(updated 5-8-20)

Lecture 2 : Examples of spaces

The first lecture was a lightning tour of the standard abstract notions of space grounded in the example $\mathbb{R}^n$, with an extended look at the connection between inner products and rotations. Today we will focus on examples.

Def^n A metric space is a pair (X, d) consisting of a set X and a function

$$d : X \times X \longrightarrow \mathbb{R}$$

satisfying the axioms:

(M1) $d(x,y) \geqslant 0 \quad \forall x,y \in X$ (non-negativity)
(M2) $d(x,y) = 0 \iff x = y \quad \forall x,y \in X$ (separation)
(M3) $d(x,y) = d(y,x) \quad \forall x,y \in X$ (symmetry)
(M4) $d(x,y) + d(y,z) \geqslant d(x,z) \quad \forall x,y,z \in X$ (triangle inequality)

Remark L2-1 (i) The empty set ϕ is a metric space ($\phi \times \phi = \phi$)

(ii) We can consider metrics taking values in other totally ordered fields (e.g. $\mathbb{Q}$), this is important for constructing the p-adic numbers, or $\mathbb{R}$ itself from $\mathbb{Q}$, but we will only consider real-valued metrics in this course.

(iii) Usually we abuse notation and say "X is a metric space" where we really mean "(X,d) is a metric space". For this reason, we also often write d_X for the metric on X, if this needs clarification.

<u>Example L2-1</u> The singleton $X = \{*\}$ is a metric space with $d(*,*) = 0$.

<u>Exercise L2-1</u> Prove that any set X becomes a metric space with the <u>discrete metric</u> defined by

$$d(x,y) = \begin{cases} 1 & x \neq y \\ 0 & x = y. \end{cases}$$

<u>Lemma L2-1</u> The following functions define metrics on $\mathbb{R}^n$

$$d_2(\underline{x}, \underline{y}) = \left\{ \sum_{i=1}^{n} (x_i - y_i)^2 \right\}^{1/2}$$

$$d_1(\underline{x}, \underline{y}) = \sum_{i=1}^{n} |x_i - y_i|$$

$$d_\infty(\underline{x}, \underline{y}) = \max\{ |x_i - y_i| \}_{i=1}^{n}$$

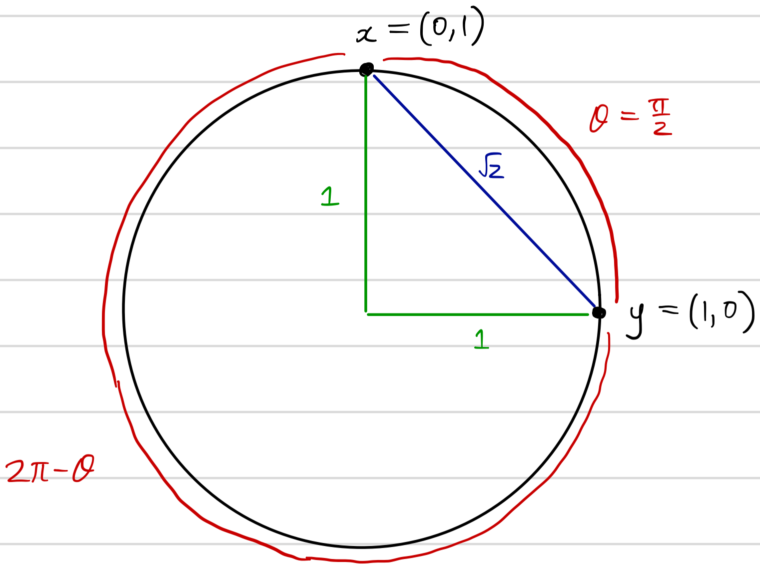
<u>Proof</u> Deferred to next lecture. $\square$

<u>Remark L2-2</u> $\mathbb{R}^0 = \{*\}$ is trivially a metric space.

<u>Exercise L2-2</u> Given a metric space (X, d) and a subset $A \subseteq X$ prove that $(A, d|_A)$ is a metric space with the <u>induced metric</u>

$$d|_A : A \times A \to \mathbb{R}, \quad d|_A(a,b) = d(a,b)$$

<u>Example L2-2</u> The circle $S^1 = \{ (x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1 \}$ becomes a metric space under the restriction of d_2, d_1 or d_∞. But there is clearly something unsatisfactory about these metrics, see the diagram overleaf.



$$d_2(x,y) = \sqrt{2}$$
$$d_1(x,y) = 2$$
$$d_\infty(x,y) = 1$$

What about $\frac{\pi}{2}$?

