updated 5/11/19

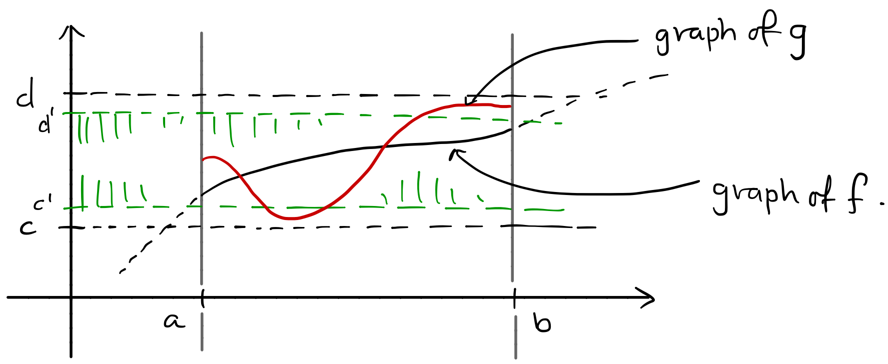
Lecture 13 : Metrics on function spaces

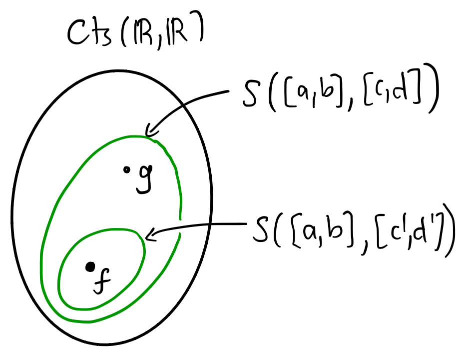
We have now defined a topology on $Cts(X,Y)$ for any pair of spaces X, Y which has special properties if X is locally compact Hausdorff. But it remains unclear how to think about the basic open sets $S(K,U)$ in this topology. In this lecture we will specialise to the case where X is compact and Y is metrisable, and, by using a metric on Y (any one will do) we can get a better handle on the compact-open topology.

Exercise L13-1 With X, Y arbitrary, we have

(i) If $K \subseteq K'$ are compact then $S(K', U) \subseteq S(K, U)$,
(ii) If $U \subseteq U'$ are open then $S(K,U) \subseteq S(K,U')$.
(iii) If K, K' are compact sets $S(K \cup K', U) = S(K,U) \cap S(K',U)$.
(iv) If U, U' are open then $S(K, U \cap U') = S(K,U) \cap S(K,U')$.

Example L13-1 With $X = \mathbb{R}$, $Y = \mathbb{R}$ the open set $S([a,b],(c,d))$ is





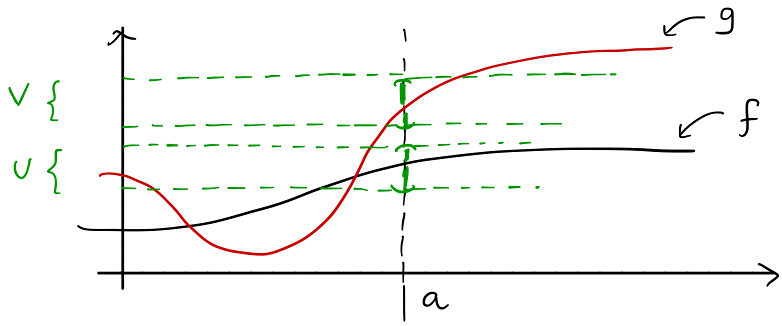
We can shrink this open neighborhood of f in $Cts(\mathbb{R},\mathbb{R})$ by <u>either</u> shrinking (c,d) to $(c',d') \subset (c,d)$ (which would for example exclude the function g indicated above) or we can expand $[a,b]$ to $[a',b'] \supset [a,b]$.

This picture raises the question: if $f \neq g$ in $Cts(X,Y)$ can we separate f,g by open neighbourhoods? That is, is $Cts(X,Y)$ Hausdorff? We know $Cts(\{*\},Y) \cong Y$ as spaces so it is certainly necessary that Y is Hausdorff.

<u>Lemma L13-1</u> If Y is Hausdorff then $Cts(X,Y)$ is Hausdorff.

