Lecture 11: Hausdorff spaces

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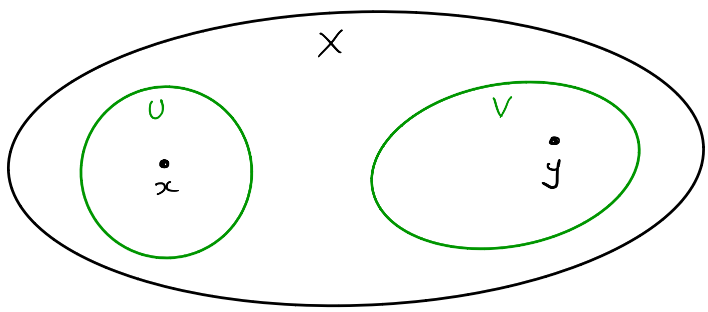
So far we have two main sources of examples of topological spaces: the underlying set of a metric space with the metric topology, and finite CW-complexes (these two classes, of course, have significant overlap). We have seen a few other examples like the Sierpiński space

$$\Sigma = \{0,1\} \qquad \mathcal{T}_\Sigma = \{\phi, \Sigma, \{1\}\}$$

but these examples seem rather "exotic". One way in which Σ is strange is that the point $1 \in \Sigma$ is <u>not closed</u> (since its complement $\{0\}$ is not open). Clearly any point in a metric topology is a closed set, and "non-closed point" seems like an oxymoron. In many parts of mathematics (with important exceptions) one is only interested in spaces in which all points are closed, and even better, any two distinct points may be <u>separated</u> by open neighborhoods:

<u>Def^n</u> A topological space X is <u>Hausdorff</u> if for any pair $x, y \in X$ of distinct points there exist open $U, V \subseteq X$ with $x \in U$, $y \in V$ and $U \cap V = \phi$.



<u>Lemma L11-1</u> If X is Hausdorff and $x \in X$ then $\{x\}$ is closed.

<u>Proof</u> For $y \in X \setminus \{x\}$ let $V_y \subseteq X$ be open with $x \notin V$, $y \in V$. Then $X \setminus \{x\} = \bigcup_y V_y$. □

<u>Lemma L11-2</u> If X is metrisable then it is Hausdorff.

<u>Proof</u> Say the topology is induced by a metric d. Given $x, y \in X$ distinct $d(x,y) > 0$ and for any $\varepsilon \leqslant \frac{1}{2} d(x,y)$ the balls $U = B_\varepsilon(x)$, $V = B_\varepsilon(y)$ will do, since

$$2\varepsilon \leqslant d(x,y) \leqslant d(x,z) + d(z,y)$$

and so if $z \in U$, so $d(z,x) < \varepsilon$ then $d(z,y) \geqslant 2\varepsilon - \varepsilon = \varepsilon$, so $z \notin V$. $\square$

Needless to say, Σ is <u>not</u> Hausdorff, and hence not metrisable.

<u>Exercise L11-1</u> Give an example of a space X in which points are closed, but which is <u>not</u> Hausdorff.

<u>Exercise L11-2</u> Prove any subspace of a Hausdorff space is Hausdorff.

There are some obvious questions: which of the standard constructions on topological spaces (product, disjoint union, quotient) preserve the Hausdorff condition? And then, who cares? What is this condition "good for"?

<u>Lemma L11-3</u> If X, Y are Hausdorff then so is $X \times Y$.

<u>Proof</u> If $(x_1, y_1), (x_2, y_2) \in X \times Y$ are distinct, without loss of generality $x_1 \neq x_2$. Then there exist open disjoint $U, V \subseteq X$ with $x_1 \in U$, $x_2 \in V$, and then $U \times Y, V \times Y \subseteq X \times Y$ are open, disjoint and $(x_1, y_1) \in U \times Y$, $(x_2, y_2) \in V \times Y$. $\square$

<u>Exercise L11-4</u> Prove that if $\{X_i\}_{i \in I}$ is a family of Hausdorff spaces that $\prod_{i \in I} X_i$ is Hausdorff.

