Lecture 20: Hilbert space

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We saw last lecture that as a consequence of the general duality theorem for L^p-spaces, the normed space $L^2(X,\mathbb{R})$ is <u>self-dual</u> with respect to the continuous linear dual, and this space therefore possesses a subtle kind of "finite-dimensionality" despite being an infinite-dimensional vector space. The concept of <u>Hilbert space</u> axiomitises this self-duality, and elaborates its consequences. We begin today's lecture with the standard definition of Hilbert space which, despite what we have just said, makes no direct mention of this self-duality (don't blame me, it's not my def^n). We then build up the theory to the point where we can prove a characterisation of Hilbert spaces as a kind of self-dual normed space (conceptually, this is the "right" def^n, at least in my opinion).

Throughout $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$, and given $\lambda \in \mathbb{F}$ we set $\bar{\lambda} = \lambda$ if $\mathbb{F} = \mathbb{R}$ and let $\bar{\lambda}$ denote the usual complex conjugate if $\mathbb{F} = \mathbb{C}$.

<u>Def^n</u> An <u>inner product space</u> $(V, \langle , \rangle)$ over $\mathbb{F}$ is an $\mathbb{F}$-vector space V together with a function $\langle , \rangle : V \times V \longrightarrow \mathbb{F}$ satisfying

$$
\begin{array}{lll}
(I1) & \langle v+w, u\rangle = \langle v,u\rangle + \langle w,u\rangle, & \\
 & \langle u, v+w\rangle = \langle u,v\rangle + \langle u,w\rangle & \forall u,v,w \in V \\
(I2) & \langle \lambda u, v\rangle = \lambda\langle u,v\rangle, & \\
 & \langle u, \lambda v\rangle = \bar{\lambda}\langle u,v\rangle & \forall u,v \in V\ \forall \lambda \in \mathbb{F} \\
(I3) & \langle u,v\rangle = \overline{\langle v,u\rangle} & \forall u,v \in V \\
(I4) & \langle u,u\rangle \geqslant 0 & \forall u \in V \\
(I5) & \langle u,u\rangle = 0 \iff u = 0 & \forall u \in V.
\end{array}
$$

We call $\langle , \rangle$ the <u>inner product</u> or <u>pairing</u> and say it is linear in the first variable and conjugate linear in the second variable.

Remark (i) Physicists write $\langle v | w \rangle$ for $\langle v, w \rangle$ and they adopt the convention that the pairing is linear in the second variable and conjugate linear in the first variable. Mathematics texts consistently use the opposite convention, as we have done. I tend to think the physicists made the right choice, but whatever: it is a convention, and it doesn't really matter, because you can just read $\langle v | w \rangle$ as $\langle w, v \rangle$.

(ii) The second lines in (I1), (I2) follow from the first lines, using (I3), so they are redundant (I include them because otherwise the def^n is oddly non-symmetric).

(iii) By (I3) $\langle u, u \rangle = \overline{\langle u, u \rangle}$ is real, so $\langle u, u \rangle \geq 0$ makes sense.

Example L20-1 (i) $(\mathbb{R}^n, \langle , \rangle)$ defined by $\langle \underline{a}, \underline{b} \rangle = \sum_{i=1}^n a_i b_i$ is a real inner product space

(ii) $(\mathbb{C}^n, \langle , \rangle)$ defined by $\langle \underline{a}, \underline{b} \rangle = \sum_{i=1}^n a_i \overline{b_i}$ is a complex inner product space (note $\langle \underline{a}, \underline{a} \rangle = \sum_i |a_i|^2$).

We call these the standard inner products on $\mathbb{R}^n$, $\mathbb{C}^n$.

Example L20-2 We proved in Lecture 4 that if $P \in M_n(\mathbb{R})$ is positive definite then $\langle \underline{a}, \underline{b} \rangle = \underline{a}^T P \underline{b}$ is an inner product on $\mathbb{R}^n$. Note that symmetry (I3) follows from $P^T = P$ since

$$\langle \underline{a}, \underline{b} \rangle = \underline{a}^T P \underline{b} = (\underline{a}^T P \underline{b})^T = \underline{b}^T P^T \underline{a} = \underline{b}^T P \underline{a} = \langle \underline{b}, \underline{a} \rangle.$$

Example L20-3 In Tutorial 2 we discussed nondegenerate bilinear forms and quadratic spaces. If $(V, \langle , \rangle)$ is a real inner product space then $\langle , \rangle$ is symmetric bilinear, and if V is finite-dimensional then $u \mapsto \langle u, - \rangle$ is an isomorphism $V \xrightarrow{\cong} V^*$, that is, the pairing is nondegenerate (we have to be more careful about what "nondegenerate" means in the infinite-dimensional case, and for that reason I will only use it for finite-dimensional spaces). To see this, note that if $u \neq 0$ then $\langle u, - \rangle$ is not the zero function by (IS), so the map $V \longrightarrow V^*$ is injective and hence an isomorphism since $\dim(V^*) = \dim(V)$.

<u>Warning</u>: for infinite-dimensional real inner product spaces the map

$$V \longrightarrow V^*, \qquad u \longmapsto \langle u, - \rangle$$

is still well-defined, linear and injective, but it is <u>never</u> surjective! (cf. the Remark on p. 4 on L19). Of course after L19 we do not expect this anyway, as the inner product leads to a norm, and we could at best hope $V \cong V^\vee$. As we will see, that in fact does hold provided V is complete.

Observe that any real inner product space is also a <u>quadratic space</u>, but not vice versa (e.g. Minkowski space does not satisfy (I4)).

The upshot of Lecture 4 and Tutorial 2 (i.e. Sylvester's law of inertia, which has a complex version as well) is that <u>all finite-dimensional inner product spaces over $\mathbb{F}$ of the same dimension are isomorphic</u>. So in a sense the only finite-dimensional examples are Example L20-1. This is elaborated more precisely in the next exercise:

<u>Def^n</u> An <u>isomorphism of inner product spaces</u> $(V, \langle -,- \rangle_V)$, $(W, \langle -,- \rangle_W)$ is an isomorphism of vector spaces $T: V \to W$ such that $\langle Tu, Tv \rangle_W = \langle u, v \rangle_V$ for all $u, v \in V$.

Exercise L20-1 (i) Prove that any pair of finite-dimensional real inner product spaces of the same dimension are isomorphic (Hint: Sylvester).

(ii) Prove that any pair of finite-dimensional complex inner product spaces of the same dimension are isomorphic.

