MAST30026 Metric and Hilbert Spaces

Assignment 3 Solutions — 2018 Semester 2

Notation and conventions

The set $\mathbb{N}$ is the set of positive integers $\{1, 2, 3, \dots\}$.

The set $\mathbb{N}_0$ is the set of nonnegative integers $\{0, 1, 2, \dots\}$.

Question 1

Exercise L11-8. Prove any metrisable space is normal.

Let $(X, \mathcal{T})$ be a metrisable topological space, and let $d\colon X \times X \longrightarrow \mathbb{R}$ be a distance function inducing the topology $\mathcal{T}$. We wish to show that X is normal.

Let us first check that the singletons in X are closed: If $x \in X$ then for every $y \in \{x\}^c$ we have $B_{d(x,y)}(y) \subseteq \{x\}^c$ (note $d(x,y) > 0$). This means that $\{x\}^c = \bigcup_{y \in \{x\}^c} B_{d(x,y)}(y)$ which is open as a union of open balls. Hence $\{x\} \subseteq X$ is closed.

Let $C, D \subseteq X$ be arbitrary disjoint closed sets. Since $C \cap D = \emptyset$, every $x \in C$ is outside $D = \overline{D}$ and hence, by Exercise L13-4(i), $x \in C$ has an open neighbourhood lying outside D. Similarly, every $y \in D$ has an open neighbourhood lying outside C. Since the topology $\mathcal{T}$ is induced by a metric, by Exercise L7-1(iii), there is no loss of generality in assuming the existence of $\varepsilon_x > 0$ for every $x \in C$ and $\rho_y > 0$ for every $y \in D$ such that

$$\begin{aligned} x \in B_{\varepsilon_x}(x) \subseteq D^c \quad \forall x \in C, \\ y \in B_{\rho_y}(y) \subseteq C^c \quad \forall y \in D. \end{aligned}$$

Let $U \coloneqq \bigcup_{x \in C} B_{\varepsilon_x/2}(x)$ and $V \coloneqq \bigcup_{y \in D} B_{\rho_y/2}(y)$. Both U and V are unions of open balls, so they are both open in X. Furthermore, $C \subseteq U$ and $D \subseteq V$. We claim that U and V are disjoint: If $z \in U \cap V$ then there exist $x \in C$ and $y \in D$ such that $d(z,x) < \varepsilon_x/2$ and $d(z,y) < \rho_y/2$. By the triangle inequality for d, we have

$$d(x,y) \leqslant d(x,z) + d(z,y) < \frac{1}{2}(\varepsilon_x + \rho_y) \leqslant \max\{\varepsilon_x, \rho_y\}.$$

Thus it must be that $d(x,y) < \varepsilon_x$ or $d(x,y) < \rho_y$. However, neither can be true, because (1) $x \in C$, while $B_{\rho_y}(y)$ lies outside C and (2) $y \in D$, while $B_{\varepsilon_x}(x)$ lies outside D. This means that $U \cap V = \emptyset$.

Thus, given arbitrary disjoint closed sets $C, D \subseteq X$, we have produced disjoint open sets $U, V \subseteq X$ such that $C \subseteq U$ and $D \subseteq V$. Furthermore, the singletons in X are closed. Therefore, X is normal.

Question 2

Exercise L11-11.

(a) Prove X is Hausdorff if and only if the diagonal $\Delta = \{(x,x) \in X \times X \mid x \in X\}$ is a closed subset of $X \times X$.

(b) Let G be a topological group (see Tutorial 4). Prove that G is Hausdorff if and only if $\{e\}$ is closed, where e is the identity element.

(c) Let G be a topological group and $H \subseteq G$ a normal subgroup. Prove that G/H is Hausdorff if and only if $H \subseteq G$ is closed.

2(a) ($\Rightarrow$) Suppose that X is Hausdorff. Take arbitrary $x, y \in X$ such that $(x,y) \in \Delta^c$. Then x and y are distinct, so, by the Hausdorff property, there exist disjoint open sets $U, V \subseteq X$ such that $x \in U$ and $y \in V$. We see that $U \times V$ is an open neighbourhood of $(x,y) \in X \times X$. Since $U \cap V = \emptyset$, the box $U \times V$ must lie outside the diagonal Δ (if $z \in X$ is such that $(z,z) \in \Delta \cap (U \times V)$ then $z \in U$ and $z \in V$, which is impossible since U and V are disjoint), so $(x,y) \in U \times V \subseteq \Delta^c$. Since $(x,y) \in \Delta^c$ was arbitrary, this shows that $\Delta \subseteq X \times X$ is closed.

($\Leftarrow$) Suppose that $\Delta \subseteq X \times X$ is closed. Take arbitrary distinct $x, y \in X$. Then $(x,y) \in \Delta^c$, and, since Δ^c is open, there exist open sets $U, V \subseteq X$ such that $(x,y) \in U \times V \subseteq \Delta^c$. As $U \times V$ lies outside the diagonal Δ, their intersection $U \cap V$ must be empty. This means that $U, V \subseteq X$ are disjoint open sets with $x \in U$ and $y \in V$. We have shown the existence of separating open neighbourhoods for arbitrary distinct $x, y \in X$, so X is Hausdorff.

(b) ($\Rightarrow$) Suppose that G is Hausdorff. Then each element in $\{e\}^c \subseteq G$ has an open neighbourhood which does not contain e (i.e. the open neighbourhood lies inside $\{e\}^c$), so, taking the union of all these open neighbourhoods, we see that $\{e\}^c$ is open. Therefore $\{e\}$ is closed.

($\Leftarrow$) Suppose that $\{e\} \subseteq G$ is closed. Consider the two maps:

$$\begin{array}{ccccc} G \times G & \xrightarrow{\ p\ } & G \times G & \xrightarrow{\ r\ } & G \\ (g,h) & \longmapsto & (g, h^{-1}) & & \\ & & (g,h) & \longmapsto & gh \end{array}$$

Since G is a topological group, inversion is continuous, so, by Exercise L12-2, the $p\colon G \times G \longrightarrow G$ is continuous. Also, $r\colon G \times G \longrightarrow G$ is continuous by the definition of a topological group. This means that $p \circ r$ is continuous. Note that $(p \circ r)(g,h) = gh^{-1}$ for all $g, h \in G$. Consider $(p \circ r)^{-1}(\{e\})$. We have

$$(p \circ r)^{-1}(\{e\}) = \{(g,h) \in G \times G \mid gh^{-1} = e\} = \{(g,h) \in G \times G \mid g = h\} = \{(g,g) \mid g \in G\},$$

which is the diagonal in $G \times G$. Since $\{e\} \subseteq G$ is closed and $p \circ r$ is continuous, $(p \circ r)^{-1}(\{e\})$ must be closed in $G \times G$. This means that the diagonal in $G \times G$ is closed, so, by part (a), G is Hausdorff.

(c) Recall (from Question 3 on Tutorial 4) that G/H has the quotient topology. Let $q\colon G \longrightarrow G/H$ be the quotient map, which is continuous by the definition of the quotient topology.

($\Rightarrow$) Suppose G/H is Hausdorff. Then, by part (b), $\{H\} \subseteq G/H$ is closed. Since q is continuous, $q^{-1}(\{H\}) = H \subseteq G$ is closed.

($\Leftarrow$) Suppose $H \subseteq G$ is closed. We will show that $\{H\} \subseteq G/H$ is closed, so that, by part (b), G/H is Hausdorff.

