Fusion in LG-models I

These lecture notes are from a mini-course on fusion of defects in topological Landau-Ginzburg (LG) models delivered at the IBS in Korea in Jan 2016. Broadly the subject matter is the theory of 2D topological field theories (TFTs) with defects, and the example of LG models.

> You can find these notes at www.therisingsea.org
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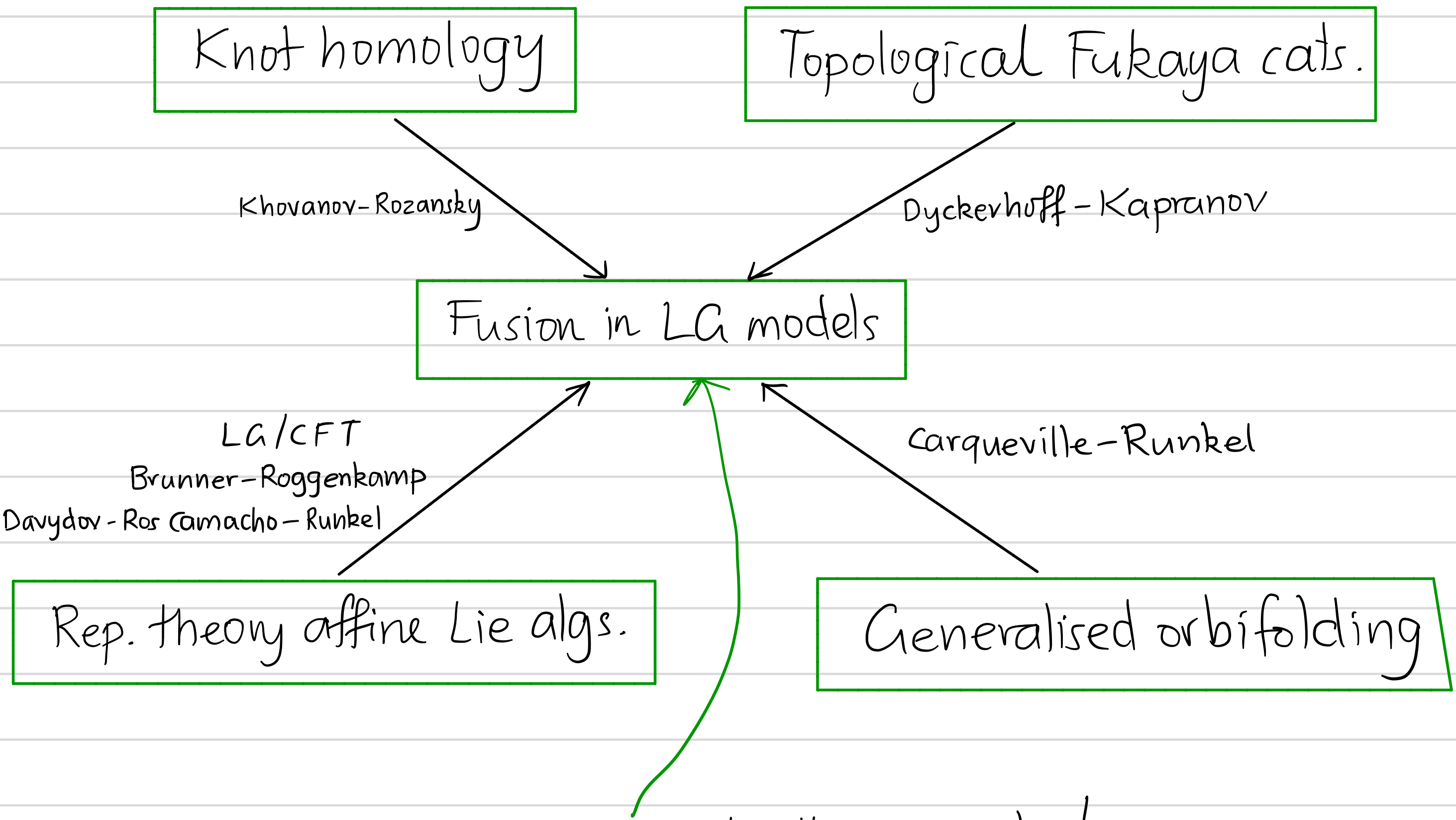
Motivation



Here we can actually compute!

We emphasise two main themes in these lectures

- As the above diagram indicates, various kinds of interesting constructions can be translated into the setting of fusion of defects in LG models, and
- This fusion is computable.