Lemma L20-1 Let $(V, \langle , \rangle)$ be an inner product space. Then $(V, \|-\|)$ is a normed space where $\|v\| = \langle v,v\rangle^{1/2}$ and for $u,v \in V$

$$|\langle u,v\rangle| \leq \|u\| \, \|v\| \qquad \text{(Cauchy – Schwarz Inequality)}.$$

Proof (N1) is clear from (I4), (I5). For (N2), we have

$$\|\lambda v\| = \langle \lambda v, \lambda v\rangle^{1/2} = \{ \lambda\bar{\lambda}\langle v,v\rangle\}^{1/2} = |\lambda| \cdot \|v\|.$$

Next we prove the Schwartz inequality. The proof is a trivial variation on the proof of Lemma L4-3: in fact our earlier proof goes unchanged for $\mathbb{F} = \mathbb{R}$. We repeat the argument here, making the necessary modifications so that it works for both $\mathbb{R}$ and $\mathbb{C}$. For any $\lambda \in \mathbb{F}$

$$\begin{aligned} 0 \leq \langle u - \lambda v, u - \lambda v\rangle &= \langle u,u\rangle - \lambda\langle v,u\rangle - \bar{\lambda}\langle u,v\rangle + \lambda\bar{\lambda}\langle v,v\rangle \\ &= \|u\|^2 + |\lambda|^2\|v\|^2 - \{ \lambda\langle v,u\rangle + \overline{\lambda\langle v,u\rangle} \} \\ &= \|u\|^2 + |\lambda|^2\|v\|^2 - 2\,\mathrm{Re}(\lambda\langle v,u\rangle). \end{aligned}$$

We may assume $v \neq 0$, and set $\lambda = \langle u,v\rangle / \langle v,v\rangle$ so that

$$\lambda\langle v,u\rangle = |\langle u,v\rangle|^2 / \|v\|^2$$

and hence

$$0 \leq \|u\|^2 + \frac{|\langle u,v\rangle|^2}{\|v\|^4} \cdot \|v\|^2 - 2\mathrm{Re}\left(\frac{|\langle u,v\rangle|^2}{\|v\|^2} \right)$$

$$= \|u\|^2 + \frac{|\langle u,v\rangle|^2}{\|v\|^2} - 2\frac{|\langle u,v\rangle|^2}{\|v\|^2}$$

$$= \|u\|^2 - \frac{|\langle u,v\rangle|^2}{\|v\|^2}$$

so $|\langle u,v\rangle|^2 \leq \|u\|^2\|v\|^2$ and hence $|\langle u,v\rangle| \leq \|u\| \cdot \|v\|$.

From the Cauchy-Schwartz inequality we deduce the triangle inequality since

$$\begin{aligned} \|u+v\|^2 &= \langle u+v, u+v\rangle = \langle u,u\rangle + \langle u,v\rangle + \langle v,u\rangle + \langle v,v\rangle \\ &= \|u\|^2 + 2\mathrm{Re}(\langle u,v\rangle) + \|v\|^2 \\ &\leq \|u\|^2 + 2|\langle u,v\rangle| + \|v\|^2 \\ &\leq \|u\|^2 + 2\|u\| \cdot \|v\| + \|v\|^2 \\ &= (\|u\| + \|v\|)^2 \end{aligned}$$

which completes the proof that $(V, \|-\|)$ is a normed space. □

Example L20-4 (i) The norm associated to the standard inner product on $\mathbb{R}^n$ is the $\|-\|_2$-norm.

(ii) The norm associated to the standard inner product on $\mathbb{C}^n$ is $\|\underline{a}\| = \{\sum_i |a_i|^2\}^{1/2}$

Def^n A Hilbert space over $\mathbb{F}$ is an inner product space $(H, \langle , \rangle)$ over $\mathbb{F}$ with the property that the associated normed space $(H, \|-\|)$ is a Banach space (that is, it is complete w.r.t. the metric $d(h_1,h_2) = \|h_1 - h_2\|$).

Remark Any inner product space is a normed space and thus a topological vector space (Ex. L8-10), and we use this structure without further comment.

Example L20-5 The standard inner products on $\mathbb{R}^n$, $\mathbb{C}^n$ make these spaces into Hilbert spaces. The completeness of $(\mathbb{R}^n, \|-\|_2)$ was explained on p. 13 of L13, and the metric induced on $\mathbb{C}^n = \mathbb{R}^{2n}$ by the standard inner product is d_2, which is complete.

Lemma L20-2 In any inner product space $(V, \langle,\rangle)$, the norm satisfies

$$\|u+v\|^2 + \|u-v\|^2 = 2\|u\|^2 + 2\|v\|^2 \qquad \text{(Parallelogram law)}$$

and if $\langle u,v\rangle = 0$ then

$$\|u+v\|^2 = \|u\|^2 + \|v\|^2 \qquad \text{(Pythagorean law)}$$

Proof We simply compute

$$\begin{aligned}\|u+v\|^2 + \|u-v\|^2 &= \langle u+v, u+v\rangle + \langle u-v, u-v\rangle \\ &= \langle u,u\rangle + 2\mathrm{Re}(\langle u,v\rangle) + \langle v,v\rangle \\ &\quad + \langle u,u\rangle - 2\mathrm{Re}(\langle u,v\rangle) + \langle v,v\rangle \\ &= 2\|u\|^2 + 2\|v\|^2.\end{aligned}$$

The same calculation also demonstrates the Pythagorean law. □

Exercise L20-2 Let $(V, \langle,\rangle)$ be an inner-product space. Prove
(i) $u = v$ iff. $\langle u,w\rangle = \langle v,w\rangle$ for all $w \in V$.
(ii) $\|u\| = \sup\{ |\langle u,v\rangle| \mid \|v\| = 1\}$.

<u>Lemma L20-3</u> Let $(V, \langle , \rangle)$ be an inner product space. Then for $u \in V$ the function $\langle -, u \rangle : V \to \mathbb{F}$ is bounded, linear and has operator norm $\|u\|$.

<u>Proof</u> By Cauchy-Schwartz $|\langle v, u\rangle| \leq \|u\| \cdot \|v\|$ which shows $\langle -, u\rangle$ is bounded and $\|\langle -, u\rangle\| \leq \|u\|$. On the other hand Ex. L20-2 (ii) shows

$$\|\langle -, u \rangle\| = \sup\{ |\langle v, u\rangle| \mid \|v\| = 1 \} = \|u\|. \square$$

It follows immediately that $\langle u, - \rangle : V \to \mathbb{F}$ is continuous, although this map is <u>not</u> linear: it is what we call conjugate linear.