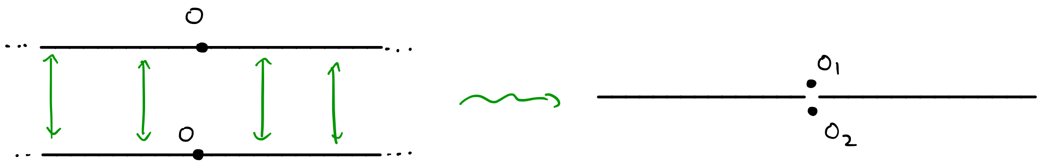
<u>Exercise L11-5</u> Prove if X, Y are Hausdorff that $X \amalg Y$ is Hausdorff.

<u>Example L11-1</u> Consider the pushout

$$\begin{tikzcd} \mathbb{R} \setminus \{0\} \arrow[r, "\iota"] \arrow[d, "\iota"'] & \mathbb{R} \arrow[d, "f"] \\ \mathbb{R} \arrow[r, "g"'] & X = (\mathbb{R} \amalg \mathbb{R})/\sim \end{tikzcd}$$

where ι is the inclusion. Here X is obtained from two copies of the line glued <u>along every point but the origin</u>. The space X is often called the "line with doubled origin". Let us write $0_1 := f(0)$ and $0_2 := g(0)$. Then as a set

$$X = \mathbb{R} \setminus \{0\} \cup \{0_1, 0_2\}.$$



The open neighborhoods of 0_1 in X are the images under the quotient map of open sets $U \amalg V \subseteq \mathbb{R} \amalg \mathbb{R}$ which are "saturated", i.e. closed under $\sim$, and for which $0 \in U$. But such an open set must contain points in the second copy of $\mathbb{R}$ arbitrarily close to 0, and thus in X there can be no open neighborhood of 0_1 avoiding every neighborhood of 0_2.

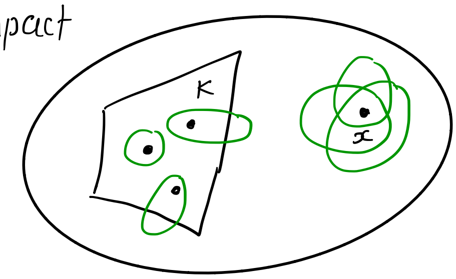
Hence X is not Hausdorff, and so the quotient of a Hausdorff space need not be Hausdorff (as $\mathbb{R} \amalg \mathbb{R}$ is Hausdorff).

Exercise L11-6 If $X \cong Y$ and X is Hausdorff then so is Y.

The Hausdorff condition is very useful, as we will see. But here are two immediately useful consequences:

Lemma L11-5 Any compact subspace of a Hausdorff space is closed.

Proof Say $K \subseteq X$ is compact, X Hausdorff. Given $x \notin K$ choose for each $k \in K$ a pair of disjoint open sets U_k, V_k with $k \in U_k$ and $x \in V_k$. The $\{U_k\}_{k \in K}$ cover K, and since it is compact finitely many, say $\{U_{k_1}, \dots, U_{k_n}\}$ will do. But then $x \in V_{k_1} \cap \cdots \cap V_{k_n}$ is an open neighborhood of x disjoint from K. This proves K is closed. $\square$



Lemma L11-6 Suppose X is compact and Y is Hausdorff. Then any continuous bijection $f : X \longrightarrow Y$ is a homeomorphism.

Proof It suffices to show that if $U \subseteq X$ is open then $f(U) \subseteq Y$ is open. But $X \setminus U$ is closed and therefore compact (Ex L9-5) and so (by Prop L9-3)

$$f(X \setminus U) = Y \setminus f(U) \qquad \text{(note } f(X) = Y!\text{)}$$

is compact, and therefore closed by Lemma L11-5. Hence the complement $f(U)$ must be open. $\square$

Exercise L11-7 (i) If $\sim$ is an equivalence relation on a space X, and $X/\sim$ is the quotient, prove there is a bijection between open sets of $X/\sim$ and open subsets U of X which are saturated, i.e. $x \sim y$ and $x \in U$ implies $y \in U$.

