updated 17/10/19

Lecture 19 : Duality and Hilbert space

Last lecture we associated to any integral pair $(X, \int_X)$ and field of scalars $\mathbb{F}$ a family of Banach spaces $\{(L^p(X,\mathbb{F}), \|-\|_p)\}_{1\le p\le\infty}$. But it remains unclear how to think about vectors in these spaces (which are, by definition, limits of Cauchy sequences of continuous $\mathbb{F}$-valued functions on X), and to put it more bluntly:

<u>Question</u>: why care about the Banach spaces $(L^p(X,\mathbb{F}), \|-\|_p)$?

Since I am not an expert in analysis, let me insert first a blatant appeal to authority by quoting from E.M. Stein and R. Shakarchi's book "Functional analysis"

> "Functional analysis, as generally understood, brought with it a change of focus from the study of functions on everyday geometric spaces such as $\mathbb{R}$, $\mathbb{R}^d$, etc., to the analysis of abstract infinite-dimensional spaces, for example, function spaces and Banach spaces. As such it established a key framework for the development of modern analysis."

OK, fine, but <u>why Banach spaces</u>? The motivation sketched at the beginning of Lecture 17 was that in order to <u>compute</u> with infinite-dimensional function spaces we need integrals (to e.g. determine coefficients of a function on S^1 with respect to the "dense basis" of trigonometric polynomials). The way these integrals are "packaged" is via the structure of a <u>norm</u> (and in a moment, the inner product), but we ran into two issues in the previous lecture:

(1) Which norm should we use? We have $\|-\|_p$ for $1\le p\le\infty$.

(2) For $1\le p<\infty$ the norm $\|-\|_p$ is not complete.

Problem (2) is serious (because we often want to construct solutions, say of DEs, by taking limits) but easily fixed by passing to the completion. So that leaves Problem (1). Today's lecture resolves this problem by explaining why the $p=2$ case is of special importance. Since we have already learned why L^∞-spaces are important, the conceptual stage in the theory of infinite-dimensional function spaces is occupied by two main actors: the Banach spaces

$$\big(L^2(X, \textstyle\int_X, \mathbb{C}),\ \|-\|_2 \big), \quad \big(\mathrm{Cts}(X, \mathbb{C}),\ \|-\|_\infty \big)$$

(depends on an integral pair) (sup-norm, does not use integrals)

The special properties of L^2-space represent a remarkable extension of our intuitions about space (which are ultimately rooted, as sketched in Lectures 1 and 2, in our perceptions of the finite-dimensional space in which we are embedded as biological agents) to <u>infinite dimensional</u> spaces. To make this precise, we begin with some characterisations of finite-dimensionality for vector spaces over a field k:

<u>Lemma L19-1</u> For a vector space V over k, the following are equivalent:

(i) V is finite-dimensional.

(ii) The canonical linear map $V \longrightarrow V^{**}$ sending v to the function $ev_v : V^* \longrightarrow \mathbb{F}$ defined by $ev_v(f) = f(v)$ is an isomorphism.

<u>Proof</u> See Tutorial 7 for dual spaces, and Tutorial 10 for the theory of bases in infinite-dimensional spaces. For (i) $\Rightarrow$ (ii) recall that if $\mathcal{B} = (v_1, \ldots, v_n)$ is a basis for V then there is a dual basis $\mathcal{B}^* = (v_1^*, \ldots, v_n^*)$ for V^* and in particular $\dim(V^*) = \dim(V)$.

Let us compute the dual of the dual basis, i.e. the basis $\mathcal{B}^{**} = (v_1^{**}, \dots, v_n^{**})$ for V^{**}. Note that if $f : V \longrightarrow k$ is linear then as a vector in V^*

$$f = \sum_{i=1}^{n} f(v_i)\, v_i^*$$

as may be checked by evaluating both sides on $v \in V$ and using linearity. But then by definition

$$v_i^{**}(f) = f(v_i) \implies v_i^{**} = ev_{v_i}.$$

Hence the map $ev_{(-)} : V \longrightarrow V^{**}$ which is clearly well-defined and linear, sends $\mathcal{B}$ to $\mathcal{B}^{**}$ and is an isomorphism.