2(c) By the definition of the quotient topology, a set $U \subseteq G/H$ is open if and only if $q^{-1}(U) \subseteq G$ is open. Taking complements, this means that a set $C \subseteq G/H$ is closed if and only if $q^{-1}(C) \subseteq G$ is closed (taking complements commutes with taking preimages). With $C = \{H\} \subseteq G/H$, we have $q^{-1}(C) = H$, which, by assumption, is a closed subset of G. This means $C = \{H\}$ must be a closed subset of G/H. By part (b), since the singleton $\{H\}$ containing the identity element of G/H is closed, G/H is Hausdorff.

Question 3

Lemma L12-1. Let $f\colon X \longrightarrow Y$ be a function (not assumed continuous) between topological spaces X, Y. The *graph* of f is

$$\Gamma_f := \{(x, y) \in X \times Y \mid y = f(x)\}.$$

Prove that

(a) If Y is Hausdorff and f is continuous, Γ_f is closed in $X \times Y$.

(b) Give a counterexample to show that if Y is *not* Hausdorff, it is not necessarily the case that the graph of a continuous function $f\colon X \longrightarrow Y$ is closed.

(c) If Y is compact and Γ_f is closed, f is continuous. (First show $X \times Y \longrightarrow X$ sends closed subsets to closed subsets, using that Y is compact.)

3(a) Assume that Y is Hausdorff and that f is continuous. Suppose we are given $(x, y) \in X \times Y \setminus \Gamma_f$. In order to show that Γ_f is closed in $X \times Y$, we must produce open subsets $U \subseteq X$ and $V \subseteq Y$ such that $(x, y) \in U \times V \subseteq \Gamma_f^c$. Note that for general subsets $A \subseteq X$ and $B \subseteq Y$, the box $A \times B$ lies outside Γ_f if and only if $f(A) \subseteq B^c$, which is if and only if $f^{-1}(B) \subseteq A^c$.

Consider the points $f(x)$ and y in Y. Since $(x, y) \notin \Gamma_f$, we have $y \neq f(x)$, and, furthermore, because Y is Hausdorff, there exist $U', V \subseteq Y$ which are open and disjoint such that $f(x) \in U'$ and $y \in V$. Take $U := f^{-1}(\overline{V})^c \subseteq X$. We know $f^{-1}(\overline{V})$ is closed as the preimage of a closed set under a continuous map, so U is open in X.

We claim that $(x, y) \in U \times V$ and that $U \times V$ lies outside Γ_f. By construction, we know $y \in V$, so, in order to show that $(x, y) \in U \times V$, it remains to show that $x \in U$. Recall that U' and V were disjoint open subsets of Y used to separate $f(x)$ and y. This means that $V \subseteq U'^c$, where $U'^c \subseteq Y$ is closed, and that $\overline{V} \subseteq U'^c$. In particular, we have $f(x) \notin \overline{V}$. We can now argue that

$$f(x) \notin \overline{V} \quad \Longleftrightarrow \quad x \notin f^{-1}(\overline{V}) \quad \Longleftrightarrow \quad x \in f^{-1}(\overline{V})^c = U,$$

which shows that $(x, y) \in U \times V$. To see that $U \times V$ lies outside Γ_f, we use the fact that $f^{-1}(V) \subseteq f^{-1}(\overline{V}) = U^c$.

We have successfully produced open sets $U \subseteq X$ and $V \subseteq Y$ such that $(x, y) \in U \times V \subseteq X \times Y \setminus \Gamma_f$. Since $(x, y) \in X \times Y \setminus \Gamma_f$ was arbitrary, Γ_f is closed in $X \times Y$.

(b) Take $X = Y = \{0, 1\}$. Let X have the discrete topology, and let Y have the indiscrete topology. Because the singletons $\{0\}$ and $\{1\}$ are not closed in Y, we see that Y is not Hausdorff.

Take the function $f\colon X \longrightarrow Y$ given by $f(0) = 0$ and $f(1) = 1$. Since X has the discrete topology (or since Y has the indiscrete topology), f is continuous. The graph of f is $\Gamma_f = \{(0, 0), (1, 1)\} \subseteq X \times Y$, while the product topology on $X \times Y$ is

$$\{\emptyset, \{0\} \times Y, \{1\} \times Y, X \times Y\}.$$

The graph Γ_f is not the complement of any open set in $X \times Y$ under the product topology, so we have provided an example where Y is not Hausdorff and where the graph of a continuous $f\colon X \longrightarrow Y$ is not closed.

(c) $\pi_X\colon X \times Y \longrightarrow X$ is closed

Assume that Y is compact. We first show that the projection map $\pi_X\colon X \times Y \longrightarrow X$, where $\pi_X(x, y) = x$, is closed. Suppose we are given a closed $E \subseteq X \times Y$ and an $x \in \pi_X(E)^c$. We will show that there exists an open $U \subseteq X$ such that $x \in U \subseteq \pi_X(E)^c$. Since $x \in \pi_X(E)^c$, we must have that $E \subseteq \pi_X^{-1}(\{x\})^c$, so

3(c) $\{x\} \times Y \subseteq E^c$. For every $y \in Y$, we have $(x, y) \in E^c$, where $E^c \subseteq X \times Y$ is open since E is closed. This means that, for every $y \in Y$, there exist open $U_y \subseteq X$ and $V_y \subseteq Y$ such that $(x, y) \in U_y \times V_y \subseteq E^c$. The family $\{V_y\}_{y \in Y}$ of open sets is a cover of Y, so, by compactness, there exists a finite set $\{y_i\}_{i=1}^n \subseteq Y$ such that $Y = \bigcup_{i=1}^n V_{y_i}$. Since $U_{y_i} \ni x$ for every $i \in \{1, 2, \ldots, n\}$, we have that $\bigcap_{i=1}^n U_{y_i} \ni x$, so

$$(x, y) \in \bigcap_{i=1}^n U_{y_i} \times V_y \subseteq E^c \quad \forall y \in Y \qquad \Longrightarrow \qquad \{x\} \times Y \subseteq \bigcap_{i=1}^n U_{y_i} \times \underbrace{Y}_{= \bigcup_{y \in Y} V_y} \subseteq E^c.$$

Take $U := \bigcap_{i=1}^n U_{y_i}$. As a finite intersection of open subsets, U itself is an open subset of X. Since $U \times Y = \pi_X^{-1}(U) \subseteq E^c$, we have $\pi_X(E) \subseteq U^c$. Thus, we have produced an open $U \subseteq X$ such that $x \in U \subseteq \pi_X(E)^c$. Since $x \in \pi_X(E)^c$ was arbitrary, we have shown that $\pi_X(E)$ is closed. Because $E \subseteq X \times Y$ was an arbitrary closed set, π_X is a closed map.

$f \colon X \longrightarrow Y$ is continuous

Now suppose further that Γ_f is closed in $X \times Y$ for the given map $f \colon X \longrightarrow Y$. Suppose we are given a closed $D \subseteq Y$. In order to show that f is continuous, we will show that $f^{-1}(D) \subseteq X$ is closed.

Since the projection $\pi_Y \colon X \times Y \longrightarrow Y$ is continuous (Exercise L7-5) and $D \subseteq Y$ is closed, the set $X \times D = \pi_Y^{-1}(D)$ is closed in $X \times Y$. Now observe that

$$\begin{aligned} f^{-1}(D) &= \{x \in X \mid f(x) \in D\} \\ &= \pi_X(\{(x, f(x)) \in X \times Y \mid f(x) \in D\}) \\ &= \pi_X(\Gamma_f \cap (X \times D)). \end{aligned}$$

Since Γ_f is closed by assumption and $X \times D = \pi_Y^{-1}(D)$ is closed, their intersection $\Gamma_f \cap (X \times D)$ is closed in $X \times Y$. Since π_X is a closed map, we see that $f^{-1}(D)$ is the image under a closed map of a closed set, and, therefore, $f^{-1}(D)$ is closed in X. As $D \subseteq Y$ was an arbitrary closed set, we have shown that $f \colon X \longrightarrow Y$ is continuous.