(ii) Let E be a nonempty set, $R \subseteq E \times E$ a subset, and define $\tilde{R}$ to be the set of all pairs (e, e') such that there exists a sequence

$$e = e_0, e_1, \ldots, e_n = e' \qquad n \geq 0$$

with $e_i \in E$ and for each $0 \leq i < n$ <u>either</u> $(e_i, e_{i+1}) \in R$ <u>or</u> $(e_{i+1}, e_i) \in R$. Prove $\tilde{R}$ is the smallest equivalence relation containing R (the $n=0$ case says $(e,e) \in \tilde{R}$ for all $e \in E$).

<u>Def^n</u> A continuous map $f: X \longrightarrow Y$ is <u>open</u> if whenever $U \subseteq X$ is open so is $f(U)$.

<u>Theorem L11-4</u> Any finite CW-complex X is compact Hausdorff.

<u>Proof</u> The compactness was Theorem L10-5. We prove X is Hausdorff by induction on a presentation of X as $X_0, X_1, \ldots, X_n = X$. Since X_0 is finite and discrete it is Hausdorff (every point is its own open neighborhood). Suppose we have shown X_{i-1} is Hausdorff. The space X_i is constructed from X_{i-1} by a pushout:

$$\begin{tikzcd}
\coprod_{\alpha \in \Lambda} S^{i-1} \arrow[r, "f"] \arrow[d, "\iota"'] & X_{i-1} \arrow[d] \\
\coprod_{\alpha \in \Lambda} D^i \arrow[r] & X_i = (X_{i-1} \amalg \coprod_\alpha D^i)/\sim
\end{tikzcd}$$

[as usual we sometimes label the copies of S^{i-1}, D^i with indices α, and write $S^{i-1}_{(\alpha)}, D^i_{(\alpha)}$]

Now a generic quotient will <u>not</u> be Hausdorff, so we need a special argument unique to <u>this</u> quotient. We use the following observations, which all follow from Exercise L11-7. Let $\rho: X_{i-1} \amalg \coprod_\alpha D^i \longrightarrow X_i$ be the quotient. We drop the ι and view $\coprod_\alpha S^{i-1}$ as a subspace of $\coprod_\alpha D^i$ (as it is!).

(0) If $x,y \in X_{i-1} \amalg \coprod_\alpha D^i$ are distinct points and $\rho(x)=\rho(y)$ then there is a sequence $x=e_0, e_1, \ldots, e_n = y$ with for $0 \le j < n$ either

$$(e_j, e_{j+1}) \text{ or } (e_{j+1}, e_j) \text{ in } \{(f(s), s) \mid s \in \coprod_\alpha S^{i-1}\}$$

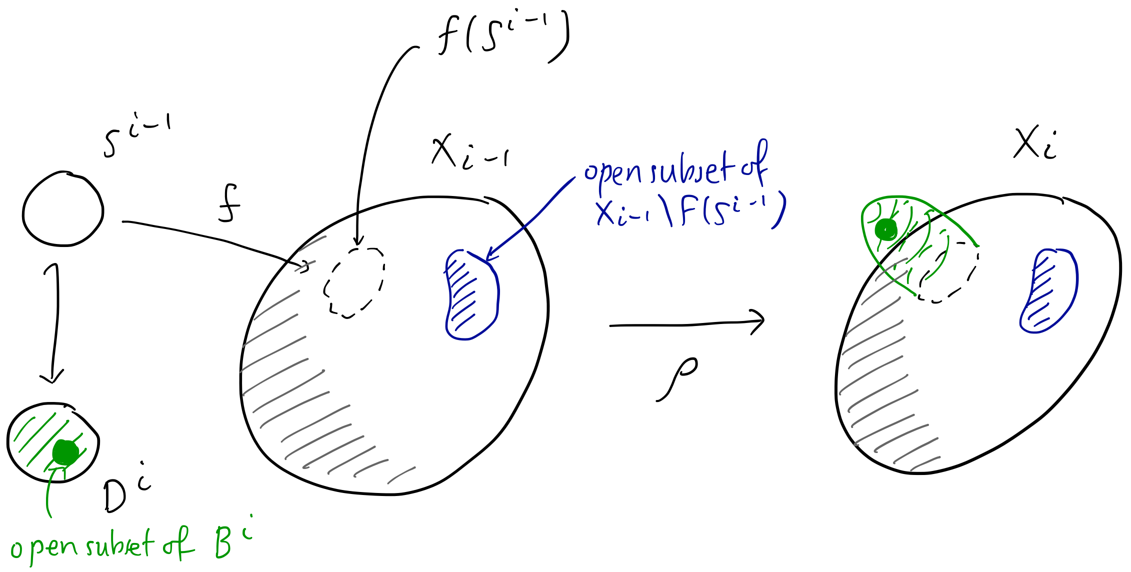
In particular if $x,y \in X_{i-1} \setminus f(\coprod_\alpha S^{i-1})$ then $\rho(x) \neq \rho(y)$, and if $x,y \in \coprod_\alpha D^i \setminus \coprod_\alpha S^{i-1}$ then $\rho(x) \neq \rho(y)$. In fact