(ii) $\Rightarrow$ (i) Without any hypothesis on V we can define $ev_{(-)} : V \to V^{**}$ and this map is clearly linear. Let $\mathcal{B} = \{v_i\}_{i \in I}$ be a basis for V (recall that a basis, that is, a linearly independent spanning set, always exists, and no matter which one we choose the index set I always has the same cardinality — this cardinal is, by def$^{\text{n}}$, the <u>dimension</u> of V). The linear maps $v_i^* : V \longrightarrow k$ sending $v \in V$ to the coefficient of v_i in the unique expression of v as a linear combination of vectors in $\mathcal{B}$ is still well-defined and linear (the $\{v_i^*\}_{i \in I}$ just do not span V^* if $\dim(V) \geq \aleph_0$). This allows us to show $ev_{(-)} : V \to V^{**}$ is injective, since

$$ev_v = 0 \iff f(v) = 0 \text{ for all } f \in V^*$$

$$\Rightarrow v_i^*(v) = 0 \text{ for all } i \in I$$

$$\Rightarrow v = 0.$$

We now show that if I is infinite, $ev_{(-)} : V \longrightarrow V^{**}$ cannot be surjective. The set $\{v_i^*\}_{i\in I}$ is linearly independent in V^* (why?) and we claim that the linear map $\chi : V \longrightarrow k$, $\chi(v_i) = 1$ for $i \in I$ is such that $\{\chi\} \cup \{v_i^*\}_{i\in I}$ is linearly independent. It suffices to show that for a finite subset $\{i_1, \ldots, i_k\} \subseteq I$ that

$$\mu\chi + \textstyle\sum_{a=1}^{k} \lambda_a v_{i_a}^* = 0 \quad \text{implies } \mu = 0.$$

But we simply take $j \in I \setminus \{i_1, \ldots, i_k\}$ (which exists since I is infinite) and evaluate on v_j to see that $\mu = 0$, as required. Now, we may extend $\{\chi\} \cup \{v_i^*\}_{i\in I}$ to a basis $\mathcal{C}$ of V^*, and define

$$F : V^* \longrightarrow k$$

on this basis by $F(\chi) = 1$ and $F(u) = 0$ for every other vector $u \in \mathcal{C}$. We claim F is not in the image of $ev_{(-)} : V \longrightarrow V^{**}$. If it were, we would have for some $\{i_1, \ldots, i_k\} \subseteq I$ and $\lambda_a \in k$, an expression

$$F = \textstyle\sum_{a=1}^{k} \lambda_a \, ev_{v_{i_a}}$$

But then $0 = F(v_{i_a}^*) = \lambda_a$ for $1 \leq a \leq k$ would imply $F = 0$, but since $F(\chi) = 1 \neq 0$ this is not the case, so no such expression can exist. This completes the proof of (ii) $\Rightarrow$ (i). $\square$

<u>Remark</u> We will not prove it, but a result of Erdös–Kaplansky says that (i), (ii) above are further equivalent to the existence of an isomorphism of vector spaces $V \cong V^*$ (see Jacobson, "Lectures in Abstract Algebra", Vol. 2 Ch. 9 §5).

The upshot is that for infinite-dimensional vector spaces, such as $\mathrm{Cts}(X,\mathbb{F})$ for X infinite, the usual linear dual is not well-behaved. Note that a finite-dimensional vector space V over $\mathbb{R}$ may be equipped with a topology making it a topological vector space (Ex. L7-15) by choosing a basis and using the resulting bijection $V \cong \mathbb{F}^n$. Moreover this topology is Hausdorff, and not only is this topology independent of the choice of basis, but this is the only way of producing finite-dimensional Hausdorff topological vector spaces:

Exercise L19-1* Prove that if V is a finite-dimensional vector space over $\mathbb{F}$ there is a unique topology on V making it a Hausdorff topological vector space (cf. the Remark on p. (12) of L17).

To make the next point we need to introduce some notation:

Def^n Given a topological vector space V over $\mathbb{F}$, we define (as sets)

$$\mathrm{Cts}(V,\mathbb{F}) = \{ f : V \longrightarrow \mathbb{F} \mid f \text{ is continuous} \}$$

$$V^* := \mathrm{Lin}(V,\mathbb{F}) = \{ f : V \longrightarrow \mathbb{F} \mid f \text{ is linear} \}$$

$$V^\vee := \mathrm{CtsLin}(V,\mathbb{F}) = \{ f : V \longrightarrow \mathbb{F} \mid f \text{ is continuous and linear} \}.$$

Lemma L19-2 If V is a finite-dimensional Hausdorff topological vector space

$$\mathrm{CtsLin}(V,\mathbb{F}) = \mathrm{Lin}(V,\mathbb{F}).$$

Proof When $V \cong \mathbb{F}^n$ as topological vector spaces this is clear, and by Ex. L19-1 this is always true. $\square$

This will not actually be used, so there is no dependence of later material on Ex. L19-1.