<u>Proof</u> Suppose $f,g : X \longrightarrow Y$ are continuous and $f(x) \neq g(x)$. Then let $U \ni f(x)$, $V \ni g(x)$ be open with $U \cap V \neq \phi$. Then $K = \{x\} \subseteq X$ is compact and $S(\{x\},U)$, $S(\{x\},V)$ are disjoint open subsets of $Cts(X,Y)$ with $f \in S(\{x\},U)$, $g \in S(\{x\},V)$. $\square$

<u>Example L13-2</u> Again with $X = Y = \mathbb{R}$, we can shrink $[a,b]$ to a point $\{a\}$, to get open subsets $S(\{a\},U)$ of $Cts(\mathbb{R},\mathbb{R})$ sufficiently fine to separate points



<u>Exercise L13-2</u> (i) Let X be a compact Hausdorff space and $\sim$ an equivalence relation on X s.t. $X/\sim$ is Hausdorff (it is automatically compact). Let $p : X \longrightarrow X/\sim$ be the quotient map. The map

$$Cts(X/\sim, Y) \xrightarrow{(-)\circ p} Cts(X,Y)$$

is continuous, by Lemma L12-1. Prove that this map is a homeomorphism onto its image, $Cts(X/\sim, Y)$ may be viewed as a subspace of $Cts(X,Y)$.

(ii) Prove that for any spaces X, Y with $X \neq \phi$ the function $Y \xrightarrow{c_{(-)}} Cts(X, Y)$ sending $y \in Y$ to $c_y : X \longrightarrow Y$ with $c_y(x) = y$ for all $x \in X$ is continuous, and induces a homeomorphism onto its image (so we may view Y as a subspace of $Cts(X, Y)$).

<u>Question</u>: When is $Cts(X, Y)$ metrisable?

In light of the previous Exercise, a necessary condition (assuming $X \neq \phi$) is that Y is metrisable. Let us consider the simplest case, which is X finite and discrete and $Y = \mathbb{R}$, so that by Exercise L12-5, we have a homeomorphism

$$\begin{aligned} Cts(\{1, \dots, n\}, \mathbb{R}) &\cong \mathbb{R}^n \\ f &\longmapsto (f(1), \dots, f(n)) \end{aligned}$$

where $\mathbb{R}^n$ has the usual topology. We know at least three metrics which induce this topology, namely d_1, d_2, d_∞, which give rise to metrics on $Cts(\{1, \dots, n\}, \mathbb{R})$ which likewise determine its topology. These are

- $d_1(f, g) = \sum_{i=1}^{n} |f(i) - g(i)|$
- $d_2(f, g) = \left\{ \sum_{i=1}^{n} |f(i) - g(i)|^2 \right\}^{1/2}$
- $d_\infty(f, g) = \sup\{ |f(i) - g(i)| \}_{i=1}^{n}$

Replacing $(\mathbb{R}, |\cdot|)$ by any metric space (Y, d_Y) and $|f(i) - g(i)|$ by $d_Y(f(i), g(i))$ similarly provides three metrics on $Cts(\{1, \dots, n\}, Y)$ inducing the compact-open topology. The real issue is how to generalise these metrics from $X = \{1, \dots, n\}$ to an arbitrary compact space.

Naturally if X is infinite we would like to replace Σ by $\int$ and define a metric d on $Cts(X, Y)$ for X compact and Y having the topology induced by a metric d_Y, using

- $d_1(f,g) = \int_X d_Y(f(x), g(x))\, d\mu$
- $d_2(f,g) = \left\{ \int_X d_Y(f(x), g(x))^2\, d\mu \right\}^{1/2}$
- $d_\infty(f,g) = \sup\{ d_Y(f(x), g(x)) \mid x \in X \}$

The "definitions" of d_1, d_2 require that we have a theory of integration on the space X, say the Riemann integral if $X = [a,b]$ or more generally a compact subset of $\mathbb{R}^n$, or more generally the Lebesgue integral, when X is given a measure μ. However the third definition always works:

Exercise L13-3 If (Y, d_Y) is a metric space prove that the metric $d_Y : Y \times Y \longrightarrow \mathbb{R}$ is uniformly continuous when $Y \times Y$ is given the product metric (see Ex. L13-8 for the definition).

Def^n Given a subset $A \subseteq X$ of a topological space X, the closure $\bar{A}$ of A and the interior A° of A are defined by

$$\bar{A} = \bigcap \{ C \subseteq X \mid C \text{ is closed and } C \supseteq A \}$$
$$A^\circ = \bigcup \{ U \subseteq X \mid U \text{ is open and } U \subseteq A \}.$$

So $\bar{A}$ is the smallest closed set containing A, and A° is the largest open set contained in A.