<u>Def^n</u> If $(V, +, a)$ is a complex vector space with action $a : \mathbb{C} \times V \to V$ the <u>complex conjugate</u> vector space $\overline{V}$ has the same underlying set V and abelian group structure $+$, but the action $\overline{a}$ defined to be

$$\begin{array}{ccccc} \mathbb{C} \times V & \xrightarrow{\overline{(-)} \times id_V} & \mathbb{C} \times V & \xrightarrow{a} & V \\ (z, v) & \longmapsto & (\overline{z}, v) & \longmapsto & a(\overline{z}, v). \end{array}$$

Less formally, in $\overline{V}$ we have $z \bullet v = \overline{z} \cdot v$ where $\bullet$ is the action of scalars in $\overline{V}$ and $\cdot$ the action in V.

<u>Exercise L20-3</u> Check that $\overline{V}$ is a $\mathbb{C}$-vector space. If $(V, \|-\|_V)$ is a normed space over $\mathbb{C}$ check $(\overline{V}, \|-\|_V)$ is a normed space (with the same norm).

<u>Def^n</u> Let V, W be $\mathbb{F}$-vector spaces. A function $T : V \to W$ is <u>conjugate linear</u> if $T(u+v) = T(u) + T(v)$ for all $u, v \in V$ and for $\lambda \in \mathbb{F}$, $u \in V$, $T(\lambda u) = \overline{\lambda}\, T(u)$.

So if $\mathbb{F} = \mathbb{R}$ there is no difference between linearity and conjugate linearity.

<u>Exercise L20-4</u> (i) Prove that a function $T : V \longrightarrow W$ is conjugate linear iff. it is <u>linear</u> viewed as a map $V \longrightarrow \overline{W}$, or $\overline{V} \longrightarrow W$.

(ii) If $(V, \langle,\rangle)$ is an inner product space then $\langle u, - \rangle : \overline{V} \longrightarrow \mathbb{F}$ is linear, bounded and has norm $\|u\|$.

<u>Lemma L20-4</u> Let $(V, \langle,\rangle)$ be an inner product space. The maps

$$\overline{V} \longrightarrow V^\vee \qquad u \longmapsto \langle -, u \rangle$$
$$V \longrightarrow \overline{V}^\vee \qquad u \longmapsto \langle u, - \rangle$$

are continuous, linear and norm-preserving.

<u>Proof</u> We prove the claims for the first map. It is well-defined and norm-preserving by Lemma L20-3. Moreover $u \longmapsto \langle -, u \rangle$ is conjugate linear in u, hence linear as a map $\overline{V} \longrightarrow V^\vee$ Continuity follows from boundedness. $\square$

A subset $X \subseteq V$ of a vector space V over $\mathbb{F}$ is <u>convex</u> if whenever $x, y \in X$ we have $\lambda x + (1-\lambda) y \in X$ for all $0 < \lambda < 1$.

<u>Lemma L20-5</u> Let $(H, \langle,\rangle)$ be a Hilbert space. If $K \subseteq H$ is closed, convex and nonempty, then for each $h \in H$ there is a unique point k in K closest to h, that is

$$\|h - k\| = d(h, K) := \inf\{ \|h - v\| \mid v \in K \}.$$

Proof Set $\alpha = d(h,K)$ and choose $k_n \in K$ such that $\|h-k_n\| \to \alpha$. Since K is convex, $\frac{1}{2}(k_n+k_m) \in K$, and hence $\|\frac{1}{2}(k_n+k_m)-h\| \geq \alpha$. By the Parallelogram law

$$\begin{aligned}\|k_n-k_m\|^2 &= \|(k_n-h)-(k_m-h)\|^2 \\ &= 2\|k_n-h\|^2+2\|k_m-h\|^2-\|k_n+k_m-2h\|^2 \\ &= 2\|k_n-h\|^2+2\|k_m-h\|^2-4\|\tfrac{1}{2}(k_n+k_m)-h\|^2 \\ &\leq 2\|k_n-h\|^2+2\|k_m-h\|^2-4\alpha^2\end{aligned}$$

which can be made arbitrarily small by making m,n large. Hence $(k_n)_{n=0}^{\infty}$ is Cauchy in H. Since H is complete $k_n \to k$ for some $k \in H$, and since K is closed $k \in K$. By continuity of the norm (Lemma L18-3) and of the vector space operations (Ex. L18-10)

$$\begin{aligned}\|h-k\| &= \|h-\lim_{n\to\infty} k_n\| \\ &= \|\lim_{n\to\infty}(h-k_n)\| \\ &= \lim_{n\to\infty}\|h-k_n\| = \alpha\end{aligned}$$

For uniqueness, if $\|h-k'\| = \alpha$ then by the previous calculation

$$\|k-k'\| \leq 2\|k-h\|^2+2\|k'-h\|^2-4\alpha^2 = 0$$

so $k=k'$. $\square$

Exercise L20-5 Let $(V,\langle,\rangle)$ be an inner product space. Prove $\langle,\rangle : V\times V \to \mathbb{F}$ is continuous, but prove that if $V \neq 0$ then it is not uniformly continuous.

Def^n If $(V, \langle , \rangle)$ is an inner product space and $u, v \in V$ we say u, v are orthogonal if $\langle u, v\rangle = 0$.

Lemma L20-6 Let $(V, \langle , \rangle)$ be an inner product space and $W \subseteq V$ a subspace. Let $v \in V$ and $w \in W$. Then the following are equivalent:

(i) $\langle v - w, y \rangle = 0$ for all $y \in W$.

(ii) w is the unique closest point to v in W.

Proof Given (i) by the Pythagorean law then for $y \in W$

$$\begin{aligned} \| v - y \|^2 &= \| (v-w) + (w-y) \|^2 \\ &= \| v - w\|^2 + \| w - y \|^2 \geqslant \| v - w \|^2, \end{aligned}$$

which proves (ii). Now suppose (ii), let $y \in W$ and $\lambda \in \mathbb{F}$. Then

$$\begin{aligned} 0 &\leq \| v - (w + \lambda y) \|^2 - \| v - w \|^2 \\ &= \| (v-w) - \lambda y \|^2 - \| v - w \|^2 \\ &= -2\,\mathrm{Re}(\langle v - w, \lambda y\rangle) + |\lambda|^2 \| y \|^2 \end{aligned}$$

Hence $2\,\mathrm{Re}(\bar{\lambda}\langle v - w, y\rangle) \leq |\lambda|^2 \|y\|^2$. If $\langle v-w, y\rangle \neq 0$ then $\|y\| \neq 0$ and we may set $\lambda = \langle v - w, y \rangle / \|y\|^2$ to get

$$2\,\mathrm{Re}(|\lambda|^2 \|y\|^2) \leq |\lambda|^2 \|y\|^2$$

which is a contradiction since $\lambda \neq 0$, $\|y\| \neq 0$. $\square$

Def^n The orthogonal complement of a subset W in an inner product space $(V, \langle , \rangle)$ is

$$W^\perp = \{ v \in V \mid \langle v, w \rangle = 0 \text{ for all } w \in W \}.$$

Note that for $w \in W$, $\langle -, w \rangle : V \to \mathbb{F}$ is continuous and linear (Lemma L20-3) so

$$W^\perp = \bigcap_{w \in W} \operatorname{Ker}(\langle -, w \rangle)$$

is a closed subspace of V.