Question 4

Exercise L12-6. Let $(S^1, +, 0)$ be the circle as a topological group (see Tutorial 4). Prove that the map $S^1 \times \mathcal{L}Y \longrightarrow \mathcal{L}Y$ sending (θ, f) to the function $\theta' \longmapsto f(\theta + \theta')$ is continuous. This map "rotates the loops" in Y.

We attempt a kind of reversed construction of the specified map using the adjunction property of the compact–open topology. Consider the following maps:

$$\begin{array}{ccccc} S^1 \times \mathcal{L}Y \times S^1 & & (\theta, f, \theta') & & \\ \big\downarrow \cong & & \big\downarrow & & \\ \mathcal{L}Y \times S^1 \times S^1 & & (f, \theta, \theta') & & \\ \big\downarrow \mathrm{id}_{\mathcal{L}Y} \times (+) & & \big\downarrow & & \\ \mathcal{L}Y \times S^1 & & (f, \theta + \theta') & & (f, \omega) \\ \big\downarrow \mathrm{ev}_{S^1, Y} & & & & \big\downarrow \\ Y & & & & f(\omega). \end{array}$$

The map $S^1 \times \mathcal{L}Y \times S^1 \longrightarrow \mathcal{L}Y \times S^1 \times S^1$ is a homeomorphism since it only swaps coordinates. The map $\mathcal{L}Y \times S^1 \times S^1 \longrightarrow \mathcal{L}Y \times S^1$ is continuous by Exercise L12-2, where we have used the fact that $+\colon S^1 \times S^1 \longrightarrow S^1$ is continuous by virtue of S^1 being a topological group. Finally, $\mathrm{ev}_{S^1,Y}\colon \mathcal{L}Y \times S^1 \longrightarrow Y$ is continuous, because S^1 is (locally) compact and Hausdorff. Thus, the composition of the above maps, which is

$$\begin{aligned} R_0\colon \qquad S^1 \times \mathcal{L}Y \times S^1 &\longrightarrow Y \\ (\theta, f, \theta') &\longmapsto f(\theta + \theta'), \end{aligned}$$

is continuous. That is, $R_0 \in \mathrm{Cts}(S^1 \times \mathcal{L}Y \times S^1, Y)$. By the adjunction property of the compact–open topology, there exists an $R \in \mathrm{Cts}(S^1 \times \mathcal{L}Y, \mathcal{L}Y)$ such that $R(\theta, f)(\theta') = R_0(\theta, f, \theta') = f(\theta + \theta')$ for every $\theta' \in S^1$. Therefore, this R is precisely the map initially specified in the question: The map R is the map $S^1 \times \mathcal{L}Y \longrightarrow \mathcal{L}Y$ sending (θ, f) to the function $\theta' \longmapsto f(\theta + \theta')$. By our construction, which used the adjunction property of the compact–open topology, R is continuous.

Question 5

Exercise L12-10. Two continuous maps $f, g\colon X \longrightarrow Y$ are *homotopic* if there exists $F\colon [0,1] \times X \longrightarrow Y$ continuous with $F(0,-) = f$ and $F(1,-) = g$. Prove that if X is locally compact Hausdorff there is a bijection between such homotopies F and paths in $\mathrm{Cts}(X,Y)$ from f to g.

By the adjunction property of the compact–open topology, because X is locally compact and Hausdorff, we know there is a bijection

$$\Psi_{[0,1],X,Y}\colon \qquad \mathrm{Cts}([0,1] \times X, Y) \longrightarrow \mathrm{Cts}([0,1], \mathrm{Cts}(X,Y)).$$

Suppose we are given $f, g \in \mathrm{Cts}(X,Y)$. Let $H \subseteq \mathrm{Cts}([0,1] \times X, Y)$ be the space of homotopies between f and g. Let $P \subseteq \mathrm{Cts}([0,1], \mathrm{Cts}(X,Y))$ be the space of paths in $\mathrm{Cts}(X,Y)$ from f to g. That is, for every $r \in \mathrm{Cts}([0,1], \mathrm{Cts}(X,Y))$, we have that $r \in P$ if and only if $r(0) = f$ and $r(1) = g$.

We claim that the restriction of $\Psi_{[0,1],X,Y}$ to H gives the desired bijection between H and P. Since $\Psi_{[0,1],X,Y}$ is a bijection, it is sufficient to show that $\Psi_{[0,1],X,Y}(H) = P$. We will show that $\Psi_{[0,1],X,Y}(H) \subseteq P$ and that $P \subseteq \Psi_{[0,1],X,Y}(H)$.

$\Psi_{[0,1],X,Y}(H) \subseteq P$

Suppose $F \in H$. We wish to show that $\Psi_{[0,1],X,Y}(F) \in P$. That is, we wish to show that

$$\Psi_{[0,1],X,Y}(F)(0)(-) = f \quad \text{and} \quad \Psi_{[0,1],X,Y}(F)(1)(-) = g.$$

This follows immediately from the definition given in class for $\Psi_{[0,1],X,Y}$, where

$$\Psi_{[0,1],X,Y}(F)(t) = F(t,-) \quad \forall t \in [0,1],$$

and the fact that $F(0,-) = f$ and $F(1,-) = g$ (from $F \in H$, i.e. from F being a homotopy between f and g). Thus, $\Psi_{[0,1],X,Y}(F) \in P$, and, as $F \in H$ was arbitrary, we have shown that $\Psi_{[0,1],X,Y}(H) \subseteq P$.

$P \subseteq \Psi_{[0,1],X,Y}(H)$

Suppose $r \in P$. Consider the map

$$[0,1] \times X \xrightarrow{r \times \mathrm{id}_X} \mathrm{Cts}(X,Y) \times X \xrightarrow{\mathrm{ev}_{X,Y}} Y.$$

By Exercise L12-2, the first map is continuous, and, because X is locally compact and Hausdorff, the second map is continuous. Define F to be the composition

$$\begin{aligned} F\colon \qquad [0,1] \times X &\longrightarrow Y \\ (t,x) &\longmapsto r(t)(x), \end{aligned}$$

which is continuous as a composition of continuous functions. Then $F(0,-) = r(0) = f$ and $F(1,-) = r(1) = g$, so $F \in H$. Finally, using the definition given in class for $\Psi_{[0,1],X,Y}$, we have $\Psi_{[0,1],X,Y}(F) = r$, so $r \in \Psi_{[0,1],X,Y}(H)$. As $r \in P$ was arbitrary, we conclude that $P \subseteq \Psi_{[0,1],X,Y}(H)$.

Question 6

Exercise L12-13. Prove that if X, Z are locally compact Hausdorff and Y is arbitrary that the bijection of Theorem L12-4

$$\mathrm{Cts}(Z\times X, Y) \xrightarrow{\Psi_{Z,X,Y}} \mathrm{Cts}(Z, \mathrm{Cts}(X,Y))$$

is a *homeomorphism* where both sides are given the compact–open topology.

We will show that $\Psi_{Z,X,Y}$ is (a) continuous and (b) open. Since $\Psi_{Z,X,Y}$ is a bijection, taking images or preimages under $\Psi_{Z,X,Y}$ commutes with taking arbitrary unions and arbitrary intersections. This means that, in order to show continuity and openness of $\Psi_{Z,X,Y}$, it suffices to consider the preimage or image of elements of a subbasis for the separate topologies.

