a}})$ the locally ringed space $(Y, \mathcal{O}_Y)$ is isomorphic to $\operatorname{Spec} A/\mathfrak{a}$.

PROOF. It follows immediately from (2) that $\operatorname{supp} \mathcal{O}_X/\mathcal{I}_{\mathfrak{a}} = V(\mathfrak{a})$ and that $(Y, \mathcal{O}_Y)$ is a locally ringed space. If $Y = \phi$ then $\mathfrak{a} = A$ in which case trivially $(Y, \mathcal{O}_Y) \cong \operatorname{Spec} A$, so assume $Y \neq \phi$. Consider the following chain of ring isomorphisms for $\mathfrak{p} \supseteq \mathfrak{a}$

$$\phi_{\mathfrak{p}} : (A/\mathfrak{a})_{\mathfrak{p}} \Longrightarrow A_{\mathfrak{p}}/\mathfrak{a}A_{\mathfrak{p}} \Longrightarrow \mathcal{O}_{X,\mathfrak{p}}/\mathcal{I}_{\mathfrak{a},\mathfrak{p}} \Longrightarrow (\mathcal{O}_X/\mathcal{I}_{\mathfrak{a}})_{\mathfrak{p}} \Longrightarrow P_{\mathfrak{p}}$$

$$\phi_{\mathfrak{p}}\left(\frac{r+\mathfrak{a}}{s+\mathfrak{a}}\right) = \left(D(s) \cap Y, \left(D(s), \dot{r/s} + \mathcal{I}_{\mathfrak{a}}(D(s))\right)\right)$$

Where $P(V) = \varinjlim_{W \supseteq V} (\mathcal{O}_X/\mathcal{I}_{\mathfrak{a}})(W)$ sheafifies to $\mathcal{O}_Y$. Let $f: Y \longrightarrow \operatorname{Spec}(A/\mathfrak{a})$ be the canonical homeomorphism, with inverse g



Let and define $f^{\#}_W : \mathcal{O}_{\operatorname{Spec}(A/\mathfrak{a})}(W) \longrightarrow \mathcal{O}_Y(f^{-1}W)$.

$$f^{\#}_W(s)(\mathfrak{p}) = \phi_{\mathfrak{p}}(s(f(\mathfrak{p})))$$

It is easy to see that $f^{\#}_W(s) \in \mathcal{O}_Y(f^{-1}W)$, and thus that $f^{\#}$ is an isomorphism. Hence $(Y, \mathcal{O}_Y) \cong \operatorname{Spec} A/\mathfrak{a}$, as claimed. Let $h: \mathcal{O}_Y \longrightarrow \operatorname{Spec}(A/\mathfrak{a}) \longrightarrow \operatorname{Spec} A$ be the composite, then the underlying map of h is the inclusion i. So we have a commutative diagram

$$\begin{tikzcd}
& \operatorname{Spec} A & \\
(Y, \mathcal{O}_Y) \arrow[ur, "i"] \arrow[rr, Rightarrow] & & \operatorname{Spec}(A/\mathfrak{a}) \arrow[ul]
\end{tikzcd}$$

In particular i is a closed immersion. $\square$

LEMMA If $\mathfrak{a} \subseteq A$ is a radical ideal then $\mathcal{I}_{\mathfrak{a}} = \mathcal{I}_{V(\mathfrak{a})}$, where $\mathcal{I}_{V(\mathfrak{a})}$ consists of sections vanishing on the closed set $V(\mathfrak{a})$ (see §3 notes)

PROOF From (1) we know that

$$\mathcal{I}_{\mathfrak{a}}(U) = \{ s \in \mathcal{O}_X(U) \mid s(\mathfrak{p}) \in \mathfrak{a}A_{\mathfrak{p}} \ \forall \mathfrak{p} \in U \}$$

And by definition

$$\begin{aligned} \mathcal{I}_{V(\mathfrak{a})}(U) &= \{ s \in \mathcal{O}_X(U) \mid \operatorname{germ}_{\mathfrak{p}} s \in \mathfrak{p}A_{\mathfrak{p}} \ \forall \mathfrak{p} \in V(\mathfrak{a}) \cap U \} \\ &= \{ s \in \mathcal{O}_X(U) \mid s(\mathfrak{p}) \in \mathfrak{p}A_{\mathfrak{p}} \ \forall \mathfrak{p} \in V(\mathfrak{a}) \cap U \} \end{aligned}$$

