Lecture 12 : Function spaces

updated 8/9/20

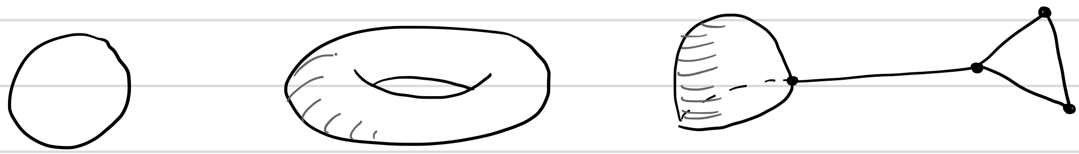
We began these lectures with the question "What is space?". I summarised some colloquial answers under three headings :

(1) Space as a stage for <u>things</u>

(2) Space as a stage for <u>motion</u>

(3) Space as a <u>channel</u> for communication

I then sketched some of the mathematical abstractions invented to formalise various aspects of our understanding of space (and time, which from Lecture 5 we know cannot be deeply separated from space). Subsequently we have understood several of these : <u>metric spaces</u>, <u>quadratic spaces</u> (Tutorial 2) and <u>topological spaces</u>. We also saw hints of a more "combinatorial" point of view on spaces in the guise of finite <u>CW-complexes</u>. This has taken us roughly half of the semester, and in all that time our examples have been things like subspaces of $\mathbb{R}^n$ or finite CW-complexes, e.g.

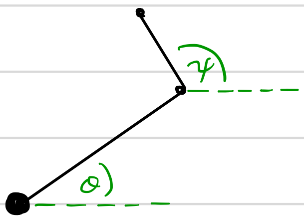


S^1 $\qquad$ $\mathbb{T} = S^1 \times S^1$ $\qquad$ CW-complex

These are clearly <u>things</u>, i.e. we have spent our time so far within paradigm (1) on the above list. We will spend the remainder of the semester developing an understanding of aspects of paradigm (2), which involves a shift to studying <u>spaces of functions</u>.

I. Motions are functions

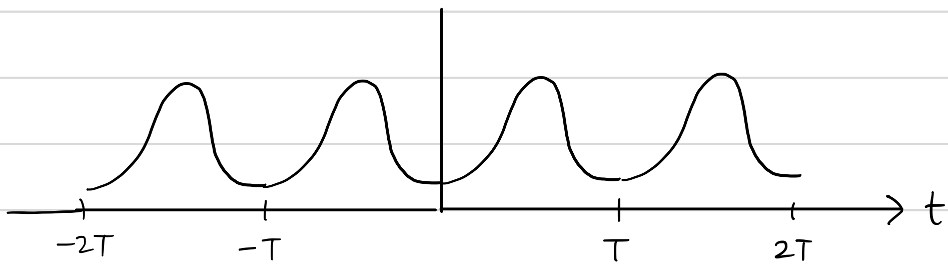
Typically motions in a space X are represented mathematically by continuous functions $\mathbb{R} \longrightarrow X$, where the domain is viewed as a time coordinate. The space X might be "physical space" e.g. $\mathbb{R}^3$ or "configuration space" as in e.g. the motion of a coupled pair of rods:



$$\gamma : \mathbb{R} \longrightarrow S^1 \times S^1$$

$$\gamma(t) = (\theta(t), \psi(t)).$$

What about periodic motion? Suppose a motion $\gamma : \mathbb{R} \longrightarrow X$ has period $T > 0$, which is to say that $\gamma(t + kT) = \gamma(t)$ for any $k \in \mathbb{Z}$.



The subset $\mathbb{Z}T := \{ kT \mid k \in \mathbb{Z} \}$ is a subgroup of $\mathbb{R}$ and by Tutorial 4 we have $\mathbb{R}/\mathbb{Z}T \cong S^1$ as topological groups, so there is a unique continuous map $\tilde{\gamma} : S^1 \longrightarrow X$ such that the diagram below commutes:

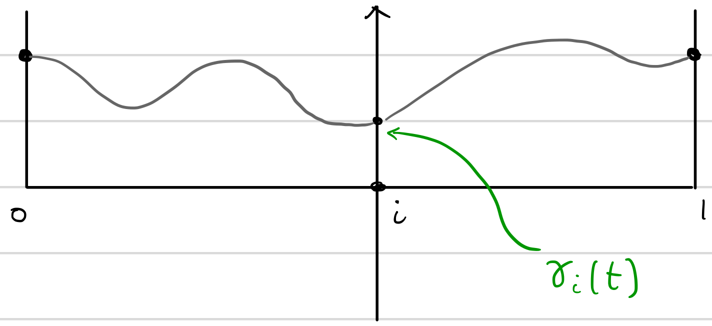
$$\begin{tikzcd}
\mathbb{R} \arrow[rr, "\gamma"] \arrow[dr, "\rho"'] & & X \\
& \mathbb{R}/\mathbb{Z}T \cong S^1 \arrow[ur, dashed, "\tilde{\gamma}"'] &
\end{tikzcd}$$

(ρ the quotient)

Two separate motions (say of particles) $\gamma_1 : \mathbb{R} \to X$ and $\gamma_2 : \mathbb{R} \to Y$, give rise to a single combined motion

$$\gamma : \mathbb{R} \longrightarrow X \times Y \qquad \gamma(t) = (\gamma_1(t), \gamma_2(t))$$

What about a <u>continuous object</u> in motion? Say a piece of string, attached at two endpoints? If we imagine the string as made up of infinitely many particles, indexed say by $i \in [0,1]$, then each particle has its own motion $\gamma_i : \mathbb{R} \longrightarrow \mathbb{R}$, let us say along a vertical axis:



Together the γ_i form a continuous function $\gamma : \mathbb{R} \longrightarrow \prod_{i \in [0,1]} \mathbb{R}$, but $X = \prod_{i \in [0,1]} \mathbb{R}$ is not the correct configuration space of a string, because the γ_i's are not independent! Of course if δ is small then

$$\gamma_i(t) \approx \gamma_{i+\delta}(t).$$

One reasonable way to say "the γ_i vary continuously in i" would be to ask that the function

$$[0,1] \times \mathbb{R} \longrightarrow \mathbb{R}, \qquad (i,t) \longmapsto \gamma_i(t)$$

was continuous. We could <u>define</u> a motion of the string to be such a function.

But wait: a configuration of the string is itself a continuous function $h : [0,1] \longrightarrow \mathbb{R}$, giving the height at any point. So we could reasonably describe a motion of the string as a continuous map into this "space" of configurations. Which raises the questions:

Q1/ Is the set of continuous maps $Cts([0,1], \mathbb{R})$ a space?

Q2/ Supposing it is, is it then the case that

$$Cts([0,1] \times \mathbb{R}, \mathbb{R}) \cong Cts(\mathbb{R}, Cts([0,1], \mathbb{R})) \quad ?$$

The answers are both Yes, as we will see.