$$\sim \; = \{(u,u) \mid u \in X_{i-1} \amalg \coprod_\alpha D^i\} \cup \{(s, f(s)), (f(s), s) \mid s \in \coprod_\alpha S^{i-1}\}$$
$$\cup \{(x,y) \mid x,y \in \coprod_\alpha S^{i-1} \text{ such that } f(x) = f(y)\}$$

(1) $f(\coprod_\alpha S^{i-1}_{(\alpha)}) = \bigcup_\alpha f(S^{i-1}_{(\alpha)})$ is compact (Prop L9-3) hence closed (using Lemma L11-5 and the inductive hypothesis that X_{i-1} is Hausdorff).

(2) The map $X_{i-1} \longrightarrow X_i$ is injective and the restriction to the complement of $\bigcup_\alpha f(S^{i-1}_{(\alpha)})$ is open.

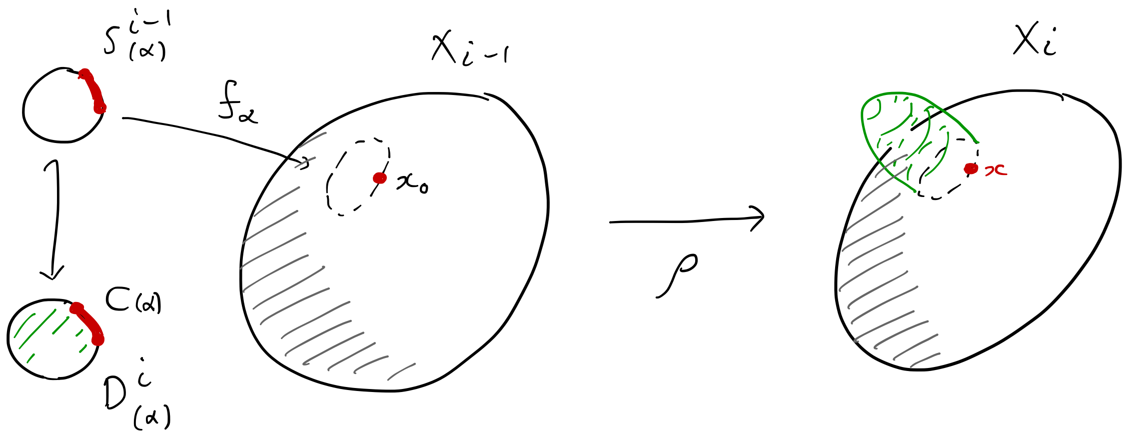
(3) The restriction of $D^i_{(\alpha)} \hookrightarrow \coprod_\alpha D^i_{(\alpha)} \longrightarrow X_i$ to the open disk $B^i \subseteq D^i$ is injective and open, where $B^i = \{\underline{x} \in \mathbb{R}^i \mid \|\underline{x}\| < 1\}$.