However if V is a Hausdorff topological vector space over $\mathbb{F}$ which is <u>not</u> finite-dimensional (e.g. $\mathrm{Cts}(X,\mathbb{F})$ as soon as X is infinite, see Ex. L17-7) then there is a difference between these two notions of a dual

$$\underbrace{\mathrm{CtsLin}(V,\mathbb{F})}_{\text{continuous linear dual}} \neq \underbrace{\mathrm{Lin}(V,\mathbb{F})}_{\text{linear dual}}$$

It stands to reason that the continuous linear dual, which uses the <u>topology</u> on V and not just the <u>linear</u> structure, might be a better notion (note that the topology is, by Ex. 19-1, not additional information beyond the linear structure in the finite-dimensional case) and that the pathology for the plain linear dual indicated by Lemma L19-1 might be avoided by switching to the "correct" notion of dual in the infinite-dimensional case. Given that the first natural class of infinite-dimensional spaces we encounter are function spaces, we must ask:

<u>Question</u> What is the continuous linear dual of $V = \mathrm{Cts}(X,\mathbb{F})$?

The best that we could hope for is to completely recover Lemma L19-1 (ii), (iii), that is, isomorphisms of topological vector spaces (X compact Hausdorff)

$$(*) \qquad \begin{aligned} \mathrm{Cts}(X,\mathbb{F}) &\xrightarrow[\cong]{\text{canonical, via eval}} \mathrm{Cts}(X,\mathbb{F})^{\vee\vee} \\ \mathrm{Cts}(X,\mathbb{F}) &\xrightarrow[\cong]{\text{noncanonical}} \mathrm{Cts}(X,\mathbb{F})^{\vee}. \end{aligned}$$

WARNING: These are vague hopes not (yet) theorems

Of course we first have to decide <u>which topology</u> we are talking about on the function space $\mathrm{Cts}(X,\mathbb{F})$ (the linear structure is always the usual one). We know for any integral pair $(X, \int_X)$ and $1 \le p \le \infty$ a norm $\|-\|_p$ on this space which makes it a topological vector space (Ex L18-10).

We would also have to decide what $\mathrm{Cts}(X,\mathbb{F})^\vee$ is as a topological vector space (so far it is just a set). Once we have fixed all these details the question of whether or not isomorphisms as in ⊛ exist becomes a precise question, which we can attempt to answer. In order to fix these details we restrict to <u>normed</u> spaces, because there the continuous linear dual has additional good properties:

<u>Defn</u> Let $(V, \|-\|_V), (W, \|-\|_W)$ be normed spaces over $\mathbb{F}$, and $T: V \to W$ a linear map. We say T is <u>bounded</u> if there exists $M \in \mathbb{R}$ with $M \geq 0$ and

$$\| T(v) \|_W \leq M \| v \|_V \quad \text{for all } v \in V. \qquad (\dagger)$$

<u>Exercise L19-2</u> Prove that T is bounded if and only if $\{ \| T(v) \|_W \mid \|v\|_V = 1 \}$ is a bounded subset of $\mathbb{R}$.

<u>Defn</u> If T is bounded then we define

$$\begin{aligned} \| T \| &= \sup \left\{ \frac{\| T(v) \|_W}{\| v \|_V} \;\middle|\; v \neq 0 \right\} \\ &= \sup \{ \| T(v) \|_W \mid \|v\|_V = 1 \} \\ &= \sup \{ \| T(v) \|_W \mid \|v\|_V \leq 1 \}. \end{aligned}$$

This is called the <u>operator norm</u>. Hence $\|T\| \geq 0$ and

$$\| T(v) \|_W \leq \|T\| \cdot \|v\|_V \quad \text{for all } v \in V$$

and moreover $\|T\|$ is the infimum of all real numbers M for which the inequality $(\dagger)$ holds.

Clearly boundedness implies uniform continuity, but the converse is also true (the following result subsumes Ex. L18-13).

