Lecture 18 : Banach space

updated 16/10/20

Last lecture we defined an integral pair $(X, \int_X)$ to be a compact Hausdorff space together with a continuous linear map $\int_X : \mathrm{Cts}(X, \mathbb{R}) \longrightarrow \mathbb{R}$ satisfying some positivity properties, and we defined a metric $d_1^{\int}$ on $\mathrm{Cts}(X, \mathbb{R})$. This begins to fulfill an analogy sketched in Lecture 13. By Ex. L17-7 the vector space $\mathrm{Cts}(X, \mathbb{R})$ is finite-dimensional iff. X is finite, in which case it is homeomorphic to $\mathbb{R}^n$ where $|X| = n$. In the finite-dimensional case, the following metrics on $\mathrm{Cts}(X, \mathbb{R})$

$$\begin{aligned}
d_1(f,g) &= \textstyle\sum_{x \in X} |f(x) - g(x)| \\
d_2(f,g) &= \left\{ \textstyle\sum_{x \in X} |f(x) - g(x)|^2 \right\}^{1/2} \\
&\;\;\vdots \\
d_p(f,g) &= \left\{ \textstyle\sum_{x \in X} |f(x) - g(x)|^p \right\}^{1/p} \\
&\;\;\vdots \\
d_\infty(f,g) &= \sup\{ |f(x) - g(x)| \mid x \in X \}
\end{aligned} \tag{1.1}$$

are all Lipschitz equivalent (see Tutorial 2) and hence determine the same topology, which is the usual topology on $\mathbb{R}^n \cong \mathrm{Cts}(X, \mathbb{R})$. Note that $(\{1, \ldots, n\}, \Sigma)$ is an integral pair, where $\Sigma : \mathbb{R}^n \to \mathbb{R}$ sends $(x_1, \ldots, x_n) \mapsto \sum_{i=1}^n x_n$. The same formulas make sense for any integral pair $(X, \int_X)$ where they read

$$\begin{aligned}
d_1(f,g) &= \textstyle\int_X |f - g| \\
d_2(f,g) &= \left\{ \textstyle\int_X |f - g|^2 \right\}^{1/2} \\
&\;\;\vdots \\
d_p(f,g) &= \left\{ \textstyle\int_X |f - g|^p \right\}^{1/p} \\
&\;\;\vdots \\
d_\infty(f,g) &= \sup\{ |f(x) - g(x)| \mid x \in X \}
\end{aligned} \tag{1.2}$$

⌜these metrics depend on $\int_X$ but we do not indicate this in the notation⌟

⌜$p \geq 1$ any real number⌟

Question : if X is infinite are these metrics Lipschitz equivalent?

Surprisingly the answer is <u>No</u>. The conceptual reason is that <u>not every integrable function is continuous</u> (this distinction is of course invisible for X finite, where every function $X \to \mathbb{R}$ is continuous and therefore also integrable). This observation explains why the infinite-dimensional case (here we mean $\mathrm{Cts}(X,\mathbb{R})$ infinite-dimensional) is much richer

At least one of these metrics we understand: $(\mathrm{Cts}(X,\mathbb{R}), d_\infty)$ is complete and its associated topology is the compact-open topology. For $1 \le p < \infty$ the metric space $(\mathrm{Cts}(X,\mathbb{R}), d_p)$ is <u>not</u> necessarily complete if X is infinite, and so it is natural to introduce the <u>L^p-space</u> (details below)

$$L^p(X,\mathbb{R}) := \text{metric space completion of } (\mathrm{Cts}(X,\mathbb{R}), d_p).$$

These spaces are the basic objects of functional analysis, and they carry (in addition to a metric) the richer structure of a <u>Banach space</u>. For $p=2$ the space $L^2(X,\mathbb{R})$ has an even richer structure: it is a <u>Hilbert space</u>. These are the canonical examples of Hilbert spaces and (the complex analogues of) these spaces give the foundational mathematical theory of quantum mechanics.

<u>Exercise L18-0</u> Prove that every integral pair $(\{1,\dots,n\}, \int)$ is of the form

$$\int f = \sum_{i=1}^{n} \lambda_i f(i)$$

for some positive constants $\lambda_1, \dots, \lambda_n \in \mathbb{R}$. The corresponding d_2 metric we have encountered in Example L4-0. Prove that the metrics $\{d_p\}_{p \ge 1}$, d_∞ of (1.2) are all Lipschitz equivalent and determine the standard topology on $\mathbb{R}^n$ (so there is nothing new obtained in the finite case by considering other integrals).

The plan now is to check that d_p is a metric, and introduce the notion of <u>completion</u> of a metric space in order to define the L^p-spaces. For this it is very convenient to introduce the notion of a <u>normed space</u>. For the following material the references are

- Hewitt, Stromberg "Real and abstract analysis" Ch. 4 (also p. 83)
- Rudin "Functional analysis" Ch. 1,
- Cheney "Analysis for applied mathematics" Ch. 1.

By a <u>field of scalars</u> $\mathbb{F}$ we will either mean $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. In both cases we have the function $|-| : \mathbb{F} \longrightarrow \mathbb{R}$ which is the absolute value if $\mathbb{F} = \mathbb{R}$ and is the modulus $|a+ib| = \sqrt{a^2+b^2}$ if $\mathbb{F} = \mathbb{C}$. We make $\mathbb{R}, \mathbb{C}$ both into topological rings with the usual operations, where as a topological space $\mathbb{C} = \mathbb{R}^2$ under the bijection $a+ib \longleftrightarrow (a,b)$. Then $\mathbb{F}$ carries the structure of a topological ring and $|-| : \mathbb{F} \longrightarrow \mathbb{R}$ is continuous. In what follows, $\mathbb{F}$ is a field of scalars.

<u>Def^n</u> A <u>norm</u> on a vector space V over $\mathbb{F}$ is a function $\|-\| : V \longrightarrow \mathbb{R}$ such that

(N1) $\|v\| \geq 0$ for all $v \in V$, and $\|v\| = 0$ if and only if $v = 0$.

(N2) $\|\lambda v\| = |\lambda| \, \|v\|$ for all $\lambda \in \mathbb{F}$ and $v \in V$.

(N3) $\|v + w\| \leq \|v\| + \|w\|$ for all $v, w \in V$. (Triangle inequality)

The pair $(V, \|-\|)$ is called a <u>normed space</u> over $\mathbb{F}$.

<u>Exercise L18-1</u> If $(V, \|-\|)$ is a normed space then (V, d) is a metric space, where

$$d(v,w) := \|v - w\|.$$

We call the metric of the previous exercise the <u>induced</u> (or <u>associated</u>) metric. Given a normed space $(V, \|-\|)$ the <u>associated topology</u> on V is the metric topology. <u>Be careful</u> : it may be that V already <u>had</u> a topology, but there is no guarantee that a generic topology $\mathcal{J}$ on a vector space V is compatible with a generic norm $\|-\|$ in the sense that $\mathcal{J}$ equals the topology associated to $\|-\|$.

