Fusion in LG models II

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In this lecture we define the bicategory $\mathcal{LG}$ and explore its basic properties. We begin with the general definition of fusion, which will be interpreted as composition of 1-morphisms in $\mathcal{LG}$.

The bicategory $\mathcal{LG}$

1-morphism
MF of $V - W$.

$$W(x) \xrightarrow{\quad X \quad} V(y)$$

objects are potentials

Some examples

$\boxed{\text{Generalised orbifolding}}$ Carqueville-Runkel

$$A = X^\vee * X \circlearrowright W_{A_{11}} \underset{X^\vee}{\overset{X}{\rightleftarrows}} W_{E_6}$$

Frobenius algebra

$$W_{A_{11}} = x_1^{12} + x_2^2, \qquad W_{E_6} = x_1^3 + x_2^4$$

$$\operatorname{mod}(A) \cong \operatorname{hmf}(W_{E_6}).$$

$\boxed{\text{Topological Fukaya category}}$ Dyckerhoff-Kapranov

$$\begin{tikzcd} Z^2 \arrow[loop below, "\mathbb{Z}/2"] \arrow[r, shift left=2] \arrow[r, shift right=2] & Z^3 \arrow[l] \arrow[loop below, "\mathbb{Z}/3"] \arrow[r, shift left=3] \arrow[r] \arrow[r, shift right=3] & Z^4 \arrow[l, shift left=1.5] \arrow[l, shift right=1.5] \arrow[loop below, "\mathbb{Z}/4"] & \cdots \end{tikzcd}$$

cocyclic object

1 General fusion

Def^{N} Given potentials $W(x)$, $V(y)$, $U(z)$ and f. rank matrix factorisations

$$Y \in hmf(\mathbb{C}[y,z], U - V)$$
$$X \in hmf(\mathbb{C}[x,y], V - W)$$

there is an (infinite rank) MF of $U - W$ over $\mathbb{C}[x,z]$

$$\left(Y \otimes_{\mathbb{C}[y]} X, \quad d_Y \otimes 1 + 1 \otimes d_X \right)$$

We write $Y * X$ for the f. rank MF homotopy equiv. to $Y \otimes_{\mathbb{C}[y]} X$ (again, it exists), and call it the <u>fusion of Y and X</u>. We depict the situation as follows:

$$W \xrightarrow{\quad X \quad} V \xrightarrow{\quad Y \quad} U.$$

Aside

The Singular code employs a "dual" language, i.e. with potentials on edges and MFs at vertices,

$$\xleftarrow{\ U(z)\ } (Y) \xleftarrow{\ V(y)\ } (X) \xleftarrow{\ W(x)\ } .$$

This is called a <u>web</u> and $Y \otimes X$ is the <u>total factorisation</u> of the web.

(see Brunner-Roggenkamp [1])

<u>Example</u> Take $W = x^d$, $V = y^d$ and write

$$V - W = y^d - x^d = \prod_{i=0}^{d-1} (y - \eta^i x) \qquad \eta = e^{2\pi i/d}$$

For any subset $S \subseteq \{0, 1, \dots, d-1\}$ we have

$$P_S = \begin{pmatrix} 0 & \prod_{i \in S} (y - \eta^i x) \\ \prod_{i \notin S} (y - \eta^i x) & 0 \end{pmatrix} \in hmf(y^d - x^d)$$

This MF is defined in the x, y variables $P_S = P_S(x, y)$ but for $S' \subseteq \{0, 1, \dots, d-1\}$ we can just as well define

$$P_{S'} = P_{S'}(y, z) \in hmf(z^d - y^d).$$

Then one can ask about the fusion

$$x^d \xrightarrow{P_S} y^d \xrightarrow{P_{S'}} z^d.$$

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The case $d = 5$, $S' = \{1, 2, 3\}$, $S = \{2\}$

$$P_{\{1,2,3\}} * P_{\{2\}} \cong P_{\{3,4,5\}}$$

Lemma $hmf(y^d - x^d)$ is a monoidal category.

$$hmf(z^d - y^d) \times hmf(y^d - x^d) \longrightarrow hmf(z^d - x^d)$$
$$(Y, X) \longmapsto Y * X$$

Theorem (Brunner-Roggenkamp [1], Davydov-Ros Camacho-Runkel [2])
For d odd, there is a monoidal equivalence

$$\text{LG} \quad \mathcal{P}_d^{gr} \xrightarrow{\cong} \mathcal{C}(N=2, d)_{NS} \quad \text{CFT}$$

monoidal subcat of $hmf^{gr}(y^d - x^d)$ gen by P_S's for S consecutive indices.

bosonic part of the $N=2$ minimal super vertex operator algebra $V(N=2,d)$, take the NS-representations.