Exercise L12-1 Let $f : X \longrightarrow Y$ be a function (not assumed continuous) between topological spaces X, Y. The graph of f is

$$\Gamma_f := \{ (x,y) \in X \times Y \mid y = f(x) \}.$$

Prove that

(i) If Y is Hausdorff and f is continuous, Γ_f is closed in $X \times Y$.

(ii) Give a counterexample to show that if Y is not Hausdorff, it is not necessarily the case that the graph of a continuous function $f : X \longrightarrow Y$ is closed.

*(iii) If Y is compact and Γ_f is closed, f is continuous.
(First show $X \times Y \longrightarrow X$ sends closed subsets to closed subsets, using that Y is compact)

So if we wanted to put time on a more even footing with space, we might also want to view a motion of the string as a special kind of closed subset of $[0,1] \times \mathbb{R} \times \mathbb{R}$, consisting of the tuples $(i, t, \gamma_i(t))$ for all $i \in [0,1]$, $t \in \mathbb{R}$.

II. Topologies on sets of functions

We will make the set $Cts(X,Y)$ of continuous maps between any two topological spaces into a topological space (the topology is called the compact-open topology) in such a way that the function

$$Cts(Z \times X, Y) \xrightarrow{\Psi_{Z,X,Y}} Cts(Z, Cts(X,Y))$$

$$\Psi_{Z,X,Y}(F)(z)(x) = F(z,x).$$

⌜If you're familiar with λ-calculus, this is $\Psi(F) = \lambda z. \lambda x. F(z,x)$ see Lambek & Scott "Introduction to higher order categorical logic"⌟

is well-defined for any triple of spaces X, Y, Z, and is a bijection whenever X is locally compact Hausdorff (the concept of local compactness is to be introduced in a moment). Any compact space is locally compact.

We refer to this bijection as the adjunction property of the compact-open topology. In particular we will have in the situation of the moving string above

$$Cts(\mathbb{R} \times [0,1], \mathbb{R}) \cong Cts(\mathbb{R}, Cts([0,1], \mathbb{R}))$$

so that both definitions of such motions agree.

<u>Remark</u> Given sets A, B write B^A for the set of all functions $A \longrightarrow B$. By definition a function $f : A \longrightarrow B$ is the same thing as an indexed family $\{f(a)\}_{a \in A}$ of elements of B, indexed by A.

Given a function $F : A \times B \longrightarrow C$ we can consider for each $a \in A$ the partial function $F(a,-) : B \longrightarrow C$ which sends $b \in B$ to $F(a,b)$. The indexed family $\{F(a,-)\}_{a \in A}$ of these partial functions is, by the above logic, <u>the same thing</u> as the function

$$\begin{aligned} A &\longrightarrow C^B \\ a &\longmapsto F(a,-) \end{aligned}$$

We denote this function by $\Lambda(F)$, i.e. $\Lambda(F) \in (C^B)^A$. We have defined

$$\Lambda : C^{A \times B} \longrightarrow (C^B)^A$$

$$\Lambda(F) = \{F(a,-)\}_{a \in A}$$

<u>or</u> $$\Lambda(F)(a) = F(a,-)$$

<u>or</u> $$\Lambda(F)(a)(b) = F(a,b).$$

We claim Λ is a bijection. Since the values of F may be recovered from $\Lambda(F)$ it is clearly injective (if $\Lambda(F) = \Lambda(G)$ then for all $a \in A, b \in B$ $F(a,b) = \Lambda(F)(a)(b) = \Lambda(G)(a)(b) = G(a,b)$). If $H \in (C^B)^A$ is given define $F : A \times B \longrightarrow C$ by $F(a,b) = H(a)(b)$ then clearly

$$\Lambda(F)(a)(b) = F(a,b) = H(a)(b)$$

Hence $\Lambda(F)(a) = H(a)$ as functions $B \longrightarrow C$, so $\Lambda(F) = H$ as functions.

<u>Def^n</u> A topological space X is <u>locally compact</u> if for every $x \in X$ there exists an open set U and compact set K with $x \in U \subseteq K$.

Clearly any compact space is locally compact.

<u>Example</u> (i) $\mathbb{R}^n$ is locally compact (but not compact) since any $x \in \mathbb{R}^n$ lies in some $(a_1, b_1) \times \cdots \times (a_n, b_n)$ which is contained in the compact set $[a_1, b_1] \times \cdots \times [a_n, b_n]$.

(ii) $\mathbb{Q}$ is <u>not</u> locally compact (so subspaces of locally compact spaces need not be locally compact).

We know from Ex L11-9 that a compact Hausdorff space is normal, hence regular. A locally compact Hausdorff space need not be normal, but it is regular:

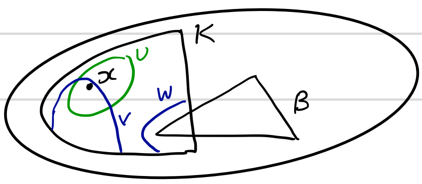
<u>Lemma L12-0</u> Suppose X is locally compact. Then

(i) If $A \subseteq X$ is closed then A is locally compact.

(ii) If X is also Hausdorff then it is regular.

<u>Proof</u> (i) Given $x \in A$ let $U \subseteq K$ be an open neighborhood of x in X contained in a compact set K. Then $x \in U \cap A \subseteq K \cap A$, and $U \cap A$ is open in A while $K \cap A \subseteq K$ is a closed subspace of a compact space, hence compact. $\square$

(ii) Let $x \in X$ and $B \subseteq X$ closed with $x \notin B$ be given, and choose $x \in U \subseteq K$ with U open and K compact. Then K is compact Hausdorff, hence regular, so we may apply regularity to $x, B \cap K$ in K to find V, W open and disjoint in K with $x \in V$ and $B \cap K \subseteq W$. Suppose V', W' are open in X with $V' \cap K = V$, $W' \cap K = W$. Then $U \cap V'$ and $W' \cup K^c$ give the required disjoint open neighborhoods of x, B in X (recall in a Hausdorff space compact sets are closed). $\square$



Until further notice, we adopt the following hypothesis:

HYPOTHESIS: Suppose that we have assigned a topology $\mathcal{T}_{X,Y}$ to $\mathrm{Cts}(X,Y)$ for every pair X, Y such that for every continuous map $F: Z \times X \longrightarrow Y$ the function

facts conditional on this hypothesis are labelled (H)

$$\begin{aligned} Z &\longrightarrow \mathrm{Cts}(X,Y) \\ z &\longmapsto F(z,-) \end{aligned}$$

is continuous, and the resulting function $\Psi_{Z,X,Y}$ is a bijection whenever X is locally compact Hausdorff.