Exercise L20-6 (i) Prove that $W^{\perp\perp} := (W^\perp)^\perp$ contains W.
(ii) Prove that $W_1 \subseteq W_2$ implies $W_2^\perp \subseteq W_1^\perp$.

Lemma L20-7 Let W be a closed vector subspace in a Hilbert space H. Then $W = W^{\perp\perp}$ and $H = W \oplus W^\perp$.

[i.e. a vector subspace of H which happens to also be a closed subset in the topology]

Proof To show $H = W \oplus W^\perp$ we need to show $W \cap W^\perp = \{0\}$ and $W + W^\perp = H$.
If $w \in W \cap W^\perp$ then $\langle w, w \rangle = 0$ so $w = 0$. If $v \in H$ and $w \in W$ is the closest point in W (which exists by Lemma L20-5) then by Lemma L20-6, $v - w \in W^\perp$ so

$$v = w + (v - w) \in W + W^\perp.$$

Finally, if $v \in W^{\perp\perp}$ then write $v = w + y$ with $w \in W$ and $y \in W^\perp$. Then

$$\langle y, y \rangle = \langle y, v - w \rangle = \langle y, v \rangle - \langle y, w \rangle = 0$$

so $y = 0$ and hence $v \in W$. $\square$

<u>Theorem L20-8</u> (Riesz representation theorem) Let $(H, \langle , \rangle)$ be a Hilbert space. If $f : H \longrightarrow \mathbb{F}$ is continuous and linear there exists a unique vector $\theta_f \in H$ with

$$f = \langle -, \theta_f \rangle.$$

<u>Proof</u> By Lemma L20-4 the map $\overline{H} \longrightarrow H^\vee$, $h \longmapsto \langle -, h \rangle$ is linear and norm-preserving and hence injective, so if h_f exists it is certainly unique. To see how we might construct θ_f, suppose we succeeded: then

$$\mathrm{Ker}(f) = \{ u \in H \mid \langle u, \theta_f \rangle = 0 \} = \{ \theta_f \}^\perp$$

But then $\{ \theta_f \} \subseteq \{ \theta_f \}^{\perp\perp} = \mathrm{Ker}(f)^\perp$.

This suggests we look in $\mathrm{Ker}(f)^\perp$ for the representing vector. So let us now proceed with the construction. If $f = 0$ take $\theta_f = 0$, otherwise $\mathrm{Ker}(f) \neq H$ and so by Lemma L20-7 we may choose a nonzero vector $u \in \mathrm{Ker}(f)^\perp$. Since $u \notin \mathrm{Ker}(f)$ we have $f(u) \neq 0$ and by rescaling we may assume $f(u) = 1$. Then notice that for $k \in H$ we have

$$k = k - f(k)u + f(k)u$$

and $f(k - f(k)u) = f(k) - f(k) = 0$ by linearity, while $f(f(k)u) = f(k)$, and so $k - f(k)u \in \mathrm{Ker}(f)$. Hence since $u \in \mathrm{Ker}(f)^\perp$

$$\langle k, u \rangle = \langle (k - f(k)u) + f(k)u, u \rangle = \langle f(k)u, u \rangle = f(k) \cdot \|u\|^2$$

Dividing by $\|u\|$ shows that $\theta_f = u / \|u\|^2$ works. $\square$

<u>Corollary L20-9</u> If $(H, \langle , \rangle)$ is a Hilbert space there are isomorphisms of normed spaces

$$\overline{H} \xrightarrow{\cong} H^\vee \qquad u \longmapsto \langle -, u \rangle$$
$$H \xrightarrow{\cong} \overline{H}^\vee \qquad u \longmapsto \langle u, - \rangle .$$

<u>Proof</u> Immediate from Lemma L20-4 and Theorem L20-8. $\square$

In the real case a Hilbert space is literally self-dual, $H \cong H^\vee$, while in the complex case $H \cong \overline{H}^\vee \cong \overline{H^\vee}$. Sometimes we introduce $V^\dagger$ to stand for the conjugated continuous linear dual $V^\dagger = \overline{V^\vee}$ so that $H \cong H^\dagger$, but there is not much need in this course to introduce yet another piece of notation.

L^2-spaces are Hilbert spaces

Next we want to check $L^2(X, \mathbb{F})$ is a Hilbert space, and from this finally deduce the isomorphism $L^2(X, \mathbb{F}) \cong \overline{L^2(X, \mathbb{F})}^\vee$ advertised in Lecture 19.

<u>Def^n</u> Given a topological space X and continuous $f : X \longrightarrow \mathbb{F}$ we denote by $\overline{f} : X \longrightarrow \mathbb{F}$ the function $\overline{f}(x) = \overline{f(x)}$ (so for $\mathbb{F} = \mathbb{R}$, $f = \overline{f}$).

Let $(X, \int_X)$ be an integral pair, $\mathbb{F}$ our field of scalars. By the same argument as p. 11 of Lecture 19, we have a bounded <u>conjugate linear</u> map for dual exponents $1 < p, q < \infty$

$$\big(\mathrm{Cts}(X, \mathbb{F}), \|-\|_q\big) \longrightarrow \big(\mathrm{Cts}(X, \mathbb{F}), \|-\|_p\big)^\vee$$
$$g \longmapsto L_{\overline{g}}$$

where $L_{\bar g}(f) = \int_X \bar g f$. Note that $\|\bar g\|_p = \|g\|_p$ for $1 \le p \le \infty$, so Hölder also shows $\|L_{\bar g}\| \le \|g\|_q$. This can be viewed as a bounded linear map into the conjugate of $(\mathrm{Cts}(X,\mathbb{F}), \|-\|_p)^\vee$. Of course if $\mathbb{F} = \mathbb{R}$ then all of this collapses to what we already did (by convention if $\mathbb{F}=\mathbb{R}$ then $\overline{V} = V$). Lemma L19-4 shows that there is a unique continuous conjugate linear map $\Phi_{q,p}$ making the diagram below commute:

$$
\begin{tikzcd}
(L^q(X,\mathbb{F}), \|-\|_q) \arrow[r, dashed, "\Phi_{q,p}"] & (L^p(X,\mathbb{F}), \|-\|_p)^\vee \arrow[d, "\cong"', "\iota_p^\vee"] \\
(\mathrm{Cts}(X,\mathbb{F}), \|-\|_q) \arrow[u, hook, "\iota_q"] \arrow[r, "L_{\overline{(-)}}"'] & (\mathrm{Cts}(X,\mathbb{F}), \|-\|_p)^\vee
\end{tikzcd}
$$

This commutativity means precisely that for $f, g \in \mathrm{Cts}(X,\mathbb{F})$, $\Phi_{q,p}(g)(f) = \int_X f\bar g$.