$$P_{m:\lambda} := P_{\{m, m+1, \ldots, m+\lambda\}}$$

$$P_{m:\lambda} * P_{n:\mu} \cong \bigoplus_{\nu = |\lambda-\mu| \text{ step } 2}^{\min(\lambda+\mu,\, 2d-4-\lambda-\mu)} P_{m+n-\frac{1}{2}(\mu+\lambda-\nu) : \nu}$$

(i.e. $\widehat{su}(2)$-fusion rules)

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2 Bicategories

A bicategory $\mathcal{B}$ has

- <u>objects</u> $a, b, c, \ldots$
- For each pair of objects a, b a <u>category</u>

$\mathcal{B}(a,b)$ — objects here are called <u>1-morphisms</u>

$$X : a \longrightarrow b$$

and morphisms are called <u>2-morphisms</u>

$$\begin{tikzcd} a \arrow[r, bend left, "X"{name=U}] \arrow[r, bend right, "Y"'{name=D}] & b \arrow[from=U, to=D, phantom, "\Downarrow\alpha"] \end{tikzcd}$$

- For each triple a, b, c a <u>functor</u> (composition)

$$\mathcal{B}(b,c) \times \mathcal{B}(a,b) \longrightarrow \mathcal{B}(a,c)$$

$$(Y : b \to c,\ X : a \to b) \longmapsto Y \circ X : a \to c$$

- A natural <u>associator</u>

$$Z \circ (Y \circ X) \xrightarrow{\cong} (Z \circ Y) \circ X$$

- For each object a, a <u>unit</u> $\Delta_a : a \to a$ and

$$\Delta_b \circ X \xrightarrow{\cong} X, \quad X \circ \Delta_a \xrightarrow{\cong} X.$$

This data is required to satisfy several <u>coherence conditions</u>.

<u>Example</u> 1 Categories, functors, nat. trans

2 Smooth projective varieties, FM kernels

<u>Def^N</u> $\mathcal{LG}$ is the bicategory with

- <u>objects</u> potentials $W(x), V(y), \ldots$
- <u>1- and 2-morphisms</u>

Karoubi completion

$$\mathcal{LG}\big(W(x), V(y) \big) = \mathrm{hmf}\big(V(y) - W(x) \big)^{\omega}$$

- <u>composition</u> = fusion

$$\begin{tikzcd} U(z) & V(y) \arrow[l, "Y"'] & W(x) \arrow[l, "X"'] \arrow[ll, bend left, "Y * X"] \end{tikzcd}$$

- <u>units</u> omitted here for lack of time.

<u>Prop</u> (Lazaroiu-McNamee, Carqueville-Runkel)
$\mathcal{LG}$ is a bicategory.

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Note If $\mathcal{B}$ is a bicategory $\mathcal{B}(a,a)$ is monoidal.

e.g. $\mathcal{LG}(x^d, x^d) = \mathrm{hmf}(y^d - x^d)^\omega$

Carqueville - Runkel [3]

Theorem (Carqueville-M[4]) In $\mathcal{LG}$ every 1-morphism has a left and right adjoint, and the bicategory is pivotal. Given a 1-morphism $X : W(x) \longrightarrow V(y)$ with $|x|, |y|$ both even, the dual is

$$X^\vee = \mathrm{Hom}_{\mathbb{C}[x,y]}(X, \mathbb{C}[x,y]) : V(y) \longrightarrow W(x)$$

$$\circlearrowleft \alpha \longmapsto -(-1)^{|\alpha|}\alpha\, dx.$$

and there is a pair of adjunctions

$$\boxed{X^\vee \dashv X \dashv X^\vee}$$

realised by morphisms with explicit formulas

$ev : X^\vee \otimes X \longrightarrow \Delta_W$ $\qquad$ $\widetilde{ev} : X \otimes X^\vee \longrightarrow \Delta_V$

$coev : \Delta_V \longrightarrow X \otimes X^\vee$ $\qquad$ $\widetilde{coev} : \Delta_W \longrightarrow X^\vee \otimes X.$

formulas involve residues, supertraces, partial derivatives of d_X and divided difference operators. In the case of quadratic potentials everything can be written in terms of Pauli matrices.