<u>Exercise L13-4</u> (i) $x \in \overline{A}$ if and only if every open neighborhood of x contains an element of A

(ii) In a metric space (X,d) we have $\overline{B_\varepsilon(x)} \subseteq \{ y \in X \mid d(x,y) \leq \varepsilon \}$.

(iii) If $f: X \longrightarrow Y$ is continuous then $f(\overline{A}) \subseteq \overline{f(A)}$.

(iv) If $A \subseteq B$ then $\overline{A} \subseteq \overline{B}$.

<u>Theorem L13-2</u> Suppose X is compact and Y is metrisable. Then for any metric d_Y inducing the topology on Y, there is an associated metric

$$d_\infty(f,g) = \sup\{ d_Y(f(x), g(x)) \mid x \in X \}$$

on $Cts(X,Y)$ whose topology is the compact-open topology.

<u>Proof</u> First we must show d_∞ is well-defined. The function

$$X \xrightarrow{\langle f,g \rangle} Y \times Y \xrightarrow{d_Y} \mathbb{R}$$
$$x \longmapsto (f(x), g(x)) \longmapsto d_Y(f(x), g(x))$$

is continuous and X is compact, so its image is compact in $\mathbb{R}$ (Prop. L9-3) and hence bounded, so the supremum exists and d_∞ is well-defined. We have to show it is a metric and that it induces the compact-open topology.

<u>d_∞ is a metric</u> (M1) Clearly $d_\infty(f,g) \geq 0$.

(M2) If $d_\infty(f,g) = 0$ then using (M1) for Y, $f(x) = g(x)$ for all $x \in X$ so $f = g$.

(M3) $$\begin{aligned} d_\infty(f,g) &= \sup\{ d_Y(f(x), g(x)) \mid x \in X \} \\ &= \sup\{ d_Y(g(x), f(x)) \mid x \in X \} \\ &= d_\infty(g,f) \end{aligned}$$

(M4) We need to show $d_\infty(f,h) \leq d_\infty(f,g) + d_\infty(g,h)$. But for $x \in X$,

$$\begin{aligned} d_Y(f(x),h(x)) &\leq d_Y(f(x),g(x)) + d_Y(g(x),h(x)) \\ &\leq d_\infty(f,g) + d_\infty(g,h) \end{aligned}$$

which proves the desired inequality.

<u>$\mathcal{T}_{d_\infty}$ is the compact-open topology</u> Let $\mathcal{T}_{X,Y}$ denote the compact-open topology. To show $\mathcal{T}_{d_\infty} \subseteq \mathcal{T}_{X,Y}$ it suffices to prove $B_\varepsilon(f) \in \mathcal{T}_{X,Y}$ for all $f \in C_b(X,Y)$ and $\varepsilon > 0$. Since f is continuous, $f(X)$ is compact, and hence $\{B_{\varepsilon/3}(y) \mid y \in f(X)\}$ has a finite subcover $\{B_{\varepsilon/3}(fx_1), \ldots, B_{\varepsilon/3}(fx_n)\}$. Define

$$K_i := \overline{f^{-1}B_{\varepsilon/3}(fx_i)}, \qquad U_i := B_{\varepsilon/2}(fx_i).$$

Then

$$\begin{aligned} f(K_i) &= f\left(\overline{f^{-1}B_{\varepsilon/3}(fx_i)}\right) \\ &\subseteq \overline{f(f^{-1}B_{\varepsilon/3}(fx_i))} \\ &\subseteq \overline{B_{\varepsilon/3}(fx_i)} \\ &\subseteq \{y \in Y \mid d(y,fx_i) \leq \varepsilon/3\} \\ &\subseteq B_{\varepsilon/2}(fx_i) = U_i. \end{aligned}$$

Hence $f \in \bigcap_{i=1}^n S(K_i,U_i) \subseteq C_b(X,Y)$. It remains to show $\bigcap_i S(K_i,U_i) \subseteq B_\varepsilon(f)$ (since then we will have shown that given $f \in Q \in \mathcal{T}_{d_\infty}$ that there is an open set Q' in $\mathcal{T}_{X,Y}$ with $f \in Q' \subseteq Q$, which shows $Q \in \mathcal{T}_{X,Y}$ for any $Q \in \mathcal{T}_{d_\infty}$).