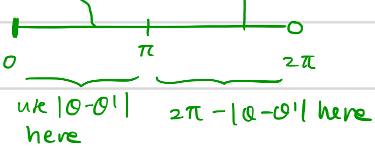
Let us now examine whether arc length gives a metric on S^1. Firstly, there is not a unique arc connecting two points, and secondly, to avoid explicitly talking about inverse trignometric functions, let us use the bijection

$$\Phi : [0,2\pi) \longrightarrow S^1, \quad \Phi(\theta) = (\cos\theta, \sin\theta).$$

Then we define ("a" for "arc") for $x,y \in S^1$

$$d_a(x,y) = \min\{\, |\Phi^{-1}(x) - \Phi^{-1}(y)|,\ 2\pi - \underbrace{|\Phi^{-1}(x) - \Phi^{-1}(y)|}_{\text{lies in } (0,2\pi]} \,\}$$

$$\text{i.e. } d_a(\Phi\theta, \Phi\theta') = \min\{\, |\theta-\theta'|,\ 2\pi - |\theta-\theta'| \,\}$$



Lemma L2-2 The pair (S^1, d_a) is a metric space.

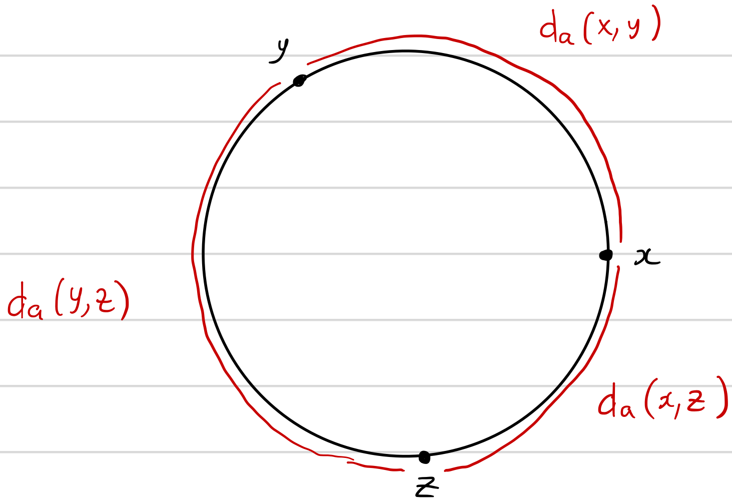
Proof (M1) is clear. For (M2), we have for $x,y \in S^1$ with $x=\Phi\theta$, $y=\Phi\theta'$

$$\begin{aligned} d_a(\Phi\theta,\Phi\theta') = 0 &\iff |\theta-\theta'| = 0 \quad (2\pi - |\theta-\theta'| \text{ cannot be zero}) \\ &\iff \theta = \theta' \\ &\iff \Phi\theta = \Phi\theta' \quad (\text{as } \Phi \text{ is a bijection}) \end{aligned}$$

For (M3) consider

$$\begin{aligned} d_a(\Phi\theta, \Phi\theta') &= \min\{|\theta-\theta'|, 2\pi - |\theta-\theta'|\} \\ &= \min\{|\theta'-\theta|, 2\pi - |\theta'-\theta|\} \\ &= d_a(\Phi\theta', \Phi\theta). \end{aligned}$$

For (M4), we have a situation like the following:



To prove that for all $x, y, z \in S^1$

$$d_a(x,y) + d_a(y,z) \geqslant d_a(x,z)$$

we can just do a case by case analysis. This is fine but a bit lame. Let us instead observe that we can reduce to the case $x = (1,0)$. Let

$$R_\psi : S^1 \longrightarrow S^1$$

be the restriction of the rotation by ψ on $\mathbb{R}^2$ in Lecture 1. Observe that

$$R_\varphi(\cos\theta, \sin\theta) = (\cos(\theta+\varphi), \sin(\theta+\varphi))$$

i.e. $R_\varphi \Phi(\theta) = \Phi(\theta+\varphi)$ where $\theta+\varphi$ is taken "modulo 2π".

Now notice that $|\theta-\theta'|$ does not change if $\theta \to \theta+\varphi$, $\theta' \to \theta'+\varphi$, so

$$d_a(R_\varphi x, R_\varphi y) = d_a(x, y) \qquad \forall x, y \in S^1$$

Hence to prove, for some fixed $x, y, z \in S^1$ that

$$d_a(x,y) + d_a(y,z) \geqslant d_a(x,z)$$

it is equivalent to prove

$$d_a(R_{-\theta}x, R_{-\theta}y) + d_a(R_{-\theta}y, R_{-\theta}z) \geqslant d_a(R_{-\theta}x, R_{-\theta}z).$$

But $R_{-\theta}x = (1,0)$, and so it suffices to prove (M4) in the case where $x = (1,0)$ and y, z are arbitrary.