Outline of Lecture 1

1. 2D TFTs with defects
2. Matrix factorisations
3. Defects in LG models and their fusion

Lecture 2 The bicategory $\mathcal{LG}$ (LG/CFT, GO)

Lecture 3 The Hard Stuff (i.e. how the code works)

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1 2D TFT with defects

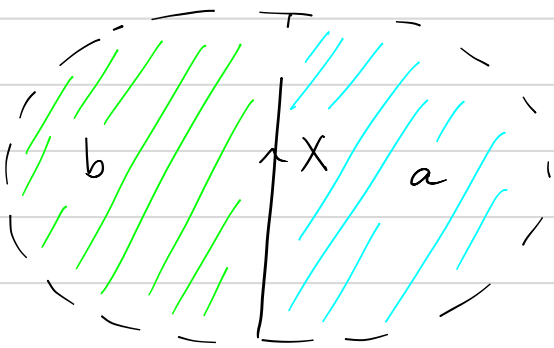
References

- I. Runkel and R. R. Suszek, Gerbe-holonomy for surfaces with defect networks, Adv. Theor. Math. Phys. 13 (2009), 1137–1219, [arXiv:0808.1419].
- A. Davydov, L. Kong, and I. Runkel, Field theories with defects and the centre functor, Mathematical Foundations of Quantum Field Theory and Perturbative String Theory, [arXiv:1107.0495].
- N. Carqueville and I. Runkel, Orbifold completion of defect bicategories, [arXiv: 1210.6363]

We fix two sets D_2 (phases) and D_1 (domain walls, or defect conditions) and a pair of functions

$$s, t : D_1 \longrightarrow D_2$$

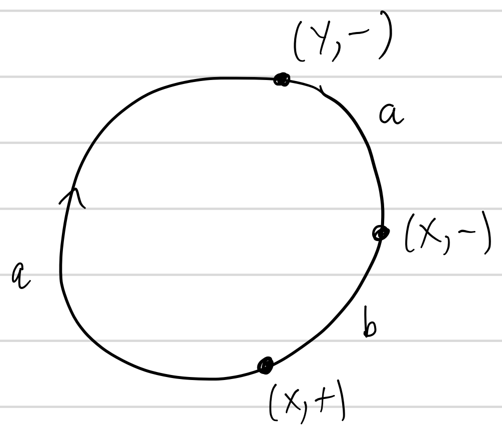
i.e. "source" and "target".



$X \in D_1 \qquad s(X) = a,\ t(X) = b.$

Def^N The category $\mathrm{Bord}^{\mathrm{def}}_{2,1}(D_2, D_1, s, t) =: \mathrm{Bord}$ has

objects

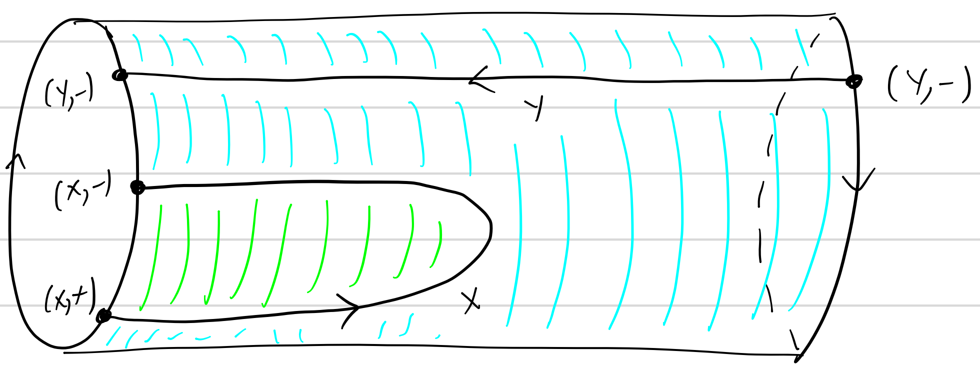


$a, b \in D_2 \quad X, Y \in D_1$

$s(X) = a,\ t(X) = b$

$s(Y) = a,\ t(Y) = a$

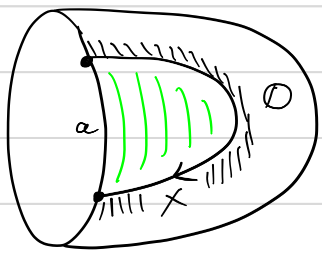
<u>morphisms</u> either a permutation (of decorated circles) or an (equivalence class of) 2D oriented compact manifolds equipped with a 1D oriented submanifold (the <u>defect graph</u>) with compatible labels