<u>Example L18-1</u> (i) $(\mathbb{R}, |-|)$ is a normed space over $\mathbb{R}$,
(ii) $(\mathbb{C}, |-|)$ is a normed space over $\mathbb{C}$.

<u>Exercise L18-2</u> Let X be compact Hausdorff. Prove that $\|-\|_\infty$ defined by

$$\| f \|_\infty = \sup\{ |f(x)| \mid x \in X\}$$

defines a normed space $(\mathrm{Cts}(X, \mathbb{R}), \|-\|_\infty)$ with associated metric d_∞. As a special case we obtain normed spaces

$$(\mathbb{R}^n, \|-\|_\infty), \qquad \|(a_1, \dots, a_n)\|_\infty = \sup\{ |a_i| \}_{i=1}^n .$$

We will soon need the complex analogue of the function space $\mathrm{Cts}(X, \mathbb{R})$, the properties of which are developed in the next Exercise.

<u>Exercise L18-3</u> For any space X, prove that $\mathrm{Cts}(X, \mathbb{C})$ is a commutative $\mathbb{C}$-algebra with the pointwise operations. That is:

(i) Prove $\mathrm{Cts}(X, \mathbb{C})$ is a $\mathbb{C}$-vector space with the pointwise operations.
(ii) Prove $\mathrm{Cts}(X, \mathbb{C})$ is a $\mathbb{C}$-algebra (see Ex. L16-2) with the pointwise operations.

Exercise L18-4 If X is locally compact Hausdorff prove that $Cts(X,\mathbb{C})$ is a topological $\mathbb{C}$-algebra with the compact-open topology (this is similar to Ex. L17-4).

Def^n Let $(X, \int_X)$ be an integral pair. The complexified integral $\int_X^{\mathbb{C}}$ is the composite

$$\begin{array}{ccccccc} Cts(X,\mathbb{C}) & \overset{\text{Ex. L12-14}}{\cong} & Cts(X,\mathbb{R}) \times Cts(X,\mathbb{R}) & \xrightarrow{\int_X \times \int_X} & \mathbb{R}\times\mathbb{R} & \xrightarrow{\cong} & \mathbb{C} \\ f & \longmapsto & (f_R, f_I) & \longmapsto & (\int_X f_R, \int_X f_I) & \longmapsto & \int_X f_R + i \int_X f_I. \end{array}$$

which is continuous and linear. Writing $f(x) = f_R(x) + i f_I(x)$ for real-valued f_R, f_I we have the explicit formula

$$\int_X^{\mathbb{C}} f = \int_X f_R + i \int_X f_I.$$

Exercise L18-5 Let X be compact Hausdorff. Prove that $\|-\|_\infty$ defined by

$$\| f \|_\infty = \sup\{ |f(x)| \mid x \in X\}$$

defines a normed space $(Cts(X,\mathbb{C}), \|-\|_\infty)$ over $\mathbb{C}$ with associated metric d_∞ and associated topology the compact-open topology. As a special case we obtain normed spaces

$$(\mathbb{C}^n, \|-\|_\infty), \qquad \|(z_1,\dots,z_n)\|_\infty = \sup\{ |z_i| \}_{i=1}^n.$$

Theorem L18-1 With either $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, $(X, \int_X)$ an integral pair, and $1 \le p < \infty$ a real number, the pair $(Cts(X,\mathbb{F}), \|-\|_p)$ is a normed space over $\mathbb{F}$ where

$$\| f \|_p := \Big\{ \int_X |f|^p \Big\}^{1/p}.$$

Moreover if $f, g \in Cts(X, \mathbb{F})$ and $1 < p, q < \infty$ with $1/p + 1/q = 1$ then

$$\| fg \|_1 \leq \| f \|_p \| g \|_q . \qquad \text{(Hölder's inequality)}$$

Proof The main part of the argument is to prove for $1 \leq p < \infty$ and $a, b > 0$

$$\inf_{t > 0} \left[\tfrac{1}{p} t^{1/p - 1} a + \left(1 - \tfrac{1}{p}\right) t^{1/p} b \right] = a^{1/p} b^{1 - 1/p} \qquad (6.1)$$

$$\inf_{0 < t < 1} \left[t^{1-p} a^p + (1-t)^{1-p} b^p \right] = (a+b)^p \qquad (6.2).$$

for which we follow L. Maligranda's paper "A simple proof of the Hölder and Minkowski inequality" 1995. We may assume $p > 1$. Let $f(t)$ be

$$f(t) = \tfrac{1}{p} t^{1/p - 1} a + \left(1 - \tfrac{1}{p}\right) t^{1/p} b \qquad t > 0$$

Then $f'(t)$ satisfies

$$f'(t) = \tfrac{1}{p}\left(\tfrac{1}{p} - 1\right) t^{1/p - 2} a + \left(1 - \tfrac{1}{p}\right) \tfrac{1}{p} t^{1/p - 1} b$$

$$= \tfrac{1}{p}\left(\tfrac{1}{p} - 1\right) t^{1/p - 2} (a - tb)$$

So f' is negative for $t < t_0 = a/b$, zero at $t_0 = a/b$ and positive for $t > t_0$. Hence f has its minimum at $t_0 = a/b$ and this is equal to

$$f(t_0) = \tfrac{1}{p} \left(\tfrac{a}{b}\right)^{1/p - 1} a + \left(1 - \tfrac{1}{p}\right) \left(\tfrac{a}{b}\right)^{1/p} b$$

$$= a^{1/p} b^{1 - 1/p}$$

which proves (6.1). For (6.2) we define g on $(0,1)$ by

$$g(t) = t^{1-p} a^p + (1-t)^{1-p} b^p.$$

We have

$$g'(t) = (1-p)t^{-p} a^p - (1-p)(1-t)^{-p} b^p$$

which vanishes only when $t = t_1 = a/a+b$. Since

$$\begin{aligned} g''(t_1) &= (1-p)(-p)t_1^{-p-1} a^p + (1-p)(-p)(1-t_1)^{-p-1} b^p \\ &= -[(1-p)p t_1^{-p-1} a^p + (1-p)p(1-t_1)^{-p-1} b^p] > 0 \end{aligned}$$

It follows that g has its local minimum at $t_1 = a/a+b$, which is equal to

$$\begin{aligned} g(t_1) = g(a/a+b) &= (a/a+b)^{1-p} a^p + \left(1-\tfrac{a}{a+b}\right)^{1-p} b^p \\ &= (a/a+b)^{1-p} a^p + (b/a+b)^{1-p} b^p = (a+b)^p \end{aligned}$$

The local minimum of g is equal to its global minimum because g is continuous and $\lim_{t\to 0^+} g(t) = \lim_{t\to 1^-} g(t) = +\infty$. This proves (6.2).