So it is clear that $\mathscr{I}_{\mathfrak{a}}(V) \subseteq \mathscr{I}_{V(\mathfrak{a})}(U)$, since if $\mathfrak{p} \in V(\mathfrak{a})$ then $\mathfrak{a}A_{\mathfrak{p}} \subseteq \mathfrak{p}A_{\mathfrak{p}}$. To show the reverse inclusion, let $s \in \mathscr{I}_{V(\mathfrak{a})}(U)$ be given, and $\mathfrak{p} \in U$. Then there are $f, a, s \in A$ with $\mathfrak{p} \in D(f) \subseteq U$ and $s \notin \mathfrak{q}$ $\forall \mathfrak{q} \in D(f)$ such that $s(\mathfrak{q}) = a/s \in A_{\mathfrak{q}}$ $\forall \mathfrak{q} \in D(f)$. For $\mathfrak{q} \in D(f) \cap V(\mathfrak{a})$ we have $a/s = s(\mathfrak{q}) \in \mathfrak{q}A_{\mathfrak{q}}$ by assumption. Hence $a \in \mathfrak{q}$. For $\mathfrak{q} \in V(\mathfrak{a}) - D(f) \cap V(\mathfrak{a})$ we have $f \in \mathfrak{q}$. Hence af belongs to every prime containing $\mathfrak{a}$, hence since $\mathfrak{a}$ is radical, $af \in \mathfrak{a}$. Then in particular $\mathfrak{p} \in D(f)$ so $f \notin \mathfrak{p}$ and

$$s(\mathfrak{p}) = a/s = af/sf \in \mathfrak{a}A_{\mathfrak{p}}$$

Since $\mathfrak{p}$ was arbitrary, we see that $s \in \mathscr{I}_{\mathfrak{a}}(U)$, as required. $\square$

NOTE (5.7) implies that any finite coproduct (or product) of quasi-coherent sheaves is quasi-coherent, and if X is noetherian the same is true for coherent sheaves.

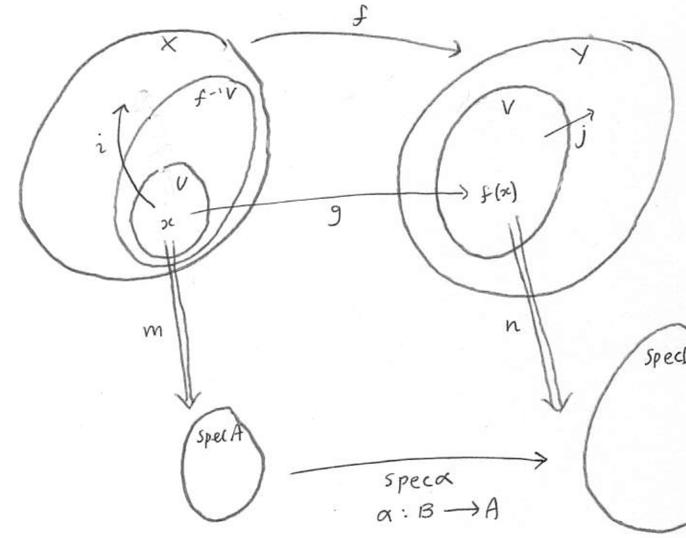
PROPOSITION 5.8 Let $f: X \longrightarrow Y$ be a morphism of schemes. Then

(a) If $\mathcal{G}$ is a quasi-coherent sheaf of $\mathcal{O}_Y$-modules, then $f^*\mathcal{G}$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules

(b) If Y is noetherian, and $\mathcal{G}$ is coherent, then $f^*\mathcal{G}$ is coherent.

(c) Assume either that X is noetherian, or f quasi-compact and separated. Then if $\mathcal{F}$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules, $f_*\mathcal{F}$ is a quasi-coherent sheaf of $\mathcal{O}_Y$-modules.

PROOF (a) Let $\mathcal{G}$ be a quasi-coherent sheaf of $\mathcal{O}_Y$-modules, and note that to show $f^*\mathcal{G}$ is quasi-coherent, it suffices to show $(f^*\mathcal{G})|_U$ is quasi-coherent on a cover of X. Let $x \in X$ be given, $f(x) \in V \subseteq Y$ an affine open neighborhood of $f(x)$ in Y and $x \in U \subseteq f^{-1}V$ an affine neighborhood of x. Let $n: V \Rightarrow \operatorname{spec} B$ and $m: U \Rightarrow \operatorname{spec} A$ be isomorphisms. Then $n_*\mathcal{G}|_V \cong \tilde{M}$ for some B-module M. consider