Suppose $g \in \bigcap_i S(K_i, U_i)$ so that $g(K_i) \subseteq U_i$ for all i. We will show

$$d_Y(g(x), f(x)) < 5\varepsilon/6$$

for all $x \in X$, then the supremum $d_\infty(g, f)$ must be $\leq 5\varepsilon/6$ and hence $< \varepsilon$, so $g \in B_\varepsilon(f)$. But since the $B_{\varepsilon/3}(fx_i)$ cover $f(X)$ there is some i s.t. $fx \in B_{\varepsilon/3}(fx_i)$ that is, $d_Y(fx, fx_i) < \varepsilon/3$, and hence $x \in K_i$. But then by hypothesis $g(x) \in U_i = B_{\varepsilon/2}(fx_i)$ and so

$$\begin{aligned} d_Y(gx, fx) &\leq d_Y(gx, fx_i) + d_Y(fx_i, fx) \\ &< \varepsilon/2 + \varepsilon/3 \\ &= 5\varepsilon/6 \end{aligned}$$

as claimed. This proves $\mathcal{T}_{d_\infty} \subseteq \mathcal{T}_{X,Y}$.

For the inclusion $\mathcal{T}_{X,Y} \subseteq \mathcal{T}_{d_\infty}$ we have to show $S(K,U) \in \mathcal{T}_{d_\infty}$ for any $K \subseteq X$ compact and $U \subseteq Y$ open. If $f \in S(K,U)$ then $f(K)$ is compact and so by Lemma L13-4 below there is $\varepsilon > 0$ with $d_Y(f(k), y) > \varepsilon$ for all $k \in K$ and $y \notin U$. We claim $B_\varepsilon(f) \subseteq S(K,U)$. If $g \in B_\varepsilon(f)$ then $d_Y(fk, gk) < \varepsilon$ for all $k \in K$, and hence $gk \in U$, so $g(K) \subseteq U$ and this proves the claim, and moreover shows that $S(K,U) \in \mathcal{T}_{d_\infty}$.

Hence $\mathcal{T}_{X,Y} = \mathcal{T}_{d_\infty}$ and the proof is complete. $\square$

<u>Corollary L13-3</u> For X compact and Y metrisable the metric topology on $\mathrm{Cts}(X,Y)$ is independent of the choice of metric on Y.

<u>Def^n</u> Let (Y,d) be a metric space, $A \subseteq Y$ a subset. Then for $y \in Y$ we define

$$d(y,A) := \inf\{ d(y,a) \mid a \in A \}.$$

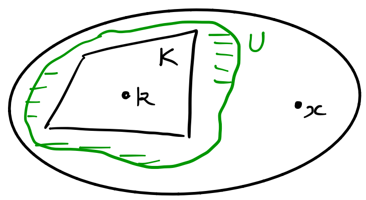
<u>Lemma L13-3</u> The function $d(-,A) : Y \longrightarrow \mathbb{R}$ is continuous.

<u>Proof</u> Given $x, y \in Y$ we have for $a \in A$

$$d(x,A) \leq d(x,a) \leq d(x,y) + d(y,a)$$

Hence $d(x,A) - d(x,y) \leq \inf\{ d(y,a) \mid a \in A \} = d(y,A)$
so $d(x,A) - d(y,A) \leq d(x,y)$ and hence $|d(x,A) - d(y,A)| \leq d(x,y)$.
(swap the role of x, y). It now follows easily that $d(-,A)$ is continuous. $\square$

<u>Lemma L13-4</u> If (Y, d_Y) is a metric space, K compact and U open with $K \subseteq U$ then there exists $\varepsilon > 0$ such that for all $k \in K$ and $x \notin U$ we have $d_Y(x,k) > \varepsilon$.



<u>Proof</u> The function $d(-,U^c)|_K : K \to \mathbb{R}$ is continuous and by the Extreme value theorem there exists $k_0 \in K$ with

$$d(k_0, U^c) = \inf\{ d(k, U^c) \mid k \in K \}.$$

The number $\varepsilon = \frac{1}{2} d(k_0, U^c) > 0$ does the job. $\square$

Exercise L13-5 Let X, Y be topological spaces with Y metrisable, and let d_Y be a metric inducing the topology on Y. For $C \subseteq X$ compact, $\varepsilon > 0$ and $f \in \mathrm{Cts}(X, Y)$ define

$$B_C(f, \varepsilon) := \{ g \in \mathrm{Cts}(X,Y) \mid \sup\{ d_Y(fx, gx) \mid x \in C \} < \varepsilon \}$$

Prove that

(i) The sets $B_C(f, \varepsilon)$ form a basis for a topology on $\mathrm{Cts}(X, Y)$, and that

(ii) This topology is the compact-open topology.