Eventually we will provide such topologies $\mathcal{T}_{X,Y}$ (the compact-open topology) and prove that the hypothesis is true in this case, so that all we are about to say will become theorems about $\mathrm{Cts}(X,Y)$ with the compact-open topology. The point of setting things up this way is that we will derive the compact-open topology as the weakest topology consistent with the hypothesis. This serves to both "explain" the topology, and at the same time the fundamental position of the adjunction property (also sometimes called the "exponential law" since $\mathrm{Cts}(X,Y)$ behaves like Y^X).

Taking $Z = \mathrm{Cts}(X,Y)$ with X locally compact Hausdorff we have

$$\mathrm{Cts}(\mathrm{Cts}(X,Y) \times X, Y) \overset{\Psi}{\cong} \mathrm{Cts}(\mathrm{Cts}(X,Y), \mathrm{Cts}(X,Y))$$

and $\Psi^{-1}(1_{\mathrm{Cts}(X,Y)})$ is the evaluation map

$$\mathrm{ev}_{X,Y} : \mathrm{Cts}(X,Y) \times X \longrightarrow Y$$
$$\mathrm{ev}_{X,Y}(f,x) = f(x)$$

which must therefore be continuous, for all X locally compact Hausdorff.

<u>Exercise L12-2</u> Given continuous maps $f : X_1 \longrightarrow Y_1$, $g : X_2 \longrightarrow Y_2$ prove that $f \times g : X_1 \times X_2 \longrightarrow Y_1 \times Y_2$ defined by $(x_1, x_2) \longmapsto (fx_1, gx_2)$ is continuous.

<u>Lemma L12-1</u> (H) (i) The <u>composition map</u>

$$c_{X,Y,Z} : \mathrm{Cts}(Y,Z) \times \mathrm{Cts}(X,Y) \longrightarrow \mathrm{Cts}(X,Z)$$
$$(g,f) \longmapsto g \circ f$$

is continuous, whenever X, Y are locally compact Hausdorff.

(ii) If $f : X \longrightarrow Y$ is continuous and Y is locally compact Hausdorff,

$$\mathrm{Cts}(Y,Z) \longrightarrow \mathrm{Cts}(X,Z), \quad g \longmapsto g \circ f$$

is continuous for any space Z.

(iii) If $g : Y \longrightarrow Z$ is continuous and X is locally compact Hausdorff

$$\mathrm{Cts}(X,Y) \longrightarrow \mathrm{Cts}(X,Z), \quad f \longmapsto g \circ f$$

is continuous.

<u>Proof</u> (i) By the adjunction property it suffices to prove

$$\mathrm{Cts}(Y,Z)\times \mathrm{Cts}(X,Y)\times X \longrightarrow Z, \qquad (g,f,x)\longmapsto g(f(x))$$

is continuous, since $\Psi_{\mathrm{Cts}(Y,Z)\times\mathrm{Cts}(X,Y),X,Z}$ applied to this map is $c_{X,Y,Z}$.
But this map is

$$\mathrm{Cts}(Y,Z)\times \mathrm{Cts}(X,Y)\times X \xrightarrow{1_{\mathrm{Cts}(Y,Z)}\times ev_{X,Y}} \mathrm{Cts}(Y,Z)\times Y \xrightarrow{ev_{Y,Z}} Z$$

which as a composite of continuous maps, is continuous.

(ii) Fixing $f\in \mathrm{Cts}(X,Y)$ in the above we have

$$\mathrm{Cts}(Y,Z)\times X \xrightarrow{1\times f} \mathrm{Cts}(Y,Z)\times Y \xrightarrow{ev_{Y,Z}} Z$$

which is continuous provided Y is locally compact Hausdorff, and the induced map $\mathrm{Cts}(Y,Z)\longrightarrow \mathrm{Cts}(X,Z)$ is $g\longmapsto g\circ f$.

(iii) Fixing $g\in \mathrm{Cts}(Y,Z)$ in the above we have

$$\mathrm{Cts}(X,Y)\times X \xrightarrow{ev_{X,Y}} Y \xrightarrow{g} Z$$

continuous, provided X is locally compact Hausdorff, and the induced map $\mathrm{Cts}(X,Y)\longrightarrow \mathrm{Cts}(X,Z)$ is $f\longmapsto g\circ f$. $\square$

(H)

Lemma L12-2 For X compact Hausdorff, the set $\{\chi_X\}$ consisting of the characteristic function of X is open in the topology $\mathcal{T}_{X,\Sigma}$ on $Cts(X,\Sigma)$ (Σ denotes the Sierpiński space $\{0,1\}$.)

Proof Since X is compact the projection map $\pi_1 : Cts(X,\Sigma)\times X \longrightarrow Cts(X,\Sigma)$ closed (see Ex L12-1) and since $ev_{X,\Sigma} : Cts(X,\Sigma)\times X \longrightarrow \Sigma$ is continuous the set $M := \pi_1(ev_{X,\Sigma}^{-1}(\{0\}))$ is closed in $Cts(X,\Sigma)$. But this set consists of those characteristic functions χ_V for $V\subseteq X$ open for which $x\in X$ exists s.t. $ev_{X,\Sigma}(\chi_V, x) = 0$, i.e. $x\notin V$. That is, $M = \{\chi_V \mid V \text{ proper}\}$ and so $M^c = \{\chi_X\}$ is open. $\square$

(H)

Lemma L12-3 If X is locally compact Hausdorff and Y is arbitrary, then for any compact $K\subseteq X$ and $U\subseteq Y$ open, the set

$$S(K,U) = \{ f\in Cts(X,Y) \mid f(K)\subseteq U\}$$

is open in the topology $\mathcal{T}_{X,Y}$ on $Cts(X,Y)$.

Proof The inclusion $f : K\longrightarrow X$ is continuous and as a subspace of a Hausdorff space K is Hausdorff, and $g = \chi_U : Y\longrightarrow \Sigma$ is continuous, so the function

$$Cts(X,Y) \xrightarrow{(-)\circ f} Cts(K,Y) \xrightarrow{\chi_U\circ(-)} Cts(K,\Sigma) \qquad (8.1)$$

$$h \longmapsto h\circ f \longmapsto \chi_U\circ h\circ f$$

is continuous, by Lemma L12-1 (ii), (iii).