Using many earlier results, we have

$$
\begin{aligned}
m_*(f^*\mathcal{G})|_U &\cong (m^{-1})^* i^* f^* \mathcal{G} \\
&\cong (m^{-1})^*(fi)^*\mathcal{G} \\
&\cong (m^{-1})^*(jg)^*\mathcal{G} \\
&\cong (m^{-1})^* g^* j^* \mathcal{G} \\
&\cong (gm^{-1})^* j^* \mathcal{G} \\
&\cong (n^{-1} \operatorname{spec}\alpha)^* \mathcal{G}|_V \\
&\cong (\operatorname{spec}\alpha)^* n_*\mathcal{G}|_V \\
&\cong (\operatorname{spec}\alpha)^* \tilde{M} \cong \widetilde{(M \otimes_B A)}
\end{aligned}
$$

So $f^*\mathcal{G}$ is quasi-coherent, as required.

(b) If Y is noetherian, $\mathcal{G}$ coherent, then in (a) by (5.4) M will be a f.g. B-module. Hence $M \otimes_B A$ is a f.g. A-module, and $f^*\mathcal{G}$ is coherent (no need for X noetherian).

(c) Let $f: X \longrightarrow Y$ be given, $\mathcal{F}$ a quasi-coherent $\mathcal{O}_X$-module. To show $f_*\mathcal{F}$ quasi-coherent, it suffices to show that each point $y \in Y$ has an open neighborhood V with $f_*\mathcal{F}|_V$ quasi-coherent. Let $y \in Y$ be given, and let V be an affine open neighborhood of y, $n: V \Rightarrow \operatorname{Spec} A$ an isomorphism.

Let $g: f^{-1}V \longrightarrow V$ be induced. If X is noetherian, so is $f^{-1}V$, and if f is quasi-compact and separated, so is g (both properties are local, as shown earlier). Now

$$
\begin{aligned}
n_*(f_*\mathcal{G}|_V) &= n_* g_* \mathcal{F}|_{f^{-1}V} \\
&= (ng)_* \mathcal{F}|_{f^{-1}V}
\end{aligned}
$$

Since $\mathcal{F}|_{f^{-1}V}$ is quasi-coherent, we have reduced to the case where Y is affine. Under either hypothesis X is quasi-compact, so we can cover X with a finite number of open affine subsets U_i (case $X=\emptyset$ is trivial). In the separated case $U_i \cap U_j$ is again affine (Ex 4.3). Call it U_{ijk}. In the noetherian case $U_i \cap U_j$ is at least quasi-compact, so we can cover it with a finite number of open affine subsets U_{ijk}. Now for any open subset $V \subseteq Y$, giving a section s of $\mathcal{F}$ over $f^{-1}V$ is the same thing as giving a collection of sections s_i of $\mathcal{F}$ over $f^{-1}V \cap U_i$ whose restrictions to the open subsets $f^{-1}V \cap U_{ijk}$ are all equal. Hence the following sequence is an exact sequence of $\mathcal{O}_Y$-modules

$$0 \longrightarrow f_*\mathcal{F} \xrightarrow{K} \bigoplus_i f_*(\mathcal{F}|_{U_i}) \xrightarrow{\alpha-\beta} \bigoplus_{i,j,k} f_*(\mathcal{F}|_{U_{ijk}}) \qquad (1)$$

where

$$\alpha_V((s_i))_{ijk} = s_i|_{f^{-1}V \cap U_{ijk}}$$

$$\beta_V((s_i))_{ijk} = s_j|_{f^{-1}V \cap U_{ijk}}$$

$$K_V(s)_i = s|_{f^{-1}V \cap U_i}$$

The coproducts in (1) are the pointwise products, which are coproducts since the index sets are finite. By abuse of notation we refer to the induced morphisms $U_i \longrightarrow Y$ and $U_{ijk} \longrightarrow Y$ by f also. It is clear from (5.2) that the $f_*(\mathcal{F}|_{U_i})$ and $f_*(\mathcal{F}|_{U_{ijk}})$ are quasi-coherent, so by (5.7) so are the coproducts. Hence by (5.7) $f_*\mathcal{F}$ is quasi-coherent, as required. □

<u>CAUTION</u> If X and Y are noetherian, it is <u>not</u> true in general that f_* of a coherent sheaf is coherent. (See Ex 5.5). However, it is true if f is a finite morphism, or a projective morphism (5.20) or more generally a proper morphism (EGA III, 3.2.1)

<u>DEFINITION</u> Let Y be a closed subscheme of a scheme X, and let $i: Y \longrightarrow X$ be the inclusion morphism. We define the <u>ideal sheaf</u> of Y, denoted $\mathcal{I}_Y$, to be the kernel of the morphism $i^\#: \mathcal{O}_X \longrightarrow i_*\mathcal{O}_Y$.