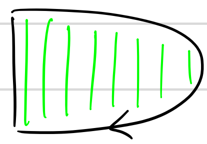
<u>Def^N</u> An <u>(oriented) 2D TFT with defects</u> is a symmetric monoidal functor

$$\mathcal{Z} : \mathrm{Bord} \longrightarrow \mathrm{Vect}_{\mathbb{C}}$$
$$\mathrm{Bord} = \mathrm{Bord}^{\mathrm{def}}_{2,1}(D_2, D_1, s, t)$$

<u>Example</u> • Open/closed 2D TFT. Fix a "trivial" phase $\mathbb{O} \in D_2$ and $a \in D_2$ and view e.g.



as



$s(X) = \mathbb{O}$, $t(X) = a$
"boundary condition"

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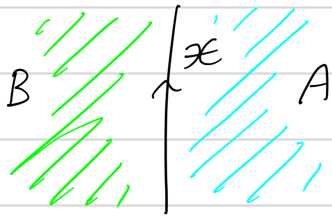
- (Conjecturally) Sigma models

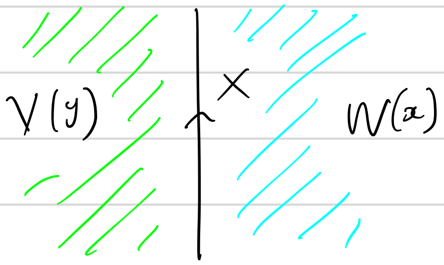
D_2 = smooth projective CY varieties



$\mathcal{X} \in D^b(\mathrm{coh}\, A \times B)$
complex of coherent sheaves

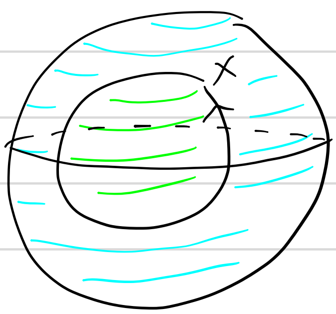
- (Conjecturally) LG models

D_2 = isolated hypersurface singularities



$X \in \mathrm{hmf}(V(y) - W(x))$
matrix factorisation

To define a TFT with defects $\mathcal{Z}$ we need e.g. to specify

$\mathcal{Z}\Big($  $\Big) \in \mathrm{Hom}_{\mathbb{C}}\big(\mathcal{Z}(\bigcirc), \mathcal{Z}(\bigcirc)\big)$

For every matrix factorisation X of $V - W$, among many other quantities. These quantities are organised by the <u>bicategory of LG models</u>, denoted $\mathcal{LG}$.

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$$\left\{ \text{2D TFT w/ defects} \right\} = \left\{ \begin{array}{c} \text{bicategory with adjoints} \\ \text{which is pivotal + ...} \end{array} \right\}$$

The vector spaces and linear maps encoded by the functor Z can be organised into a pivotal bicategory with additional structure, and it is expected that any "sufficiently nice" bicategory arises in this way (although, as far as I know, the exact list of conditions is not known).

Anyway, at least conceptually we view the bicategory of $\mathcal{LG}$ models (Lecture 2) as encoding the associated defect TFT.

2 Matrix factorisations

Let $W \in \mathbb{C}[x] = \mathbb{C}[x_1, \dots, x_n]$ be a potential, i.e. the critical points of W are isolated.

Def^N A matrix factorisation of W is a $\mathbb{Z}_2$-graded free $\mathbb{C}[x]$-module $X = X^0 \oplus X^1$ with an odd operator $d_X : X \longrightarrow X$ such that $d_X^2 = W \cdot 1_X$

$$d_X = \begin{pmatrix} 0 & d_X^1 \\ d_X^0 & 0 \end{pmatrix}$$

Def^N Let X, Y be MFs of W. Then

$$\mathrm{Hom}(X,Y) = \mathrm{Hom}(X,Y)^0 \oplus \mathrm{Hom}(X,Y)^1$$

is a $\mathbb{Z}_2$-graded complex with differential ∂ defined by

$$\partial(f) = d_Y f - (-1)^{|f|} f d_X.$$

Def^N $mf(W)$ = DG category of finite rank MFs of W
$hmf(W) = H^0 mf(W)$, homotopy category.

Note MFs were introduced by Eisenbud as part of his investigation of free resolutions over complete intersection singularities

Lemma hmf(W) has f.dimensional Hom-spaces.