<u>$\|-\|_p$ is a norm</u> (N1) Clearly $\|f\|_p \geqslant 0$, and if $\|f\|_p = 0$ then $\int_X |f|^p = 0$ and hence by definition of an integral pair $f = 0$. We have by linearity of $\int_X$

$$\begin{aligned} \|\lambda f\|_p = \left\{\int_X |\lambda f|^p\right\}^{1/p} &= \left\{\int_X |\lambda|^p |f|^p\right\}^{1/p} \\ &= \left\{|\lambda|^p \int_X |f|^p\right\}^{1/p} = |\lambda| \, \|f\|_p \end{aligned}$$

It remains to prove (N3), that is, $\|f+g\|_p \leq \|f\|_p + \|g\|_p$. By (6.2) for $0<t<1$

$$(a+b)^p \leqslant t^{1-p} a^p + (1-t)^{1-p} b^p$$

Hence for $0<t<1$

$$\|f+g\|_p^p = \int_X |f+g|^p$$

$$\text{Lemma L17-1 (i)} \quad \leqslant \int_X \left(|f|+|g|\right)^p$$

$$\text{(6.2)} \quad \leqslant \int_X \left[t^{1-p}|f|^p + (1-t)^{1-p}|g|^p \right]$$

$$= t^{1-p} \int_X |f|^p + (1-t)^{1-p} \int_X |g|^p$$

$$= t^{1-p} \|f\|_p^p + (1-t)^{1-p} \|g\|_p^p$$

Taking the infimum over $0<t<1$ and using (6.2) yields

$$\|f+g\|_p^p \leqslant \left(\|f\|_p + \|g\|_p \right)^p$$

which proves (N3). Finally, to prove the Hölder inequality note that by (6.1)

$$a^{1/p} b^{1-1/p} \leqslant \tfrac{1}{p} t^{1/p-1} a + \left(1-\tfrac{1}{p}\right) t^{1/p} b$$

for all $t>0$ and hence

$$\| \, |f|^{1/p} |g|^{1-1/p} \|_1 = \int_X |f|^{1/p} |g|^{1-1/p}$$

$$\leq \int_X \left[\tfrac{1}{p} t^{1/(p-1)} |f| + (1-\tfrac{1}{p}) t^{1/p} |g| \right]$$

$$= \tfrac{1}{p} t^{1/(p-1)} \int_X |f| + (1-\tfrac{1}{p}) t^{1/p} \int_X |g|$$

$$= \tfrac{1}{p} t^{1/(p-1)} \| f \|_1 + (1-\tfrac{1}{p}) t^{1/p} \| g \|_1$$

Taking the infimum over all $t > 0$ yields by (6.1)

$$\| \, |f|^{1/p} |g|^{1-1/p} \|_1 \leq \| f \|_1^{1/p} \| g \|_1^{1-1/p}$$

which is Hölder's inequality (replace f by $|f|^p$, g by $|g|^q$). □

Corollary L18-2 For any integral pair $(X, \int_X)$ the pair $(Ct(X, \mathbb{F}), d_p)$ is a metric space for $1 \leq p < \infty$ where

$$d_p(f, g) = \left\{ \int_X |f-g|^p \right\}^{1/p}.$$

Proof Immediate from Exercise L18-1 and Theorem L18-1. □

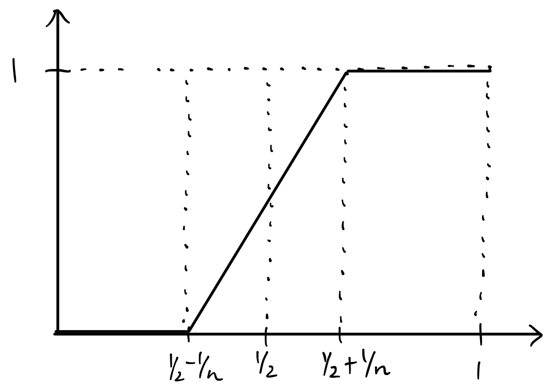
Exercise L18-6 If $[c,d] \subseteq [a,b]$ the restriction function

$$(Ct([a,b], \mathbb{F}), d_p) \longrightarrow (Ct([c,d], \mathbb{F}), d_p)$$

is continuous for all $1 \leq p \leq \infty$. (The $p=\infty$ case is Lemma L12-1).

As has already been mentioned, for $1 \leq p < \infty$ the spaces $(Ct(X, \mathbb{F}), d_p)$ need not be complete, as the following counter-example shows:

<u>Example L18-2</u> Consider $X = [0,1]$ and the sequence of functions $f_n : X \longrightarrow \mathbb{R}$ given for $n \geqslant 4$ by

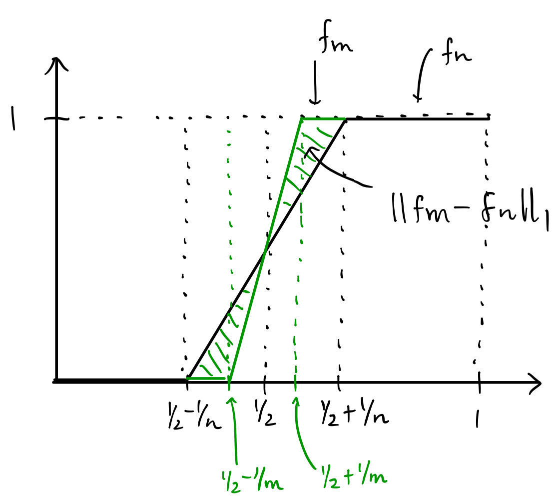


$$f_n(x) = \begin{cases} 0 & 0 \leq x \leq \frac{1}{2} - \frac{1}{n} \\ \frac{n}{2}\left(x - \frac{1}{2} + \frac{1}{n}\right) & \frac{1}{2} - \frac{1}{n} < x < \frac{1}{2} + \frac{1}{n} \\ 1 & \frac{1}{2} + \frac{1}{n} \leq x \leq 1 \end{cases}$$

The sequence $(f_n)_{n=4}^{\infty}$ converges <u>pointwise</u> to

$$f(x) = \begin{cases} 1 & x > \frac{1}{2} \\ \frac{1}{2} & x = \frac{1}{2} \\ 0 & x < \frac{1}{2} \end{cases}$$

but this convergence is certainly not uniform (as the uniform limit of continuous functions is continuous). So $(f_n)_{n=4}^{\infty}$ <u>does not converge</u> in $(Cts(X,\mathbb{R}), d_\infty)$ (if it converged, it would have to be to f) and hence is not Cauchy (as this space is complete). However we claim the sequence <u>is</u> Cauchy in $(Cts(X,\mathbb{R}), d_p)$ but still does not converge, where throughout $1 \leq p < \infty$.



Observe that for $m \geqslant n \geqslant 4$, $|f_m - f_n| \leq |f - f_n|$ so

$$d_p(f_m, f_n) = \| f_m - f_n \|_p = \left\{ \int_X | f_m - f_n |^p \right\}^{1/p}$$

$$\leq \left\{ \int_X | f - f_n |^p \right\}^{1/p}$$

Note f is Riemann integrable since it is piecewise continuous, see e.g. T. Tao "Analysis" Prop 11.5.6

$$= \left\{ 2 \int_{1/2}^{1/2 + 1/n} \left\{ 1 - \tfrac{n}{2}\left(x - \tfrac{1}{2} + \tfrac{1}{n}\right) \right\}^p dx \right\}^{1/p}$$

$$= 2^{1/p} \left\{ \int_{1/2}^{1/2 + 1/n} \left\{ \tfrac{1}{2} + \tfrac{n}{2}\left(\tfrac{1}{2} - x\right) \right\}^p dx \right\}^{1/p}$$

$$= 2^{1/p} \left\{ \frac{2}{n} \int_0^{1/2} u^p \, du \right\}^{1/p}$$

$$= 2^{2/p} \cdot n^{-1/p} \left\{ \int_0^{1/2} u^p \, du \right\}^{1/p}$$

which goes to zero as $n \to \infty$, proving that $(f_n)_{n=0}^{\infty}$ is Cauchy in $(Cts(X, \mathbb{R}), d_p)$. Next we show that the sequence $(f_n)_{n=0}^{\infty}$ cannot converge w.r.t. d_p which will show this space is not complete.