<u>PROPOSITION 5.9</u> Let X be a scheme. For any closed subscheme Y of X, the corresponding ideal sheaf $\mathcal{I}_Y$ is a quasi-coherent sheaf of ideals on X, which is independent of the choice of closed immersion. If X is noetherian, $\mathcal{I}_Y$ is coherent. Conversely, any quasi-coherent sheaf of ideals on X is the ideal sheaf of a uniquely determined closed subscheme of X.

<u>PROOF</u> Suppose a closed subscheme is represented by two closed immersions $i: Y \longrightarrow X$ and $k: Z \longrightarrow X$. Say $j: Z \Longrightarrow Y$ is an isomorphism with $ij = k$. Then $j^\#: \mathcal{O}_Y \longrightarrow j_*\mathcal{O}_Z$ is an isomorphism, and the following diagram commutes

$$\begin{tikzcd}
\mathcal{O}_X \arrow[r, "i^\#"] \arrow[dr, "k^\#"'] & i_*\mathcal{O}_Y \arrow[d, Rightarrow, "i_*j^\#"] \\
& k_*\mathcal{O}_Z = i_*j_*\mathcal{O}_Z
\end{tikzcd}$$

So $\operatorname{Ker} i^\# = \operatorname{Ker} k^\#$ and i, j both determine the same ideal sheaf. To see that $\mathcal{I}_Y$ is quasi-coherent, note that i is quasi-compact (any closed immersion is (Ex 3.13)) and separated (see §4) so by (5.8) $i_*\mathcal{O}_Y$ is quasi-coherent on X. By definition $i^\#: \mathcal{O}_X \longrightarrow i_*\mathcal{O}_Y$ is a morphism of $\mathcal{O}_X$-modules, so by (5.7) $\mathcal{I}_Y$ is quasi-coherent. If X is noetherian, then for any open affine $U \cong \operatorname{Spec} A$ of X the ring A is noetherian, so the ideal $\mathcal{I}_Y(U)$ is finitely-generated, so $\mathcal{I}_Y$ is coherent.

Conversely let X be a scheme, $\mathcal{I}$ a quasi-coherent sheaf of ideals, $\mathcal{O}_X/\mathcal{I}$ the quotient sheaf of rings on X. Let Y be the support of $\mathcal{O}_X/\mathcal{I}$. That is,

$$\begin{aligned}
Y &= \{x \in X \mid (\mathcal{O}_X/\mathcal{I})_x \neq 0\} \\
&= \{x \in X \mid \mathcal{O}_{X,x}/\mathcal{I}_x \neq 0\} \\
&= \{x \in X \mid \mathcal{I}_x \neq \mathcal{O}_{X,x}\} \\
&= \{x \in X \mid \text{is some section } s \in \mathcal{O}_X(U) \text{ over a neighborhood of } x \text{ s.t. } s|_V \notin \mathcal{I}(V) \text{ for every open } V \subseteq U \text{ with } x \in V\}
\end{aligned}$$

[See our Induced Reduced structure notes in §4]

Our first task is to show that Y is closed. Let $\cup_i V_i$ be an affine open cover of X. It would suffice to show that $Y \cap V_i$ is closed in V_i for all i. The restriction of $\mathcal{I}$ to V_i is quasi-coherent, and $\mathcal{I}(V_i)$ is identified with an ideal $\mathfrak{a} \subseteq A$ s.t. $\mathcal{I} \cong \tilde{\mathfrak{a}}$ on $\operatorname{Spec} A$. To be more precise, let $n : V_i \Rightarrow \operatorname{Spec} A$ be the isomorphism, and Ω the sheaf of ideals of $\mathcal{O}_{\operatorname{Spec} A}$ with $n_* \mathcal{I} \cong \Omega$. Then let $\mathfrak{a} \subseteq A$ correspond to $\Omega \subseteq \mathcal{O}_{\operatorname{Spec} A}(\operatorname{Spec} A)$

