# Metric and Hilbert Spaces (MAST30026)
# Assignment 2 (2021)

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Tutorial: 10am Wednesday with Daniel Murfet

## Exercise L11-1 (Not Hausdorff)

Define the cofinite topology $\mathcal{T}$ on the integers $\mathbb{Z}$ by

$$U \in \mathcal{T} \iff U = \varnothing \text{ or } U = \mathbb{Z} \setminus S \text{ for some finite subset } S \subseteq \mathbb{Z}$$

This is a valid topology since

(i) $\varnothing \in \mathcal{T}$ by definition and $\mathbb{Z} = \mathbb{Z} \setminus \varnothing \in \mathcal{T}$ as the empty set is finite.

(ii) For any collection of finite subsets $(S_i)_{i \in \mathbb{N}}$ of $\mathbb{Z}$ we have

$$\bigcup_{i \in \mathbb{N}} (\mathbb{Z} \setminus S_i) = \mathbb{Z} \setminus \left( \bigcap_{i \in \mathbb{N}} S_i \right) \in \mathcal{T}$$

with the set inclusion justified by the intersection of finite sets being finite.

(iii) For any finite collection of finite subsets $(S_1, \dots, S_n)$ of $\mathbb{Z}$ we have

$$\bigcap_{i=1}^{n} (\mathbb{Z} \setminus S_i) = \mathbb{Z} \setminus \left( \bigcup_{i=1}^{n} S_i \right) \in \mathcal{T}$$

with the set inclusion justified by the finite union of finite sets being finite.

We now claim that the cofinite topology on $\mathbb{Z}$ has closed points but is not Hausdorff.

*Proof.* (Points are closed) Let $x \in \mathbb{Z}$. Since the set $\{x\}$ is finite then $\mathbb{Z} \setminus \{x\}$ is open in $\mathcal{T}$ and hence $\{x\} = \mathbb{Z} \setminus (\mathbb{Z} \setminus \{x\})$ is closed in $\mathcal{T}$.

($\mathcal{T}$ is not Hausdorff) Assume for the purpose of contradiction that $\mathcal{T}$ is Hausdorff. Then for any $x, y \in \mathbb{Z}$ there exists there exists disjoint open sets $U, V \in \mathcal{T}$ such that $x \in U$ and $y \in V$.

Since $U$ and $V$ are open, then by the definition of $\mathcal{T}$ there exist finitely many integers $u_1, \dots, u_m$ and $v_1, \dots, v_n$ such that $U = \mathbb{Z} \setminus \{u_1, \dots, u_m\}$ and $V = \mathbb{Z} \setminus \{v_1, \dots, v_n\}$.

This implies $U \cap V = \mathbb{Z} \setminus \{u_1, \dots, u_m, v_1, \dots, v_n\}$ is non-empty as the removal of finitely many elements from an infinite set leaves an infinite set.

However, this contradicts that $U$ and $V$ are disjoint and hence $\mathcal{T}$ is not Hausdorff.

$\square$

## Exercise L11-12 (Doubled origin)

*Proof.* Let $f : X \longrightarrow \mathbb{R}$ be a function and assume for the purpose of contradiction that $f$ is continuous and $f(O_1) \neq f(O_2)$.

Since $\mathbb{R}$ is Hausdorff, then there exists open and disjoint neighborhoods $V_1$ and $V_2$ of $f(O_1)$ and $f(O_2)$ respectively.

It then follows that since $f$ is continuous and the preimage of disjoint sets is disjoint, then $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are disjoint and open neighborhoods of $f^{-1}(O_1)$ and $f^{-1}(O_2)$ respectively.

However, this is a contradiction, as example L11-1 shows that $O_1$ and $O_2$ have no disjoint neighborhoods.

Hence there does not exist a continuous function $f : X \longrightarrow \mathbb{R}$ such that $f(O_1) \neq f(O_2)$.