Suppose f_n converges in $(Cts(X, \mathbb{R}), d_p)$ to g (so g is continuous). Then for $[c,d] \subseteq (\tfrac{1}{2}, 1]$ we have by Ex. L18-6 that $f_n|_{[a,b]} \longrightarrow g|_{[a,b]}$ as $n \to \infty$ and this shows that $g(x) = 1$ for $x > \tfrac{1}{2}$. Similarly $g(x) = 0$ for $x < \tfrac{1}{2}$. So g cannot be continuous at $x = \tfrac{1}{2}$, and this contradiction shows f_n cannot converge.

<u>Exercise L18-7</u> Give a counterexample to show that convergence $f_n \longrightarrow f$ in $(Cts([a,b], \mathbb{R}), d_p)$ for $1 \leq p < \infty$ does <u>not</u> imply pointwise convergence. (Hint: ) (However $f_n \to f$ does imply pointwise convergence "almost everywhere" in a precise sense we will define later).

<u>Exercise L18-8</u> Let $(V, \|-\|)$ be a normed space. Prove

$$\|0\| = 0, \qquad \big|\, \|x\| - \|y\| \,\big| \leq \|x - y\| \qquad \forall x, y \in V.$$

The second identity is sometimes called the <u>reverse triangle inequality</u>.

<u>Lemma L18-3</u> Let $(V, \|-\|)$ be a normed space, and d the associated metric. Then $\|-\|: V \longrightarrow \mathbb{R}$ is uniformly continuous with respect to this metric.

<u>Proof</u> We have $\big|\, \|v\| - \|w\| \,\big| \leq \|v - w\| = d(v, w)$ so this is clear. $\square$

<u>Def^n</u> A <u>Banach space</u> is a normed vector space $(V, \|-\|)$ with the property that the associated metric space (V, d) is complete.

<u>Example L18-3</u> (i) For any compact Hausdorff space X the normed spaces

$$(\mathrm{Cts}(X, \mathbb{R}), \|-\|_\infty), \quad (\mathrm{Cts}(X, \mathbb{C}), \|-\|_\infty)$$

of Ex. L18-2, Ex. L18-5 are Banach spaces, by Corollary L13-6, where $\mathbb{F}$ is respectively $\mathbb{R}$ and $\mathbb{C}$. We denote these Banach spaces by

$$L^\infty(X, \mathbb{F}) := (\mathrm{Cts}(X, \mathbb{F}), \|-\|_\infty).$$

(ii) In particular $(\mathbb{R}^n, \|-\|_\infty), (\mathbb{C}^n, \|-\|_\infty)$ are Banach spaces.

However it is not necessarily the case (by Example L18-2) that $(\mathrm{Cts}(X, \mathbb{F}), \|-\|_p)$ is a Banach space for $1 \leq p < \infty$. But there is a canonical way to "convert" a normed space into a Banach space, called the <u>completion</u>, which we will now develop.

We have seen in Lecture 15 (see Remark L15-2) the fundamental role that limits of sequences of functions play in applied mathematics, and this explains our preference for Banach spaces over normed spaces. To define the completion of a normed space to a Banach space we first study the completion of a metric space. For that, we will need the following technical lemmas.

<u>Lemma L18-4</u> Let $(X, d_X), (Y, d_Y)$ be metric spaces with (Y, d_Y) complete, and let $A \subseteq X$ be a dense subset. If $f : A \longrightarrow Y$ is uniformly continuous then there is a unique continuous map $F : X \longrightarrow Y$ making

$$\begin{tikzcd} X \arrow[r, dashed, "F"] & Y \\ A \arrow[u, hook] \arrow[ur, "f"'] & \end{tikzcd}$$

commute. Moreover, this unique F is itself uniformly continuous.

<u>Proof</u> Uniqueness follows from Lemma L17-2 so we need only prove existence. Given $x \in X$ choose a sequence $(x_n)_{n=0}^{\infty}$ in A with $x_n \longrightarrow x$. This sequence is Cauchy in A and so $(f x_n)_{n=0}^{\infty}$ is Cauchy in Y (the image of a Cauchy sequence under a <u>uniformly</u> continuous map is Cauchy). Set $Fx := \lim_{n \to \infty} f x_n$.

If $(x_n')_{n=0}^{\infty}$ is another sequence converging to x, it will be equivalent as a Cauchy sequence to $(x_n)_{n=0}^{\infty}$ (that is, $\forall \varepsilon > 0 \; \exists N \, (n \geq N \Rightarrow d_X(x_n, x_n') < \varepsilon)$). We claim $(f x_n)_{n=0}^{\infty}$ is equivalent to $(f x_n')_{n=0}^{\infty}$. Given $\varepsilon > 0$ there is by uniform continuity a $\delta > 0$ such that $d_X(y, z) < \delta$ implies $d_Y(fy, fz) < \varepsilon$. Choose N such that for $n \geq N$ we have $d_X(x_n, x_n') < \delta$, then $d_Y(f x_n, f x_n') < \varepsilon$, proving the claim. Hence $\lim_{n \to \infty} f x_n = \lim_{n \to \infty} f x_n'$ and so Fx is well-defined. Clearly $F|_A = f$ since if $x \in A$ we may choose a constant sequence $x_n = a$.

It only remains to show F is uniformly continuous. Let $\varepsilon > 0$ be given. We have to produce $\delta > 0$ such that $d_X(x,y) < \delta$ implies $d_Y(\lim_{n\to\infty} fx_n, \lim_{n\to\infty} fy_n) < \varepsilon$ where $(x_n)_{n=0}^{\infty}$, $(y_n)_{n=0}^{\infty}$ are Cauchy sequences converging to x, y respectively.