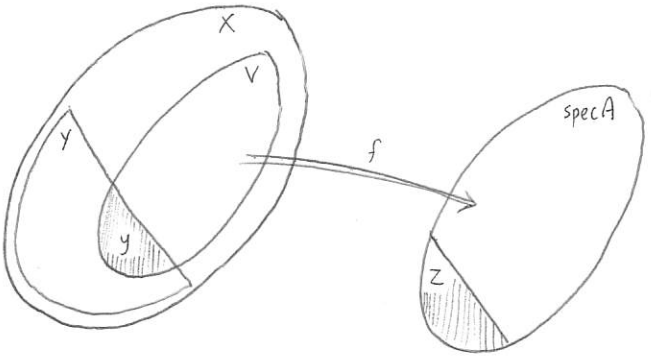
$$
\begin{aligned}
n(Y \cap V_i) &= \{ \mathfrak{p} \in \operatorname{Spec} A \mid (\mathcal{O}_{\operatorname{Spec} A}/\Omega)_{\mathfrak{p}} \neq 0 \} \\
&= \{ \mathfrak{p} \in \operatorname{Spec} A \mid \mathcal{O}_{\operatorname{Spec} A, \mathfrak{p}}/\Omega_{\mathfrak{p}} \neq 0 \}
\end{aligned}
$$

We claim the isomorphism $\mathcal{O}_{\operatorname{Spec} A, \mathfrak{p}} \Rightarrow A_{\mathfrak{p}}$ identifies $\Omega_{\mathfrak{p}}$ and $\mathfrak{a} A_{\mathfrak{p}}$. (We showed this in an earlier note). Hence $\mathcal{O}_{\operatorname{Spec} A, \mathfrak{p}}/\Omega_{\mathfrak{p}} \cong A_{\mathfrak{p}}/\mathfrak{a} A_{\mathfrak{p}}$ which is 0 iff. $\mathfrak{a} \not\subseteq \mathfrak{p}$, so $n(Y \cap V_i) = V(\mathfrak{a})$, which is closed. Hence the support is closed

Let $i : Y \longrightarrow X$ be the inclusion. Then $i^{-1}(\mathcal{O}_X/\mathcal{I})$ is a sheaf of rings on Y, and

$$
i^{-1}(\mathcal{O}_X/\mathcal{I})_y \cong (\mathcal{O}_X/\mathcal{I})_y \cong \mathcal{O}_{X,y}/\mathcal{I}_y
$$

For all $y \in Y$. Hence $(Y, i^{-1}(\mathcal{O}_X/\mathcal{I}))$ is a locally ringed space, since $\mathcal{I}_y \subset \mathcal{O}_{X,y}$ $\forall y \in Y$. Put $\mathcal{O}_Y = i^{-1}(\mathcal{O}_X/\mathcal{I})$. We claim that $(Y, \mathcal{O}_Y)$ is a scheme. Let $y \in Y$ be given and let $y \in V \subseteq X$ be an open affine, with $f : V \longrightarrow \operatorname{Spec} A$ an isomorphism.



$$
\begin{tikzcd}
Y \arrow[r, "i"] & X \\
Y \cap U \arrow[u, "j"] \arrow[r, "n"] \arrow[d, Rightarrow, "f'"'] & U \arrow[u, "k"'] \arrow[d, Rightarrow, "f"] \\
Z \arrow[r, "t"'] & \operatorname{Spec} A
\end{tikzcd}
$$

Morphisms of spaces

As above, let $\Omega = \mathcal{I}_{\mathfrak{a}}$ be the quasi-coherent sheaf of ideals on $\operatorname{Spec} A$ with $f_* \mathcal{I}|_V \cong \Omega$. The following diagram commutes:

$$
\begin{tikzcd}
f_* \mathcal{I}|_V \arrow[r] & f_* \mathcal{O}_X|_V \\
\Omega \arrow[u, Rightarrow] \arrow[r] & \mathcal{O}_{\operatorname{Spec} A} \arrow[u]
\end{tikzcd}
$$

From which it follows that $f_*(\mathcal{O}_X/\mathcal{I}|_V) \cong \mathcal{O}_{\operatorname{Spec} A}/\Omega$. Hence

$$
\begin{aligned}
\mathcal{O}_Y|_{Y \cap V} &\cong j^{-1} i^{-1} \mathcal{O}_X/\mathcal{I} \\
&\cong n^{-1} k^{-1} \mathcal{O}_X/\mathcal{I} \\
&\cong n^{-1}(\mathcal{O}_X/\mathcal{I}|_V) \\
&\cong n^{-1} f^{-1} f_*(\mathcal{O}_X/\mathcal{I}|_V) \\
&\cong (fn)^{-1} \mathcal{O}_{\operatorname{Spec} A}/\Omega \\
&\cong f'^{-1} t^{-1} \mathcal{O}_{\operatorname{Spec} A}/\Omega
\end{aligned}
$$

Since $t^{-1} \mathcal{O}_{\operatorname{Spec} A}/\Omega \cong \operatorname{Spec} A/\mathfrak{a}$ it follows that $(Y \cap V, \mathcal{O}_Y|_{Y \cap V}) \cong \operatorname{Spec} A/\mathfrak{a}$, as required. Hence $(Y, \mathcal{O}_Y)$ is a scheme.