<u>Lemma L19-3</u> Let $(V, \|-\|_V)$, $(W, \|-\|_W)$ be normed spaces over $\mathbb{F}$ and $T: V \longrightarrow W$ a linear map. Then the following are equivalent

(i) T is bounded.
(ii) T is uniformly continuous.
(iii) T is continuous.
(iv) T is continuous at 0.

<u>Proof</u> Here continuity means with respect to the associated metrics. For (i) $\Rightarrow$ (ii) note that

$$\begin{aligned} d_W(Tv, Tv') = \| Tv - Tv' \|_W &= \| T(v-v') \|_W \\ \le M \|v - v'\|_V &= M d_V(v, v'), \end{aligned}$$

so T is uniformly continuous. (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (iv) are immediate (continuity at $v_0 \in V$ means $\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall v \in V \; (d_V(v, v_0) < \delta \Rightarrow d_W(Tv, Tv_0) < \varepsilon))$. For (iv) $\Rightarrow$ (i) suppose for $\varepsilon = 1$ that $\|v\|_V < \delta$ implies $\|Tv\|_W < 1$. For any $v \neq 0$,

$$\left\| \frac{\delta}{2\|v\|_V} v \right\|_V = \delta/2 < \delta$$

so $\left\| T\left(\frac{\delta}{2\|v\|_V} v \right) \right\|_W < 1$, or what is the same

$$\| T(v) \|_W < \frac{2}{\delta} \|v\|_V$$

so T is bounded. $\square$

The lemma means that if a topological vector space V arises from a normed space $(V, \|-\|_V)$ then the continuous linear maps $V \longrightarrow \mathbb{F}$ are precisely the <u>bounded</u> linear maps (here $\mathbb{F}$ is given the $|-|$ norm, which is the absolute value in the case of $\mathbb{R}$ and the modulus for $\mathbb{C}$).

<u>Exercise L19-3</u> Let $(V, \|-\|_V)$ be a normed space over $\mathbb{F}$. Prove that the pair $(\mathrm{CtsLin}(V, \mathbb{F}), \|-\|)$ consisting of continuous linear maps with the pointwise $\mathbb{F}$-vector space structure, and the operator norm $\|-\|$, is a Banach space.

<u>Def^n</u> Given a normed space $(V, \|-\|_V)$ we call the Banach space $(\mathrm{CtsLin}(V, \mathbb{F}), \|-\|)$ the <u>continuous linear dual</u> (or just continuous dual) of V, and denote it by $(V^\vee, \|-\|_{V^\vee})$.

<u>Exercise L19-4</u> Let $(V, \|-\|_V)$, $(W, \|-\|_W)$ be normed spaces and $T : V \longrightarrow W$ a bounded linear operator. Prove that

$$T^\vee : W^\vee \longrightarrow V^\vee$$
$$T^\vee(g) = g \circ T$$

is a bounded linear operator with $\|T^\vee\| \leq \|T\|$. Prove that $(-)^\vee$ is a functor, that is, $(\mathrm{id}_V)^\vee = \mathrm{id}_{V^\vee}$ and if $S : W \longrightarrow U$ is bounded and linear then $(S \circ T)^\vee = T^\vee \circ S^\vee$.

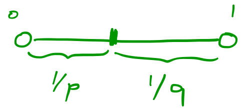
<u>Exercise L19-5</u> Prove $\delta_V : V \longrightarrow V^{\vee\vee}$, $\delta_V(\omega) = ev_\omega$ is continuous and linear, where $ev_\omega(f) = f(\omega)$ for $f : V \longrightarrow \mathbb{F}$ continuous and linear.
(This map is actually always norm-preserving and hence injective, but we will not use this).

Def^n An <u>isomorphism</u> of normed spaces $(V, \|-\|_V)$, $(W, \|-\|_W)$ is a vector space isomorphism $T: V \longrightarrow W$ such that $\|T(v)\|_W = \|v\|_V$ for all $v \in V$. In this case we write $V \cong W$ and say V, W are <u>isomorphic</u> as normed spaces.

<u>Exercise L19-6</u> (i) Let $\iota : (V, \|-\|_V) \longrightarrow (\hat{V}, \|-\|_{\hat{V}})$ be the completion of a normed space, and prove $\iota^\vee : (\hat{V})^\vee \longrightarrow V^\vee$ is an isomorphism.

(ii) Let $(V, \|-\|_V)$, $(W, \|-\|_W)$ be normed spaces and $T: V \to W$ an isomorphism of normed spaces. Then $T^\vee : W^\vee \longrightarrow V^\vee$ is also an isomorphism of normed spaces.