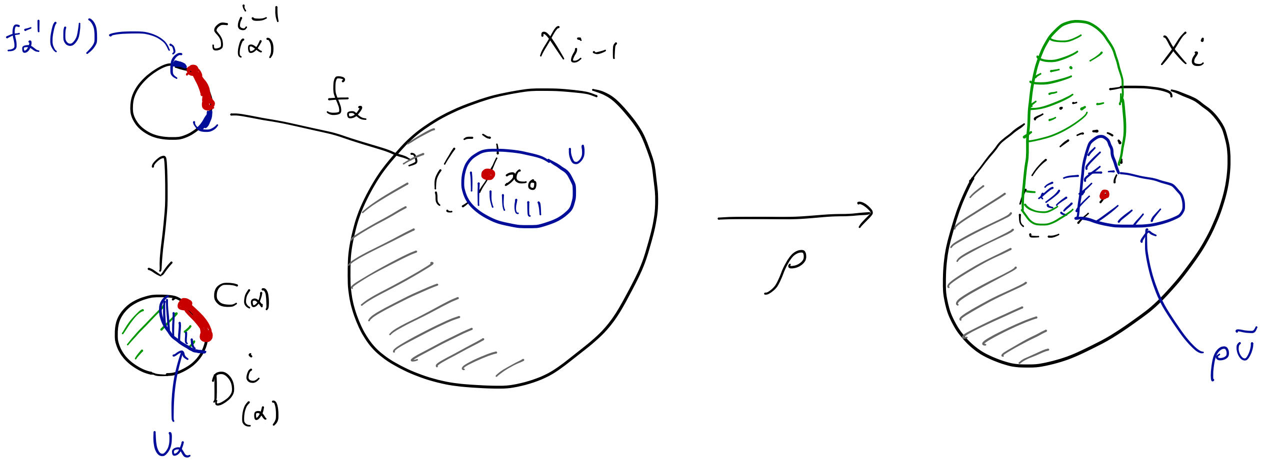
Let $x, y \in X_i$ be distinct.

<u>Easy cases</u> x, y both lie in the image of one of the injective open maps from (2), (3) above (perhaps different maps). Since B^i, X_{i-1} are Hausdorff and these maps are open, it is clear we can find open neighborhoods of x, y which are disjoint.

Otherwise at least one of the points lies in the image of $\amalg_\alpha S^{i-1}$, say x.



Now $\rho^{-1}(x)$ contains precisely one point x_0 of X_{i-1}, and for each α some closed subset $C_{(\alpha)} := \rho^{-1}(x) \cap D^i_{(\alpha)}$ of the boundary sphere (maybe $C_{(\alpha)} = \phi$, but it can be nonempty for multiple α and is nonempty for <u>at least</u> one α). Given an open neighborhood U of x_0 in X_{i-1} each $f_\alpha^{-1}(U) \subseteq S^{i-1}_{(\alpha)}$ is an open neighborhood of $f_\alpha^{-1}(x_0)$:

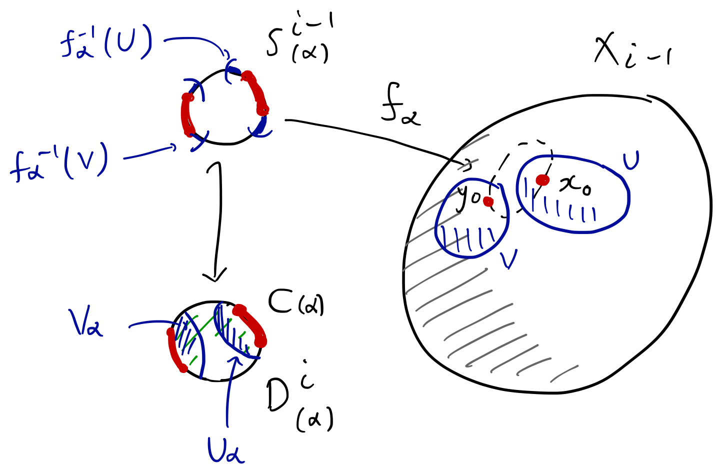


Since $S^{i-1}_{(\alpha)} \subseteq D^i_{(\alpha)}$ has the subspace topology, we can find an open subset $U_\alpha \subseteq D^i_{(\alpha)}$ such that $U_\alpha \cap S^{i-1}_{(\alpha)} = f_\alpha^{-1}(U)$. Then

$$\tilde{U} := U \cup \bigcup_{\alpha\in\Lambda} U_\alpha$$

is an open subset of $X_{i-1} \amalg \coprod_\alpha D^i_{(\alpha)}$ and $\rho^{-1}(\rho\tilde{U}) = \tilde{U}$ so $\rho\tilde{U}$ is open, and it is a neighborhood of x in X_i. Note we chose U arbitrarily, and U_α arbitrarily subject to $U_\alpha \cap S^{i-1}_{(\alpha)} = f_\alpha^{-1}(U)$.