By Lemma L12-2 the set $\{\chi_K\} \subseteq \mathrm{Cts}(K, \Sigma)$ is open and hence the preimage under (8.1) is open, which is

$$\{ h \in \mathrm{Cts}(X,Y) \mid \chi_U \circ h \circ f = \chi_K \}$$

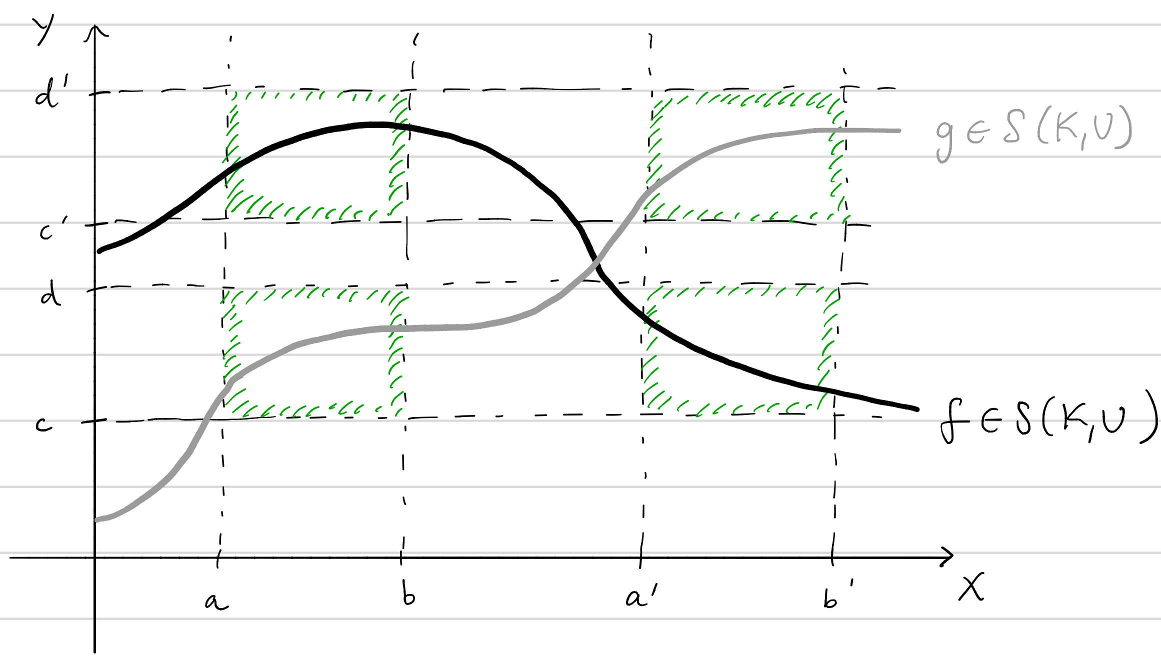
$$= \{ h \in \mathrm{Cts}(X,Y) \mid \chi_U((h \circ f)(k)) = 1 \text{ for all } k \in K \}$$

$$= \{ h \in \mathrm{Cts}(X,Y) \mid h(f(k)) \in U \text{ for all } k \in K \}$$

$$= \{ h \in \mathrm{Cts}(X,Y) \mid h(K) \subseteq U \}$$

$$= S(K,U). \quad \square$$

<u>Example</u> Let $X = Y = \mathbb{R}$ and $K = [a,b] \cup [a',b']$ with $b < a'$ and $U = (c,d) \cup (c',d')$ with $d < c'$. Then $S(K,U)$ consists of those continuous functions passing through the "windows" defined by the pair (K,U):



Now we drop the HYPOTHESIS and start again!

First we introduce the notion of a sub-basis:

Defn The topology on a set X generated by a collection S of subsets of X (possibly empty) is the intersection

$$\langle S \rangle := \bigcap \{ \mathcal{T} \mid \mathcal{T} \text{ is a topology on } X \text{ and } \mathcal{T} \supseteq S \}.$$

By definition if $\mathcal{T}$ is a topology and $\mathcal{T} \supseteq S$ then $\mathcal{T} \supseteq \langle S \rangle$.

Defn Let $(X, \mathcal{T})$ be a topological space. A sub-basis for $\mathcal{T}$ is a subset $Q \subseteq \mathcal{T}$ such that $\langle Q \rangle = \mathcal{T}$.

Exercise L12-3 (i) Prove that if $\{\mathcal{T}_i\}_{i \in I}$ is a collection of topologies on a single set X that $\bigcap_{i \in I} \mathcal{T}_i$ is a topology on X.

(ii) With the above notation prove that $U \in \langle S \rangle$ if and only if U can be written as a union of sets, each of which is a finite intersection of elements of S.

(the convention is that $X \in \langle S \rangle$ can be written as the intersection of no elements of S, i.e. the "empty intersection" is the whole space)

(iii) If $f: X \longrightarrow Y$ is a function and S is a sub-basis for the topology on Y, then f is continuous iff. $f^{-1}(U) \subseteq X$ is open for every $U \in S$.

<u>Def^n</u> Let X, Y be topological spaces. The <u>compact-open topology</u> $\mathcal{T}_{X,Y}$ on $\mathrm{Cts}(X,Y)$ is the topology generated by the set

$$\{ S(K,U) \}_{K \subseteq X \text{ compact},\, U \subseteq Y \text{ open}},$$

where $S(K,U) = \{ f \mid f(K) \subseteq U \}$. More explicitly, a subset $W \subseteq \mathrm{Cts}(X,Y)$ is open if and only if it is a union of sets, each of which is of the form

$$S(K_1,U_1) \cap \cdots \cap S(K_n,U_n). \qquad (K_i \text{ compact}, U_i \text{ open})$$

<u>Remark</u> A special case of interest is when X is compact Hausdorff, where by Ex L9-5 and Lemma L11-5 the compact subsets $K \subseteq X$ are precisely the closed subsets.

Lemmas L12-2, L12-3 show that the compact-open topology is the <u>weakest</u> topology (at least when X is locally compact Hausdorff) which is consistent with the adjunction property. It is an important theorem that in fact the adjunction property holds for this topology.

<u>Theorem L12-4</u> With $\mathcal{T}_{X,Y}$ the compact-open topology, the earlier hypothesis is fulfilled, that is: for any continuous map $F: Z \times X \longrightarrow Y$ the map $z \longmapsto F(z,-)$ is a continuous map $Z \longrightarrow \mathrm{Cts}(X,Y)$ and for X locally compact Hausdorff there is a bijection

$$\mathrm{Cts}(Z \times X, Y) \xrightarrow{\Psi_{Z,X,Y}} \mathrm{Cts}(Z, \mathrm{Cts}(X,Y)).$$

$$\Psi_{Z,X,Y}(F)(z)(x) = F(z,x)$$

We will delay the proof for a minute, to examine some consequences. Some we have already elaborated: as a consequence of the Theorem, Lemma L12-1 and Lemma L12-2 now become absolute facts, conditional on nothing, and the evaluation map

$$ev_{X,Y} : \mathrm{Cts}(X,Y) \times X \longrightarrow Y$$
$$ev_{X,Y}(f,x) = f(x)$$

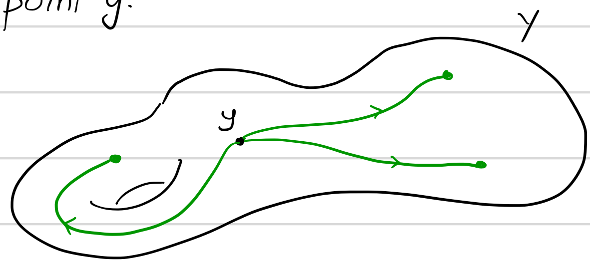
is continuous whenever X is locally compact Hausdorff.