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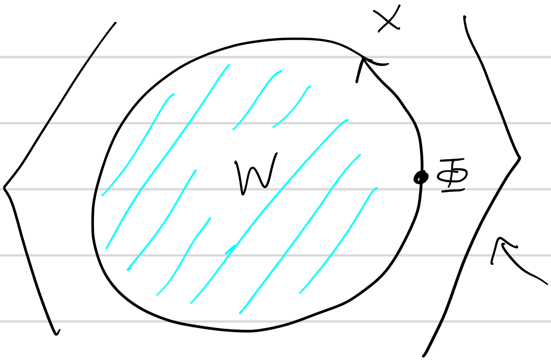
Example (Disc correlator)

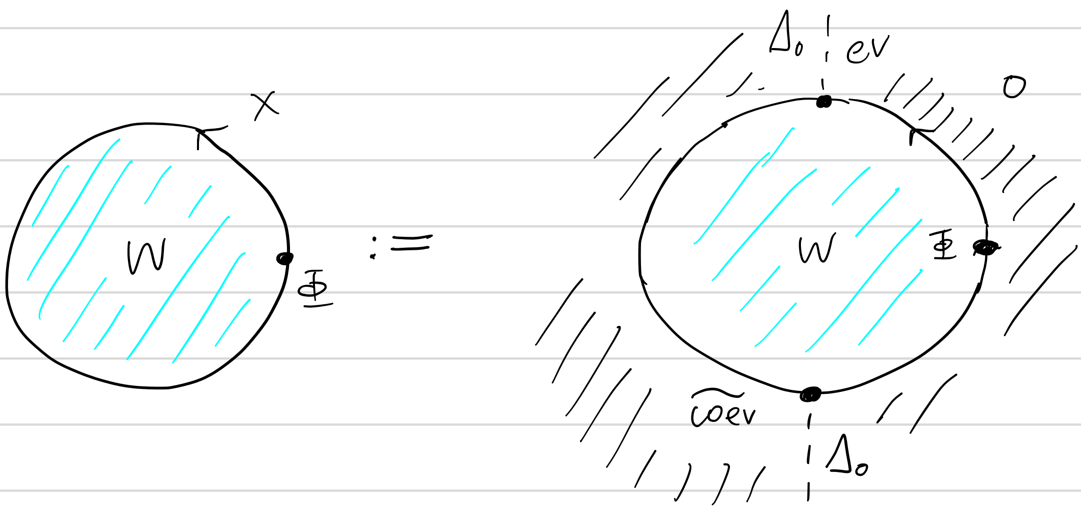


$X \in hmf(W(x))$

$\Phi : X \longrightarrow X$ morphism

computed in open top. LG model
i.e. apply Z, gives a scalar

This correlator can be represented (and calculated) in $\mathcal{LG}$ using (rigorously formulated) string diagrams



In string diagrams, regions are labelled with objects, 1-dimensional data by 1-morphisms and vertices with 2-morphisms. Here O is a potential and $\Delta_0 = \mathbb{C}$, so the string diagram denotes a linear map

$$\mathbb{C} = \Delta_0 \xrightarrow{\widetilde{\text{coev}}} X^\vee \otimes X \xrightarrow{1 \otimes \Phi} X^\vee \otimes X \xrightarrow{\text{ev}} \Delta_0 = \mathbb{C}$$

In this case the formulas for ev, $\widetilde{\mathrm{coev}}$ yield

$$\mathrm{Res}_{\mathbb{C}[x]/\mathbb{C}}\left(\frac{\mathrm{str}(\Phi\,\partial_{x_1}(d_x)\cdots\partial_{x_n}(d_x))\,\underline{dx}}{\partial_{x_1}W \;\cdots\; \partial_{x_n}W}\right)$$

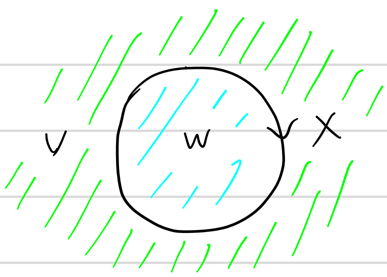
i.e. the <u>Kapustin–Li formula</u>.

<u>Example</u> For a potential W, in the category $\mathcal{LG}(W,W)$ the monoidal unit is Δ_W and

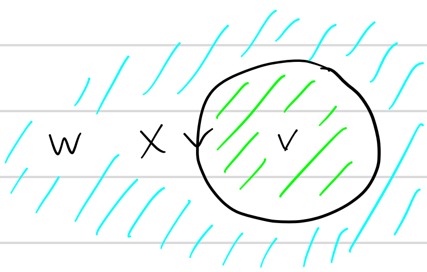
$$\mathrm{Hom}(\Delta_W, \Delta_W) \cong \mathbb{C}[x]/\partial W =: \mathrm{Jac}W$$

<u>Example</u> (Quantum dimension)

Given $X : W(x) \to V(y)$ with $|x|, |y|$ even we define the left and right <u>quantum dimensions</u>:



$$\dim_r X \in \mathrm{Hom}(\Delta_V, \Delta_V) \cong \mathrm{Jac}W$$



$$\dim_\ell X \in \mathrm{Hom}(\Delta_W, \Delta_W) \cong \mathrm{Jac}V$$

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For example if $|y| = m$ then

$$\dim_r X = (-1)^{\binom{m+1}{2}} \operatorname{Res}_{\mathbb{C}[x,y]/\mathbb{C}[y]} \left(\frac{\operatorname{str}\left(\partial_{x_1}(d_x)\cdots\partial_{x_n}(d_x)\,\partial_{y_1}(d_x)\cdots\partial_{y_m}(d_x)\right)}{\underbrace{\partial_{x_1}W \cdots \partial_{x_n}W}_{J_V}} \, \underline{dx} \right)$$

For more background on the formulas (and how to compute residues) see [4].