For general topological spaces A and B, we define

$$S_{A,B}(K,U) \coloneqq \{f \in \mathrm{Cts}(A,B) \mid f(K) \subseteq U\}, \quad K \subseteq A \text{ compact},\ U \subseteq B \text{ open},$$

which is as per the notation in class, except that we now make a distinction as to what the underlying topological spaces are.

We will use some auxiliary results throughout. Some of their proofs will be included as part (c) at the end.

6(a) Our aim is to show that

$$\Psi_{Z,X,Y} \in \mathrm{Cts}(\mathrm{Cts}(Z\times X,Y), \mathrm{Cts}(Z,\mathrm{Cts}(X,Y))).$$

Consider the following two maps (both from Theorem L12-4):

$$\begin{array}{c} \mathrm{Cts}(\mathrm{Cts}(Z\times X,Y)\times Z\times X, Y) \\ \Big\downarrow \Psi_{\mathrm{Cts}(Z\times X,Y)\times Z,X,Y} \\ \mathrm{Cts}(\mathrm{Cts}(Z\times X,Y)\times Z, \mathrm{Cts}(X,Y)) \\ \Big\downarrow \Psi_{\mathrm{Cts}(Z\times X,Y),Z,\mathrm{Cts}(X,Y)} \\ \mathrm{Cts}(\mathrm{Cts}(Z\times X,Y), \mathrm{Cts}(Z,\mathrm{Cts}(X,Y))). \end{array}$$

Since X and Z are both locally compact Hausdorff, both maps are bijections. Consider the evaluation map

$$\begin{aligned} \mathrm{ev}_{Z\times X,Y}: \qquad \mathrm{Cts}(Z\times X,Y)\times Z\times X &\longrightarrow Y \\ (F,z,x) &\longmapsto F(z,x). \end{aligned}$$

We claim that $\mathrm{ev}_{Z\times X,Y}$ is continuous and that, under the two maps above, we have

$$(\Psi_{\mathrm{Cts}(Z\times X,Y),Z,\mathrm{Cts}(X,Y)} \circ \Psi_{\mathrm{Cts}(Z\times X,Y)\times Z,X,Y})(\mathrm{ev}_{Z\times X,Y}) = \Psi_{Z,X,Y},$$

which will show that $\Psi_{Z,X,Y}$ is a continuous map from $\mathrm{Cts}(Z\times X,Y)$ to $\mathrm{Cts}(Z,\mathrm{Cts}(X,Y))$.

$\mathrm{ev}_{Z\times X,Y}$ is continuous

From class, we know that $\mathrm{ev}_{A,B}$ is continuous as soon as A is locally compact Hausdorff, so we will show that $Z\times X$ is locally compact Hausdorff. Note that $Z\times X$ is Hausdorff as a product of Hausdorff spaces (Lemma L11-3), so it remains to show that $Z\times X$ is locally compact. To this end, we have the following result. The proof will be provided in part (c).

6(a) **Lemma 6.1**

The product of two locally compact spaces is locally compact.

Applying Lemma 6.1 to Z and X, we find that $Z \times X$ is locally compact (in addition to already being Hausdorff). This shows that $\mathrm{ev}_{Z\times X,Y}$ is continuous.

$\Psi_{Z,X,Y}$ is continuous

Let us consider what happens to $\mathrm{ev}_{Z\times X,Y}$ after the application of $\Psi_{\mathrm{Cts}(Z\times X,Y)\times Z,X,Y}$. Call the output Θ_1. We know that Θ_1 is a continuous function from $\mathrm{Cts}(Z \times X, Y) \times Z$ to $\mathrm{Cts}(X, Y)$. Moreover, given a continuous $F\colon Z \times X \longrightarrow Y$ and a $z \in Z$, we have, by the definition of $\Psi_{\mathrm{Cts}(Z\times X,Y)\times Z,X,Y}$, that $\Theta_1(F, z)$ is the continuous function from X to Y where

$$\begin{aligned}\Theta_1(F,z)(x) &= \underbrace{\mathrm{ev}_{Z\times X,Y}(F,z,x)}_{\text{Think of as } \mathrm{ev}_{Z\times X,Y}((F,z),x)} \\ &= F(z,x) \in Y \quad \forall x \in X.\end{aligned} \tag{6.1}$$

Next, let us consider what happens to Θ_1 after the application of $\Psi_{\mathrm{Cts}(Z\times X,Y),Z,\mathrm{Cts}(X,Y)}$. Call the output Θ_2. Then Θ_2 is a continuous function from $\mathrm{Cts}(Z \times X, Y)$ to $\mathrm{Cts}(Z, \mathrm{Cts}(X, Y))$. Given a continuous $F\colon Z \times X \longrightarrow Y$, we have, by the definition of $\Psi_{\mathrm{Cts}(Z\times X,Y),Z,\mathrm{Cts}(X,Y)}$, that $\Theta_2(F)$ is the continuous function from Z to $\mathrm{Cts}(X, Y)$ where

$$\Theta_2(F)(z) = \Theta_1(F,z) \in \mathrm{Cts}(X,Y) \quad \forall z \in Z.$$

This means that, given a $z \in Z$, the symbol $\Theta_2(F)(z)$ denotes the continuous function from X to Y where

$$\begin{aligned}\Theta_2(F)(z)(x) &= \Theta_1(F,z)(x) \\ &= \mathrm{ev}_{Z\times X,Y}(F,z,x) \quad \text{by (6.1)} \\ &= F(z,x) \in Y \qquad \forall x \in X.\end{aligned}$$

Letting z vary over Z, we have that $\Theta_2(F)(z)(x) = F(z, x)$ for every $(z, x) \in Z \times X$. However, this means that $\Theta_2(F) = \Psi_{Z,X,Y}(F)$ as elements of $\mathrm{Cts}(Z, \mathrm{Cts}(X, Y))$! As $F \in \mathrm{Cts}(Z \times X, Y)$ was arbitrary, we have that

$$\Theta_2 = \Psi_{Z,X,Y}$$

as (set) maps from $\mathrm{Cts}(Z \times X, Y)$ to $\mathrm{Cts}(Z, \mathrm{Cts}(X, Y))$. Since Θ_2 is continuous by construction, it must be that $\Psi_{Z,X,Y}$ is continuous also!

(b) ### Properties of $S_{A,B}(-,-)$

In this response, we will use some properties of $S_{A,B}(-,-)$. The properties are more set-theoretic than topological.

Lemma 6.2

Let A and B be arbitrary topological spaces. Then $S_{A,B}(-,-)$ has the following properties:

(a) Anti-distributive over finite unions in the first argument: If $\{K_i\}_{i=1}^n$ is a finite collection of compact subsets of A and $U \subseteq B$ is open then $S_{A,B}(\bigcup_{i=1}^n K_i, U) = \bigcap_{i=1}^n S_{A,B}(K_i, U)$.

(b) Distributive over finite intersections in the second argument: If $K \subseteq A$ is compact and $\{U_i\}_{i=1}^n$ is a finite collection of open subsets of B then $S_{A,B}(K, \bigcap_{i=1}^n U_i) = \bigcap_{i=1}^n S_{A,B}(K, U_i)$.

(c) Inclusion-reversing in the first argument: If $K \subseteq K'$ are two compact subsets of A and $U \subseteq B$ is open then $S_{A,B}(K, U) \supseteq S_{A,B}(K', U)$.

(d) Inclusion-preserving in the first argument: If $K \subseteq A$ is compact and $U \subseteq U'$ are two open subsets

6(b) of B then $S_{A,B}(K,U) \subseteq S_{A,B}(K,U')$.