Def^n For a real number $1 < p < \infty$ the equation (note $1<p \iff 0 < \frac{1}{p} < 1$)



$$\frac{1}{p} + \frac{1}{q} = 1$$

has a unique solution $q = p/p-1$ in $(1, \infty)$, called the <u>dual exponent</u>. Note that p is then the dual exponent of q.

We claim there is a deep relation (a duality) between L^p and L^q spaces for dual exponents p, q. This is partly a consequence of Hölder's inequality (Theorem L18-1) which says $\|fg\|_1 \le \|f\|_p \|g\|_q$ for $f, g \in Cts(X, \mathbb{F})$, $(X, \int_X)$ any integral pair. To see how this connects L^p to L^q, recall that $Cts(X, \mathbb{R})$ (the complex case involves conjugation, so we leave it to later) is an algebra, so there is a function

$$\begin{array}{ccccc} Cts(X,\mathbb{R}) \times Cts(X,\mathbb{R}) & \xrightarrow{- \cdot -} & Cts(X,\mathbb{R}) & \xrightarrow{\int_X} & \mathbb{R} \\ (f,g) & \longmapsto & fg & \longmapsto & \int_X fg. \end{array}$$

which is bilinear, and, if we give $Cts(X,\mathbb{R})$ the compact-open topology, also <u>continuous</u>, so we obtain (by Lecture 12) a continuous map

$$Cts(X,\mathbb{R}) \longrightarrow Cts(Cts(X,\mathbb{R}),\mathbb{R}), \quad g \longmapsto \{ f \mapsto \textstyle\int_X fg \}$$

where all topologies involved are compact-open. <u>What happens</u> if we use the topologies associated to the $\|-\|_p$ norms instead? To prove the map

$$\begin{aligned} (Cts(X,\mathbb{R}), \|-\|_q) &\longrightarrow (Cts(X,\mathbb{R}), \|-\|_p)^\vee \\ g &\longmapsto \{ f \mapsto \textstyle\int_X fg \} \end{aligned} \tag{11.1}$$

is well-defined we need to show that $L_g := \int_X (-)g$ is continuous (it is clearly linear) as a map of normed spaces

$$L_g : (Cts(X,\mathbb{R}), \|-\|_p) \longrightarrow (\mathbb{R}, |-|).$$

By Lemma L19-3 it suffices to show L_g is <u>bounded</u>, but by Hölder

$$|L_g(f)| = \left|\int_X fg\right| \leq \int_X |fg| = \|fg\|_1 \leq \|f\|_p \|g\|_q = M\|f\|_p$$

where $M = \|g\|_q$. This shows that not only is L_g bounded, but $\|L_g\| \leq \|g\|_q$. The map $g \mapsto L_g$ is linear and bounded (by what we have just said) hence continuous, so we have shown the existence of a map (11.1) relating the normed spaces for dual exponents p, q. Now by abstract nonsense this extends to a continuous linear map from $L^q(X,\mathbb{R})$ to the dual of $L^p(X,\mathbb{R})$.

<u>Lemma L19-4</u> For an integral pair $(X, \int_X)$ and dual exponents $1 < p, q < \infty$ there is a unique continuous linear map $\Phi_{q,p}$ making

$$\begin{tikzcd}
(L^q(X,\mathbb{R}), \|-\|_q) \arrow[r, dashed, "\Phi_{q,p}"] & (L^p(X,\mathbb{R}), \|-\|_p)^\vee \arrow[d, "\iota_p^\vee", "\cong"'] \\
(\mathrm{Cts}(X,\mathbb{R}), \|-\|_q) \arrow[u, hook, "\iota_q"] \arrow[r, "L_{(-)}"'] & (\mathrm{Cts}(X,\mathbb{R}), \|-\|_p)^\vee
\end{tikzcd}$$

where the bottom row is $g \longmapsto L_g$, the left hand vertical map is the inclusion, and the right hand vertical map is the dual of the inclusion (see Ex. L19-5).