<u>Def^n</u> Given an integral pair $(X, \int_X)$ we define

$$\langle , \rangle : L^p(X,\mathbb{F}) \times L^q(X,\mathbb{F}) \longrightarrow \mathbb{F}$$

by the formula $\langle f, g\rangle := \Phi_{q,p}(g)(f)$.

<u>Lemma L20-10</u> Given Cauchy sequences $(f_n)_{n=0}^\infty$, $(g_n)_{n=0}^\infty$ in $\mathrm{Cts}(X,\mathbb{F})$ with respect to $\|-\|_p$, $\|-\|_q$ resp., we have

$$\left\langle \lim_{n\to\infty} f_n, \lim_{m\to\infty} g_m \right\rangle = \lim_{n\to\infty}\lim_{m\to\infty} \int_X f_n \overline{g_m} = \lim_{m\to\infty}\lim_{n\to\infty} \int_X f_n \overline{g_m}$$

<u>Proof</u> Set $f = \lim_{n\to\infty} f_n$, $g = \lim_{m\to\infty} g_m$. Since $\Phi_{q,p}(g) : L^p(X,\mathbb{F}) \longrightarrow \mathbb{F}$ is continuous

$$\langle f, g\rangle = \Phi_{q,p}(g)(f) = \lim_{n\to\infty} \Phi_{q,p}(g)(f_n)$$

By Ex. L19-5 the map $ev_{f_n} : L^p(X,\mathbb{F})^\vee \longrightarrow \mathbb{F}$ is continuous, with respect to the operator norm topology on $L^p(X,\mathbb{F})^\vee$. Hence

$$\begin{aligned}\Phi_{q,p}(g)(f_n) &= ev_{f_n}\left(\Phi_{q,p}(g)\right)\\ &= ev_{f_n}\left(\lim_{m\to\infty}\Phi_{q,p}(g_m)\right)\\ &= \lim_{m\to\infty} ev_{f_n}\left(\Phi_{q,p}(g_m)\right)\\ &= \lim_{m\to\infty}\Phi_{q,p}(g_m)(f_n).\end{aligned}$$

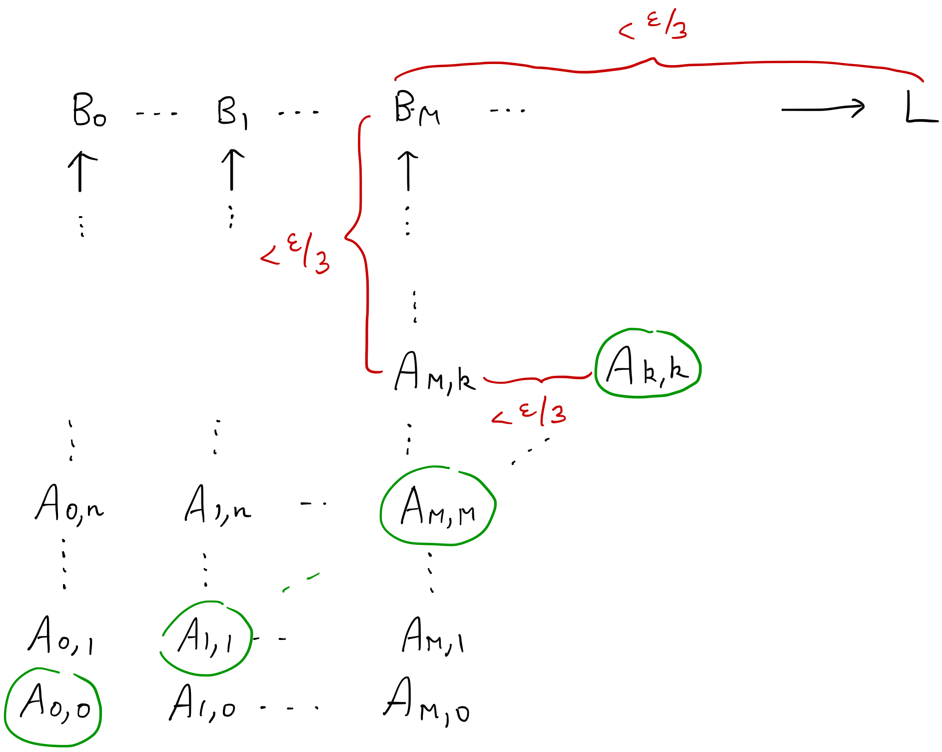
combining these calculations gives the first equality and performing them in the other order gives the second. $\square$

<u>Lemma L20-11</u> Let $(A_{m,n})_{m,n=0}^{\infty}$ be a doubly-indexed set of scalars in $\mathbb{F}$, such that $\lim_{m\to\infty}\lim_{n\to\infty} A_{m,n} = L$ and the convergence $A_{m,n} \longrightarrow \lim_{n\to\infty} A_{m,n}$ is uniform in m in the following sense:

$$\forall \varepsilon>0\ \exists K\ \forall m,m',n \geq K\left(\,|A_{m,n}-A_{m',n}|<\varepsilon\right).$$

Then $\lim_{n\to\infty} A_{n,n} = L$.

<u>Proof</u> Set $B_m = \lim_{n\to\infty} A_{m,n}$ and given $\varepsilon>0$ let M be sufficiently large that both $|B_m - L| < \varepsilon/3$ for $m \geq M$ and for $m,m',n \geq M$ we have $|A_{m,n} - A_{m',n}| < \varepsilon/3$. Since $A_{M,n} \longrightarrow B_M$ there is $M' \geq M$ with $|A_{M,k} - B_M| < \varepsilon/3$ for $k \geq M'$. The situation is as shown below:



Then for $k \geq M'$

$$
\begin{aligned}
|A_{k,k} - L| &\leq |A_{k,k} - A_{M,k}| + |A_{M,k} - B_M| + |B_M - L| \\
&< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon
\end{aligned}
$$

as claimed. $\square$

In the situation of Lemma L20-10:

<u>Lemma L20-12</u> $\left\langle \lim_{n\to\infty} f_n, \lim_{n\to\infty} g_n \right\rangle = \lim_{n\to\infty} \int_X f_n \overline{g_n}$.