Next, consider the function

$$T : S^1 \longrightarrow S^1 \qquad T(a_1, a_2) = (a_1, -a_2).$$

and observe that

$$d_a(x,y) = d_a(Tx, Ty) \qquad \forall x, y \in S^1$$

because $T(\cos\theta, \sin\theta) = (\cos(-\theta), \sin(-\theta))$ and $|\theta-\theta'| = |-\theta-(-\theta')|$.

OK, so we have to prove for all $y, z \in S^1$ that

$$d_a((1,0), y) + d_a(y, z) \geq d_a((1,0), z). \qquad (6.1)$$

This is equivalent to

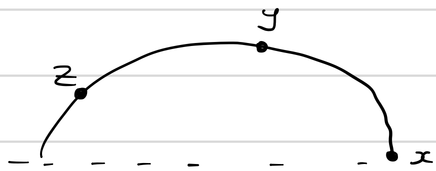
$$d_a((1,0), Ty) + d_a(Ty, Tz) \geq d_a((1,0), Tz) \qquad (6.2)$$

So we may assume y lies in the upper hemisphere! i.e. that $\theta' = \Phi^{-1}(y)$ lies in $[0, \pi]$, because if it doesn't then Ty does, and we can just prove (6.2) instead. But if $\theta' \in [0, \pi]$ then

$$d_a((1,0), y) = \theta'$$

and we just have two cases depending on $\theta'' = \Phi^{-1}(z)$

$\boxed{\text{Case 1}}$ $0 \leq \theta'' \leq \pi$, i.e.



both y, z in the upper hemisphere

Then $0 \leq |\theta' - \theta''| \leq \pi$ so

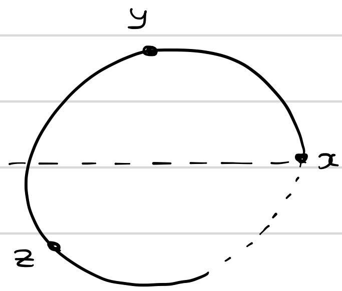
$$d_a((1,0), y) + d_a(y, z) = \theta' + |\theta' - \theta''|$$

Now either $\theta' \geq \theta''$ in which case

$$\theta' + |\theta' - \theta''| = 2\theta' - \theta'' \geq \theta'' = d_a((1,0), z)$$

or $\theta' < \theta''$ in which case $\theta' + |\theta' - \theta''| = \theta'' = d_a((1,0), z)$.

$\boxed{\text{Case 2}}$ $\pi < \theta'' < 2\pi$ i.e.



Now $|\theta' - \theta''| = \theta'' - \theta'$ and so

$$d_a((1,0),y) + d_a(y,z) = \theta' + \min\{\theta'' - \theta', 2\pi - (\theta'' - \theta')\}$$

$$= \min\{\theta'', 2\pi + 2\theta' - \theta''\}.$$

$$\geq 2\pi - \theta'' = d_a((1,0),z).$$

which completes the proof that d_a is a metric. $\square$

Exercise L2-3 Give a direct proof of Lemma L2-2 by dividing into cases, as follows: given $x, y, z \in S^1$ set $\theta = \Phi^{-1}(x)$, $\theta' = \Phi^{-1}(y)$ and $\theta'' = \Phi^{-1}(z)$. Consider the following three statements:

$\boxed{P_1}$ $|\theta - \theta'| \leq \pi$

$\boxed{P_2}$ $|\theta' - \theta''| \leq \pi$

$\boxed{P_3}$ $|\theta - \theta''| \leq \pi$

Each is either true or false for a particular triple (x,y,z), and this means there are $2^3 = 8$ cases (e.g. P_1, P_2 true but P_3 false). Prove each case individually, and in this way prove the lemma.

<u>Remark L2-3</u> Observe that $d_a(x,y) \leq \pi$ for all $x, y \in S^1$, in contradistinction to $\mathbb{R}^n$ where the metric (d_1, d_2 or d_∞) has no upper bound. In this sense (S^1, d_a) is really a new example.

<u>Exercise L2-4</u> Is it true that for distinct x, y, z that

$$d_a(x,y) + d_a(y,z) + d_a(z,x) = 2\pi \ ?$$

(i.e. don't be fooled by pictures like the one on p.4)