<u>Proof</u> The composite $(\iota_p^\vee)^{-1} \circ L_{(-)}$ is continuous and linear, so this is immediate from Theorem L18-9. □

We will prove the $p=2$ special case of the following theorem (marked Z for WARNING because the theorem is not intended as a valid node in our knowledge graph) using the structure theory of Hilbert spaces:

<u>Theorem</u>^{Z} (Duality for L^p-spaces) The map Φ is an isomorphism of Banach spaces

$$\Phi_{q,p} : L^q(X,\mathbb{R}) \xrightarrow{\cong} L^p(X,\mathbb{R})^\vee.$$

Duality for L^p-spaces represents an extension of duality for finite-dimensional vector spaces to the infinite-dimensional setting, with the <u>linear dual</u> replaced by the <u>continuous linear dual</u>. Let us now elaborate some immediate consequences, which will suffice to flesh out the general story of L^p-spaces. As has been stated above, we will only <u>use</u> the $p=2$ case, for which we will provide a proof.

For any pair of dual exponents $1 < p, q < \infty$ we have an isomorphism of normed spaces

$$L^q(X,\mathbb{R}) \xrightarrow[\cong]{\Phi_{q,p}} L^p(X,\mathbb{R})^\vee \xrightarrow[\cong]{\{\Phi_{p,q}^\vee\}^{-1}} L^q(X,\mathbb{R})^{\vee\vee} \qquad (13.1)$$

using Ex. L19-6, so for $1 < q < \infty$ the L^q-space is isomorphic to its double dual. Moreover this isomorphism is precisely the canonical map described in Ex. L19-5, which means that the L^p-spaces are what is called <u>reflexive</u>, as explained below.

<u>Exercise L19-7</u> Prove that (13.1) coincides with the canonical map of Ex. L19-5 with $V = L^q(X,\mathbb{R})$, i.e. show that for $\omega \in L^q(X,\mathbb{R})$

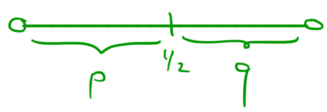
$$\Phi_{q,p}(\omega) = \mathrm{eval}_\omega \circ \Phi_{p,q}$$

as elements of $L^p(X,\mathbb{R})^\vee$.

<u>Def^n</u> A Banach space $(V, \|-\|_V)$ is called <u>reflexive</u> if the canonical map $V \longrightarrow V^{\vee\vee}$ is an isomorphism of normed spaces.

As recalled in Lemma L19-1, a vector space over $\mathbb{F}$ is reflexive (with respect to the ordinary linear dual) if and only if it is finite-dimensional, so reflexivity of L^p-spaces should be understood as a kind of finiteness (although of a much more subtle kind). For the linear dual another condition characterising finite-dimensionality of V was the existence of an isomorphism $V \cong V^*$. When does this happen for L^p-spaces with respect to the continuous linear dual? The only fixed point of $p \mapsto q = p/p-1$ in $(1,\infty)$ is

$$p^2 - p = p \iff p - 1 = 1 \iff p = 2.$$



so the only case of <u>self-duality</u> emerging directly from the theorem is

$$\Phi_{2,2} : L^2(X,\mathbb{R}) \xrightarrow{\cong} L^2(X,\mathbb{R})^\vee$$
$$f \longmapsto \int_X (-)\cdot f$$

Moreover one should not expect <u>any</u> isomorphism $L^p(X,\mathbb{R}) \cong L^p(X,\mathbb{R})^\vee$ as soon as $p \neq 2$ and X is infinite (Ch. XII of Banach's book "Theory of linear operators", specifically the Lemma on p. 122 shows that for $X = [a,b]$ if $L^p(X,\mathbb{R}) \cong L^p(X,\mathbb{R})^\vee$, and hence $L^p(X,\mathbb{R}) \cong L^q(X,\mathbb{R})$ where q is the dual exponent, then $p = q = 2$. Probably the argument generalises, but I have not tried).

<u>Exercise L19-8</u> (i) Prove that for all $f, g \in L^2(X,\mathbb{R})$

$$\Phi_{2,2}(f)(g) = \Phi_{2,2}(g)(f)$$

(ii) Prove that for all $f \in L^2(X,\mathbb{R})$, $\Phi_{2,2}(f)(f) = \|f\|_2^2$.

In summary, for vector spaces V

$$V \text{ finite-dimensional} \iff V \text{ reflexive} \iff V \text{ self-dual}$$

(i.e. $V \xrightarrow{\cong} V^{**}$, $\omega \longmapsto ev_\omega$) (i.e. $\exists\ V \cong V^*$)

while for normed spaces and continuous linear duals the conditions of reflexivity and self-duality are not equivalent: <u>all</u> the L^p-spaces are reflexive, but only L^2-spaces have the additional property of being self-dual. This property of being self-dual is so remarkable that these self-dual Banach spaces are given a special name: <u>Hilbert spaces</u>.