Pointwise and Uniform convergence

When X is compact and Y is metrisable we know that $\mathrm{Cts}(X, Y)$ is metrisable, with metric d_∞. What does it mean for a sequence of functions $(f_n)_{n=0}^{\infty}$ to converge to f in this metric?

Def^n Let X be a set, (Y, d_Y) a metric space, $(f_n)_{n=0}^{\infty}$ a sequence of functions $f_n : X \to Y$, and $f : X \to Y$ a function. Then

- $(f_n)_{n=0}^{\infty}$ is pointwise convergent to f if for every $x \in X$ the sequence $(f_n(x))_{n=0}^{\infty}$ converges to $f(x)$ in Y. That is,

$$\forall x \in X \; \forall \varepsilon > 0 \; \exists N \in \mathbb{N} \, (n \geq N \Rightarrow d_Y(f_n x, fx) < \varepsilon)$$

- $(f_n)_{n=0}^{\infty}$ is uniformly convergent to f if

$$\forall \varepsilon > 0 \; \exists N \in \mathbb{N} \; \forall x \in X (n \geq N \Rightarrow d_Y(f_n x, fx) < \varepsilon).$$

Since uniform convergence is equivalent to

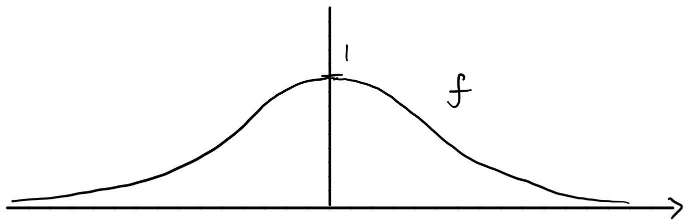
$$\forall \varepsilon > 0 \; \exists N \in \mathbb{N} \; \big(n \geqslant N \Longrightarrow \sup\{ d_Y(f_n x, fx) \mid x \in X \} < \varepsilon \big)$$

it is clear that if X is compact and (Y, d_Y) is a metric space, then a sequence of continuous functions $f_n : X \to Y$ converges to a continuous function $f : X \to Y$ in the metric space $(\mathrm{Cts}(X,Y), d_\infty)$ <u>if and only if</u> $(f_n)_{n=0}^{\infty}$ converges to f uniformly in the above sense. So at least when X is compact, the <u>compact-open topology captures the notion of uniform convergence</u>.

<u>Example L13-3</u> The sequence $(f_n : [0,1] \to \mathbb{R})_{n=1}^{\infty}$ given by $f_n(x) = x/n$ converges uniformly to the constant function $f(x) \equiv 0$, since $\sup\{ |f_n(x)| \mid x \in [0,1] \} = 1/n$, which we can make arbitrarily small.

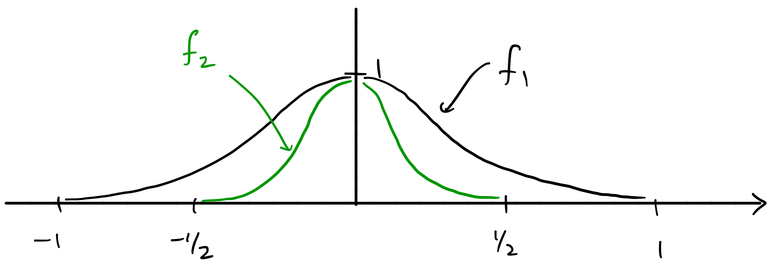
Uniform convergence implies pointwise convergence, but the reverse is false.