<u>Proof</u> Set $A_{m,n} = \int_X f_n \overline{g_m} = \Phi_{q,p}(g_m)(f_n)$. We have to check the uniformity hypothesis of Lemma L20-11. Since $(\Phi_{q,p}(g_m))_{m=0}^{\infty}$ is Cauchy in $L^p(X,\mathbb{F})^\vee$ with respect to the operator norm, given $\delta > 0$ we can find K s.t. for $m, m' \geq K$

$$\| \Phi_{q,p}(g_m) - \Phi_{q,p}(g_{m'}) \| < \delta$$

Thus for any n, we have, provided $\|f_n\|_p \neq 0$,

$$|A_{m,n} - A_{m',n}| = |\Phi_{q,p}(g_m)(f_n) - \Phi_{q,p}(g_{m'})(f_n)| < \delta \|f_n\|_p$$

Set $f = \lim_{n\to\infty} f_n$, $g = \lim_{m\to\infty} g$. If $\|f\|_p = 0$ then $f = 0$ so by linearity $\langle f, g\rangle = 0$, and by Hölder's inequality

$$\begin{aligned} \left|\int_X f_n \overline{g_n}\right| &\leq \int_X |f_n \overline{g_n}| \\ &= \|f_n \overline{g_n}\|_1 \\ &\leq \|f_n\|_p \|g_n\|_q \end{aligned}$$

Hence by continuity of the norm $\lim_{n\to\infty} \int_X f_n \overline{g_n} = 0$ also. So we may assume $\|f\|_p \neq 0$. Now let $\varepsilon > 0$ be given and find K such that for $n \geq K$ we have $\|f_n\|_p < \|f\|_p + \varepsilon$. Then take $\delta = \varepsilon / \|f\|_p + \varepsilon$ in the above, and increase K if necessary, so that for $m, m', n \geq K$

$$|A_{m,n} - A_{m',n}| < \delta \|f_n\|_p = \varepsilon \cdot \frac{\|f_n\|_p}{\|f\|_p + \varepsilon} < \varepsilon. \quad \square$$

<u>Theorem L20-13</u> For any integral pair $(X, \int_X)$ the tuple $(L^2(X,\mathbb{F}), \langle , \rangle)$ is a Hilbert space with associated normed space $(L^2(X,\mathbb{F}), \|-\|_2)$, where the pairing is

$$\langle \lim_{n\to\infty} f_n, \lim_{n\to\infty} g_n \rangle = \lim_{n\to\infty} \int_X f_n \overline{g_n}. \qquad (17.1)$$

<u>Proof</u> Axioms (I1), (I2) follow from conjugate linearity of $\Phi_{2,2}$. By Lemma L20-12 the given formula (17.1) agrees with $\Phi_{2,2}(g)(f)$. For (I3) we compute

$$\overline{\langle g, f\rangle} = \overline{\lim_{n\to\infty}\int_X g_n \overline{f_n}}$$

$$= \lim_{n\to\infty} \overline{\int_X g_n \overline{f_n}} \qquad \text{(conjugation is continuous)}$$

$$= \lim_{n\to\infty} \int_X \overline{g_n \overline{f_n}} \qquad \text{(def}^{\text{N}}\text{ of complexified integral)}$$

$$= \lim_{n\to\infty} \int_X \overline{g_n} f_n$$

$$= \langle f, g\rangle.$$

For this calculation we could have just as well used the original "double limit" presentation of $\langle f,g\rangle$, but for the next step we genuinely need the result of Lemma L20-12. We have

$$\langle f, f\rangle = \lim_{n\to\infty}\int_X f_n \overline{f_n} \qquad \text{(Lemma L20-12)}$$

$$= \lim_{n\to\infty}\int_X |f_n|^2$$

$$= \lim_{n\to\infty} \| f_n \|_2^2$$

$$= \| f \|_2^2 \qquad \text{(continuity of } (-)^2, \|-\|_2)$$

Since we already know $(L^2(X,\mathbb{F}), \|-\|_2)$ is a normed space, this proves (I4), (I5) so $\langle,\rangle$ defines an inner product space. Moreover we have just shown the underlying norm is $\|-\|_2$ which is complete by construction, so $(L^2(X,\mathbb{F}), \langle,\rangle)$ is a Hilbert space. □

<u>Corollary L20-14</u> The function

$$\Phi_{2,2} : L^2(X,\mathbb{F}) \longrightarrow \overline{L^2(X,\mathbb{F})}^{\vee}$$

$$\Phi_{2,2}(g) = \langle -, g \rangle$$

is an isomorphism of normed spaces.

<u>Proof</u> Immediate from Corollary L20-9 and Theorem L20-13. □

Why is it useful to know that Hilbert spaces in general, and L^2-spaces in particular, are self-dual? Because it is generally easier to construct <u>functionals</u> (i.e. elements of $H^\vee$) than <u>vectors</u> (i.e. elements of H). One important application of this principle is the construction of adjoints, but here we will use the idea to give a "friendlier face" to the vectors of $L^2(X,\mathbb{F})$ (which up till now were just abstract Cauchy sequences).

By the universal property of the completion of a normed space we know that any continuous linear $\theta : Cts(X,\mathbb{F}) \longrightarrow \mathbb{F}$ (with respect to $\|-\|_2$ on the domain) extends uniquely to a continuous linear $\Theta : L^2(X,\mathbb{F}) \longrightarrow \mathbb{F}$ (Theorem L18-9), as in:

$$\begin{tikzcd} L^2(X,\mathbb{F}) \arrow[r, "\Theta"] & \mathbb{F} \\ Cts(X,\mathbb{F}) \arrow[u, hook, "\iota"] \arrow[ur, "\theta"'] & \end{tikzcd} \qquad \begin{tikzcd} L^2(X,\mathbb{F})^\vee \arrow[d, "\iota^\vee", "\cong"'] & \ni \Theta \arrow[d, mapsto] \\ Cts(X,\mathbb{F})^\vee & \ni \Theta \circ \iota = \theta \end{tikzcd}$$

With a little extra checking, this shows $\iota^\vee$ is an isomorphism of normed spaces

$$\iota^\vee : L^2(X,\mathbb{F})^\vee \xrightarrow{\ \cong\ } Cts(X,\mathbb{F})^\vee$$

Combined with Corollary L20-14 we have an isomorphism of normed spaces

$$L^2(X,\mathbb{F}) \xrightarrow[\cong]{\Phi_{2,2}} \overline{L^2(X,\mathbb{F})}^{\vee} \xrightarrow[\cong]{\iota^{\vee}} \overline{\operatorname{Cts}(X,\mathbb{F})}^{\vee}$$

$$g \longmapsto \langle -, g\rangle \longmapsto \langle -, g\rangle\big|_{\operatorname{Cts}(X,\mathbb{F})}$$