Proof $\partial_{x_i} W \cdot 1_X \in Hom(X,X)$ $\quad W = d_X \cdot d_X$

$$\partial_{x_i}(W) = \partial_{x_i}(d_X) d_X + d_X \partial_{x_i}(d_X)$$

$$\therefore \partial_{x_i}(W) \cdot 1_X \simeq 0.$$

$\Rightarrow H^0 Hom(X,X)$ is a f.g. $\mathbb{C}[x]/(\partial W)$-module $\therefore$ f.dim (W is a potential). □

Example • $W = x^3, \quad X = \mathbb{C}[x] \oplus \mathbb{C}[x]$

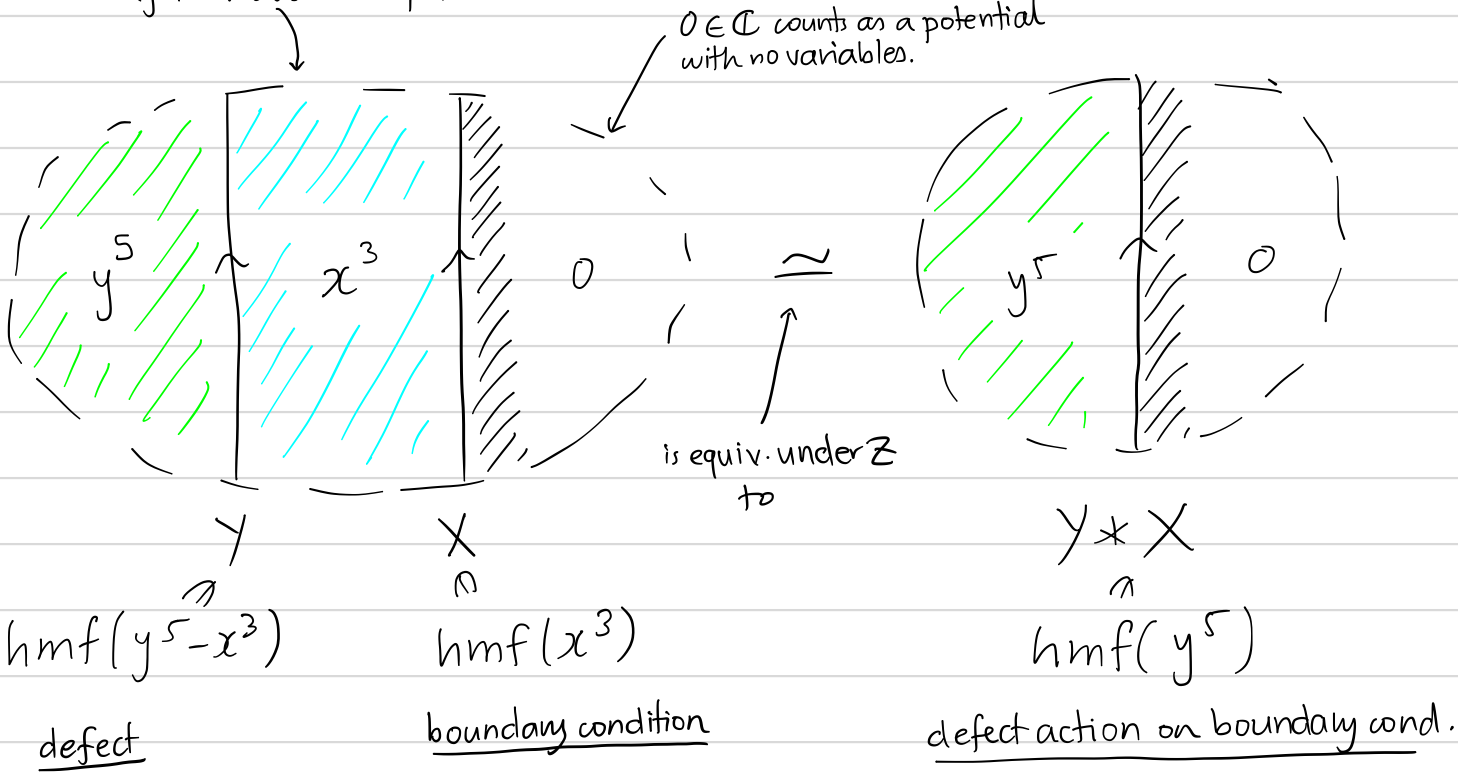
$$d_X := \begin{pmatrix} 0 & x^2 \\ x & 0 \end{pmatrix}$$

• $W = y^5 - x^3, \quad Y = \mathbb{C}[x,y]^{\oplus 2} \oplus \mathbb{C}[x,y]^{\oplus 2}$

$$d_Y = \begin{pmatrix} 0 & 0 & x^2 & -y \\ 0 & 0 & y^4 & -x \\ -x & y & 0 & 0 \\ -y^4 & x^2 & 0 & 0 \end{pmatrix}$$

③ Defects and fusion

imagine this as a local patch on a bordism



If one begins with the TFT Z, then the fusion $Y * X$ is the f.rank MF of y^5 behaving in all correlators as a pair of parallel lines labelled Y, X. Luckily this has a simple mathematical description, which we take as primary.