□

# Exercise L12-4 (Constant loops)

*Proof.* We want to show that

$$\text{const} \in \mathsf{Cts}(Y, \mathscr{L}Y) = \mathsf{Cts}(Y, \mathsf{Cts}(S^1, Y))$$

Since $S^1$ is compact Hausdorff (by L12-1) and so locally compact Hausdorff, then we will have shown this to be true by the adjunction property if we can find a function $f \in \mathsf{Cts}(Y \times S^1, Y)$ so that const $= \Psi(f)$ where $\Psi$ is the bijection of theorem L12-4:

$$\Psi : \mathsf{Cts}(Y \times S^1, Y) \longrightarrow \mathsf{Cts}(Y, \mathsf{Cts}(S^1, Y))$$

Denote $\pi_Y : Y \times S^1 \longrightarrow Y$ as the projection map $(y, \theta) \longmapsto y$, which we know is continuous by exercise L7-5

$$\text{i.e. } \pi_Y \in \mathsf{Cts}(Y \times S^1, Y)$$

Since for all $(y, \theta) \in Y \times S^1$ we have

$$\text{const}(y)(\theta) = y = \pi_Y(y, \theta) = \Psi(\pi_Y)(y)(\theta)$$

then const $= \Psi(\pi_Y)$ and so by our reasoning above

$$\text{const} \in \mathsf{Cts}(Y, \mathscr{L}Y)$$

i.e. const is continuous.

□

## Exercise L12-14 (Product preserving)

*Proof.* The canonical bijection $\Phi$ of lemma L7-2 will be a homeomorphism if we can show it is both continuous and an open map.

To first show it is continuous, we define the maps,

$$F_Y : \mathsf{Cts}(X, Y \times Z) \longrightarrow \mathsf{Cts}(X, Y) \text{ and } F_Z : \mathsf{Cts}(X, Y \times Z) \longrightarrow \mathsf{Cts}(X, Z)$$

by

$$F_Y := \pi_Y \circ \mathrm{id}_{\mathsf{Cts}(X,Y\times Z)} \text{ and } F_Z := \pi_Z \circ \mathrm{id}_{\mathsf{Cts}(X,Y\times Z)}$$

where $\pi_Y$ and $\pi_Z$ are the projection maps from $Y \times Z$ to $Y$ and $Z$ respectively.

Now by definition $\Phi = F_Y \times F_Z$ and since $\pi_Y$ and $\pi_Z$ are continuous (by exercise L7-5) and $X$ is locally compact Hausdorff, then lemma 12-1 (iii) implies $F_Y$ and $F_Z$ are continuous and hence $\Phi$ is continuous as it is the product of continuous functions.

To show $\Phi$ is open, it suffices to show by exercise 12-3 (iii) that $\Phi\, S(K, U)$ is open for any compact $K \subseteq X$ and open $U \subseteq Y \times Z$ as the collection of all such sets $S(K, U)$ form a sub-basis for the compact-open topology on $\mathsf{Cts}(X, Y \times Z)$.

Let $K \subseteq X$ be any compact subset of $X$ and let $U \subseteq Y \times Z$ be any open subset of $Y \times Z$ so that $U = U_Y \times U_Z$ where $U_Y \subseteq Y$ and $U_Z \subseteq Z$ are open.

Now

$$\begin{aligned}
\Phi\, S(K, U) &= \Phi\, S(K, U_Y \times U_Z) \\
&= \{\, (F_Y f, F_Z f) \mid f \in S(K, U_Y \times U_Z) \,\} \\
&= \{\, (\pi_Y \circ (f_Y, f_Z), \pi_Z \circ (f_Y, f_Z)) \mid (f_Y, f_Z) \in S(K, U_Y) \times S(K, U_Z) \,\} \\
&= \{\, (f_Y, f_Z) \mid f_Y \in S(K, U_Y) \text{ and } f_Z \in S(K, U_Z) \,\} \\
&= S(K, U_Y) \times S(K, U_Z)
\end{aligned}$$

which is open in the compact-open topology on $\mathsf{Cts}(X, Y) \times \mathsf{Cts}(X, Z)$.

Hence the canonical bijection $\Phi$ is also open and continuous and therefore a homeomorphism.

$\square$