<u>Example L13-4</u> Consider the function $f(x) = e^{-x^2}$, with graph



We can concentrate this bump onto a compact subset $[-1/n, 1/n] \subseteq \mathbb{R}$ by precomposing with the continuous map $(-1/n, 1/n) \to \mathbb{R}$ sending $x \longmapsto (1 - n^2x^2)^{-1/2}$ and extending by zero outside of $(-n, n)$ to define a sequence of continuous functions ($n \geqslant 1$ an integer)

$$f_n : [-1,1] \longrightarrow \mathbb{R} \qquad f_n(x) = \begin{cases} \exp\left(\dfrac{-1}{1-n^2x^2}\right) & -1/n < x < 1/n \\ 0 & \text{otherwise} \end{cases}$$



Clearly we have for each $-1 \leq x \leq 1$ a limit $\lim_{n \to \infty} f_n(x) = \delta_0(x)$ meaning 0 if $x \neq 0$ and 1 if $x = 0$. So there is pointwise convergence $f_n \longrightarrow \delta_0$. However this convergence is not uniform. We can show this directly, but it is also an immediate consequence of the next theorem.

A uniform limit of continuous functions is continuous:

Theorem L13-5 Let X be a topological space and (Y, d_Y) a metric space. If $f: X \to Y$ is the uniform limit of a sequence $(f_n : X \to Y)_{n=0}^{\infty}$ each member of which is continuous then f is also continuous.

Proof Let $V \subseteq Y$ be open, and let $x_0 \in f^{-1}V$ be given, with say $B_\varepsilon(fx_0) \subseteq V$. Then by uniformity there exists $N > 0$ such that for $n \geqslant N$ and $x \in X$

$$d_Y(f_n x, fx) < \varepsilon/3.$$

Set $y := f_N x_0$. Since f_N is continuous, the set $f_N^{-1} B_{\varepsilon/3}(y)$ is open, and if x lies in this open set then

$$\begin{aligned} d_Y(fx, fx_0) &\leq d_Y(fx, f_N x) + d_Y(f_N x, y) + d_Y(y, fx_0) \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon \end{aligned}$$

Hence $x_0 \in f_N^{-1} B_{\varepsilon/3}(y) \subseteq f^{-1} B_\varepsilon(fx_0) \subseteq f^{-1}V$ so $f^{-1}V$ is open. $\square$

Since for X compact and (Y, d_Y) a metric space convergence in $Cts(X,Y)$ means uniform convergence, and uniform limits of continuous maps are continuous, this suggests that as long as Cauchy sequences in Y converge, Cauchy sequences in $Cts(X,Y)$ ought also to converge.

Recall that a sequence $(a_n)_{n=0}^{\infty}$ in a metric space (A,d) is <u>Cauchy</u> if for every $\varepsilon > 0$ there exists $N > 0$ such that for all $n, m \geq N$ we have $d(a_n, a_m) < \varepsilon$. Any convergent sequence is Cauchy (Ex. L9-3), but the converse is not true: for example $(1/n)_{n=1}^{\infty}$ is a Cauchy sequence in $(0,1)$ which does not converge.

<u>Exercise L13-6</u> Any Cauchy sequence in (A,d) is bounded (as a subset).

<u>Exercise L13-7</u> Any Cauchy sequence which contains a convergent subsequence is itself convergent.

<u>Def^n</u> A metric space (A,d) is <u>complete</u> if every Cauchy sequence in A converges.

<u>Example L13-5</u> $\mathbb{R}$ with its usual metric is complete (by definition, the way we have set things up, in Tutorial 5).

<u>Exercise L13-8</u> Let $(A_1, d_1), \ldots, (A_n, d_n)$ be metric spaces and with $A = \prod_{i=1}^{n} A_i$ define $d : A \times A \longrightarrow \mathbb{R}$ by $d\big((a_i)_{i=1}^{n}, (b_i)_{i=1}^{n}\big) = \sum_i d_i(a_i, b_i)$. Prove that

(i) (A,d) is a metric space

(ii) The topology on A induced by d is the product topology on $\prod_{i=1}^{n} A_i$ (giving each A_i its metric topology).

(iii) If each (A_i, d_i) is complete so is (A,d).

Since $(0,1)$ is homeomorphic to $\mathbb{R}$, but $(\mathbb{R}, |\cdot|)$ is complete while $((0,1), |\cdot|)$ is not, this shows completeness is <u>not</u> a topological property. It is genuinely a property of the <u>metric</u> (so except in some exceptional circumstances, like topological groups, we only talk about completeness of a metric).

<u>Exercise L13-9</u> (i) Prove that if two metrics d_1, d_2 on A are Lipschitz equivalent (see Tutorial 3) then (A, d_1) is complete if and only if (A, d_2) is complete.