Note that the last two properties are consequences of the first two. The requirement of the collections $\{K_i\}_{i=1}^n$ and $\{U_i\}_{i=1}^n$ to be finite is so that the union and intersection, respectively, remain compact and open, respectively.

$\Psi_{Z,X,Y}$ is open

Suppose we are given a compact $G \subseteq Z \times X$ and an open $V \subseteq Y$. We wish to show that

$$\Psi_{Z,X,Y}(S_{Z\times X,Y}(G,V))$$

is open in $\mathrm{Cts}(Z, \mathrm{Cts}(X,Y))$. Let $R \in \Psi_{Z,X,Y}(S_{Z\times X,Y}(G,V))$ be given. Then, equivalently, we wish to produce an open neighbourhood of R contained in $\Psi_{Z,X,Y}(S_{Z\times X,Y}(G,V))$.

The idea will be to cover G with finitely many boxes $L \times H$, where $L \subseteq Z$ and $H \subseteq X$ are compact. This is motivated by the fact that the image of $S_{Z\times X,Y}(L\times H, V)$ under $\Psi_{Z,X,Y}$ is simply $S_{Z,\mathrm{Cts}(X,Y)}(L, S_{X,Y}(H,V))$ as well as the fact that $S_{A,B}(-,-)$ is anti-distributive over finite unions and inclusion-reversing in the first argument. We have to be careful, however, that L and H are small enough so that $S_{Z,\mathrm{Cts}(X,Y)}(L, S_{X,Y}(H,V))$ is large enough to still contain R.

Fix an arbitrary $(z,x) \in G$. Let $R_0 \in S_{Z\times X,Y}(G,V)$ be the unique function such that $R = \Psi_{Z,X,Y}(R_0)$. Since $R_0\colon Z \times X \longrightarrow Y$ is continuous, $T := R_0^{-1}(V) \subseteq Z \times X$ is open. Note that $(z,x) \in G \subseteq T$.

Since Z and X are both locally compact Hausdorff, the product $Z \times X$ is also locally compact Hausdorff (Lemma L11-3 and Lemma 6.1). Therefore, $Z \times X$ is regular (Lemma L12-0(ii)), and since (z,x) is outside the closed set T^c, there exist disjoint open sets $T_{z,x}^{(1)}, T_{z,x}^{(2)} \subseteq Z \times X$ such that $(z,x) \in T_{z,x}^{(1)}$ and $T^c \subseteq T_{z,x}^{(2)}$, so we can write

$$(z,x) \in T_{z,x}^{(1)} \subseteq {T_{z,x}^{(2)}}^c \subseteq T.$$

We will see later that ${T_{z,x}^{(2)}}^c$ is a device to ensure our boxes $L \times H$ are small enough.

Since Z and X are both locally compact, there exist an open $W_z^{(1)} \subseteq Z$, an open $U_x^{(1)} \subseteq X$, a compact $L_z \subseteq Z$, and a compact $H_x \subseteq X$ such that

$$z \in W_z^{(1)} \subseteq L_z \quad \text{and} \quad x \in U_x^{(1)} \subseteq H_x.$$

Since $T_{z,x}^{(1)}$ is open and contains (z,x), there exist an open $W_{z,x}^{(2)} \subseteq Z$ and an open $U_{z,x}^{(2)} \subseteq X$ such that

$$(z,x) \in W_{z,x}^{(2)} \times U_{z,x}^{(2)} \subseteq T_{z,x}^{(1)}.$$

Let $W_{z,x} := W_z^{(1)} \cap W_{z,x}^{(2)}$ and $U_{z,x} := U_x^{(1)} \cap U_{z,x}^{(2)}$. Both $W_{z,x} \subseteq Z$ and $U_{z,x} \subseteq X$ are open. Our candidates for L and H are going to be (some finite number of the) $\overline{W}_{z,x}$ and $\overline{U}_{z,x}$. Let us proceed to show that they are appropriate candidates.

In the next part of the argument, we will use the following result. It is proved in part (c).

Lemma 6.3

Let A and B be topological spaces, and let $C \subseteq A$ and $D \subseteq B$ be arbitrary subsets. Then, in the product space $A \times B$, we have $\overline{C \times D} = \overline{C} \times \overline{D}$.

By the definition of the open sets $W_{z,x}$ and $U_{z,x}$, we have

$$(z,x) \in W_{z,x} \times U_{z,x} \subseteq {T_{z,x}^{(2)}}^c \cap (L_z \times H_x) \subseteq T.$$

First, observe that ${T_{z,x}^{(2)}}^c \cap (L_z \times H_x)$ is closed in $Z \times X$: (1) ${T_{z,x}^{(2)}}^c$ is closed due to $T_{z,x}^{(2)}$ being open and (2) by Lemma L10-2, $L_z \times H_x$ is compact as a product of the compact spaces L_z and H_x, so $L_z \times H_x$ is a

6(b) compact subspace of the Hausdorff space $Z \times X$ and hence closed (Lemma L11-5). Thus, if we consider the closure of $W_{z,x} \times U_{z,x}$ and use Lemma 6.3 then we have

$$(z,x) \in W_{z,x} \times U_{z,x} \subseteq \overline{W_{z,x} \times U_{z,x}} = \overline{W}_{z,x} \times \overline{U}_{z,x} \subseteq T_{z,x}^{(2)^c} \cap (L_z \times H_x) \subseteq T.$$

One consequence is that the closed set $\overline{W}_{z,x}$ is inside the compact space L_z, so, by Exercise L9-5, $\overline{W}_{z,x}$ is compact. Similarly, $\overline{U}_{z,x}$ is compact. Importantly, $\overline{W}_{z,x} \times \overline{U}_{z,x} \subseteq T = R_0^{-1}(V)$, so that

$$R_0(\overline{W}_{z,x} \times \overline{U}_{z,x}) \subseteq V \qquad \Longleftrightarrow \qquad R \in S_{Z,\mathrm{Cts}(X,Y)}(\overline{W}_{z,x}, S_{X,Y}(\overline{U}_{z,x}, V)). \tag{6.2}$$

This is what we were referring to in mentioning above that the boxes $L \times H$ needed to be small enough.

We now let (z,x) vary over G. This means that $\{W_{z,x} \times U_{z,x}\}_{(z,x)\in G}$ is an open cover of the compact space G, so, by compactness, we reduce to a finite subcover $\{W_{z_i,x_i} \times U_{z_i,x_i}\}_{i=1}^n$ for some finite set $\{(z_i,x_i)\}_{i=1}^n \subseteq G$. The closures of the open boxes in this finite subcover are precisely our choices of $L \times H$.