<u>Case I</u> y also lies in the image of $\coprod_\alpha S^{i-1}_{(\alpha)}$. Let y_0 be the unique preimage in X_{i-1} and find (using X_{i-1} Hausdorff) open neighborhoods $x_0 \in U$ and $y_0 \in V$ s.t. $U \cap V = \phi$. Then running the above construction also for y and V, we have $f_\alpha^{-1}(U) \cap f_\alpha^{-1}(V) = f_\alpha^{-1}(U\cap V) = \phi$ and we may choose $U_\alpha, V_\alpha \subseteq D^i_{(\alpha)}$ disjoint with U_α as above and $V_\alpha \cap S^{i-1}_{(\alpha)} = f_\alpha^{-1}(V)$ (why? see Ex L11-10)



Then

$$\tilde{V} := V \cup \bigcup_{\alpha\in\Lambda} V_\alpha$$

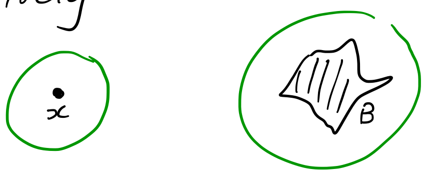
gives an open neighbourhood $\rho\tilde{V}$ of y disjoint from $\rho\tilde{U}$.

<u>Case II</u> y is not in the image of $\coprod_\alpha S^{i-1}_{(\alpha)}$. Since $\bigcup_\alpha f(S^{i-1}_{(\alpha)})$ is closed we can find an open neighborhood V of y disjoint from U which does not meet $\bigcup_\alpha f(S^{i-1}_{(\alpha)})$. Then ρV is an open neighborhood of y disjoint from $\rho\tilde{U}$, completing the proof $\square$

There are some stronger separation axioms that are also in common use:

<u>Def^n</u> Suppose one-point sets are closed in X. Then X is called

- <u>regular</u> if for each pair of a point x and $B \subseteq X$ closed disjoint from x, there exist disjoint open sets containing x and B respectively



- <u>normal</u> if for each pair of disjoint closed sets $A, B \subseteq X$ there exist disjoint open sets containing A, B respectively.

<u>Remark</u> It is clear that normal $\Longrightarrow$ regular $\Longrightarrow$ Hausdorff.

<u>Exercise L11-8</u> Prove any metrisable space is normal.

<u>Exercise L11-9</u> Prove any compact Hausdorff space is normal. (<u>Hint</u>: use the proof of Lemma L11-5).

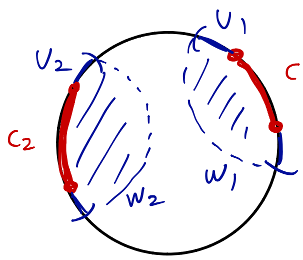
Hence a finite CW-complex is normal.

Exercise L11-10 (i) For $i \geq 1$, $0 < h < 1$, prove $\gamma : S^{i-1} \times [0,h) \longrightarrow D^i$ defined by

$$\gamma(\underline{x}, \lambda) = (1-\lambda)\underline{x}$$

gives a homeomorphism of $S^{i-1} \times [0,h)$ with an open neighborhood of S^{i-1} in D^i (usually called a <u>collar</u>).

(ii) Fill in the final detail in the proof of Theorem L11-4, by showing that if $C_1, C_2 \subseteq S^{i-1}$ are disjoint closed sets, and $U_1, U_2 \subseteq S^{i-1}$ are disjoint open sets with $U_j \supseteq C_j$ there exist disjoint open subsets W_1, W_2 of D^i such that $W_j \cap S^{i-1} = U_j$ for $j \in \{1,2\}$.



Exercise L11-11 (i) Prove X is Hausdorff if and only if the diagonal $\Delta = \{(x,x) \in X \times X \mid x \in X\}$ is a closed subset of $X \times X$.