Since $\mathbb{R}^n \cong \mathbb{R} \times \cdots \times \mathbb{R}$ is locally compact we deduce that there is no difference between a real-valued function of multiple variables and functions which return functions which .... return functions:

$$\begin{aligned}
\mathrm{Cts}(\mathbb{R}^3, \mathbb{R}) &\cong \mathrm{Cts}(\mathbb{R}^2 \times \mathbb{R}, \mathbb{R}) \\
&\cong \mathrm{Cts}(\mathbb{R}^2, \mathrm{Cts}(\mathbb{R},\mathbb{R})) \\
&\cong \mathrm{Cts}(\mathbb{R} \times \mathbb{R}, \mathrm{Cts}(\mathbb{R},\mathbb{R})) \\
&\cong \mathrm{Cts}(\mathbb{R}, \mathrm{Cts}(\mathbb{R}, \mathrm{Cts}(\mathbb{R},\mathbb{R})))
\end{aligned}$$

Example L12-1 Let Y be any space.

(i) $[0,1]$ is compact Hausdorff and if $y \in Y$ is a fixed "basepoint" the subspace $P_y Y$ of $\mathrm{Cts}([0,1], Y)$ with the compact-open topology of functions $f : [0,1] \to Y$ with $f(0) = y$ is called the path space of Y with base point y.

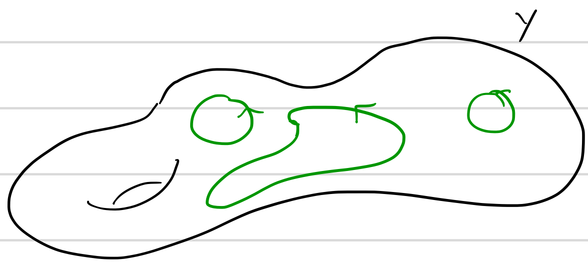


The evaluation map $ev_{[0,1],Y} : \mathrm{Cts}([0,1], Y) \times [0,1] \longrightarrow Y$ is continuous and hence the "evaluation at 1" map

$$P_y Y \xhookrightarrow{\text{inclusion}} \mathrm{Cts}([0,1], Y) \xrightarrow{ev(-,1)} Y$$

is also continuous. Its image is the set of points connected by a path to the basepoint $y \in Y$.

(ii) S^1 is compact Hausdorff and $\mathcal{L}Y := \mathrm{Cts}(S^1, Y)$ with the compact-open topology is called the free loop space of Y.



Amusing but probably meaningless: loops upon loops!

$$\mathrm{Cts}(S^1, \mathrm{Cts}(S^1, Y)) \cong \mathrm{Cts}(\mathbb{T}, Y).$$

<u>Exercise L12-4</u> Let $\mathcal{L}Y$ denote the free loop space of Y. Prove the function $\mathrm{const} : Y \longrightarrow \mathcal{L}Y$ sending $y \in Y$ to the constant loop at y (i.e. $\theta \in S^1 \longmapsto y \in Y$) is continuous.

<u>Exercise L12-5</u> Prove that if X is finite and discrete (hence compact Hausdorff) that there is a homeomorphism

$$\begin{aligned} \mathrm{Cts}(X,Y) &\xrightarrow{\cong} \textstyle\prod_{x\in X} Y \\ f &\longmapsto (f(x))_{x\in X} \end{aligned}$$

which identifies $S(x,U)$ with $\pi_x^{-1}(U)$ where $\pi_x : \prod_{x\in X} Y \to Y$ is the projection.

<u>Exercise L12-6</u> Let $(S^1, +, 0)$ be the circle as a topological group (see Tutorial 4). Prove that the map $S^1 \times \mathcal{L}Y \longrightarrow \mathcal{L}Y$ sending (θ, f) to the function $\theta' \longmapsto f(\theta+\theta')$ is continuous. This map "rotates the loops" in Y.

<u>Def^n</u> Given a topological space $(X, \mathcal{J})$ we define $\mathrm{Open}(X)$ to be the topological space whose points are open subsets of X, with a basis for the topology given by sets $\uparrow K$ where $K \subseteq X$ is compact and

$$\uparrow K = \{ U \in \mathcal{J} \mid U \supseteq K \}.$$

Note that ϕ is compact and $\uparrow \phi = \mathcal{J}$.

<u>Exercise L12-7</u> Prove the set $\{ \uparrow K \mid K \subseteq X \text{ compact} \}$ is a basis for a topology on $\mathcal{J}$.

<u>Lemma L12-6</u> For any space X there is a homeomorphism

$$\begin{aligned} \mathrm{Cts}(X, \Sigma) &\longrightarrow \mathrm{Open}(X) \\ f &\longmapsto f^{-1}(\{1\}) \end{aligned}$$

<u>Proof</u> By Lemma L6-2 we already know this map is a bijection. The topology on $\mathrm{Cts}(X,\Sigma)$ is generated by the sets $S(K,U)$ with $K \subseteq X$ compact and $U \subseteq \Sigma$ open, so $U \in \{\phi, \Sigma, \{1\}\}$. We have

$$S(K,\phi) = \phi, \quad S(K,\Sigma) = \mathrm{Cts}(X,\Sigma)$$

so in fact the topology is generated by the sets (using that every continuous map $X \longrightarrow \Sigma$ is the characteristic function χ_V of a unique open set $V \subseteq X$)

$$\begin{aligned} S(K,\{1\}) &= \{\chi_V \mid \chi_V(K) \subseteq \{1\}\} \\ &= \{\chi_V \mid V \supseteq K\}. \\ &= \{\chi_V \mid V \in \uparrow K\}. \end{aligned}$$

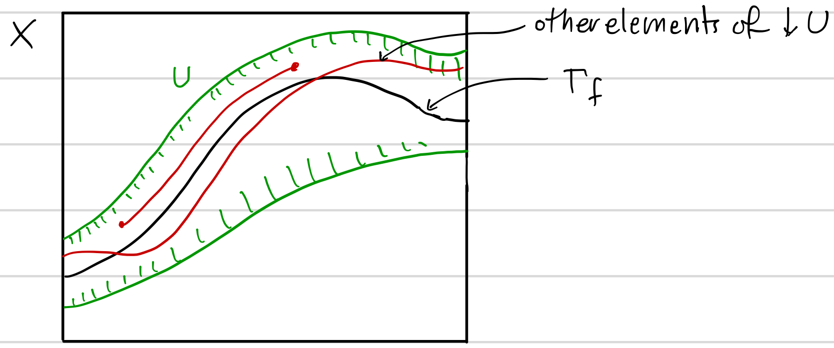
This corresponds to a basis for the topology on $\mathrm{Open}(X)$, completing the proof $\square$