But f is uniformly continuous: let $\delta > 0$ be such that $d_X(a,a') < \delta$ implies $d_Y(fa, fa') < \varepsilon/3$. Then if for $x, y \in X$ arbitrary we have $d_X(x,y) < \delta/3$ we may find N s.t. for $n \geqslant N$ both $d_X(x_n, x) < \delta/3$ and $d_X(y_n, y) < \delta/3$ so

$$
\begin{aligned}
d_X(x_n, y_n) &\leq d_X(x_n, x) + d_X(x,y) + d_X(y, y_n) \\
&< \delta/3 + \delta/3 + \delta/3 \\
&= \delta
\end{aligned}
$$

Hence $d_Y(fx_n, fy_n) < \varepsilon/3$. Now, possibly by increasing N, we can also arrange for $d_Y(fx_n, Fx) < \varepsilon/3$ and $d_Y(fy_n, Fy) < \varepsilon/3$ for $n \geqslant N$ so finally, for any $n \geqslant N$

$$
\begin{aligned}
d_Y(Fx, Fy) &\leq d_Y(Fx, fx_n) + d_Y(fx_n, fy_n) + d_Y(fy_n, Fy) \\
&< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 \\
&= \varepsilon.
\end{aligned}
$$

Overall this shows that if $d_X(x,y) < \delta/3$ then $d_Y(Fx, Fy) < \varepsilon$, as required. $\square$

<u>Example L18-4</u> The subset $A = (0,1]$ is dense in $[0,1]$ but $\sin(\frac{1}{x}) : A \longrightarrow \mathbb{R}$ cannot be extended to a continuous function $[0,1] \longrightarrow \mathbb{R}$, so the Lemma does not necessarily hold if f is not uniformly continuous.

<u>Exercise L18-9</u> If $A \subseteq X$, $B \subseteq Y$ are dense subsets, then $A \times B$ is dense in $X \times Y$.

<u>Exercise L18-10</u> If $(V, \|-\|)$ is a normed vector space over $\mathbb{F}$ then with the topology associated to $\|-\|$, V is a topological $\mathbb{F}$-vector space.

<u>Lemma L18-5</u> In the context of Lemma L18-4 suppose X, Y are topological $\mathbb{F}$-vector spaces (in their metric topologies) and that A is a vector subspace of X. Then if $f : A \longrightarrow Y$ is linear, so is the extension F.

<u>Proof</u> To say that F is linear is to say that the diagrams

$$\begin{tikzcd} X \times X \arrow[r, "+"] \arrow[d, "F \times F"'] & X \arrow[d, "F"] \\ Y \times Y \arrow[r, "+"] & Y \end{tikzcd} \qquad \begin{tikzcd} \mathbb{R} \times X \arrow[r, "\cdot"] \arrow[d, "1 \times F"'] & X \arrow[d, "F"] \\ \mathbb{R} \times Y \arrow[r, "\cdot"] & Y \end{tikzcd}$$

commute. But $F \circ +,\ + \circ (F \times F) : X \times X \longrightarrow Y$ are continuous maps which agree on a dense subspace $A \times A \subseteq X \times X$ (since $F|_A = f$ is linear) so they are equal by Lemma L17-2. The argument for the other diagram is similar (here we have used Ex. L17-4). $\square$

<u>Here in outline is the strategy</u>: for $1 \le p < \infty$, and an integral pair $(X, \int_X)$

- From the normed space $(Cts(X, \mathbb{F}), \|-\|_p)$ take the metric space $(Cts(X, \mathbb{F}), d_p)$.

- Define $L^p(X, \mathbb{F})$ as a <u>metric space</u> to be the completion of $(Cts(X, \mathbb{F}), d_p)$

- Lift the norm $\|-\|_p$ on $Cts(X, \mathbb{F})$ to $L^p(X, \mathbb{F})$ so as to make this a <u>Banach space</u>.

<u>Lemma L18-6</u> Let (X, d_X) be a metric space. The relation $(f_n)_{n=0}^{\infty} \sim (g_n)_{n=0}^{\infty}$ iff. $(d(f_n, g_n))_{n=0}^{\infty}$ converges to zero is an equivalence relation on the set of Cauchy sequences in X.

<u>Proof</u> First we prove that $\sim$ is an equivalence relation on the set of Cauchy sequences (this was addressed in a slightly different setting in Tutorial 5 Q3). The relation is clearly reflexive and symmetric, and if $(x_n)_{n=0}^{\infty} \sim (y_n)_{n=0}^{\infty}$, $(y_n)_{n=0}^{\infty} \sim (z_n)_{n=0}^{\infty}$ then $d(x_n, y_n) \to 0$, $d(y_n, z_n) \to 0$ and $d(x_n, z_n) \leq d(x_n, y_n) + d(y_n, z_n)$ so $d(x_n, z_n) \longrightarrow 0$ and hence $(x_n)_{n=0}^{\infty} \sim (z_n)_{n=0}^{\infty}$. So $\sim$ is an equivalence relation. $\square$

<u>Exercise L18-11</u> If $f : (X, d_X) \longrightarrow (Y, d_Y)$ is uniformly continuous then (a) if $(x_n)_{n=0}^{\infty}$ is Cauchy in X then $(fx_n)_{n=0}^{\infty}$ is Cauchy in Y (b) if $(x_n)_{n=0}^{\infty}, (x_n')_{n=0}^{\infty}$ are Cauchy sequences in X then $(x_n)_{n=0}^{\infty} \sim (x_n')_{n=0}^{\infty}$ implies $(fx_n)_{n=0}^{\infty} \sim (fx_n')_{n=0}^{\infty}$ in Y.

<u>Theorem L18-7</u> Let (X, d) be a metric space, and let $\hat{X}$ denote the set of Cauchy sequences modulo the equivalence relation $\sim$ defined above. Then $(\hat{X}, \hat{d})$ is a metric space with metric

$$\hat{d}\big((f_n)_{n=0}^{\infty}, (g_n)_{n=0}^{\infty} \big) := \lim_{n \to \infty} d(f_n, g_n).$$

This metric space is complete, the canonical map $(X, d) \longrightarrow (\hat{X}, \hat{d})$ sending x to $(x)_{n=0}^{\infty}$ is injective, distance preserving and has dense image. We call $(\hat{X}, \hat{d})$ the <u>completion</u> of X.

<u>Proof</u> • <u>$(d(f_n, g_n))_{n=0}^{\infty}$ is Cauchy</u> $((f_n, g_n))_{n=0}^{\infty}$ is Cauchy in $X \times X$ and $d : X \times X \longrightarrow \mathbb{R}$ is uniformly continuous (Ex. L13-3) so this follows from the fact that the uniformly continuous image of a Cauchy sequence is Cauchy (Ex. L18-11). Hence the limit $\hat{d}((f_n)_{n=0}^{\infty}, (g_n)_{n=0}^{\infty}) \in \mathbb{R}$ exists.

- $\underline{\hat{d} \text{ is well-defined}}$: suppose $(f_n)_{n=0}^{\infty} \sim (f_n')_{n=0}^{\infty}$, $(g_n)_{n=0}^{\infty} \sim (g_n')_{n=0}^{\infty}$. Then

$$\big((f_n, g_n) \big)_{n=0}^{\infty} \sim \big((f_n', g_n') \big)_{n=0}^{\infty} \quad \text{in } X \times X.$$

Hence by Ex L18-11,

$$\big(d(f_n, g_n) \big)_{n=0}^{\infty} \sim \big(d(f_n', g_n') \big)_{n=0}^{\infty}$$

which implies $\lim_{n\to\infty} d(f_n, g_n) = \lim_{n\to\infty} d(f_n', g_n')$, so $\hat{d}$ is well-defined.