We now have

$$G \subseteq \bigcup_{i=1}^n (W_{z_i,x_i} \times U_{z_i,x_i}) \subseteq \bigcup_{i=1}^n \left(\overline{W}_{z_i,x_i} \times \overline{U}_{z_i,x_i}\right).$$

Note that the last set is compact as a finite union of compact sets (Exercise L10-4). Applying $S_{Z\times X,Y}(-,V)$ to the compact sets, and noting that $S_{A,B}(-,-)$ reverses inclusion in the first argument, we have that

$$S_{Z\times X,Y}\left(\bigcup_{i=1}^n \left(\overline{W}_{z_i,x_i} \times \overline{U}_{z_i,x_i}\right), V\right) \subseteq S_{Z\times X,Y}(G,V).$$

Now, $S_{A,B}(-,-)$ is anti-distributive over finite unions in the first argument, so we have

$$\bigcap_{i=1}^n S_{Z\times X,Y}(\overline{W}_{z_i,x_i} \times \overline{U}_{z_i,x_i}, V) \subseteq S_{Z\times X,Y}(G,V),$$

and, applying $\Psi_{Z,X,Y}$ to both sides, we have

$$\bigcap_{i=1}^n S_{Z,\mathrm{Cts}(X,Y)}(\overline{W}_{z_i,x_i}, S_{X,Y}(\overline{U}_{z_i,x_i}, V)) \subseteq \Psi_{Z,X,Y}(S_{Z\times X,Y}(G,V))$$

Note that the set on the left-hand side is a finite intersection of subbasis elements, so it is open in $\mathrm{Cts}(Z, \mathrm{Cts}(X,Y))$. Using (6.2), we altogether have that

$$R \in \underbrace{\bigcap_{i=1}^n S_{Z,\mathrm{Cts}(X,Y)}(\overline{W}_{z_i,x_i}, S_{X,Y}(\overline{U}_{z_i,x_i}, V))}_{\text{open in } \mathrm{Cts}(Z,\mathrm{Cts}(X,Y))} \subseteq \Psi_{Z,X,Y}(S_{Z\times X,Y}(G,V)).$$

Since $R \in \Psi_{Z,X,Y}(S_{Z\times X,Y}(G,V))$ was arbitrary, we have shown that $\Psi_{Z,X,Y}(S_{Z\times X,Y}(G,V))$ is open in $\mathrm{Cts}(Z,\mathrm{Cts}(X,Y))$. Since $G \subseteq Z \times X$ (compact) and $V \subseteq Y$ (open) were arbitrary, we have shown that the bijection $\Psi_{Z,X,Y}$ sends every set in a subbasis for the compact–open topology on $\mathrm{Cts}(Z\times X, Y)$ to an open set in $\mathrm{Cts}(Z,\mathrm{Cts}(X,Y))$. Thus, $\Psi_{Z,X,Y}$ is an open map. Note that the injectivity of $\Psi_{Z,X,Y}$ is important in ensuring that taking the image commutes with finite intersections.

(c) We provide proofs to some of the auxiliary results we have used.

Lemma 6.1

The product of two locally compact spaces is locally compact.

6(c) **Proof**

Suppose we are given two locally compact topological spaces A and B and an arbitrary $(a,b) \in A \times B$. We will show that there exists an open neighbourhood of (a,b) which is itself contained in a compact subset of $A \times B$. By local compactness of A and B, there exist an open $U \subseteq A$, an open $V \subseteq B$, a compact $H \subseteq X$, and a compact $K \subseteq Y$ such that

$$a \in U \subseteq H \quad \text{and} \quad b \in V \subseteq K.$$

Immediately, we see that $U \times V$ is an open neighbourhood of (a,b) in $A \times B$ and that $U \times V \subseteq H \times K$. Furthermore, $H \times K$ is compact as a product of two compact space (Lemma L10-2). Thus, we have shown that there exists an open neighbourhood of (a,b) contained in a compact subset of $A \times B$. As $(a,b) \in A \times B$ was arbitrary, this means that $A \times B$ is locally compact. □

Lemma 6.3

Let A and B be topological spaces, and let $C \subseteq A$ and $D \subseteq B$ be arbitrary subsets. Then, in the product space $A \times B$, we have $\overline{C \times D} = \overline{C} \times \overline{D}$.

Proof

First, let us verify that $\overline{C} \times \overline{D}$ is closed in $A \times B$. Denoting by $\pi_A \colon A \times B \longrightarrow A$ and $\pi_B \colon A \times B \longrightarrow B$ the projection maps, we note that

$$\overline{C} \times \overline{D} = (\overline{C} \times B) \cap (A \times \overline{D}) = \pi_A^{-1}(\overline{C}) \cap \pi_B^{-1}(\overline{D}).$$

Since π_A and π_B are continuous (Exercise L7-5) and $\overline{C} \subseteq A$ and $\overline{D} \subseteq B$ are closed, both $\pi_A^{-1}(\overline{C})$ and $\pi_B^{-1}(\overline{D})$ are closed sets in $A \times B$, so their intersection $\pi_A^{-1}(\overline{C}) \cap \pi_B^{-1}(\overline{D}) = \overline{C} \times \overline{D}$ is also closed in $A \times B$.

Now, since the set $\overline{C} \times \overline{D}$ is closed and contains $C \times D$, we must have that $\overline{C \times D} \subseteq \overline{C} \times \overline{D}$.

Next, we show that if a point (a,b) is outside $\overline{C \times D}$ then (a,b) is also outside $\overline{C} \times \overline{D}$. Because $\overline{C \times D}$ is closed, there must exist open $U \subseteq A$ and $V \subseteq B$ such that

$$(a,b) \in U \times V \subseteq \overline{C \times D}^c.$$

We claim that $U \subseteq \overline{C}^c$ or $V \subseteq \overline{D}^c$, so that $(a,b) \in U \times V \subseteq (\overline{C} \times \overline{D})^c$. Suppose for a contradiction that $U \cap \overline{C} \neq \emptyset$ and $V \cap \overline{D} \neq \emptyset$. Then U is an open neighbourhood of a point in $\overline{C}$, so, by Exercise L13-4(i), we have $U \cap C \neq \emptyset$. Similarly, we have $V \cap D \neq \emptyset$. However, this means that $U \times V$ is not completely outside $\overline{C \times D}$, because there is a point in $C \times D$ that is also in $U \times V$. We have therefore arrived at a contradiction. Thus, given a point (a,b) outside $\overline{C \times D}$, it must be that (a,b) is also outside $\overline{C} \times \overline{D}$. This shows that $\overline{C \times D} \supseteq \overline{C} \times \overline{D}$.

Altogether, we have shown that $\overline{C \times D} = \overline{C} \times \overline{D}$. □

Question 7

Exercise L13-3. If (Y, d_Y) is a metric space prove that the metric $d_Y : Y \times Y \longrightarrow \mathbb{R}$ is continuous when $Y \times Y$ is given the product topology.

Note that an alternate proof can be given using the result of Question 8(b).

Suppose we are given reals $a < b$. Since the topology on $\mathbb{R}$ has the collection of bounded open intervals as a basis, it suffices to show that $(d_Y)^{-1}((a, b)) \subseteq Y \times Y$ is open.

If $b \leqslant 0$ then, since d_Y is nonnegative, we have $(d_Y)^{-1}((a, b)) = \emptyset$, which is open in $Y \times Y$.

If $a < 0 < b$ then, since d_Y is nonnegative, we have

$$(d_Y)^{-1}((a, b)) = (d_Y)^{-1}([0, b)) = \{(y, z) \in Y \times Y \mid d_Y(y, z) < b\}$$

Suppose $y, z \in Y$ are such that $d_Y(y, z) < b$. Let $\varepsilon := b - d_Y(y, z) > 0$. We claim that

$$(y, z) \in B_{\varepsilon/2}(y) \times B_{\varepsilon/2}(z) \subseteq (d_Y)^{-1}([0, b)).$$

Take an arbitrary $(r, s) \in B_{\varepsilon/2}(y) \times B_{\varepsilon/2}(z)$. By the triangle inequality for d_Y, we have

$$d_Y(r, s) \leqslant d_Y(r, y) + d_Y(y, z) + d_Y(z, s) < \frac{\varepsilon}{2} + d_Y(y, z) + \frac{\varepsilon}{2} = b,$$

which shows that $B_{\varepsilon/2}(y) \times B_{\varepsilon/2}(z) \subseteq (d_Y)^{-1}([0, b))$ as claimed. As $B_{\varepsilon/2}(y) \times B_{\varepsilon/2}(z)$ is open under the product topology on $Y \times Y$, we have shown that $(d_Y)^{-1}((a, b)) \subseteq Y \times Y$ is open if $a < 0 < b$.