<u>Def</u>

<u>Def^n</u> Let X be a Hausdorff topological space, and let $\mathrm{Closed}(X)$ be the set of closed subsets of X with a basis for the topology given by the sets $\downarrow U$ for $U \subseteq X$ open <u>with U^c compact</u>, where

$$\downarrow U = \{Z \text{ closed in } X \mid Z \subseteq U\}.$$

<u>Example L12-2</u> Let $X = [0,1] \times [0,1]$ and $\Gamma_f \subseteq X$ the graph of a continuous function $f : [0,1] \longrightarrow [0,1]$ so that Γ_f is closed, and so determines a point of $\mathrm{Closed}(X)$. Let us denote this point by $[\Gamma_f]$. For every open set $U \subseteq X$ containing Γ_f we get a basic open neighborhood $\downarrow U$ of $[\Gamma_f]$ in $\mathrm{Closed}(X)$:



In particular

$$U_\varepsilon = \{ (x,y) \in [0,1]^2 \mid d_2(f(x), y) < \varepsilon \}$$

gives an open neighborhood $\downarrow U_\varepsilon$ of Γ_f, which contains Γ_g for $g : [0,1] \longrightarrow [0,1]$ another function iff. for all $x \in [0,1]$ we have $d_2(f(x), g(x)) < \varepsilon$.

<u>Exercise L12-8</u> If X is Hausdorff the bijection $\mathrm{Open}(X) \longrightarrow \mathrm{Closed}(X)$ sending U to U^c is a homeomorphism.

Let Y be Hausdorff. Then the diagonal $\Delta \subseteq Y \times Y$ is closed (Ex. L11-11) and hence Δ^c is open and we have the continuous map

$$\chi_{\Delta^c} : Y \times Y \longrightarrow \Sigma .$$

By the Theorem, the corresponding map

$$\mathrm{sing} : Y \longrightarrow \mathrm{Cts}(Y, \Sigma)$$

is continuous, where $\mathrm{sing}(y) = \chi_{\{y\}^c}$. If we think of $\mathrm{Cts}(Y,\Sigma)$ as the set of closed sets of Y then this continuous map sends a point to the closed singleton set containing it. That is, using Lemma L12-6 and Ex L12-8 there is a continuous function

$$Y \longrightarrow \mathrm{Cts}(Y,\Sigma) \cong \mathrm{Open}(Y) \cong \mathrm{Closed}(Y)$$

$$y \longmapsto \{y\} \in \mathrm{Closed}(Y)$$

Sending a continuous map to its graph gives an injective map from $\mathrm{Cts}(X,Y)$ (for Y Hausdorff) to $\mathrm{Closed}(X \times Y)$. As Example L12-2 shows, the neighborhoods of a graph in the latter space have a clear intuitive content. The next theorem gives some conditions under which this is a homeomorphism.

Theorem L12-7 Let X be locally compact Hausdorff, Y compact Hausdorff. Then

$$\mathrm{Cts}(X,Y) \xrightarrow{\Gamma_{(-)}} \mathrm{Closed}(X \times Y)$$
$$f \longmapsto \Gamma_f$$

sending a function to its graph, is continuous and injective. Denote its image by $\mathcal{G}_{X,Y} \subseteq \mathrm{Closed}(X \times Y)$, the "space of graphs" with its subspace topology. The map $\Gamma_{(-)}$ induces a homeomorphism

$$\mathrm{Cts}(X,Y) \xrightarrow{\cong} \mathcal{G}_{X,Y}.$$

Proof By Lemma L10-2 and L11-3 the product $X\times Y$ is locally compact Hausdorff (to see local compactness, given $x\in X, y\in Y$ choose $x\in U\subseteq K$, $y\in V\subseteq L$ with U,V open and K,L compact. Then $(x,y)\in U\times V\subseteq K\times L$ and $K\times L$ is compact).

By Ex L12-1, $\Gamma_f\subseteq X\times Y$ is closed, so the map is well-defined. Consider the continuous map

$$\mathrm{Cts}(X,Y)\times X\times Y \xrightarrow{ev_{X,Y}\times 1_Y} Y\times Y \xrightarrow{\chi_{\Delta^c}} \Sigma.$$

This sends (f,x,y) to $(f(x),y)$ and then to $0\in\Sigma$ if $f(x)=y$ and $1\in\Sigma$ if $f(x)\neq y$. It corresponds under Ψ to a continuous map

$$\mathrm{Cts}(X,Y)\longrightarrow \mathrm{Cts}(X\times Y,\Sigma)$$

sending $f:X\to Y$ to $\{(x,y)\mapsto \delta_{y\neq f(x)}\}$ which is precisely the characteristic function of the complement of Γ_f. Now composing with the homeomorphisms

$$\mathrm{Cts}(X\times Y,\Sigma)\cong \mathrm{Open}(X\times Y)\cong \mathrm{Closed}(X\times Y)$$

we find the graph map $\Gamma_{(-)}$ is continuous. It is clearly injective.

Let $\mathcal{G}\subseteq \mathrm{Closed}(X\times Y)$ denote the subspace of graphs, i.e. the image of $\Gamma_{(-)}$. To prove that there is a homeomorphism $\mathrm{Cts}(X,Y)\longrightarrow\mathcal{G}$ it suffices to show that the image of $S(K,U)$ is open in $\mathcal{G}$ for $K\subseteq X$ compact and $U\subseteq Y$ open. This is a little subtle. Note that the inclusion $K\times Y\xrightarrow{\iota} X\times Y$ is continuous and by Lemma L12-1 the composition map

$$\begin{aligned} \mathrm{Cts}(X \times Y, \Sigma) &\longrightarrow \mathrm{Cts}(K \times Y, \Sigma) \\ f &\longmapsto f \circ \iota \end{aligned}$$

is continuous, since $K \times Y$ is locally compact Hausdorff. There is a commutative diagram of functions

$$\begin{tikzcd} \mathrm{Cts}(X \times Y, \Sigma) \arrow[r, "(-)\circ\iota"] \arrow[d, "\cong"'] & \mathrm{Cts}(K \times Y, \Sigma) \arrow[d, "\cong"] \\ \mathrm{Closed}(X \times Y) \arrow[r, "\iota^{-1}(-)"'] & \mathrm{Closed}(K \times Y) \end{tikzcd}$$

where the vertical maps are the homeomorphisms obtained from Lemma L12-6 and Ex L12-8. The bottom map sends a closed subset $Z \subseteq X \times Y$ to $\iota^{-1}(Z) = Z \cap (K \times Y)$. Commutativity of this diagram expresses $\iota^{-1}(-)$ as a composite of continuous maps, so it is continuous. Hence the open subset $K \times U \subseteq K \times Y$ determines an open set $\downarrow(K \times U)$ in the topology on $\mathrm{Closed}(K \times Y)$ (here we use that Y is <u>compact</u>, so $K \times Y$ is compact and hence $(K \times U)^c$ is also compact) and so

$$\iota^{-1}\big(\downarrow(K \times U)\big) = \{ Z \subseteq X \times Y \text{ closed} \mid Z \cap (K \times Y) \subseteq K \times U \}.$$

is open in $\mathrm{Closed}(X \times Y)$. Finally, this proves that

$$\begin{aligned} \mathcal{G} \cap \iota^{-1}(\downarrow(K \times U)) &= \{ \Gamma_f \mid \Gamma_f \cap (K \times Y) \subseteq K \times U \} \\ &= \{ \Gamma_f \mid f(K) \subseteq U \} \end{aligned}$$

is open in the subspace topology on $\mathcal{G}$, which is what we need to show. $\square$