We define a closed immersion $i: Y \longrightarrow X$ by $i^\#_U : \mathcal{O}_X(U) \longrightarrow \mathcal{O}_Y(Y \cap U)$

$$i^\#_U(s) = (U, s + \mathcal{I}(U))$$

This is the composite $\mathcal{O}_X(U) \longrightarrow \mathcal{O}_X(U)/\mathcal{I}(U) \longrightarrow (\mathcal{O}_X/\mathcal{I})(U) \longrightarrow \varinjlim_{W \supseteq Y \cap U} (\mathcal{O}_X/\mathcal{I})(W) \longrightarrow \mathcal{O}_Y(Y \cap U)$, hence is a morphism of rings. It is clearly a morphism of sheaves $\mathcal{O}_X \longrightarrow i_* \mathcal{O}_Y$. One checks that the following diagram commutes for $y \in Y$:

$$\begin{tikzcd} \mathcal{O}_{X,y} \arrow[r, "i^\#_y"] \arrow[ddr] & \mathcal{O}_{Y,y} \arrow[d, Rightarrow] \\ & (\mathcal{O}_X/\mathcal{I})_y \arrow[d, Rightarrow] \\ & \mathcal{O}_{X,y}/\mathcal{I}_y \end{tikzcd} \qquad (1)$$

showing that i is a morphism of schemes. To show that i is a closed immersion, it suffices to note that $i^\#_y$ is surjective $\forall y \in Y$. (see notes in §3)

Next we show that the ideal sheaf of $i: Y \longrightarrow X$ is $\mathcal{I}$. It suffices to show that the following sequence of $\mathcal{O}_X$-modules is exact:

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_X \xrightarrow{i^\#} i_* \mathcal{O}_Y \longrightarrow 0$$

And we can check exactness of this sequence on stalks. If $x \in X - Y$ then $(i_* \mathcal{O}_Y)_x = 0$ (see Ex 1.19) and $\mathcal{I}_x \longrightarrow \mathcal{O}_{X,x}$ is an isomorphism (since $x \notin Y$ means by definition $\mathcal{I}_x = \mathcal{O}_{X,x}$), so the sequence of stalks is exact. If $y \in Y$ then the sequence becomes

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{I}_y \arrow[r] & \mathcal{O}_{X,y} \arrow[r] \arrow[ddr] & (i_* \mathcal{O}_Y)_y \arrow[r] \arrow[d, Rightarrow] & 0 \\ & & & \mathcal{O}_{Y,y} \arrow[d, Rightarrow] & \\ & & & \mathcal{O}_{X,y}/\mathcal{I}_y & \end{tikzcd} \qquad (2)$$

The composite $\mathcal{O}_{X,y} \longrightarrow (i_* \mathcal{O}_Y)_y \longrightarrow \mathcal{O}_{Y,y}$ is $i^\#_y$, and from (1) we know the triangle in (2) commutes. So the sequence of stalks is exact $\forall x \in X$, and hence $\mathcal{I} = \operatorname{Ker} i^\#$, as desired.

To complete the proof, we need to show that if $Z \longrightarrow X$ is a closed subscheme with ideal sheaf $\mathcal{I}$, then $(Y, i^{-1}(\mathcal{O}_X/\mathcal{I}))$ and $Z \longrightarrow X$ determine the same closed subscheme. We may assume $Z \subseteq X$ is closed and the underlying map of $j: Z \longrightarrow X$ is the inclusion. So we have an exact sequence

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_X \longrightarrow j_* \mathcal{O}_Z \longrightarrow 0$$

So there is an isomorphism $\mathcal{O}_X/\mathcal{I} \xrightarrow{k} j_* \mathcal{O}_Z$ induced by the presheaf morphism

$$P \longrightarrow j_* \mathcal{O}_Z \qquad \ulcorner P(U) = \mathcal{O}_X(U)/\mathcal{I}(U) \lrcorner$$
$$s + \mathcal{I}(U) \longmapsto j^\#_U(s)$$

which is clearly an isomorphism of sheaves of rings. Thus $\operatorname{Supp} \mathcal{O}_X/\mathcal{I} = Z$ and also

$$\varepsilon : j^{-1} j_* \mathcal{O}_Z \longrightarrow \mathcal{O}_Z$$

is an isomorphism (this is true whenever j is the inclusion of a subspace). So we have an isomorphism of sheaves of rings on Z:

$$j^{-1}(\mathcal{O}_X/\mathcal{I}) \xRightarrow{j^{-1}k} j^{-1} j_* \mathcal{O}_Z \xRightarrow{\varepsilon} \mathcal{O}_Z$$