- $\underline{\hat{d} \text{ is a metric}}$: (M1) The set $[0,\infty)$ contains $d(f_n, g_n)$ for all $n \in \mathbb{N}$, and hence also the limit since this set is closed. Hence $\hat{d}\big((f_n)_{n=0}^{\infty}, (g_n)_{n=0}^{\infty} \big) \geqslant 0$.

  (M2) says $\hat{d}\big((f_n)_{n=0}^{\infty}, (g_n)_{n=0}^{\infty} \big) = 0 \iff (f_n)_{n=0}^{\infty} \sim (g_n)_{n=0}^{\infty}$ which is true by definition of $\sim$.

  (M3) says $\hat{d}$ is symmetric, which is clear.

  (M4) Let Cauchy sequences $(f_n)_{n=0}^{\infty}, (g_n)_{n=0}^{\infty}, (h_n)_{n=0}^{\infty}$ be given. Then each element of the sequence

$$\big(d(f_n, h_n) - d(f_n, g_n) - d(g_n, h_n) \big)_{n=0}^{\infty}$$

lies in $(-\infty, 0]$ by (M4) for X. Hence the limit also lies in $(-\infty, 0]$. But $\mathbb{R}$ is a topological group, so we can exchange the order of this limit and the arithmetic operations to find

$$\lim_{n\to\infty} d(f_n, h_n) - \lim_{n\to\infty} d(f_n, g_n) - \lim_{n\to\infty} d(g_n, h_n) \leq 0,$$

which proves (M4) for $\hat{X}$.

- <u>X is a dense subspace of $\hat{X}$</u> Define $\iota : X \longrightarrow \hat{X}$ by $\iota(x) = (x)_{n=0}^{\infty}$ (i.e. the constant sequence). This is clearly distance preserving hence injective. Let $F = (f_n)_{n=0}^{\infty}$ be a Cauchy sequence in X. Then $(\iota(f_n))_{n=0}^{\infty}$ is a sequence <u>in $\hat{X}$</u> and

$$\hat{d}(\iota(f_m), F) = \lim_{n\to\infty} d(f_m, f_n). \qquad (18.1)$$

Given $\varepsilon > 0$ let N be such that for $m,n \geqslant N$ we have $d(f_n, f_m) < \varepsilon$. Then for $m \geqslant N$ we have $\hat{d}(\iota(f_m), (f_n)_{n=0}^{\infty}) \leqslant \varepsilon$ which suffices to show that $\iota(f_n) \to F$ in $\hat{X}$, so $\iota(X) \subseteq \hat{X}$ is dense.

- <u>$\hat{X}$ is complete</u> Let $(F_n)_{n=0}^{\infty}$ be a Cauchy sequence in $\hat{X}$, say $F_n = (x_a^{(n)})_{a=0}^{\infty}$. Since the image of $\iota : X \longrightarrow \hat{X}$ is dense, we may find for each $n > 0$ an element $y_n \in X$ with $\hat{d}(\iota(y_n), F_n) < 1/n$. We claim that $F = (y_n)_{n=0}^{\infty}$ is Cauchy and that $F_n \longrightarrow F$ with respect to $\hat{d}$.

To see that F is Cauchy let $\varepsilon > 0$ be given and use that $(F_n)_{n=0}^{\infty}$ is Cauchy to find N such that for $m,n \geqslant N$ we have $\hat{d}(F_m, F_n) < \varepsilon/3$. By increasing N we may also assume $N > 3/\varepsilon$. Since $\hat{d}$ has already been shown to be a metric, we may use the triangle inequality to calculate for $m,n \geqslant N$ (and hence both $1/n < \varepsilon/3$ and $1/m < \varepsilon/3$)

$$\begin{aligned}
\hat{d}(\iota(y_n), \iota(y_m)) &\leq \hat{d}(\iota(y_n), F_n) + \hat{d}(F_n, \iota(y_m)) \\
&\leq \hat{d}(\iota(y_n), F_n) + \hat{d}(F_n, F_m) + \hat{d}(F_m, \iota(y_m)) \\
&< 1/n + \varepsilon/3 + 1/m \\
&< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.
\end{aligned}$$

Since $X \longrightarrow \hat{X}$ is distance preserving this shows $F = (y_n)_{n=0}^{\infty}$ is Cauchy in X, so that F is a valid element of $\hat{X}$. It remains to show $F_n \longrightarrow F$.

By our proof that X is dense we know that $(\iota(y_n))_{n=0}^{\infty}$ converges, as a sequence in $\hat{X}$, to F. Given $\varepsilon > 0$ we may therefore find N such that for all $n \geq N$, $\hat{d}(\iota(y_n), F) < \varepsilon/2$. We may also assume $N > 2/\varepsilon$. Then for $n \geq N$ (so $1/n < \varepsilon/2$)

$$\begin{aligned}\hat{d}(F_n, F) &\leq \hat{d}(F_n, \iota(y_n)) + \hat{d}(\iota(y_n), F)\\ &< 1/n + \varepsilon/2\\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon.\end{aligned}$$

This completes the proof that $\hat{X}$ is complete. $\square$

<u>Example L18-5</u> $(\mathbb{Q}, |-|)$ is a metric space whose completion is isometric to $(\mathbb{R}, |-|)$ (of course it may be that in your personal mathematical universe this is an <u>equality</u>, but for a "Dedekind cut" guy/gal it is only an isometry).

<u>Lemma L18-8</u> If (Y, d_Y) is a complete metric space and $f : (X, d) \longrightarrow (Y, d_Y)$ is uniformly continuous there is a unique continuous map F making

$$\begin{tikzcd}(\hat{X}, \hat{d}) \arrow[r, dashed, "F"] & (Y, d_Y)\\ (X, d) \arrow[u, hook] \arrow[ur, "f"'] & \end{tikzcd}$$

commute, and F is uniformly continuous.

<u>Proof</u> Immediate from Lemma L18-4 and Theorem L18-7. $\square$

<u>Exercise L18-12</u> Given metric spaces $(X, d_X), (Y, d_Y)$ prove that

$$(X \times Y)^\wedge \longrightarrow X^\wedge \times Y^\wedge$$
$$\big((x_n, y_n) \big)_{n=0}^\infty \longmapsto \big((x_n)_{n=0}^\infty, (y_n)_{n=0}^\infty \big)$$

is a distance preserving bijection.