If $a \geqslant 0$ then take arbitrary $y, z \in Y$ where $a < d_Y(y, z) < b$. Let $\rho := \min\{b - d_Y(y, z), d_Y(y, z) - a\} > 0$. We claim that

$$(y, z) \in B_{\rho/2}(y) \times B_{\rho/2}(z) \subseteq (d_Y)^{-1}((a, b)).$$

Take an arbitrary $(r, s) \in B_{\rho/2}(y) \times B_{\rho/2}(z)$. By the triangle inequality for d_Y, we have

$$d_Y(y, z) \leqslant d_Y(y, r) + d_Y(r, s) + d_Y(s, z) < \frac{\rho}{2} + d_Y(r, s) + \frac{\rho}{2},$$

so that

$$d_Y(r, s) > d_Y(y, z) - \rho \geqslant d_Y(y, z) - (d_Y(y, z) - a) = a.$$

By the triangle inequality again, we also have

$$\begin{aligned} d_Y(r, s) &\leqslant d_Y(r, y) + d_Y(y, z) + d_Y(z, s) \\ &< \frac{\rho}{2} + d_Y(y, z) + \frac{\rho}{2} \\ &= d_Y(y, z) + \rho \\ &\leqslant d_Y(y, z) + b - d_Y(y, z) = b. \end{aligned}$$

We have shown that, for arbitrary $(r, s) \in B_{\rho/2}(y) \times B_{\rho/2}(z)$, we have $a < d_Y(r, s) < b$. This shows that $B_{\rho/2}(y) \times B_{\rho/2}(z) \subseteq (d_Y)^{-1}((a, b))$ as claimed. Since $B_{\rho/2}(y) \times B_{\rho/2}(z)$ is open, we have shown that $(d_Y)^{-1}((a, b)) \subseteq Y \times Y$ is open if $a \geqslant 0$.

In all three cases of $b \leqslant 0$, $a < 0 < b$, as well as $a \geqslant 0$, we have shown that $(d_Y)^{-1}((a, b)) \subseteq Y \times Y$ is open. This means that d_Y is continuous.

Question 8

Exercise L13-8. Let $(A_1, d_1), \dots, (A_n, d_n)$ be metric spaces and with $A = \prod_{i=1}^n A_i$ define $d\colon A \times A \longrightarrow \mathbb{R}$ by $d((a_i)_{i=1}^n, (b_i)_{i=1}^n) = \sum_i d_i(a_i, b_i)$. Prove that

(a) (A, d) is a metric space.

(b) The topology on A induced by d is the product topology on $\prod_{i=1}^n A_i$ (giving each A_i its metric topology).

(c) If each (A_i, d_i) is complete so is (A, d).

8(a) Nonnegativity and symmetry of d are inherited from the nonnegativity (a sum of nonnegative numbers is nonnegative) and symmetry of each d_i. Next, we show that d separates distinct elements: Suppose $\boldsymbol{a}, \boldsymbol{b} \in A$ are such that $d(\boldsymbol{a}, \boldsymbol{b}) = 0$. Then $\sum_i d_i(a_i, b_i) = 0$, and, since each $d_i(a_i, b_i)$ is nonnegative, we must have $d_i(a_i, b_i) = 0$ for every $i \in \{1, 2, \dots, n\}$. Since each d_i separates distinct elements in A_i, we have that $a_i = b_i$ for every $i \in \{1, 2, \dots, n\}$, so that $\boldsymbol{a} = \boldsymbol{b}$. This shows that d separates distinct elements of A.

It remains to show that d satisfies the triangle inequality. Take arbitrary $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in A$. Using the triangle inequality for each d_i, we have that $d_i(a_i, b_i) + d_i(b_i, c_i) \geqslant d_i(a_i, c_i)$. Summing in i over $\{1, 2, \dots, n\}$, we have

$$\underbrace{\sum_{i=1}^n [d_i(a_i, b_i) + d_i(b_i, c_i)]}_{=d(\boldsymbol{a},\boldsymbol{b})+d(\boldsymbol{b},\boldsymbol{c})} \geqslant \sum_{i=1}^n d_i(a_i, c_i) = d(\boldsymbol{a}, \boldsymbol{c}),$$

so that d satisfies the triangle inequality. Since d is nonnegative, is symmetric, separates distinct elements of A, and satisfies the triangle inequality, (A, d) is a metric space.

(b) Let $\mathcal{T}_d$ denote the topology on A induced by d, and let $\mathcal{T}$ denote the product topology on A. To show that $\mathcal{T}_d = \mathcal{T}$, we only have to show that the sets comprising a basis for $\mathcal{T}_d$ is contained in $\mathcal{T}$ (which shows $\mathcal{T}_d \subseteq \mathcal{T}$) and that the sets comprising a basis for $\mathcal{T}$ is contained in $\mathcal{T}_d$ (which shows $\mathcal{T} \subseteq \mathcal{T}_d$).

Write $B_\varepsilon(\boldsymbol{a})$ to denote the open ball centred at $\boldsymbol{a} \in A$ of radius $\varepsilon > 0$, and, for each $i \in \{1, 2, \dots, n\}$, write $B_\varepsilon^{(i)}(a_i)$ to denote the open ball centred at $a_i \in A_i$ of radius $\varepsilon > 0$.

Recall (Exercise L7-1(iii)) that a basis for $\mathcal{T}_d$ is

$$\mathcal{B}_d := \{B_\varepsilon(\boldsymbol{a}) \mid \boldsymbol{a} \in A,\ \varepsilon > 0\},$$

while a basis for $\mathcal{T}$, which is the product topology for a finite product of spaces $A = \prod_{i=1}^n A_i$, is

$$\mathcal{B} := \left\{ \prod_{i=1}^n U_i \;\middle|\; U_i \subseteq A_i \text{ open} \right\}.$$

$\mathcal{B}_d \subseteq \mathcal{T}$

Suppose we are given an $\boldsymbol{a} \in A$ and an $\varepsilon > 0$. We will show that $B_\varepsilon(\boldsymbol{a}) \in \mathcal{T}$.

Let $\boldsymbol{b} \in B_\varepsilon(\boldsymbol{a})$ be arbitrary, i.e. $\boldsymbol{b} \in A$ and $d(\boldsymbol{a}, \boldsymbol{b}) < \varepsilon$. Let $\delta := \varepsilon - d(\boldsymbol{a}, \boldsymbol{b}) > 0$. Note that if $\boldsymbol{c} \in A$ and $d(\boldsymbol{b}, \boldsymbol{c}) < \delta$ then, by the triangle inequality, we have

$$d(\boldsymbol{a}, \boldsymbol{c}) \leqslant d(\boldsymbol{a}, \boldsymbol{b}) + d(\boldsymbol{b}, \boldsymbol{c}) < d(\boldsymbol{a}, \boldsymbol{b}) + \delta = \varepsilon \quad \implies \quad \boldsymbol{c} \in B_\varepsilon(\boldsymbol{a}).$$

8(b) That is, $B_\delta(\boldsymbol{b}) \subseteq B_\varepsilon(\boldsymbol{a})$. Take $\prod_{i=1}^n B_{\delta/n}^{(i)}(b_i) \in \mathcal{B}$. We will show that

$$\boldsymbol{b} \in \prod_{i=1}^n B_{\delta/n}^{(i)}(b_i) \subseteq B_\varepsilon(\boldsymbol{a}).$$

That $\boldsymbol{b} \in \prod_{i=1}^n B_{\delta/n}^{(i)}(b_i)$ can be seen from the fact that each b_i is an element of $B_{\delta/n}^{(i)}(b_i)$. For the other part, we will show that $\prod_{i=1}^n B_{\delta/n}^{(i)}(b_i) \subseteq B_\delta(\boldsymbol{b})$ and then use the fact that $B_\delta(\boldsymbol{b}) \subseteq B_\varepsilon(\boldsymbol{a})$.