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<u>Theorem</u> (Carqueville–Runkel) If $X : W \to V$ has $\dim_r(X)$ invertible then

$$A := X^\vee * X \;:\; W \longrightarrow W$$

is a separable symmetric Frobenius algebra in $\mathcal{LG}(W,W)$ and there is an equivalence

$$\mathrm{mod}(A) \cong \mathrm{hmf}(V).$$

One says V is a <u>generalised orbifold</u> (GO) of W, and writes $W \xrightarrow{GO} V$.

<u>Example</u> (Carqueville–Ros Camacho–Runkel)

$\boxed{d \text{ even}}$

$$\overset{A}{\circlearrowright} W_{A_{d-1}} \xrightarrow{GO} W_{D_{d/2+1}}$$

$$W_{A_{d-1}} = x_1^d + x_2^2 \qquad W_{D_{d/2+1}} = x_1^{d/2} + x_1 x_2^2$$

$$\circlearrowright A_{11} \xrightarrow[X]{GO} E_6 \qquad A_{17} \xrightarrow{GO} E_7 \qquad A_{29} \xrightarrow{GO} E_8$$

$A = X^\vee * X$

In all cases one can compute (via Singular) that the Frobenius algebra $A = X^\vee * X$ is a sum of P_S's. (see [5])

e.g. for A_{11}/E_6, $A \cong \Delta \oplus P_{\{-3,-2,\dots,2,3\}}$

Open Questions

- What is the geometric/quiver explanation for these relations, and the particular P_S summands?

- Can we combine Khovanov-Rozansky homology (A-type) and GO to define D, E-type knot homology?

- Can one "orbifold" the Dyckerhoff-Kapranov cocyclic object?

<u>Conjecture</u> Unimodular singularities of the same central charge are orbifold equivalent (see [6]).

$$Q_{10}: x^4+y^3+xz^2 \sim K_{14}: x^4z+y^3+z^2$$

- The Frobenius algebras A which implement the GOs between ADE singularities are all sums of P_S-type defects. What are the structure maps? (e.g. product, coproduct).

This is a good example of where explicit formulas for units and counits in $\mathcal{LG}$ come in handy: the product on $A = X^\vee \otimes X$ is

$$A \otimes A = X^\vee \otimes X \otimes X^\vee \otimes X \xrightarrow{1 \otimes \widetilde{ev} \otimes 1} X^\vee \otimes \Delta \otimes X \cong X^\vee \otimes X = A$$

given in terms of residues, similar to KL formula.

ADE curve singularities

$$V^{(\mathrm{A}_{d-1})} = x_1^d + x_2^2 \qquad c = 3 - 3 \cdot \frac{2}{d} \qquad (d \geqslant 2)$$

$$V^{(\mathrm{D}_{d+1})} = x_1^d + x_1 x_2^2 \qquad c = 3 - 3 \cdot \frac{2}{2d} \qquad (d \geqslant 3)$$

$$V^{(\mathrm{E}_6)} = x_1^3 + x_2^4 \qquad c = 3 - 3 \cdot \frac{2}{12}$$

$$V^{(\mathrm{E}_7)} = x_1^3 + x_1 x_2^3 \qquad c = 3 - 3 \cdot \frac{2}{18}$$

$$V^{(\mathrm{E}_8)} = x_1^3 + x_2^5 \qquad c = 3 - 3 \cdot \frac{2}{30}$$

central charges

$$c_W = \sum_i (1 - |x_i|) \quad \text{s.t. } |W| = 2.$$

References

[1] Brunner, Roggenkamp "B-type defects in LG models" arXiv:0707.0922.

[2] Davydov, Ros Camacho, Runkel "N=2 minimal conformal field theories and matrix bifactorisations of x^d" arXiv:1409.2144.

[3] N. Carqueville and I. Runkel, On the monoidal structure of matrix bi-factorisations, J. Phys. A: Math. Theor. 43 (2010), 275401, [arXiv:0909.4381].

[4] Carqueville, Murfet "Adjunctions and defects in Landau-Ginzburg models" arXiv:1208.1481

[5] Carqueville, Velez "Calabi-Yau completions and orbifold equivalences" arXiv:1509.0088.

[6] Newton, Ros Camacho "strangely dual orbifold equivalence I"