Let $(V, \|-\|_V)$ be a normed space over $\mathbb{F}$, (V, d_V) the associated metric space and $(\hat{V}, \hat{d}_V)$ its completion. We define vector space operations $+ : \hat{V} \times \hat{V} \longrightarrow \hat{V}$ and $a : \mathbb{F} \times \hat{V} \longrightarrow \hat{V}$ by

$$(v_n)_{n=0}^\infty + (w_n)_{n=0}^\infty = (v_n + w_n)_{n=0}^\infty, \qquad \lambda \cdot (v_n)_{n=0}^\infty = (\lambda v_n)_{n=0}^\infty .$$

It is easily checked these operations are well-defined on equivalence classes and make $\hat{V}$ into an $\mathbb{F}$-vector space, with zero vector $(0)_{n=0}^\infty$. Note that given Cauchy sequences $(v_n)_{n=0}^\infty, (v_n')_{n=0}^\infty$ in V

$$(v_n)_{n=0}^\infty \sim (v_n')_{n=0}^\infty \iff d(v_n, v_n') \longrightarrow 0 \iff \| v_n - v_n' \|_V \longrightarrow 0 \qquad (18.1)$$

Since (V, d_V) is dense in $(\hat{V}, \hat{d}_V)$ and $\|-\|_V : V \longrightarrow \mathbb{F}$ is uniformly continuous (Lemma L18-3) there is a unique (uniformly) continuous map $\|-\|_{\hat{V}} : \hat{V} \longrightarrow \mathbb{F}$ making the diagram

$$\begin{tikzcd} \hat{V} \arrow[r, dashed, "\|-\|_{\hat{V}}"] & \mathbb{F} \\ V \arrow[u, hook, "\iota"] \arrow[ur, "\|-\|_V"'] & \end{tikzcd}$$

⌜note $\mathbb{R}, \mathbb{C}$ are both complete⌟

commute. Moreover given $(v_n)_{n=0}^\infty \in \hat{V}$, we have ⌜using the "X is dense" part of the proof of Theorem L18-7 to see $(v_n)_{n=0}^\infty = \lim_{n\to\infty} v_n$⌟

$$\| (v_n)_{n=0}^\infty \|_{\hat{V}} = \| \lim_{n\to\infty} v_n \|_{\hat{V}} = \lim_{n\to\infty} \| v_n \|_V$$

really $\iota(v_n)$ where $\iota : V \to \hat{V}$ is canonical.

<u>Def^n</u> Given normed spaces $(V, \|-\|_V), (W, \|-\|_W)$ we say $f: V \to W$ is <u>norm preserving</u> if $\|f(v)\|_W = \|v\|_V$ for all $v \in V$.

<u>Exercise L18-13</u> If $(V, \|-\|_V), (W, \|-\|_W)$ are normed spaces and $f: V \to W$ is linear, then f is continuous (w.r.t. the associated metrics) iff. it is uniformly continuous.

<u>Theorem L18-9</u> Let $(V, \|-\|_V)$ be a normed space over $\mathbb{F}$. Then $(\hat{V}, \|-\|_{\hat{V}})$ as defined above is a Banach space. The inclusion $(V, \|-\|_V) \xrightarrow{\iota} (\hat{V}, \|-\|_{\hat{V}})$ is norm-preserving and linear, and the image is dense in $\hat{V}$. If $(W, \|-\|_W)$ is a Banach space and $f: V \to W$ is continuous and linear, there is a unique continuous map F making

$$\begin{tikzcd} \hat{V} \arrow[r, dashed, "F"] & W \\ V \arrow[u, hook, "\iota"] \arrow[ur, "f"'] & \end{tikzcd}$$

⌜The existence of this unique lifting F is called the <u>universal property</u> of the <u>completion</u>,

commute, and this F is linear. We call $(\hat{V}, \|-\|_{\hat{V}})$ the <u>completion</u> of $(V, \|-\|_V)$.

<u>Proof</u> So far we have a vector space $\hat{V}$ and a function $\|-\|_{\hat{V}} \to \mathbb{F}$, which is uniformly continuous with respect to $\hat{d}$. For (N1) we have, since $[0, \infty)$ is closed

$$\|(v_n)_{n=0}^\infty\|_{\hat{V}} = \lim_{n\to\infty} \|v_n\|_V \geq 0.$$

If $\|(v_n)_{n=0}^\infty\|_{\hat{V}} = 0$ then $(v_n)_{n=0}^\infty \sim (0)_{n=0}^\infty$ by (18.1). For (N2),

$$\begin{aligned} \|\lambda (v_n)_{n=0}^\infty\|_{\hat{V}} &= \|(\lambda v_n)_{n=0}^\infty\|_{\hat{V}} = \lim_{n\to\infty} \|\lambda v_n\|_V \\ &= |\lambda| \lim_{n\to\infty} \|v_n\|_V = |\lambda| \, \|(v_n)_{n=0}^\infty\|_{\hat{V}} \end{aligned}$$

For (N3) we compute

$$\|(v_n)_{n=0}^{\infty}+(w_n)_{n=0}^{\infty}\|_{\hat{V}}-\|(v_n)_{n=0}^{\infty}\|_{\hat{V}}-\|(w_n)_{n=0}^{\infty}\|_{\hat{V}}$$

$$=\lim_{n\to\infty}\|v_n+w_n\|_V-\lim_{n\to\infty}\|v_n\|_V-\lim_{n\to\infty}\|w_n\|_V$$

$$=\lim_{n\to\infty}\left(\|v_n+w_n\|_V-\|v_n\|_V-\|w_n\|_V\right)\leqslant 0.$$

So $(\hat{V},\|-\|_{\hat{V}})$ is a normed space. The metric $\tilde{d}$ associated to $\|-\|_{\hat{V}}$ is

$$\begin{aligned}\tilde{d}((v_n)_{n=0}^{\infty},(w_n)_{n=0}^{\infty}) &= \|(v_n)_{n=0}^{\infty}-(w_n)_{n=0}^{\infty}\|_{\hat{V}}\\ &= \lim_{n\to\infty}\|v_n-w_n\|_V\\ &= \lim_{n\to\infty} d_V(v_n,w_n)\\ &= d_{\hat{V}}((v_n)_{n=0}^{\infty},(w_n)_{n=0}^{\infty}).\end{aligned}$$

That is, $\tilde{d}=d_{\hat{V}}$. Since by construction $(\hat{V},d_{\hat{V}})$ is complete, $(\hat{V},\|-\|_{\hat{V}})$ is a Banach space. The inclusion ι is obviously linear and norm preserving, and has dense image by Theorem L18-7. If $(W,\|-\|_W)$ is Banach and $f:V\longrightarrow W$ is continuous and linear. By Ex. L18-13 it is uniformly continuous, so we may apply Lemma L18-4, L18-5 and Ex. L18-10 to obtain the desired extension F. $\square$

<u>Def^n</u> (L^p-spaces) Let $(X,\int_X)$ be an integral pair and $\mathbb{F}$ a field of scalars. For $1\leq p<\infty$ a real number we define the Banach space $L^p(X,\int_X,\mathbb{F})$ (or just $L^p(X,\mathbb{F})$ if the chosen integral is clear) to be the completion of the normed space $(Cts(X,\mathbb{F}),\|-\|_p)$. We also write $\|-\|_p$ for the norm on $L^p(X,\mathbb{F})$, called the <u>p-norm</u>. Note $Cts(X,\mathbb{F})$ is canonically a vector subspace of $L^p(X,\mathbb{F})$ and $\overline{Cts(X,\mathbb{F})}=L^p(X,\mathbb{F})$.

Remark $L^\infty(X,\mathbb{F})$ (no dependence on an integral) was defined in Example L18-3.