Let $\boldsymbol{c} \in \prod_{i=1}^n B_{\delta/n}^{(i)}(b_i)$ be arbitrary. Observe:

$$\begin{aligned} & \boldsymbol{c} \in \prod_{i=1}^n B_{\delta/n}^{(i)}(b_i) & \\ \iff \quad & d_i(b_i, c_i) < \delta/n & \forall i \in \{1, 2, \ldots, n\} \\ \implies \quad & \underbrace{\sum_{i=1}^n d_i(b_i, c_i)}_{=d(\boldsymbol{b},\boldsymbol{c})} < \delta & \\ \iff \quad & \boldsymbol{c} \in B_\delta(\boldsymbol{b}). & \end{aligned}$$

Letting $\boldsymbol{c}$ vary over $\prod_{i=1}^n B_{\delta/n}^{(i)}(b_i)$, we have that

$$\boldsymbol{b} \in \underbrace{\prod_{i=1}^n B_{\delta/n}^{(i)}(b_i)}_{\in \mathcal{B}} \subseteq B_\delta(\boldsymbol{b}) \subseteq B_\varepsilon(\boldsymbol{a}).$$

Since $\boldsymbol{b} \in B_\varepsilon(\boldsymbol{a})$ was arbitrary, this shows that $B_\varepsilon(\boldsymbol{a}) \in \mathcal{T}$. As $\boldsymbol{a} \in A$ and $\varepsilon > 0$ were arbitrary, we have shown that $\mathcal{B}_d \subseteq \mathcal{T}$.

$\mathcal{B} \subseteq \mathcal{T}_d$

For each $i \in \{1, 2, \ldots, n\}$, take arbitrary open sets $U_i \subseteq A_i$. We will show that $\prod_i U_i \in \mathcal{T}_d$.

Let $\boldsymbol{a} \in \prod_i U_i$ be arbitrary. For every $i \in \{1, 2, \ldots, n\}$, because $U_i \subseteq A_i$ is open and the topology on A_i is induced by d_i, there exists $\varepsilon_i > 0$ such that $B_{\varepsilon_i}^{(i)}(a_i) \subseteq U_i$. Let $\varepsilon := \bigwedge_{i=1}^n \varepsilon_i > 0$ (the smallest of the ε_i). We claim that $B_\varepsilon(\boldsymbol{a}) \subseteq \prod_{i=1}^n B_{\varepsilon_i}^{(i)}(a_i)$. Take an arbitrary $\boldsymbol{b} \in B_\varepsilon(\boldsymbol{a})$. Observe:

$$\begin{aligned} & \boldsymbol{b} \in B_\varepsilon(\boldsymbol{a}) & \\ \iff \quad & \sum_{i=1}^n d_i(a_i, b_i) < \varepsilon & \\ \implies \quad & d_i(a_i, b_i) < \varepsilon \leqslant \varepsilon_i & \forall i \in \{1, 2, \ldots, n\} \\ \implies \quad & b_i \in B_{\varepsilon_i}^{(i)}(a_i) & \forall i \in \{1, 2, \ldots, n\} \\ \iff \quad & \boldsymbol{b} \in \prod_{i=1}^n B_{\varepsilon_i}^{(i)}(a_i). & \end{aligned}$$

Letting $\boldsymbol{b}$ vary over $\in B_\varepsilon(\boldsymbol{a})$, we conclude that

$$\boldsymbol{a} \in \underbrace{B_\varepsilon(\boldsymbol{a})}_{\in \mathcal{T}_d} \subseteq \prod_{i=1}^n B_{\varepsilon_i}^{(i)}(a_i) \subseteq \prod_{i=1}^n U_i.$$

Since $\boldsymbol{a} \in \prod_i U_i$ was arbitrary, this shows that $\prod_i U_i \in \mathcal{T}_d$. As the $U_i \subseteq A_i$ were arbitrary open sets, we have shown that $\mathcal{B} \subseteq \mathcal{T}_d$.

8(b) Conclusion

Since $\mathcal{B}_d \subseteq \mathcal{T}$ and $\mathcal{B} \subseteq \mathcal{T}_d$ and $\mathcal{B}$ and $\mathcal{B}_d$ are bases for $\mathcal{T}$ and $\mathcal{T}_d$, respectively, we have that $\mathcal{T} = \mathcal{T}_d$.

(c) Assume that each (A_i, d_i) is a complete metric space. Let $(\boldsymbol{a}_m)_{m\in\mathbb{N}}$ be an arbitrary Cauchy sequence in (A, d). We will show that $(\boldsymbol{a}_m)_{m\in\mathbb{N}}$ converges in (A, d). We will write $\boldsymbol{a}_m = (a_{m,1}, a_{m,2}, \dots, a_{m,n})$.

The idea will be to show that, for every $i \in \{1, 2, \dots, n\}$, the sequence $(a_{m,i})_{m\in\mathbb{N}}$ is a Cauchy sequence in (A_i, d_i). We then use completeness of (A_i, d_i) to obtain a candidate for the ith coordinate of (what we will show is) the limit of $(\boldsymbol{a}_m)_{m\in\mathbb{N}}$.

Note that, for every $\boldsymbol{b}, \boldsymbol{c} \in A$ and every $i \in \{1, 2, \dots, n\}$, we have $d_i(b_i, c_i) \leqslant d(\boldsymbol{b}, \boldsymbol{c})$. Apply this result with terms of the sequence $(\boldsymbol{a}_m)_{m\in\mathbb{N}}$ in A in place of $\boldsymbol{b}$ and $\boldsymbol{c}$: Since $(\boldsymbol{a}_m)_{m\in\mathbb{N}}$ is d-Cauchy, we see that the sequence $(a_{m,i})_{m\in\mathbb{N}}$ in A_i is d_i-Cauchy. By the completeness of (A_i, d_i), this means that there exists $\widetilde{a}_i \in A_i$ such that $a_{m,i} \to \widetilde{a}_i$ in (A_i, d_i) as $m \to \infty$.

With $\widetilde{\boldsymbol{a}} := (\widetilde{a}_1, \widetilde{a}_2, \dots, \widetilde{a}_n) \in A$, we claim that $\boldsymbol{a}_m \to \widetilde{\boldsymbol{a}}$ in (A, d) as $m \to \infty$.

$$\begin{aligned} d(\boldsymbol{a}_m, \widetilde{\boldsymbol{a}}) &= \sum_{i=1}^{n} d_i(a_{m,i}, \widetilde{a}_i) \\ \lim_{m\to\infty} d(\boldsymbol{a}_m, \widetilde{\boldsymbol{a}}) &= \lim_{m\to\infty} \sum_{i=1}^{n} d_i(a_{m,i}, \widetilde{a}_i) \\ &= \sum_{i=1}^{n} \lim_{m\to\infty} d_i(a_{m,i}, \widetilde{a}_i) = 0, \end{aligned}$$

where the last equality is because $a_{m,i} \to \widetilde{a}_i$ in (A_i, d_i) as $m \to \infty$ for every $i \in \{1, 2, \dots, n\}$. This shows that $\boldsymbol{a}_m \to \widetilde{\boldsymbol{a}}$ in (A, d) as $m \to \infty$. Since $(\boldsymbol{a}_m)_{m\in\mathbb{N}}$ was an arbitrary Cauchy sequence in (A, d), we have shown that (A, d) is a complete metric space.