Denote by i the morphism $(Z, j^{-1}(\mathcal{O}_X/\mathcal{I})) \longrightarrow (X, \mathcal{O}_X)$. We need only show the following commutes:

$$\begin{tikzcd} & (X, \mathcal{O}_X) & \\ (Z, j^{-1}(\mathcal{O}_X/\mathcal{I})) \arrow[ur, "i"] & & (Z, \mathcal{O}_Z) \arrow[ul, "j"'] \arrow[ll, Leftarrow] \end{tikzcd}$$

$$\begin{tikzcd} & \mathcal{O}_X(U) \arrow[dl, "i^{\#}_U"'] \arrow[dr, "j^{\#}_U"] & \\ j^{-1}(\mathcal{O}_X/\mathcal{I})(Z\cap U) \arrow[rr] & & \mathcal{O}_Z(Z\cap U) \end{tikzcd}$$

Given $s \in \mathcal{O}_X(U)$, $i^{\#}_U(s) = (U, \dot{s} + \mathcal{I}(U))$ which becomes $(U, \dot{j}^{\#}_U(s))$ in $(j^{-1}j_*\mathcal{O}_Z)(Z\cap U)$ and then $j^{\#}_U(s)|_{Z\cap U} = j^{\#}_U(s)$ in $\mathcal{O}_Z(Z\cap U)$, as required. Hence $Z \longrightarrow X$ is equal as a closed subscheme to the closed immersion induced by its ideal sheaf. So we have defined a bijection between closed subschemes of X and quasi-coherent sheaves of ideals. $\square$

To summarise, let X be any scheme:

Closed subschemes		Quasi-coherent sheaves of ideals
$j: Y \longrightarrow X$	$\longmapsto$	$\mathcal{I} = Ker(j^{\#}: \mathcal{O}_X \longrightarrow j_*\mathcal{O}_Y)$
Let $Y = supp(\mathcal{O}_X/\mathcal{I})$, $\mathcal{O}_Y = i^{-1}\mathcal{O}_X/\mathcal{I}$ where $i: Y \longrightarrow X$ inclusion. Then $(Y, \mathcal{O}_Y) \longrightarrow (X, \mathcal{O}_X)$ is a closed subscheme.	$\longleftarrow$	$\mathcal{I}$

$\mathcal{I} = 0$ is the closed subscheme 1_X

<u>COROLLARY 5.10</u> If $X = Spec A$ is an affine scheme, there is a bijection between ideals $\mathfrak{a} \subseteq A$ and closed subschemes Y of X, given by $\mathfrak{a} \longmapsto Spec(A/\mathfrak{a}) \longrightarrow Spec(A)$. In particular every closed subscheme of an affine scheme is affine.

<u>PROOF</u> The bijection follows from (5.9) and our note on ideal sheaves on affine schemes. In (5.9), the closed subscheme $j: Spec(A/\mathfrak{a}) \longrightarrow Spec(A)$ is associated with the sheaf of ideals $\mathcal{I}$ with

$$\mathcal{I}(Spec(A)) = Ker(\mathcal{O}_{SpecA}(SpecA) \longrightarrow \mathcal{O}_{SpecA/\mathfrak{a}}(Spec\, A/\mathfrak{a}))$$

Now $a \in A$ corresponds to a global section of $\mathcal{I}$ ~~iff~~ $a \in \mathfrak{a}$, so our earlier note associates $\mathcal{I}$ with $\mathfrak{a}$, showing that the bijection has the desired form. $\square$

<u>COROLLARY</u> Let $f: X \longrightarrow Y$ be a closed immersion with associated ideal sheaf $\mathcal{I}$. The closed image of f is precisely the support of $\mathcal{O}_Y/\mathcal{I}$. That is,

$$f(X) = supp(\mathcal{O}_X/\mathcal{I}) \quad (= supp(f_*\mathcal{O}_X))$$

<u>NOTE</u> The bijection above is between closed subschemes (closed subset + scheme structure) and q.c. sheaves of ideals since two closed subschemes can be distinct but have the same image (for example $Spec A \longrightarrow Spec A$ and $Spec\, A/\mathfrak{N} \longrightarrow Spec A$, $\mathfrak{N}$ the nilradical). This example suggests that if X is a reduced scheme then the only surjective closed subscheme $Y \longrightarrow X$ is $X \Longrightarrow X$. This follows from the fact that if $t \in \mathcal{O}_X(V)$ has $germ_x t \in \mathfrak{m}_x$ $\forall x \in V$, then $t = 0$. (see our §3 notes on induced reduced schemes).