To summarise: the vectors of $L^p(X,\mathbb{F})$ are Cauchy sequences $(f_n)_{n=0}^\infty$ of continuous functions $X \longrightarrow \mathbb{F}$, where the "Cauchy-ness" is tested with respect to the d_p-metric, and we identify Cauchy sequences $(f_n)_{n=0}^\infty$, $(g_n)_{n=0}^\infty$ in $L^p(X,\mathbb{F})$ iff.

$$\int_X |f_n - g_n|^p \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.$$

The vector space operations are

$$(f_n)_{n=0}^\infty + (g_n)_{n=0}^\infty = (f_n + g_n)_{n=0}^\infty, \qquad \lambda \cdot (f_n)_{n=0}^\infty = (\lambda f_n)_{n=0}^\infty.$$

We identify $\mathrm{Cts}(X,\mathbb{F})$ with the subset of constant sequences, and view the rest of $L^p(X,\mathbb{F})$ as limits of sequences in this subset. If you are squeamish about manipulating equivalence classes of Cauchy sequences, you should probably get over it: it's not much worse than manipulating real numbers. We will see more elementary ways of thinking about vectors in $L^p(X,\mathbb{F})$ later.

$$\begin{tikzcd}
 & \underline{\text{Banach spaces}} & \underline{\text{metric spaces}} \\
 & (L^1(X,\mathbb{F}), \|-\|_1) \arrow[r, squiggly] & (L^1(X,\mathbb{F}), d_1) \\
(X, \int_X) \arrow[ur] \arrow[r] \arrow[dr] \arrow[ddr] & (L^2(X,\mathbb{F}), \|-\|_2) \arrow[r, squiggly] & (L^2(X,\mathbb{F}), d_2) \\
 & \vdots & \\
 & (L^p(X,\mathbb{F}), \|-\|_p) \arrow[r, squiggly] & (L^p(X,\mathbb{F}), d_p) \\
 & \vdots & \\
 & (L^\infty(X,\mathbb{F}), \|-\|_\infty) \arrow[r, squiggly] & (\mathrm{Cts}(X,\mathbb{F}), d_\infty) \\
 & \| & \\
 & (\mathrm{Cts}(X,\mathbb{F}), \|-\|_\infty) &
\end{tikzcd}$$

These are all new, except in the case of X finite where they all collapse to $\mathbb{F}^{|X|}$.

This we know

<u>Exercise L18-14</u> In the context of Theorem L18-7, prove that the canonical map $X \longrightarrow \hat{X}$ is an embedding (i.e. a homeomorphism onto its image, where the image has the subspace topology).

<u>Exercise L18-15</u>* Let $1 \le p < q \le \infty$. Show that for any integral pair $(X, \int_X)$ and $f \in Cts(X, \mathbb{F})$ that (Hint: Hölder inequality with $g = 1$)

$$\| f \|_p \le \| f \|_q \cdot V^{\frac{1}{p} - \frac{1}{q}}$$

⌜for $q = \infty$ read the RHS as $\|f\|_\infty V^{1/p}$⌟

where $V = \int_X 1$. Hence show that the identity function

$$\big(Cts(X, \mathbb{F}), d_q \big) \longrightarrow \big(Cts(X, \mathbb{F}), d_p \big)$$

is continuous. By Theorem L18-9 there is a unique continuous linear map F making the diagram

$$\begin{tikzcd} L^q(X, \mathbb{F}) \arrow[r, dashed, "F"] & L^p(X, \mathbb{F}) \\ Cts(X, \mathbb{F}) \arrow[u, "\text{can}"] \arrow[r, "id"'] & Cts(X, \mathbb{F}) \arrow[u, "\text{can}"'] \end{tikzcd}$$

d_q-completion $\qquad\qquad$ d_p-completion

commute, where the vertical maps are the canonical inclusions into the completion. Prove that this F is injective, and give a counter-example to show that in general F is <u>not</u> an embedding of topological spaces (careful: if X is finite it is a homeomorphism).

⌜Sometimes you'll see people write $L^q \subseteq L^p$, but be careful!⌟

Solutions to selected exercises

$\boxed{\text{L18-9}}$ Suppose $Q \supseteq A \times B$ is closed and $Q \neq X \times Y$. Let $(x,y) \in X \times Y \setminus Q$. Since this is an open set we may find $U \subseteq X, V \subseteq Y$ open with

$$(x,y) \in U \times V \subseteq Q^c.$$

Now since A is dense in X, $U \cap A \neq \phi$ say $a \in U \cap A$. Then since $\{a\} \times V \subseteq Q^c$ we must have $V \subseteq B^c$ which is a contradiction. Hence $Q = X \times Y$.

$\boxed{\text{L18-10}}$ We prove first that $+: V \times V \longrightarrow V$ is continuous at (v_0, w_0), by calculating

$$\begin{aligned} \| +(v_1,w_1) - +(v_0,w_0) \| &= \| v_1 - v_0 + w_1 - w_0 \| \\ &\leq \| v_1 - v_0 \| + \| w_1 - w_0 \| \end{aligned}$$

with respect to the product metric on $V \times V$ this is

$$= d_{V \times V}\big((v_1,w_1), (v_0,w_0) \big)$$

so continuity is clear. Similarly for scalar multiplication

$$\begin{aligned} \| \lambda_1 v_1 - \lambda_0 v_0 \| &= \| \lambda_1 v_1 - \lambda_0 v_1 + \lambda_0 v_1 - \lambda_0 v_0 \| \\ &= \| (\lambda_1 - \lambda_0) v_1 + \lambda_0 (v_1 - v_0) \| \\ &\leq \| (\lambda_1 - \lambda_0) v_1 \| + \| \lambda_0 (v_1 - v_0) \| \\ &= |\lambda_1 - \lambda_0| \, \| v_1 \| + |\lambda_0| \, \| v_1 - v_0 \| \\ &= |\lambda_1 - \lambda_0| \, \| v_1 - v_0 + v_0 \| + |\lambda_0| \, \| v_1 - v_0 \| \\ &\leq |\lambda_1 - \lambda_0| \, \| v_1 - v_0 \| + |\lambda_1 - \lambda_0| \, \| v_0 \| \\ &\quad + |\lambda_0| \, \| v_1 - v_0 \| \end{aligned}$$

Given $\varepsilon > 0$ choose

$$\delta \leqslant \min\left\{ \sqrt{\frac{\varepsilon}{3}}, \frac{\varepsilon}{3\|v_0\|}, \frac{\varepsilon}{3|\lambda_0|} \right\}$$

Then if $d_{\mathbb{F}\times V}((\lambda_0, v_0), (\lambda_1, v_1)) < \delta$ we have

$$|\lambda_0 - \lambda_1| + \|v_0 - v_1\| < \delta$$

and hence $|\lambda_0 - \lambda_1| < \delta$, $\|v_0 - v_1\| < \delta$ and so

$$\begin{aligned}
\|\lambda_1 v_1 - \lambda_0 v_0\| &\leqslant |\lambda_1 - \lambda_0| \, \|v_1 - v_0\| + |\lambda_1 - \lambda_0| \, \|v_0\| \\
&\quad + |\lambda_0| \, \|v_1 - v_0\| \\
&< \delta^2 + \delta\|v_0\| + |\lambda_0|\delta \\
&\leqslant \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon
\end{aligned}$$

as required.