(ii) Let G be a topological group (see Tutorial 4). Prove that G is Hausdorff if and only if $\{e\}$ is closed, where e is the identity element.

(iii) Let G be a topological group and $H \subseteq G$ a normal subgroup. Prove G/H is Hausdorff if and only if $H \subseteq G$ is closed.

Exercise L11-12 In the context of Example L11-1, prove that there does not exist a continuous function $f : X \longrightarrow \mathbb{R}$ with $f(O_1) \neq f(O_2)$.

Aside on the Hausdorff condition

The following statement from Munkres is typical:

" Topologies in which one-point sets are not closed, or in which sequences can converge to more than one point, are considered by many mathematicians to be somewhat strange. They are not really very interesting, for they seldom occur in other parts of mathematics. "

Exercise L11-13* Let k be an algebraically closed field. An affine variety (see §I of Hartshorne "Algebraic Geometry") is a closed subset of $\mathbb{A}^n := k^n$ which cannot be written as a union of two proper closed subsets, where k^n is given the topology in which sets of the form

$$Z(T) = \{ P \in \mathbb{A}^n \mid f(P) = 0 \text{ for all } P \in T \}$$

for $T \subseteq k[x_1, \ldots, x_n]$, are closed (this is called the Zariski topology). Prove an affine variety is Hausdorff iff. it is a singleton (this is Exercise 1.7 (d) of Hartshorne).

So, I suppose algebraic geometers are strange. More seriously, as hinted at on p. 5 of Lecture 6, the purpose of a topology on X is to tell you which maps out of X are continuous, and in algebraic geometry there is a mismatch between the underlying topology of our spaces (which is "bad") and the topology which is implicit in the category of spaces we use (which is better).

To be more precise, even though varieties are rarely Hausdorff (and the situation for schemes is even worse!) algebraic geometers are effectively still working most of the time with "Hausdorff" spaces, since we ask for our schemes to be separated, which means the image of the diagonal

$$X \longrightarrow X \times X$$

is closed in $X \times X$ (see Exercise L11-11). Note this is a morphism of schemes, so this not the same as saying the underlying space of X is Hausdorff (but it is the "right" notion of Hausdorff for our purposes).

So Munkres is right, conceptually, that "everyone" likes Hausdorff spaces, if that term is appropriately interpreted. However he is flat wrong about non-closed points: those are great!

Remark For a discussion of non-closed points in algebraic geometry see Example 2.3.4 of Hartshorne or Eisenbud, Harris "Geometry of schemes".

Solutions to selected exercises

<u>Ex LII-10</u> Define $Q_h^i := \{ \underline{x} \in \mathbb{R}^i \mid 1-h < \| \underline{x} \| \leq 1 \}$. This is an open subset of D^i, as $\{ \underline{x} \mid \| \underline{x} \| > 1-h \} \subseteq \mathbb{R}^i$ is open ($\|-\| : \mathbb{R}^i \to \mathbb{R}$ is continuous). Clearly γ is continuous and

$$\| \gamma(\underline{x}, \lambda) \| = 1 - \lambda > 1-h$$

so γ factors via a continuous map $S^{i-1} \times [0,h) \longrightarrow Q_h^i$. Define $\mathcal{T} : Q_h^i \longrightarrow S^{i-1} \times [0,h)$ by

$$\mathcal{T}(\underline{y}) = \left(\frac{1}{\| \underline{y} \|} \underline{y} ,\ 1 - \| \underline{y} \| \right).$$

This is continuous and gives a two-sided inverse to $S^{i-1} \times [0,h) \to Q_h^i$.

(ii) Using the notation of (i), set $W_j := \gamma(U_j \times [0,h))$. This is an open subset of Q_h^i and thus of D^i, and $W_1 \cap W_2 = \phi$ since $(U_1 \times [0,h)) \cap (U_2 \times [0,h)) = \phi$. Moreover,

$$\begin{aligned} W_j \cap S^{i-1} &= \gamma(U_j \times [0,h)) \cap \gamma(S^{i-1} \times \{0\}) \\ &= \gamma([U_j \times [0,h)] \cap (S^{i-1} \times \{0\})) \\ &= \gamma(U_j \times \{0\}) \\ &= U_j \end{aligned}$$