Solutions to selected exercises

L19-5 Let $(V, \|-\|_V)$ be a normed space. For $w \in V$ we first need to show

$$ev_w : V^\vee \longrightarrow \mathbb{F}$$
$$ev_w(f) = f(w)$$

is continuous and linear. It is clearly linear. Now any $f \in V^\vee$ is bounded, and so

$$|f(w)| \leq \|f\| \cdot \|w\|_V$$

Hence

$$|ev_w(f)| = |f(w)| \leq \|f\| \cdot \|w\|_V$$

which shows that ev_w is bounded and moreover $\|ev_w\| \leq \|w\|_V$. So by Lemma L19-3, ev_w is continuous, thus $ev_w \in V^{\vee\vee}$. Now, this shows the function $V \longrightarrow V^{\vee\vee}$, $w \longmapsto ev_w$ is well-defined and it is clearly linear. Once again, to show it is continuous it suffices to show it is bounded, but this follows from $\|ev_w\| \leq \|w\|_V$.

L19-7 By Lemma L17-2 it suffices to prove this for $w \in Cts(X, \mathbb{R})$ and also to check $\Phi_{q,p}(w)$, $evalw \circ \Phi_{p,q}$ agree on the subset of $f \in Cts(X, \mathbb{R}) \subseteq L^p(X, \mathbb{R})$. But then

$$\Phi_{q,p}(w)(f) = \int_X wf$$

$$evalw(\Phi_{p,q}(f)) = evalw\left(\int_X f \cdot (-)\right) = \int_X fw$$

which completes the proof.

$\boxed{L19\text{-}6}$ The linear map $\iota : V \longrightarrow \hat{V}$ is bounded, so we have a bounded linear map

$$\iota^\vee : \hat{V}^\vee \longrightarrow V^\vee \qquad\qquad \iota^\vee(f) = f \circ \iota .$$

Note that for $F : \hat{V} \longrightarrow \mathbb{R}$ continuous and linear, the diagram

$$\begin{tikzcd}
\hat{V} \arrow[r, "F"] & \mathbb{R} \\
V \arrow[u, "\iota"] \arrow[ur, "\iota^\vee(F) = F\circ\iota"']
\end{tikzcd}$$

commutes. To show $\iota^\vee$ is surjective, let $f : V \longrightarrow \mathbb{R}$ be continuous and linear, hence by Theorem L18-9 there exists a unique continuous linear $F : \hat{V} \longrightarrow \mathbb{R}$ such that $\iota^\vee(F) = f$, so $\iota^\vee$ is surjective. Injectivity follows from the uniqueness part of the aforementioned Theorem. So $\iota^\vee$ is an isomorphism of vector spaces. It remains to show $\|\iota^\vee(F)\| = \|F\|$ for all F, or, setting $f = F \circ \iota$, to show $\|f\| = \|F\|$ in the circumstance of the above diagram.

<u>$\|f\| \le \|F\|$</u> note that for $v \in V$

$$\frac{|F(\iota(v))|}{\|\iota(v)\|_{\hat{V}}} = \frac{|f(v)|}{\|v\|_V}$$

Hence

$$\|f\| = \sup\left\{ \frac{|f(v)|}{\|v\|_V} \;\middle|\; v \in V \setminus \{0\} \right\} \le \sup\left\{ \frac{|F(w)|}{\|w\|_{\hat{V}}} \;\middle|\; w \in \hat{V} \setminus \{0\} \right\} = \|F\| .$$

$\underline{\|F\| \leq \|f\|}$ Given $\omega \in \hat{V}$, say $\omega = \lim_{n\to\infty} \omega_n$ with $\omega_n \in V$. Then

$$
\begin{aligned}
|F(\omega)| &= |\lim_{n\to\infty} f(\omega_n)| \\
&= \lim_{n\to\infty} |f(\omega_n)| \\
&\leq \lim_{n\to\infty} \|f\| \, \|\omega_n\|_V \\
&= \|f\| \cdot \lim_{n\to\infty} \|\omega_n\|_V \\
&= \|f\| \cdot \|\omega\|_{\hat{V}}
\end{aligned}
$$

This shows $\|F\| \leq \|f\|$. $\square$