Spelled out explicitly, this says that for every continuous linear $\mathcal{O}: \operatorname{Cts}(X,\mathbb{F}) \longrightarrow \mathbb{F}$ (with respect to $\|-\|_2$) there is a Cauchy sequence $(g_m)_{m=0}^{\infty}$ in $\operatorname{Cts}(X,\mathbb{F})$ with

$$\mathcal{O}(f) = \lim_{m\to\infty} \langle f, g_m\rangle = \lim_{m\to\infty} \int_X f\,\overline{g_m} \qquad \forall f \in \operatorname{Cts}(X,\mathbb{F})$$

Moreover the equivalence class of this Cauchy sequence is <u>unique</u>, and we may denote it $g_{\mathcal{O}} \in L^2(X,\mathbb{F})$. So if we can construct interesting $\mathcal{O}$'s, we can get interesting vectors in $L^2(X,\mathbb{F})$. One obvious supply of $\mathcal{O}$'s is integrating against a continuous function: given $g \in \operatorname{Cts}(X,\mathbb{F})$

$$\mathcal{O}_g(f) = \int_X f\,\overline{g} \quad \Longrightarrow \quad g_{\mathcal{O}_g} = g \in L^2(X,\mathbb{F}).$$

This doesn't tell us anything, but it suggests a means of constructing more interesting examples:

<u>Lemma L20-15</u> Suppose $g: [a,b] \longrightarrow \mathbb{F}$ is a function which is Riemann integrable on $[a,b]$. Then with $X = [a,b]$

$$\mathcal{O}_g : \operatorname{Cts}(X,\mathbb{F}) \longrightarrow \mathbb{F}, \quad \mathcal{O}_g(f) = \int_{[a,b]} f\,\overline{g}$$

is continuous and linear, i.e. $\mathcal{O}_g \in \operatorname{Cts}(X,\mathbb{F})^{\vee}$ (with respect to $\|-\|_2$).

<u>Proof</u> Linearity is a basic property of the integral. Continuity with respect to $\|\cdot\|_2$ follows from the Hölder inequality (the proof of which goes through in the present case, with g Riemann integrable but not necessarily continuous) since

$$\begin{aligned} \left| \int f\bar{g} - \int f'\bar{g} \right| &= \left| \int (f-f')\bar{g} \right| \\ &\leq \int |(f-f')\bar{g}| \\ &\leq \|f-f'\|_2 \|g\|_2 \end{aligned}$$

$\left[\|g\|_2 = \left\{ \int_{[a,b]} |g|^2 \right\}^{1/2} \right]$

In fact this shows $\mathcal{O}_g$ is bounded, and $\|\mathcal{O}_g\| \leq \|g\|_2$. $\square$

Let $\hat{g}$ denote the representing element for $\mathcal{O}_g$ in $L^2(X, \mathbb{F})$, so that for $f \in Cts(X, \mathbb{F})$

$$\langle f, \hat{g} \rangle = \int_{[a,b]} f\bar{g}.$$

pairing in $L^2(X,\mathbb{F})$ Riemann integral of a non-continuous f^n

This defines a function $g \longmapsto \hat{g}$ from integrable functions to $L^2(X, \mathbb{F})$, which is just the inclusion of $Cts(X, \mathbb{F})$ when restricted to continuous functions.

<u>Exercise L20-7</u> Prove that $\|\hat{g}\|_2 = \int_{[a,b]} |g|^2$.

One is therefore tempted to think of integrable functions as a <u>subset</u> of $L^2(X, \mathbb{F})$, but :

<u>Def^n</u> A Riemann integrable function $g: [a,b] \longrightarrow \mathbb{R}$ is <u>zero almost everywhere</u> (or zero a.e.) if $\int_{[a,b]} fg = 0$ for all $f \in Cts(X, \mathbb{R})$. Two Riemann integrable functions g, g' are <u>equal almost everywhere</u> if $g - g'$ is zero a.e. We extend these def^{n}s to complex-valued functions in the obvious way, i.e. $g: [a,b] \longrightarrow \mathbb{C}$ is zero a.e. iff. $\mathrm{Re}(g)$, $\mathrm{Im}(g)$ are zero a.e.

There is another characterisation of "almost everywhere" in terms of sets of measure zero, but that is beyond the scope of this course.

<u>Lemma L20-16</u> The kernel of the linear map

$$\begin{aligned} \{\text{integrable functions } [a,b] \to \mathbb{F}\} &\longrightarrow L^2([a,b],\mathbb{F}) \\ g &\longmapsto \hat{g} \end{aligned}$$

is the set of those g which are zero almost everywhere.

<u>Proof</u> By definition $\hat{g} = 0$ in $L^2(X,\mathbb{F})$ if and only if $\int_{[a,b]} f\bar{g} = 0$ for all f continuous, which means g is zero a.e. $\square$

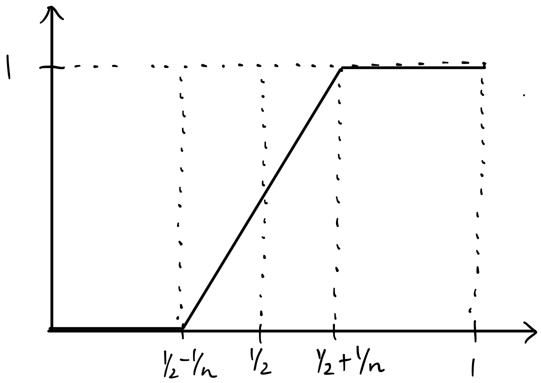
<u>Example L20-6</u> The function $g : [a,b] \longrightarrow \mathbb{R}$, $g(a) = 1$, $g(x) = 0$ for $x > a$ is Riemann integrable but clearly $\int_{[a,b]} fg = 0$ for all continuous f, since $fg = f(a)\cdot g$, and $\int_{[a,b]} g = 0$. So g is a.e. zero and hence $\hat{g} = 0$.

In conclusion, we have a diagram of injective linear maps $(X = [a,b])$

$$\begin{tikzcd} & L^2(X,\mathbb{F}) & \\ Cts(X,\mathbb{F}) \arrow[ur, hook, "\iota"] \arrow[rr, hook] & & \{\text{Integrable } g : X \to \mathbb{F}\}/\text{a.e. equality} \arrow[ul, "g \mapsto \hat{g}"'] \end{tikzcd}$$

However not every element of $L^2(X,\mathbb{F})$ can be obtained as $\hat{g}$! The notion of Riemann integrability is artifically restrictive, the correct notion is <u>Lebesgue integrability</u>, and it is true that every vector in $L^2(X,\mathbb{F})$ represents a Lebesgue integrable function.