<u>Proof of Theorem L12-4</u> First we prove that for arbitrary spaces X, Y, Z and any continuous function $F : Z \times X \longrightarrow Y$ the function

$$\Psi(F) : Z \longrightarrow Cts(X,Y)$$

$$\Psi(F)(z)(x) = F(z,x)$$

is well-defined and continuous, when $Cts(X,Y)$ is given the compact open topology. To check $\Psi(F)$ is well-defined means to check that $x \longmapsto F(z,x)$ is actually continuous as a function $X \longrightarrow Y$. But $F(z,-)$ is the composite of continuous functions

$$X \cong \{z\} \times X \xhookrightarrow{\text{inclusion}} Z \times X \xrightarrow{F} Y$$

hence continuous, so $\Psi(F)(z) \in Cts(X,Y)$. To see the assignment $z \longmapsto F(z,-)$ is continuous, it suffices by Ex L12-3 (iii) to show that for $K \subseteq X$ compact and $U \subseteq Y$ open that

$$\{ z \in Z \mid F(z,-) \in S(K,U) \}$$

$$= \{ z \in Z \mid \forall k \in K \ F(z,k) \in U \}$$

$$= \{ z \in Z \mid \{z\} \times K \subseteq F^{-1}(U) \}$$

is an open subset of Z. Suppose $\{z_0\} \times K \subseteq F^{-1}(U)$, and choose for $k \in K$ an open neighborhood $A_k \times B_k$ of (z_0, k) in $F^{-1}(U)$. The $\{B_k\}_{k \in K}$ cover K in X, and there is a finite subcover $\{B_{k_1}, \ldots, B_{k_r}\}$. Then for $w \in \bigcap_{i=1}^r A_{k_i}$ we have $\{w\} \times K \subseteq (\bigcap_{i=1}^r A_{k_i}) \times (\bigcup_{i=1}^r B_{k_i}) \subseteq F^{-1}(U)$.

Hence $z_0 \in \bigcap_{i=1}^{r} A_{k_i} \subseteq \{ z \in Z \mid \{z\} \times K \subseteq F^{-1}(U) \}$ so that this set is open in Z. As a result, for X, Y, Z arbitrary we have a well-defined function

$$\mathrm{Cts}(Z \times X, Y) \xrightarrow{\Psi_{Z,X,Y}} \mathrm{Cts}(Z, \mathrm{Cts}(X,Y)).$$

$$\Psi_{Z,X,Y}(F)(z)(x) = F(z,x)$$

This function is clearly injective, since F may be recovered from $\Psi(F)$. Now, assuming X locally compact Hausdorff we prove $\Psi_{Z,X,Y}$ is surjective.

Let $f : Z \longrightarrow \mathrm{Cts}(X,Y)$ continuous be given and define

$$F : Z \times X \longrightarrow Y, \qquad F(z,x) := f(z)(x).$$

We need only show F is continuous: then clearly $\Psi_{Z,X,Y}(F) = f$ and we are done. Note that continuity of f tells us precisely that for $K \subseteq X$ compact and $U \subseteq Y$ open, the set

$$\begin{aligned} &\{ z \in Z \mid f(z) \in S(K,U) \} \\ &= \{ z \in Z \mid f(z)(k) \in U \text{ for all } k \in K \} \\ &= \{ z \in Z \mid F(z,k) \in U \text{ for all } k \in K \} \\ &= \{ z \in Z \mid \{z\} \times K \subseteq F^{-1}(U) \} \end{aligned} \qquad (*)$$

is open in Z. We have to employ this somehow to prove F is continuous.

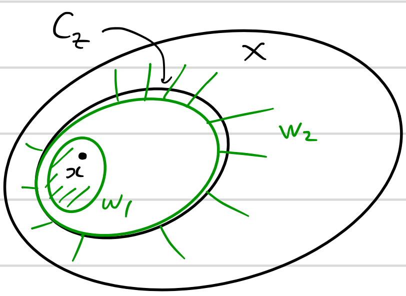
So we take $U \subseteq Y$ open and $(z,x) \in F^{-1}(U)$. Then

$$C_z := \{p \in X \mid (z,p) \in F^{-1}(U)\}$$

$$= \{p \in X \mid F(z,p) \in U\}$$

$$= \{p \in X \mid f(z)(p) \in U\}$$

is open in X since $f(z) : X \to Y$ is assumed continuous. Moreover $x \in C_z$ by hypothesis. Since X is locally compact Hausdorff it is regular (Lemma L12-0) and applying regularity to the pair x, C_z^c we find there exists $W_1, W_2 \subseteq X$ open with $x \in W_1$, $C_z^c \subseteq W_2$ and $W_1 \cap W_2 = \emptyset$.



This shows there is a closed set $K := W_2^c$ with the property that $x \in K \subseteq C_z$ and $x \in W_1 \subseteq K$. Since X is locally compact there is T open and L compact with $x \in T \subseteq L$. Set $\tilde{K} := L \cap K$. Then $\tilde{K}$ is closed in L hence compact, and

$$(z,x) \in \{z\} \times \tilde{K} \subseteq F^{-1}(U).$$

We also shrink W_1 to the open neighborhood $\tilde{W}_1 := W_1 \cap T$ of x in $\tilde{K}$. Now, applying $(*)$ overleaf to the compact set $\tilde{K}$ we find that $V := \{w \in Z \mid \{w\} \times \tilde{K} \subseteq F^{-1}(U)\}$ is <u>open</u> in Z. But then

$$(z,x) \in V \times \tilde{W}_1 \subseteq F^{-1}(U)$$

which shows $F^{-1}(U)$ is open and completes the proof. $\square$

Exercise L12-9 In this exercise Y is compact Hausdorff.

(i) Prove that the singleton map of p. 17 induces a homeomorphism of Y with the subspace of $\mathrm{Closed}(Y)$ consisting of singletons.