(ii) Prove that if (A, d) is complete and $B \subseteq A$ is closed then (B, d) is also complete.

In particular Ex. L13-8 plus L13-5 gives that $(\mathbb{R}^n, d_1)$, $(\mathbb{R}^n, d_2)$ and $(\mathbb{R}^n, d_\infty)$ are all complete, as they are Lipschitz equivalent (Tutorial 2 Q10).

<u>Lemma L13-6</u> Any compact metric space (A, d) is complete.

<u>Proof</u> If $(a_n)_{n=0}^{\infty}$ is Cauchy then by compactness it contains a convergent subsequence, and by Ex. L13-7 the original sequence converges. $\square$

<u>Corollary L13-6</u> If X is compact and (Y, d_Y) is a complete metric space, then the metric space $(\mathrm{Cts}(X,Y), d_\infty)$ is also complete.

<u>Proof</u> Let $(f_n)_{n=0}^{\infty}$ be Cauchy in $\mathrm{Cts}(X,Y)$, with respect to d_∞. Then for each $x \in X$ the sequence $(f_n x)_{n=0}^{\infty}$ is Cauchy in Y and we define $f(x)$ to be the limit. Then $f : X \longrightarrow Y$ is a function. If we can show that the sequence $(f_n)_{n=0}^{\infty}$ converges to f uniformly then by Theorem L13-5 f must be continuous, and it is then immediate that $f_n \longrightarrow f$ with respect to d_∞, and the proof is complete.

Given $\varepsilon > 0$ choose N s.t. for $m, n \geqslant N$ and $x \in X$ we have

$$d_Y(f_m x, f_n x) < \varepsilon/2$$

(This we may do since $(f_n)_{n=0}^{\infty}$ is Cauchy w.r.t. d_∞). For any fixed $x \in X$ convergence $f_n(x) \longrightarrow f(x)$ means we can find $m(x) \geqslant N$ such that

$$d_Y(f_{m(x)} x, f x) < \varepsilon/2$$

and then for any $n \geqslant N$ we have for all $x \in X$

$$\begin{aligned} d_Y(f_n x, f x) &\leq d_Y(f_n x, f_{m(x)} x) + d_Y(f_{m(x)}, f x) \\ &< \varepsilon/2 + \varepsilon/2 < \varepsilon. \end{aligned}$$

Hence $f_n \longrightarrow f$ uniformly and the proof is complete. $\square$

<u>Example L13-6</u> If X is compact then $\mathrm{Cts}(X, \mathbb{R})$ is complete (by default when we say such a thing, we mean it with respect to the d_∞ metric on $\mathrm{Cts}(X, \mathbb{R})$ associated to $|-|$ on $\mathbb{R}$).

<u>Exercise L13-10</u> Suppose X is compact and Y is metrisable, with d_Y^1, d_Y^2 being Lipschitz equivalent metrics inducing the topology. Prove that the two associated metrics d_∞^1, d_∞^2 on $\mathrm{Cts}(X, Y)$ are also Lipschitz equivalent.

The importance of completeness of function spaces can hardly be overstated, since it allows us to construct <u>exact</u> solutions of differential equations by taking a limit of <u>approximate</u> solutions, as we will see in the next two lectures.

Solutions to selected exercises

$\boxed{\text{L13-3}}$ We have

$$d(x,y) \leq d(x,z) + d(z,y)$$

$$\therefore \ d(x,y) - d(z,y) \leq d(x,z)$$

Exchanging the vole of x, z yields $d(z,y) - d(x,y) \leq d(z,x) = d(x,z)$, so

$$|d(x,y) - d(z,y)| \leq d(x,z). \qquad (*)$$

Hence

$$\begin{aligned} |d(x_1,x_2) - d(y_1,y_2)| &= |d(x_1,x_2) - d(x_1,y_2) \\ &\qquad + d(x_1,y_2) - d(y_1,y_2)| \\ &\leq |d(x_1,x_2) - d(x_1,y_2)| \\ &\quad + |d(x_1,y_2) - d(y_1,y_2)| \\ &\overset{\text{by } (*)}{\leq} d(x_2,y_2) + d(x_1,y_1). \end{aligned}$$

which proves $d_Y : Y \times Y \longrightarrow \mathbb{R}$ is uniformly continuous, where we give $Y \times Y$ the product metric of Ex. L13-8.