Lecture 10: Compactness III

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Which spaces are compact? So far we know that closed and bounded subsets of $\mathbb{R}$ (e.g. intervals $[a,b]$) are compact (using Bolzano–Weierstrass plus Theorem L9-2). This is not a very impressive list. Let us improve it. We will begin to use "space" to mean "topological space" unless otherwise indicated.

<u>Lemma L10-1</u> If $\sim$ is an equivalence relation on a compact space X then $X/\sim$ is compact.

<u>Proof</u> The quotient map $\rho: X \longrightarrow X/\sim$ is continuous so this follows from Proposition L9-3. $\square$

<u>Exercise L10-1</u> If X and Y are homeomorphic then X is compact iff. Y is compact.

So at least $S^1 \cong [0,1]/\sim$ is compact. If we knew $[0,1]\times[0,1]$ was compact we could use the same trick to show e.g. the torus is compact.

<u>Lemma L10-2</u> If X, Y are compact spaces then $X \times Y$ is compact.

<u>Proof</u> By Lemma L9-1 it suffices to show any open cover of $X \times Y$ of the form $\{U_i \times V_i\}_{i \in I}$ with $U_i \subseteq X$, $V_i \subseteq Y$ open has a finite subcover. If either X, Y is empty then $X \times Y = \phi$ is compact, so suppose $X \neq \phi$, $Y \neq \phi$ and choose some $x \in X$. Then $\{x\} \times Y \subseteq X \times Y$ is covered by the $U_i \times V_i$, and since $\{x\} \times Y \cong Y$ is compact there is $J_x \subseteq I$ finite such that (here we are using Ex L10-1 and Ex L9-4, to be more precise)

$$\{x\} \times Y \subseteq \bigcup_{j \in J_x} U_j \times V_j$$

We may assume $x \in U_j$ for all $j \in J_x$ (otherwise $U_j \times V_j$ is superfluous and we may remove j from J_x) and so x lies in the open set $U_x := \bigcap_{j \in J_x} U_j$. Note $U_x \times Y \subseteq \bigcup_{j \in J_x} U_j \times V_j$. Since $\{U_x\}_{x \in X}$ is an open cover of X and X is compact, there is a finite subcover $\{U_{x_1}, \dots, U_{x_r}\}$. But then

$$X \times Y = \bigcup_{i=1}^{r} U_{x_i} \times Y \subseteq \bigcup_{i=1}^{r} \bigcup_{j \in J_{x_i}} U_j \times V_j. \quad \square$$

We deduce that the torus $\mathbb{T} = [0,1] \times [0,1]/\sim$ is compact. It is true that <u>any</u> product (even infinite ones) of compact spaces is compact, but this is much harder to prove – it is called Tychonoff's Theorem (alas, we do not have time to prove this). Clearly by induction we deduce from the lemma that any <u>finite</u> product of compact spaces is compact.

<u>Theorem L10-3</u> (Heine-Borel) A subset of $\mathbb{R}^n$ is compact iff. it is closed and bounded.

<u>Proof</u> If $Y \subseteq \mathbb{R}^n$ is compact then it is sequentially compact (Thm L9-2) and so closed and bounded by the same argument as in Bolzano-Weierstrass (see Exercise L8-2). If Y is bounded then $Y \subseteq [a,b]^n$ for some a, b (Exercise L8-4) and if Y is additionally closed then it is a closed subspace of a compact space, hence compact (Exercise L9-5). $\square$

<u>Exercise L10-2</u> A function $f: X \longrightarrow Y$ between topological spaces is continuous iff. $f^{-1}(C)$ is closed for every closed set $C \subseteq Y$.

<u>Corollary L10-4</u> The n-disk $D^n \subseteq \mathbb{R}^n$ and n-sphere $S^n \subseteq \mathbb{R}^{n+1}$ are compact.

<u>Proof</u> We need only show these subspaces are closed. The easy way is to note that $f = \|-\| : \mathbb{R}^n \longrightarrow \mathbb{R}$ is continuous, so e.g.

$$D^n := f^{-1}([0,1])$$

is closed, and similarly for S^n. $\square$

<u>Exercise L10-3</u> Recall real projective space $\mathbb{R}\mathbb{P}^n$ from p. 14 of Lecture 7. Show $\mathbb{R}\mathbb{P}^n$ is compact by exhibiting it as a quotient of S^n.

<u>Lemma L10-5</u> If X, Y are compact spaces then $X \amalg Y$ is compact.

<u>Proof</u> If $\{U_i\}_{i \in I}$ is an open cover of $X \amalg Y$ then (identifying X, Y with subspaces of $X \amalg Y$) we have $U_i = (U_i \cap X) \cup (U_i \cap Y)$. Choose $J \subseteq I$ finite such that $\{U_j \cap X\}_{j \in J}$ covers X and $\{U_j \cap Y\}_{j \in J}$ covers Y. Then $\{U_j\}_{j \in J}$ covers $X \amalg Y$. $\square$

<u>Exercise L10-4</u> Let X be a space and $Y_1, \dots, Y_n$ compact subsets. Then $Y_1 \cup \dots \cup Y_n$ is compact.

<u>Theorem L10-5</u> Any finite CW-complex is compact.

<u>Proof</u> By definition if X is a finite CW-complex there is a sequence of spaces $X_0, X_1, \dots, X_{n-1}, X_n = X$ with X_0 finite and discrete, and each X_i obtained from X_{i-1} via a pushout

$$\begin{tikzcd}
\coprod_{\alpha \in \Lambda} S^{i-1} \arrow[r] \arrow[d] & X_{i-1} \arrow[d] \\
\coprod_{\alpha \in \Lambda} D^i \arrow[r] & X_i = (X_{i-1} \amalg \coprod_\alpha D^i)/\sim
\end{tikzcd}$$

We prove X is compact by induction: X_0 is clearly compact, and if X_{i-1} is compact then so is $X_{i-1} \amalg \coprod_{\alpha \in \Delta} D^i$ by Corollary L10-4 and the previous lemma (it is important Δ is finite!). But then the quotient X_i is also compact by Lemma L10-1, which completes the inductive step. $\square$

That is a respectable supply of compact spaces!