Example L20-7 In Example L18-2 we considered $X = [0,1]$ and the sequence of functions $f_n : X \longrightarrow \mathbb{R}$ given for $n \geq 4$ by



$$f_n(x) = \begin{cases} 0 & 0 \leq x \leq 1/2 - 1/n \\ n/2\,(x - 1/2 + 1/n) & 1/2 - 1/n < x < 1/2 + 1/n \\ 1 & 1/2 + 1/n \leq x \leq 1 \end{cases}$$

$$f(x) = \begin{cases} 1 & x > 1/2 \\ 1/2 & x = 1/2 \\ 0 & x < 1/2 \end{cases}$$

The function f is not continuous, but it is Riemann integrable, and we claim

$$\hat{f} = (f_n)_{n=0}^{\infty} \text{ in } L^2(X, \mathbb{R}).$$

But by Ex. L20-7

$$\| \hat{f} - f_n \|_2^2 = \langle \hat{f} - f_n, \hat{f} - f_n \rangle$$

$$= \int_X |f|^2 + \| f_n \|^2 - 2\langle f_n, \hat{f} \rangle$$

$$= \tfrac{1}{2} + \int_0^1 f_n(x)^2\, dx - 2\int_0^1 f_n(x) f(x)\, dx$$

$$= \tfrac{1}{2} + \int_{1/2 - 1/n}^{1/2 + 1/n} \frac{n^2}{4}\left(x - \tfrac{1}{2} + \tfrac{1}{n}\right)^2 dx + \left(1 - \tfrac{1}{2} - \tfrac{1}{n}\right)$$

$$- 2\int_{1/2}^{1/2 + 1/n} \frac{n}{2}\left(x - \tfrac{1}{2} + \tfrac{1}{n}\right) dx - 2\cdot\left(1 - \tfrac{1}{2} - \tfrac{1}{n}\right)$$

$$= \frac{1}{n} + \frac{n^2}{4}\left[\tfrac{1}{3}\left(x - \tfrac{1}{2} + \tfrac{1}{n}\right)^3 \right]_{1/2 - 1/n}^{1/2 + 1/n} - n\left[\tfrac{1}{2}\left(x - \tfrac{1}{2} + \tfrac{1}{n}\right)^2 \right]_{1/2}^{1/2 + 1/n}$$

$$= \frac{1}{n} + \frac{n^2}{12}\left(\frac{2}{n}\right)^3 - \frac{n}{2}\left(\left[\frac{2}{n}\right]^2 - \left[\frac{1}{n}\right]^2\right)$$

$$= \frac{1}{n} + \frac{2}{3}\cdot\frac{1}{n} - \frac{n}{2}\cdot\frac{3}{n^2}$$

$$= 1/6n$$

This converges to zero, proving the claim.

Indeed this example suggests the correct general strategy for constructing a Cauchy sequence in $\mathrm{Cts}([a,b],\mathbb{R})$ giving the representative element $\hat{g}$ in $L^2([a,b],\mathbb{R})$ representing a general piecewise-continuous function g on $[a,b]$.

<u>Example L20-8</u> Let $(S^1, \int_{S^1})$ be the integral pair of Example L17-2, so $S^1 = [0,2\pi]/\sim$ and $\int_{S^1}$ is ($\rho : [0,2\pi] \longrightarrow S^1$ being the quotient)

$$\mathrm{Cts}(S^1,\mathbb{R}) \xrightarrow{(-)\circ\rho} \mathrm{Cts}([0,2\pi],\mathbb{R}) \xrightarrow{\int_{[0,2\pi]}} \mathbb{R}$$

We have shown there is an isomorphism

$$\begin{aligned} L^2(S^1,\mathbb{F}) &\xrightarrow{\cong} \overline{L^2(S^1,\mathbb{F})^\vee} \\ g &\longmapsto \langle -, g\rangle. \end{aligned}$$

It follows from Lemma L20-15 that if $g : [0,2\pi] \longrightarrow \mathbb{F}$ is integrable then the following function is continuous and linear

$$\mathcal{O}_g : \mathrm{Cts}(S^1,\mathbb{F}) \longrightarrow \mathbb{F}$$

$$\mathcal{O}_g(f) = \int_{[0,2\pi]} (f\circ\rho)\cdot\bar{g}$$

and is therefore represented by a unique $\hat{g} \in L^2(S^1; \mathbb{F})$ with the property that $\langle f, \hat{g} \rangle = \int_{[0,2\pi]} (f \circ \rho)\bar{g}$ for all $f \in Cts(S^1; \mathbb{F})$.
Note that while <u>continuous</u> functions on S^1 are in bijection with periodic functions on $\mathbb{R}$, an integrable function on S^1 does not need to care about "matching" at the glueing site: we think of g as being an integrable function on S^1.

Solutions to selected exercises

$\boxed{L20\text{-}5}$ We give $V \times V$ the product metric. Then

$$
\begin{aligned}
|\langle a_1, b_1\rangle - \langle a_2, b_2\rangle| &\leq |\langle a_1, b_1\rangle - \langle a_2, b_1\rangle \\
&\qquad + \langle a_2, b_1\rangle - \langle a_2, b_2\rangle| \\
&\leq |\langle a_1, b_1\rangle - \langle a_2, b_1\rangle| + |\langle a_2, b_1\rangle - \langle a_2, b_2\rangle| \\
&= |\langle a_1 - a_2, b_1\rangle| + |\langle a_2, b_1 - b_2\rangle| \\
&\leq \|a_1 - a_2\| \cdot \|b_1\| + \|a_2\| \, \|b_1 - b_2\| \\
&\leq \|a_1 - a_2\| \cdot \|b_1\| + \|a_2 - a_1 + a_1\| \cdot \|b_1 - b_2\| \\
&\leq \|a_1 - a_2\| \cdot \|b_1\| + \|a_1 - a_2\| \cdot \|b_1 - b_2\| \\
&\qquad + \|a_1\| \cdot \|b_1 - b_2\|
\end{aligned}
$$

Let us prove $\langle , \rangle$ is continuous at (a_1, b_1). Let $\varepsilon > 0$ be given, find $\delta > 0$ such that $|w| < \delta$, $|z| < \delta$ implies $|wz| < \varepsilon/3$ with also

$$\delta < \varepsilon / 3\|b_1\| \quad , \quad \delta < \varepsilon / 3\|a_1\|.$$

Then $\|a_1 - a_2\| + \|b_1 - b_2\| < \delta$ implies

$$
\begin{aligned}
|\langle a_1, b_1\rangle - \langle a_2, b_2\rangle| &< \delta \cdot \|b_1\| + \varepsilon/3 + \delta \cdot \|a_1\| \\
&< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon. \quad \square
\end{aligned}
$$