Prop Let $W(x) \in \mathbb{C}[x]$, $V(y) \in \mathbb{C}[y]$ be potentials and

$$Y \in \mathrm{hmf}(\mathbb{C}[x,y], V - W)$$
$$X \in \mathrm{hmf}(\mathbb{C}[x], W)$$

Then $\left(Y \underset{\mathbb{C}[x]}{\otimes} X,\ d_Y \otimes 1 + 1 \otimes d_X \right)$ is a MF of V.

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<u>Proof</u> $(d_Y \otimes 1 + 1 \otimes d_X)^2$

$$= d_Y^2 \otimes 1 + 1 \otimes d_X^2$$

$$+ (d_Y \otimes 1)(1 \otimes d_X) + (1 \otimes d_X)(d_Y \otimes 1)$$

$$= (V - W) \cdot 1_{Y \otimes X} + W \cdot 1_{Y \otimes X}$$

$$+ d_Y \otimes d_X - d_Y \otimes d_X$$

$$= V \cdot 1_{Y \otimes X}. \quad \square$$

<u>Note</u> $Y = \mathbb{C}[x,y]^{\oplus r} \quad X = \mathbb{C}[x]^{\oplus s}$

$\therefore \; Y \underset{\mathbb{C}[x]}{\otimes} X = \mathbb{C}[x,y]^{\oplus rs}$ is <u>infinite rank</u> over $\mathbb{C}[y]$

$$= \mathbb{C}[y]^{\oplus rs} \oplus x\,\mathbb{C}[y]^{\oplus rs} \oplus \cdots$$

$$\mathbb{C}[y]^{\oplus rs} \oplus x\,\mathbb{C}[y]^{\oplus rs} \oplus \cdots$$

$$d_{Y \otimes X} = \begin{pmatrix} y^2 & 1 & 0 & & \\ 0 & 0 & y+y^2 & 0 & \\ & & & \vdots & \cdots \\ 1 & 0 & 0 & \vdots & \\ 0 & 1 & 0 & 0 & \\ 0 & 0 & 1 & 0 & \\ & & \vdots & & \end{pmatrix}$$

The "remnant" x's contribute I's to the ∞ matrix of $d_{Y \otimes X}$ over $\mathbb{C}[y]$.

Brunner–Roggenkamp [1]

Thm/Def^N (Khovanov–Rozansky [2], Dyckerhoff–M [3])

There exists a f. rank MF $Y * X$ of $V(y)$ over $\mathbb{C}[y]$ and a homotopy equivalence

$$Y * X \simeq Y \otimes_{\mathbb{C}[x]} X$$

Note

- True as stated in the graded case
- In the ungraded case, only true over $\mathbb{C}[[y]]$. But over $\mathbb{C}[y]$ it is still true $Y \otimes X$ is a summand of a f. rank MF.

Question What is $Y * X$? i.e. how to describe it as a matrix given Y, X?

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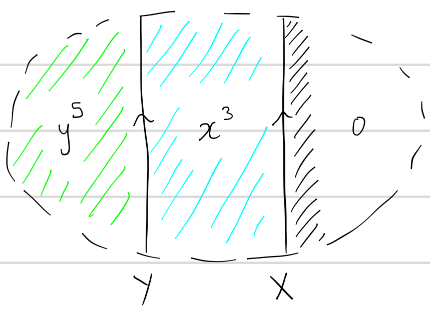
At this point I showed how to use Singular to compute $Y * X$ (https://github.com/dmurfet/mf) in the example from earlier, namely



$$X := \begin{pmatrix} 0 & x^2 \\ x & 0 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & 0 & -x & y \\ 0 & 0 & -y^4 & x^2 \\ x^2 & -y & 0 & 0 \\ y^4 & x & 0 & 0 \end{pmatrix}$$

$$Y * X = \begin{pmatrix} 0 & 0 & 0 & y \\ 0 & 0 & y & 0 \\ 0 & y^4 & 0 & 0 \\ y^4 & 0 & 0 & 0 \end{pmatrix}$$

Further reading

[1] Brunner, Roggenkamp "B-type defects in Landau-Ginzburg models" arXiv:0707.0922.

[2] Khovanov, Rozansky "Matrix factorisations and link homology" arXiv:0404.1268.

[3] Dyckerhoff, Murfet "Pushing forward matrix factorisations" arXiv:1102.2957.

[4] Carqueville, Murfet "Computing Khovanov-Rozansky homology and defect fusion" arXiv:1108.1081 (see for more background on the code).