NOTE Closed Subschemes and Ideal Sheaves

Let X be a scheme. We say a closed subscheme $i : Y \longrightarrow X$ precedes another closed subscheme $j : Y' \longrightarrow X$, written $i \leq j$, if i factors through j. By (5.9) there is a bijection between closed subschemes of X and quasi-coherent sheaves of ideals on X. In particular, the former class is actually a set. The relation $\leq$ makes the closed subschemes into a partially ordered set. The set of quasi-coherent sheaves of ideals on X is partially ordered by saying $\mathcal{J} \subseteq \mathcal{K}$ iff. $\mathcal{J}(U) \subseteq \mathcal{K}(U)$ $\forall U \subseteq X$.

PROPOSITION The map from closed subschemes to quasi-coherent sheaves of ideals

$$(Y \xrightarrow{i} X) \longmapsto \operatorname{Ker}(\mathcal{O}_X \longrightarrow i_* \mathcal{O}_Y)$$

is an anti isomorphism of partially ordered sets. This is a bijection which reverses $\leq$: so Given closed subschemes $i : Y \longrightarrow X$ and $j : Y' \longrightarrow X$ we have

$$i \leq j \quad \text{iff.} \quad \mathcal{J}_j \subseteq \mathcal{J}_i$$

PROOF This follows from (5.9) and our notes on factorisation through closed immersions in §3. □

Note that for q.c. sheaves of ideals $\mathcal{J}, \mathcal{K}$ we have $\mathcal{J} \subseteq \mathcal{K}$ iff. $\mathcal{J}$ preceedes $\mathcal{K}$ as a subobject of $\mathcal{O}_X$ in $\mathcal{O}_X$-Mod.

NOTE Scheme-Theoretic Image

This note is a solution to Ex 3.11d). See also V.1 of Eisenbud & Harris for examples.

PROPOSITION Let $f : X \longrightarrow Y$ be a morphism with X noetherian. Then there is a closed immersion $Z \longrightarrow Y$ with the property that f factors through $Z \longrightarrow Y$, and if f factors through any other closed immersion $Z' \longrightarrow Y$ then $Z \longrightarrow Y$ factors through $Z' \longrightarrow Y$ also.

PROOF Since X is noetherian, by (5.8) $f_* \mathcal{O}_X$ is a quasi-coherent $\mathcal{O}_Y$-module. Let $\mathscr{I}$ be the kernel of $f^\# : \mathcal{O}_Y \longrightarrow f_* \mathcal{O}_X$. By (5.7) $\mathscr{I}$ is a quasi-coherent sheaf of ideals. Using (5.9) let $Z \longrightarrow Y$ be the closed immersion determined by $\mathscr{I}$. By (5.9) the ideal sheaf of $Z \longrightarrow Y$ is $\mathscr{I}$, so by our notes on "Factorisation through closed Immersions" f factors uniquely through $Z \longrightarrow Y$. If $Z' \longrightarrow Y$ is another closed immersion through which f factors, then $\mathscr{K} \subseteq \mathscr{I}$ where $\mathscr{K}$ is the ideal sheaf of $Z' \longrightarrow Y$. It follows that $Z \longrightarrow Y$ factors through $Z' \longrightarrow Y$. $\square$

We call the closed immersion $Z \longrightarrow Y$ the scheme-theoretic image of $f : X \longrightarrow Y$. The noetherian assumption on X is only used to show $f_* \mathcal{O}_X$ is quasi-coherent, so we could also have f quasi-compact and separated (5.8).

LEMMA Let $f : X \longrightarrow Y$ be a morphism with X noetherian, $Z \xrightarrow{i} Y$ the scheme image. Then $Z = \overline{f(X)}$.

PROOF Of course $f(X) \subseteq Z$. By definition i is the inclusion of a closed subset $Z = \mathrm{Supp}(\mathcal{O}_X/\mathscr{I})$ where $\mathscr{I}$ is the kernel of $f^\# : \mathcal{O}_Y \longrightarrow f_* \mathcal{O}_X$. Since Z is closed $\overline{f(X)} \subseteq Z$. Suppose $y \in Z - \overline{f(X)}$. Then $Y - \overline{f(X)}$ is open so it follows that $(f_* \mathcal{O}_X)_y = 0$, contradicting the assumption that $z \in \mathrm{Supp}(\mathcal{O}_X/\mathscr{I}) = \mathrm{Supp}(f_* \mathcal{O}_X)$. $\square$