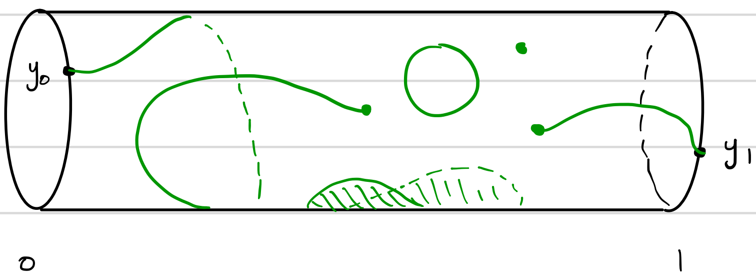
So we may identify Y as a subspace of $\mathrm{Closed}(Y)$.

(ii) Prove that there is a bijection between paths

$$[0,1] \xrightarrow{f} \mathrm{Closed}(Y)$$

beginning at $y_0 \in Y$ (so $f(0) = \{y_0\}$) and ending at $y_1 \in Y$ (so $f(1) = \{y_1\}$) and closed subsets $F \subseteq [0,1] \times Y$ with $F \cap (\{0\} \times Y) = \{(0, y_0)\}$ and $F \cap (\{1\} \times Y) = \{(1, y_1)\}$.

In the case $Y = S^1$ an example of such a path is



Exercise L12-10 Two continuous maps $f, g : X \longrightarrow Y$ are <u>homotopic</u> if there exists $F : [0,1] \times X \longrightarrow Y$ continuous with $F(0,-) = f$ and $F(1,-) = g$. Prove that if X is locally compact Hausdorff there is a bijection between such homotopies F and <u>paths</u> in $\mathrm{Cts}(X,Y)$ from f to g.

The requirement in Theorem L12-7 that Y be compact (rather than just locally compact) is not such a big deal. There is a construction (the <u>one-point compactification</u>) which produces from a locally compact Hausdorff space Y a compact Hausdorff space $\tilde{Y}$ such that Y is homeomorphic to an open subset $U \subseteq \tilde{Y}$ and $\tilde{Y} \setminus U$ is a single point ($Y = \mathbb{R}$ and $\tilde{Y} = S^1$ is the canonical example).

<u>Exercise L12-11</u> Suppose X is locally compact Hausdorff and that $Y_1 \subseteq Y_2$ is a subspace, with inclusion $\iota : Y_1 \longrightarrow Y_2$. Prove that the continuous map

$$\begin{aligned} \mathrm{Cts}(X, Y_1) &\longrightarrow \mathrm{Cts}(X, Y_2) \\ f &\longmapsto \iota \circ f \end{aligned}$$

is a homeomorphism onto its image, so that we may identify $\mathrm{Cts}(X, Y_1)$ with the subspace of $f \in \mathrm{Cts}(X, Y_2)$ with image contained in Y_1.

Hence if X, Y are locally compact Hausdorff, and $\tilde{Y}$ is the one-point compactification of Y, we have that $\mathrm{Cts}(X, Y)$ embeds as a subspace of $\mathcal{G}_{X,\tilde{Y}}$:

$$\mathrm{Cts}(X, Y) \hookrightarrow \mathrm{Cts}(X, \tilde{Y}) \xrightarrow[\cong]{\text{Thm L12-7}} \mathcal{G}_{X,\tilde{Y}} \hookrightarrow \mathrm{Closed}(X \times \tilde{Y}). \quad (*).$$

where continuous $f : X \longrightarrow Y$ is sent to the closed set $\Gamma_{\iota \circ f} = \{(x, fx) \mid x \in X\}$.

<u>Exercise L12-12</u> The map $\mathbb{R} \longrightarrow (0,1)$, $x \longmapsto \tanh(x)$ is a homeomorphism, and composing with $(0,1) \hookrightarrow [0,1] \longrightarrow [0,1]/\sim = S^1$ embeds $Y = \mathbb{R}$ as a subspace of S^1 with complement a point. With $X = (0,1)$ and $f : X \longrightarrow Y$ given by $f(x) = \frac{1}{x}$ sketch the closed subset of $X \times \tilde{Y} = (0,1) \times S^1$ associated to f by $(*)$.

<u>Exercise L12-13*</u> Prove that if X, Z are locally compact Hausdorff and Y is arbitrary that the bijection of Theorem L12-4

$$\mathrm{Cts}(Z \times X, Y) \xrightarrow{\Psi_{Z,X,Y}} \mathrm{Cts}(Z, \mathrm{Cts}(X,Y))$$

is a <u>homeomorphism</u> where both sides are given the compact-open topology.

<u>Exercise L12-14</u> Prove that if X is locally compact Hausdorff and Y, Z are arbitrary then the canonical bijection

$$\mathrm{Cts}(X, Y \times Z) \longrightarrow \mathrm{Cts}(X,Y) \times \mathrm{Cts}(X,Z)$$

of Lemma L7-2 is a homeomorphism.

Solutions to selected exercises

Ex L12-13 If X, Z are locally compact Hausdorff so is $X \times Z$ and hence

$$ev_{Z\times X, Y} : Z \times X \times Cts(Z\times X, Y) \longrightarrow Y$$

is continuous. It follows that

$$\begin{aligned} Z \times Cts(Z\times X, Y) &\longrightarrow Cts(X, Y) \\ (z, F) &\longmapsto ev_{Z\times X, Y}(z, -, F) \end{aligned}$$

is continuous. But $ev_{Z\times X,Y}(z,-,F) = F(z,-)$, and so associated to this map is precisely $\Psi_{Z,X,Y}$ which is therefore continuous. It remains to show that the inverse $\Psi^{-1}_{Z,X,Y}$ is continuous. But (with $\tau_{Z,X} : Z\times X \to X\times Z$ is the swap)

$$\begin{tikzcd} Z \times X \times Cts(Z, Cts(X,Y)) \arrow[d, "\tau_{Z,X}\times 1"', "\cong"] & Y \\ X \times Z \times Cts(Z, Cts(X,Y)) \arrow[r, "1_X \times ev_{Z, Cts(X,Y)}"] & X \times Cts(X,Y) \arrow[u, "ev_{X,Y}"'] \end{tikzcd}$$

is continuous since both Z, X are locally compact Hausdorff. Associated to this is a continuous map

$$\begin{aligned} Cts(Z, Cts(X,Y)) &\longrightarrow Cts(Z\times X, Y) \\ \gamma &\longmapsto [ev_{X,Y}\circ(1_X \times ev_{Z,Cts(X,Y)})\circ \tau_{Z,X}\times 1](-,-,\gamma) \\ &\qquad\qquad \| \\ &\qquad \{(z,x) \mapsto \gamma(z)(x)\} \end{aligned}$$

But this is $\Psi^{-1}_{Z,X,Y}$ so we are done. $\square$